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On the robust solution of Convection-Diffusion Equation coupled with Stokes equation

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1 Introduction

The dynamics of concentration of small particles ruled by external velocity flow field plays a significant role for research and applications in natural science and industry. A big variety of such problems are described mathematically as convection-diffusion equations coupled with Stokes equation.

Convection-dominated is the most common case in such problems. Standard numerical schemes for simulation of concentration lead to spurious oscillations and instabilities of the discrete approximation. A commonly used for such problems SUPG (streamline-upwind Petrov-Galerkin) method is compared with more recently developed EAFE (edge-averaged finite element) scheme. For the latter method we also derive error analysis for non-stationary case.

In section 2 we introduce a mathematical description of the problem. Two ways of constructing a robust numerical solution of the state equation are discussed in Section 3. Section 4 is devoted to corresponding numerical experiments.

2 Mathematical model

2.1 Problem statement

In the simple, as well as rather general case in the applications the velocity flow field satisfies a Stokes equation:

$$\begin{cases} \triangle \vec{V} - \nabla p = 0, & \text{in } \Omega \\ \nabla \cdot \vec{V} = 0, & \text{in } \Omega \\ \vec{V} = \vec{g}(\cdot,\, u(t)), & \text{on } \partial\Omega \end{cases} \tag{1}$$

Here V is a velocity, p — pressure, $\vec{g}(x)$ is some prescribed boundary velocity. One can see that velocity flow $\vec{V}(\cdot) = \vec{V}(\cdot, u(t))$ at any given time t is a function of time-dependant control parameters $u = u(t)$, due to its Dirichlet boundary conditions $\vec{g}(x, u)$.

We have a convection-diffusion equation for the concentration of particles $c(x,t)$ with a diffusion coefficient ε and velocity flow $\vec{V}$ (which satisfies (1)):

$$\begin{cases} \frac{\partial}{\partial t}c + \vec{V}\cdot\nabla c = \varepsilon\triangle c, & (x,t)\in\Omega\times(0,T] \\ c(x,0) = c_0(x), & x\in\Omega \\ \big(\vec{V}c - \varepsilon\nabla c\big)\cdot\vec{n} = 0, & (x,t)\in\partial\Omega\times(0,T] \end{cases} \tag{2}$$

We can write out time-space variational formulation of the problem (2). For this we introduce $W = L^2\big(0,T;H^1(\Omega)\big)$, $X = \{\phi\in W : \phi_t \in W^*\}$, $Q = \Omega\times(0,T)$. By definition $\phi\in W$ means that $\|\phi(t)\|_{H^1(\Omega)}\in L^2(0,T)$. To ensure existence and boundedness of all items in the time-space variational formulation, we assume

$$c(x,t)\in X, \quad \varphi(x,t)\in W.$$

Using divergence theorem and incompressibility of vector flow $\vec{V}$ (i.e. $\nabla\cdot\vec{V} = 0$), for any $\varphi\in W$ we obtain:

$$\begin{aligned} 0 &= \langle \tfrac{\partial}{\partial t}c,\varphi\rangle_{W^*,W} + \langle \vec{V}\cdot\nabla c - \varepsilon\triangle c,\varphi\rangle_{L^2(Q)} \\ &= \langle \tfrac{\partial}{\partial t}c,\varphi\rangle_{W^*,W} + \langle \nabla\cdot(\vec{V}c - \varepsilon\nabla c),\varphi\rangle_{L^2(Q)} = \\ &= \langle \tfrac{\partial}{\partial t}c,\varphi\rangle_{W^*,W} + \langle \varphi,(\vec{V}c - \varepsilon\nabla c)\cdot\vec{n}\rangle_{L^2(\partial\Omega\times(0,T))} - \langle \vec{V}c - \varepsilon\nabla c,\nabla\varphi\rangle_{L^2(Q)} \end{aligned}$$

$$\Rightarrow \begin{cases} \langle \frac{\partial}{\partial t}c,\varphi\rangle_{W^*,W} - \big\langle \vec{V}c - \varepsilon\nabla c,\nabla\varphi\big\rangle_{L^2(Q)} = 0 \\ c(0) = c_0 \end{cases} \tag{3}$$

2.2 Properties of Stokes equation

We formulate a proposition given in [1] for Stokes system (1). We define

$$L^2_0(\partial\Omega) = \{\vec{f}\in L^2(\partial\Omega)\colon \int_{\partial\Omega}\vec{f}\cdot\vec{n} = 0\}.$$

Proposition 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\vec{g}\in L^2_0(\partial\Omega)$. Then system (1) admits a unique solution $(\vec{V},p)\in\big[H^{1/2}(\Omega)\cap C^\infty(\Omega)\big]\times C^\infty(\Omega)$ and there exists a constant $C_\Omega > 0$ s.t.*

$$\|\vec{V}\|_{H^{1/2}(\Omega)} \le C_\Omega\|\vec{g}\|_{L^2(\partial\Omega)}. \tag{4}$$

The domain Ω of the problem (1)-(2) is shown on the Figure 1.

2.3 Existence, uniqueness and regularity of solution in the convection-diffusion problem

Due to analytical form of $\vec{V}(x,u)$ that will be shown in the last section, for any uniformly bounded control: $\operatorname{ess}\sup_{t\in[0,T]}|u(t)|\le C_u$, velocity flow is uniformly bounded: $\|\vec{V}(t)\|_{C^\infty(\Omega)}\le\tilde{C}_u$, $\forall t\ge 0$. In consequence of this property, henceforth we will assume $u_1(t),u_2(t)\in L^\infty[0,T]$.

Now consider a backward Euler time discretization of variational system (3):

$$\begin{cases} \langle\hat{c}(t_k) - \hat{c}(t_{k-1}),\varphi\rangle_{L^2(\Omega)} - \Delta t\big\langle\vec{V}(t_k)\hat{c}(t_k) - \varepsilon\nabla\hat{c}(t_k),\nabla\varphi\big\rangle_{L^2(\Omega)} = 0 \\ \hat{c}(t_0) = c_0 \end{cases} \tag{5}$$

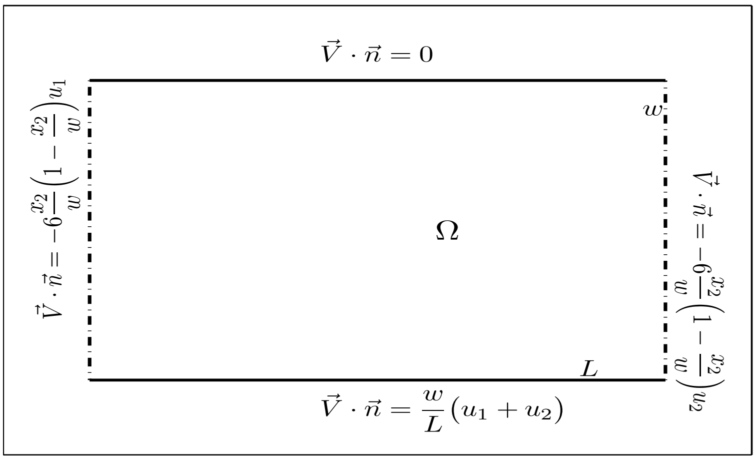

Figure 1: Domain Ω of the system and boundary conditions

for any $\varphi \in H^1(\Omega)$, where $\Delta t = \frac{T}{m}$, $t_k = k\Delta t$, $k = 1, \dots, m$.

Since in the numerical implementation we are considering this scheme, to ensure its approximation property we need the following

Assumption 1. *For $m \to \infty \Leftrightarrow \Delta t \to 0$ discrete approximation $\hat{c}(x, t_k)$ (i.e. solution of (5)) converges to its continuous analogue $c(x, t)$ (as solution of (3)) at every time step t_k, $k = 1, \dots, m$:*

$$\|\hat{c}(t_k) - c(t_k)\|_{H^1(\Omega)} \to 0.$$

For any fixed k, (5) can be regarded as convection-diffusion-reaction equation (in variational form) for $\hat{c}(t_k)$ with right hand side $\hat{c}(t_{k-1})$. Control functions $u_1(t), u_2(t) \in L^\infty[0, T]$, as a result, are approximated by piecewise-constant (in time) functions on a finite time grid $0 = t_0 < t_1 < \ldots < t_m = T$.

C^∞-boundedness of $\vec{V}(\cdot, t_k)$ ensures the following monotonicity property mentioned in [2]:

- if $\hat{c}(t_{k-1}) \geq 0$ on Ω, then $\hat{c}(t_k) \geq 0$ on Ω.

This implies, together with Assumption 1, the following result.

Proposition 2.

1. *Consider initial condition $c_0 \geq 0$, $\forall x \in \Omega$ ($c_0 \in H^1(\Omega)$). Then there exists non-negative solution $c(x, t) \geq 0$, $\forall x \in \Omega$, $t \geq 0$.*

2. *The problem (5) of finding $\hat{c}(t_k) \in H^1(\Omega)$ is well-posed and uniquely solvable.*

Proof.

1. It follows from monotonicity and approximation properties of semi-discrete solution $\hat{c}(x, t_k)$.

2. Since mass conservation property $\int_\Omega c(x, t)\, dx = \int_\Omega c_0(x)\, dx$, $t \geq 0$, holds, non-negativeness of $c(x, t)$ yields the uniqueness of solution of considered problem (3): for initial condition $c_0 \equiv 0$ there exists only zero solution $c(x, t) \equiv 0$, $\forall x \in \Omega$, $t \geq 0$.

Well-posedness of solution of (5) follows from Lemma 6.2 in [2].

□

Taking into account these properties, we can formulate the following result about regularity of variational system (3).

Proposition 3. *For any control function $u(t) \in L^\infty[0,T]$ the problem (3) of finding $c(\cdot,\cdot) \in X$ is uniquely solvable. The problem (5) of finding its approximation $\hat{c}(x,t)$ on any fixed time grid $\{t_k\}_{1\le k\le m}$ is well-posed.*

3 Numerical solution of convection-diffusion equation

3.1 The SUPG method

Let $\mathcal{T}_h = \{T_e\}_e$ be a nondegenerate shape-regular subdivision of Ω. As usual, we denote h_e to be the diameter of element Ω_e and h the maximum diameter of elements in $\mathcal{T}_h$.

Let $p \ge 2$ be a positive integer. Define the finite element approximating space

$$V_h = \{v_h \in H^1(\Omega) : \forall T_e \in \mathcal{T}_h(\Omega), v_h|_{T_e} \in \mathbb{P}^p(T_e)\},$$

where $\mathbb{P}^p(T_e)$ is the set of polynomials of total degree less than or equal to p. Let $(\cdot,\cdot)_{T_e}$ denote the L^2 inner-product over $T_e \in \mathcal{T}_h$.

Finite element formulation for convection-diffusion equation (3) consists of seeking continuous in time $c_h(t) \in V_h$, satisfying $\forall v \in V_h$:

$$\begin{cases} (\partial_t c_h, v)_\Omega - (\vec{V}(t)c_h - \varepsilon\nabla c_h, \nabla v)_\Omega = 0, \quad t \in [0,T] \\ (c_h(0), v)_\Omega = (c_0, v)_\Omega. \end{cases} \tag{6}$$

Here $(\cdot,\cdot)_\Omega = \sum\limits_{T_e\in\mathcal{T}_h} (\cdot,\cdot)_{T_e}$.

We consider a backward Euler in time discretization of finite element formulations above. For $c_h, v \in V_h$, define the bilinear form

$$a_{t_k}(c_h, v) = -\Delta t \cdot (\vec{V}(t_k)c_h - \varepsilon\nabla c_h, \nabla v)_\Omega.$$

Thus, (6) converts to the following scheme:

$$\begin{cases} \big(c_h(t_k) - c_h(t_{k-1}), v\big)_\Omega + a_{t_k}(c_h(t_k), v) = 0, \\ (c_h(0), v)_\Omega = (c_0, v)_\Omega \end{cases} \tag{7}$$

for $k = 1, \dots, m$, where $t_k = k\Delta t$, $\Delta t = \frac{T}{m}$.

It is well known that in the case of dominant convection, standard Galerkin finite element formulation (7) becomes instable and its solution shows spurious oscillations in the whole domain.

One of possible ways out of this problem is to use a streamline-upwind Petrov-Galerkin (SUPG) method, which is designed for convection-dominated problems. SUPG method adds a consistent diffusion term in streamline direction

$$\sum_{\Omega_e\in\mathcal{T}_h} \tau_e \Delta t\big(\text{"residual"}, \vec{V}\cdot\nabla v\big)_{\Omega_e}. \tag{8}$$

Here $\{\tau_e\}_e$ is a set of parameters depending on the mesh cells $T_e \in \mathcal{T}_h$, and the residual is the difference of the left and right hand side of backward Euler scheme:

$$\text{"residual"} = c_h(t_k) - c_h(t_{k-1}) + \Delta t\big(-\varepsilon \triangle c_h(t_k) + \vec{V} \cdot \nabla c_h(t_k)\big).$$

Inserting (8) into (7) leads to

$$\begin{cases} \big(c_h(t_k) - c_h(t_{k-1}), v\big)_\Omega + a_{t_k}(c_h(t_k), v) + \sum\limits_{T_e \in \mathcal{T}_h} \tau_e \Delta t\big(c_h(t_k) - c_h(t_{k-1}), \vec{V} \cdot \nabla v\big)_{T_e} \\ + \sum\limits_{T_e \in \mathcal{T}_h} \tau_e (\Delta t)^2 \big(-\varepsilon \Delta c_h(t_k) + \vec{V} \cdot \nabla c_h(t_k), \vec{V} \cdot \nabla v\big)_{T_e} = 0, \\ (c_h(0), v)_\Omega = (c_0, v)_\Omega \end{cases} \tag{9}$$

The crucial question in the application of the SUPG stabilization is the choice of parameters $\{\tau_e\}_e$. Among different proposals of parameters presented in [3] we have used the most convenient one for numerical computation:

$$\tau_e^{\text{Cod}} = \frac{h_e^2}{4 \triangle t \varepsilon + 2 h_e \Delta t \|\vec{V}\|_2 + h_e^2}.$$

Here $\|\vec{V}\|_2$ is Euclidean norm of a vector $\vec{V} \in \mathbb{R}^2$, h_e is a size of the mesh cell Ω_e (in our problem it is chosen as a size of the mesh cell in the direction of convection, as it is recommended in [3]).

It is also mentioned in [3] that the coupling of the SUPG method to implicit time stepping schemes leads to a stable discretization, regardless of the length of the time step, but spurious oscillations may be expected for small time steps.

3.2 EAFE scheme

To formulate edge-averaged finite element scheme, we follow the basic article [2]. We introduce notations for shape regular triangulation $\mathcal{T}_h$: for a given $T_e \in \mathcal{T}_h$

- q_j $(1 \le j \le 3)$: the vertices of T_e;
- E_{ij} or simply E: the edge connecting two vertices q_i and q_j;
- $\delta_E \phi = \phi(q_i) - \phi(q_j)$, for any continuous function ϕ on $E = E_{ij}$;
- $\tau_E = \delta_E x = q_i - q_j$, a directional vector of E.

Defining stiffness coefficients $(a_{ij}^{T_e})$ for any given $T_e \in \mathcal{T}_h$:

$$\forall c_h, v_h \in V_h : \int_{T_e} \nabla c_h \cdot \nabla v_h \, dx = \sum_{i,j} a_{ij}^{T_e} c_h(q_i) v_h(q_j),$$

we obtain the identity

$$\int_{T_e} \nabla c_h \cdot \nabla v_h \, dx = -\sum_{i<j} a_{ij}^{T_e} (c_h(q_i) - c_h(q_j))(v_h(q_i) - v_h(q_j)).$$

It yields the following expression for bilinear form

$$\int_\Omega \nabla c_h \cdot \nabla v_h \, dx = \sum_{T_e \in \mathcal{T}_h} \sum_{E \subset T_e} \omega_E^{T_e} \delta_E c_h \, \delta_E v_h, \tag{10}$$

where $\omega_E^{T_e} = -a_{ij}^{T_e}$ with E connecting the vertices q_i and q_j.

We return to backward Euler time discretized analog (5). It represents convection-diffusion-reaction system with respect to $\hat{c}(t_k)$ (for a fixed k). For convenience we multiply its both sides by Δt so that it has a right-hand side $\langle \hat{c}(t_{k-1}), \varphi \rangle_{L^2(\Omega)}$.

Given $T_e \in \mathcal{T}_h$ and an edge $E \in T_e$, we define a function $\psi_E^{T_e}$ by

$$\frac{\partial \psi_E^{T_e}}{\partial \tau_E} = -\frac{1}{|\tau_E|}\varepsilon^{-1}(\vec{V} \cdot \tau_E).$$

Let $\tilde{\varepsilon}_E^{T_e}(\vec{V})$ be the corresponding harmonic average of $\varepsilon e^{-\psi_E^{T_e}}$:

$$\tilde{\varepsilon}_E^{T_e}(\vec{V}) = \Big[\frac{1}{|\tau_E|}\int_E \varepsilon^{-1} e^{\psi_E^{T_e}}\, ds\Big]^{-1}.$$

We set V_h to be the usual piecewise linear finite element space: $V_h \subset H^1(\Omega)$. First we approximate the flux $j(c) = \varepsilon \nabla c - \vec{V} c$ over each simplex T_e by a constant vector $j_{T_e}(c)$. Then from Lemma 3.1 in [2] we have that

$$j_{T_e}(c) \cdot \tau_E \approx \tilde{\varepsilon}_E^{T_e}(\vec{V})\delta_E(e^{\psi_E} c).$$

By (10) for any $v_h \in V_h$ we get

$$\int_{T_e} j_{T_e}(c)\cdot\nabla v_h\, dx = \sum_{E\subset T_e} \omega_E^{T_e}(j_{T_e}(c)\cdot\tau_E)\delta_E v_h \approx \sum_{E\subset T_e} \omega_E^{T_e}\tilde{\varepsilon}_E^{T_e}(\vec{V})\delta_E(e^{\psi_E} c)\delta_E v_h.$$

Thus the approximating bilinear form can be defined as

$$a_h(c_h, v_h) = \Delta t \sum_{E\in\mathcal{T}_h} \omega_E \tilde{\varepsilon}_E(\vec{V})\delta_E(e^{\psi_E} c_h)\delta_E v_h + (c_h, v_h)_\Omega.$$

Here $(\cdot,\cdot)_\Omega = \sum\limits_{T_e\in\mathcal{T}_h} (\cdot,\cdot)_{L^2(T_e)}$. According to continuity of $\vec{V}$ on Ω we also get $\tilde{\varepsilon}_E^{T_e} \equiv \tilde{\varepsilon}_E$.

For any time step t_k $(k = 1, \ldots, m)$ we have the resulting finite element scheme: find $c_h(t_k)$ such that

$$a_h[t_k](c_h(t_k), v_h) = (c_h(t_{k-1}), v_h)_\Omega, \tag{11}$$

where $a_h[t_k](c_h(t_k), v_h)$ corresponds to velocity flow field $\vec{V}(\cdot) = \vec{V}(\cdot, t_k)$. For convenience we will further use notation $a_h(c_h(t_k), v_h)$ assuming that a_h may depend on t_k.

It is shown in [2] that the considered finite element solution $c_h(t_k)$ satisfies already mentioned monotonicity property.

For convenience we introduce notations

$$\|\cdot\|_{1,\Omega}^2 = \sum_{T_e\in\mathcal{T}_h} \|\cdot\|_{H^1(T_e)}^2, \quad |\cdot|_{1,\Omega}^2 = \sum_{T_e\in\mathcal{T}_h} |\cdot|_{H^1(T_e)}^2, \quad \|\cdot\|_{0,\Omega}^2 = \sum_{T_e\in\mathcal{T}_h} \|\cdot\|_{L^2(T_e)}^2.$$

Denoting by c_I the nodal interpolation of $\hat{c}(x,t)$ (as a solution of (5)) at a given time moment t_k, we formulate the following convergence result for scheme (11).

Proposition 4. *Assume that for all $T_e \in \mathcal{T}_h$ we have $\hat{c} \in H^1(T_e)$, $j(\hat{c}) \equiv \varepsilon\nabla\hat{c} - \vec{V}\hat{c} \in [H^1(T_e)]^2$. Then the following estimate holds:*

$$\|c_I(t_k) - c_h(t_k)\|_{1,\Omega} \le Ch\Big\{\big((\Delta t)^2|j(\hat{c}(t_k))|^2_{1,\Omega} + |\hat{c}(t_k)|^2_{1,\Omega}\big)^{1/2} + |\hat{c}(t_{k-1})|_{1,\Omega}\Big\} + \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega}.$$

Proof. The estimates given below follow from Lemmata 6.1-6.2 in [2]:

$$|a(\hat{c}, v_h) - a_h(c_I, v_h)| \le C_1 h\Big\{(\Delta t)^2|j(\hat{c})|^2_{1,\Omega} + |\hat{c}|^2_{1,\Omega}\Big\}^{1/2}\|v_h\|_{1,\Omega}, \tag{12}$$

$$\sup_{v_h\in V_h} \frac{a_h(c_h, v_h)}{\|v_h\|_{1,\Omega}} \ge c_0\|c_h\|_{1,\Omega}, \quad \forall c_h \in V_h. \tag{13}$$

Here we are considering the bilinear form $a(\cdot,\cdot)$ that corresponds to the semi-discrete formulation (5):

$$a(\hat{c}, v_h) = \Delta t(\varepsilon\nabla\hat{c} - \vec{V}c, \nabla v_h)_\Omega + (c_h, v_h)_\Omega.$$

Since $a_h(c_h, v_h)$ has a term $(c_h, v_h)_\Omega$, we can estimate $c_0 \ge 1$. Using (12), (13) and an inverse estimate for the interpolation error $\hat{c}(t_{k-1}) - c_I(t_{k-1})$, we have

$$\begin{aligned}
&\|c_I(t_k) - c_h(t_k)\|_{1,\Omega} \le \frac{|a_h(c_I(t_k), v_h) - a_h(c_h(t_k), v_h)|}{c_0\|v_h\|_{1,\Omega}}\\
&\le \frac{1}{c_0\|v_h\|_{1,\Omega}}\big(|a(\hat{c}(t_k), v_h) - a_h(c_I(t_k), v_h)| + |a(\hat{c}(t_k), v_h) - a_h(c_h(t_k), v_h)|\big)\\
&= \frac{1}{c_0\|v_h\|_{1,\Omega}}\big(|a(\hat{c}(t_k), v_h) - a_h(c_I(t_k), v_h)| + |(\hat{c}(t_{k-1}) - c_h(t_{k-1}), v_h)_\Omega|\big)\\
&\le \frac{C_1}{c_0}h\Big\{(\Delta t)^2|j(\hat{c}(t_k))|^2_{1,\Omega} + |\hat{c}(t_k)|^2_{1,\Omega}\Big\}^{1/2}\\
&+ \frac{1}{c_0\|v_h\|_{1,\Omega}}\big(|(\hat{c}(t_{k-1}) - c_I(t_{k-1}), v_h)_\Omega| + |(c_I(t_{k-1}) - c_h(t_{k-1}), v_h)_\Omega|\big)\\
&\le C_2 h\Big\{(\Delta t)^2|j(\hat{c}(t_k))|^2_{1,\Omega} + |\hat{c}(t_k)|^2_{1,\Omega}\Big\}^{1/2}\\
&+ \|\hat{c}(t_{k-1}) - c_I(t_{k-1})\|_{0,\Omega} + \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega}\\
&\le C_2 h\Big\{(\Delta t)^2|j(\hat{c}(t_k))|^2_{1,\Omega} + |\hat{c}(t_k)|^2_{1,\Omega}\Big\}^{1/2}\\
&+ C_3 h|\hat{c}(t_{k-1})|_{1,\Omega} + \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega},
\end{aligned}$$

which completes the proof of the estimate after taking $C = \max(C_2, C_3)$. ☐

According to this result we can formulate a theorem about approximation error of the scheme (11) at any given time $t \in [0, T]$.

Theorem 1. *Under conditions of the previous proposition with additional consistency and approximation assumptions for initial conditions:*

$$\int_\Omega \hat{c}(x,0)\ dx = \int_\Omega c_h(x,0)\ dx, \quad \|c_I(0) - c_h(0)\|_{0,\Omega} \le \tilde{C}_0 h,$$

for any $t \in \{t_1, \dots, t_m\}$ *the following estimate holds:*

$$\|c_I(t) - c_h(t)\|_{1,\Omega} \le \tilde{C}h\Big\{(\Delta t) \max_{\tau\in[0,t]} |j(\hat{c}(\tau))|_{1,\Omega} + \max_{\tau\in[0,t]} |\hat{c}(\tau)|_{1,\Omega}\Big\} + \tilde{C}_0 h. \quad (14)$$

Proof. Provided by Proposition 4, it is sufficient to estimate properly the term $\|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega}$ in the upper bound of $\|c_I(t_k) - c_h(t_k)\|_{1,\Omega}$.

Denote $\bar{c} = \int_\Omega (c_I(t_{k-1}) - c_h(t_{k-1}))\, dx$. From the mass conservation property for the functions $\hat{c}(x,t)$ and $c_h(x,t)$ and the consistency assumption of the theorem we deduce $\bar{c} = \int_\Omega (c_I(t_{k-1}) - \hat{c}(t_{k-1}))\, dx$. Using Cauchy-Schwarz inequality we can estimate

$$|\bar{c}| \le |\Omega|^{1/2} \cdot \|\hat{c}(t_{k-1}) - c_I(t_{k-1})\|_{0,\Omega}. \quad (15)$$

The interpolation error $\hat{c}(t_{k-1}) - c_I(t_{k-1})$, under the given regularity assumptions for $\hat{c}(t_{k-1})$, satisfies the estimate

$$\|\hat{c}(t_{k-1}) - c_I(t_{k-1})\|_{0,\Omega} \le C_4 h |\hat{c}(t_{k-1})|_{1,\Omega}. \quad (16)$$

Using Poincare's inequality together with (15) and (16), we obtain

$$\begin{aligned} \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega} &\le \|c_I(t_{k-1}) - c_h(t_{k-1}) - \bar{c}\|_{0,\Omega} + \|\bar{c}\|_{0,\Omega} \\ &\le C_\Omega |c_I(t_{k-1}) - c_h(t_{k-1})|_{1,\Omega} + |\Omega|^{1/2} \cdot |\bar{c}| \\ &\le C_\Omega |c_I(t_{k-1}) - c_h(t_{k-1})|_{1,\Omega} + |\Omega| \cdot C_4 h |\hat{c}(t_{k-1})|_{1,\Omega} \end{aligned}$$

and thus we have

$$(1+C_\Omega)\|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega} \le C_\Omega \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{1,\Omega} + |\Omega| \cdot C_4 h |\hat{c}(t_{k-1})|_{1,\Omega},$$

which is equivalent to

$$\|c_I(t_{k-1}) - c_h(t_{k-1})\|_{0,\Omega} \le \eta \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{1,\Omega} + C_5 h |\hat{c}(t_{k-1})|_{1,\Omega}, \quad (17)$$

where $\eta = \frac{C_\Omega}{1+C_\Omega} < 1$.

It is easily seen that (17) together with Proposition 4 yields the following recurrent inequality:

$$\begin{aligned} \|c_I(t_k) - c_h(t_k)\|_{1,\Omega} \le C_6 h\Big\{ \big((\Delta t)^2 |j(\hat{c}(t_k))|^2_{1,\Omega} + |\hat{c}(t_k)|^2_{1,\Omega}\big)^{1/2} + |\hat{c}(t_{k-1})|_{1,\Omega}\Big\} \\ + \eta \|c_I(t_{k-1}) - c_h(t_{k-1})\|_{1,\Omega}. \end{aligned}$$

Finally, if we let $\tilde{C} = \frac{2C_6}{1-\eta}$, this recurrent inequality results in the error estimate (14). □

For completeness we have to show that there exist initial condition $c_h(0) \in V_h$ which satisfies consistency and approximation assumptions of the proven theorem. Define

$$c_h(0) = c_I(0) \frac{\int_\Omega \hat{c}(x,0)\, dx}{\int_\Omega c_I(x,0)\, dx},$$

then we deduce consistency assumption $\int_\Omega c_h(x,0)\, dx = \int_\Omega \hat{c}(x,0)\, dx$. Using inequalities (15) and (16) for $t_{k-1} = 0$, we have

$$\begin{aligned} \|c_I(0) - c_h(0)\|_{0,\Omega} &= \|c_I(0)\|_{0,\Omega} \frac{|\int_\Omega \big(c_I(x,0) - \hat{c}(x,0)\big) dx|}{\int_\Omega c_I(x,0) dx} \\ &= |\bar{c}| \frac{\|c_I(0)\|_{0,\Omega}}{\int_\Omega c_I(x,0) dx} \le C_4 |\Omega|^{1/2} |\hat{c}(0)|_{1,\Omega} \frac{\|c_I(0)\|_{0,\Omega}}{\int_\Omega c_I(x,0) dx} h, \end{aligned}$$

whence approximation assumption holds for $\tilde{C}_0 = C_4 \frac{\|c_I(0)\|_{0,\Omega}}{\int_\Omega c_I(x,0) dx} |\hat{c}(0)|_{1,\Omega}$.

4 Numerical experiments

We present here some numerical results for considered numerical schemes with simplified (compared with practical tasks) parameters of equation.

In the current setting we consider a certain case of boundary conditions for velocity flow $\vec{V}$ shown on Figure 1. The Stokes equation (1) will read in this case as

$$\begin{cases} \nabla \cdot \left(\nu \nabla \vec{V}\right) = \nabla p \\ \nabla \cdot \vec{V} = 0 \\ \langle n, \vec{V} \rangle|_{\partial\Omega_{membrane}} = \dfrac{w \cdot (u_1 + u_2)}{L} \\ \langle n, \vec{V} \rangle|_{\partial\Omega_{walls}} = 0 \\ \langle n, \vec{V} \rangle|_{\partial\Omega_{inlet}} = -6 \cdot \dfrac{x_2}{w} \cdot \left(1 - \dfrac{x_2}{w}\right) \cdot u_1 \\ \langle n, \vec{V} \rangle|_{\partial\Omega_{outlet}} = -6 \cdot \dfrac{x_2}{w} \cdot \left(1 - \dfrac{x_2}{w}\right) \cdot u_2 \\ \langle \tau, \vec{V} \rangle|_{\partial\Omega} = 0 \end{cases} \tag{18}$$

when we can write out an exact solution $\vec{V}(x, u)$. We have

$$\begin{aligned} V_1 &= -6(u_1 + u_2)\frac{x_1}{L} \cdot \frac{x_2}{w}\left(1 - \frac{x_2}{w}\right) + 6u_1\frac{x_2}{w}\left(1 - \frac{x_2}{w}\right), \\ V_2 &= -(u_1 + u_2)\left(1 - 3\left(\frac{x_2}{w}\right)^2 + 2\left(\frac{x_2}{w}\right)^3\right) \cdot \frac{w}{L} \cdot (x_1 > 0) \cdot (x_1 < L) \\ p &= p_0 - 12 \cdot \frac{u_1 \cdot \nu}{w} \cdot \frac{L}{w} \cdot \frac{x_1}{L} + 6 \cdot \frac{(u_1 + u_2) \cdot \nu}{L} \cdot \left(\frac{L^2}{w^2} \cdot \frac{x_1^2}{L^2} + \frac{x_2}{w} \cdot \left(1 - \frac{x_2}{w}\right)\right), \end{aligned}$$

where p_0 is some constant parameter.

Parameters of the system are the following:

- $L = 5$, $w = 0.05$ – sizes of rectangular domain Ω,
- $\varepsilon = 4 \cdot 10^{-6}$ – diffusion coefficient.

A sketch of the domain with initial condition is shown on Figure 2.

Triangular mesh $\mathcal{T}_h$ of the domain Ω was selected uniform, with the numbers of partitions

$$n_x = 3300, \quad n_y = 250$$

along x-axis and y-axis respectively. Control parameters u are selected such that only the part of the mesh $\mathcal{T}_h$ on subdomain

$$\Omega_\theta = [0, \theta \cdot L] \times [0, w], \quad \theta = 0.3,$$

was used for evaluating $c(x, t)$ (as its values on the rest of the domain are physically negligible).

In the subsequent plots of solutions (constructed by isovalues) it is clearly seen that EAFE and SUPG approaches produce relatively similar (for example, the error between them can be estimated in $\| \cdot \|_{1,\Omega}$) robust solutions $c_h(t_k)$ before boundary layer is formed (due to velocity flow $\vec{V}$ with properly selected

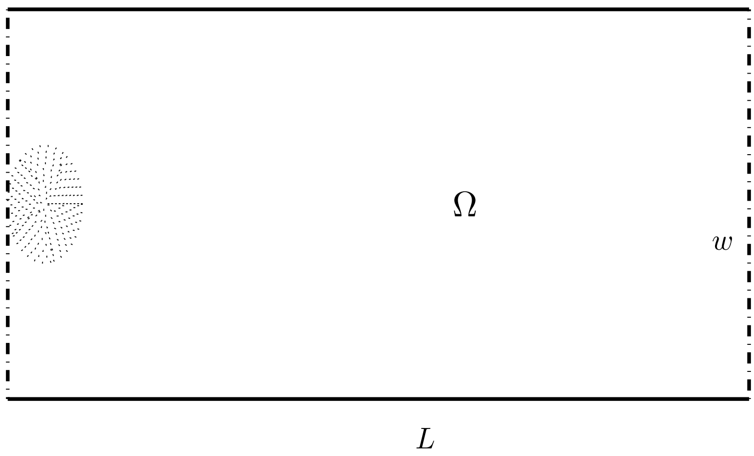

Figure 2: Domain Ω of the system and initial conditions

control parameters $u(t)$). When this regime of forming boundary layer is kept (what can be done by setting $u_1 > 0$, $u_2 > 0$), oscillations of discrete solution are observed for SUPG method.

On the other hand, EAFE scheme as a monotone scheme is free from these defects. This property can be considered as one of its main advantages.

5 Acknowledgements

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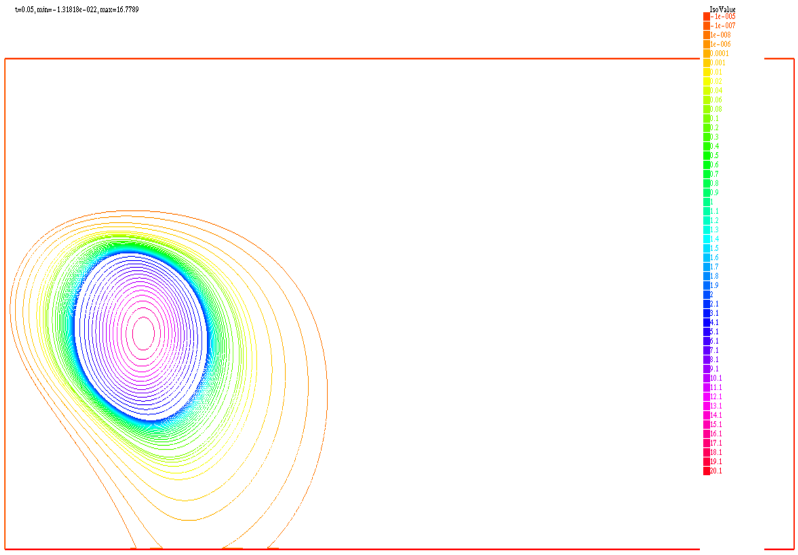

Figure 3: $c(x,t)$ for $t = 0.05$, $u = (2.1,\ 8.5)$, obtained by EAFE method

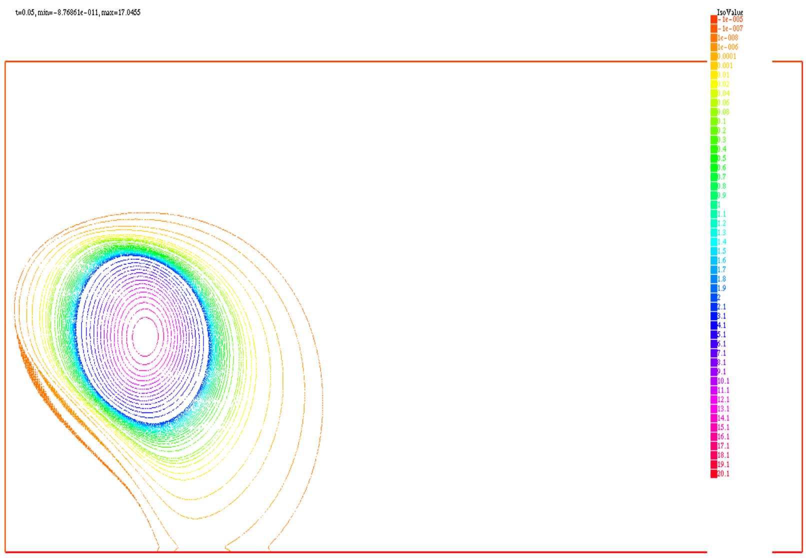

Figure 4: $c(x,t)$ for $t = 0.05$, $u = (2.1,\ 8.5)$, obtained by SUPG method

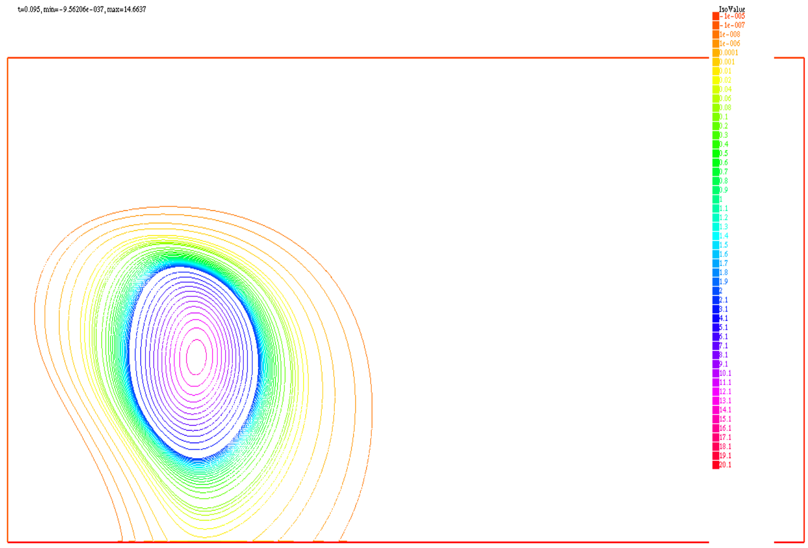

Figure 5: $c(x,t)$ for $t = 0.095$, $u = (2.1,\ 8.5)$, obtained by EAFE method

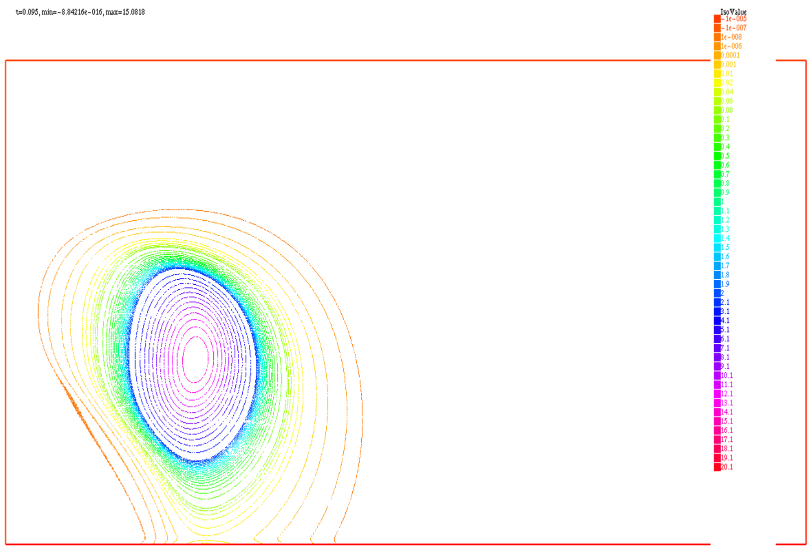

Figure 6: $c(x,t)$ for $t = 0.095$, $u = (2.1,\ 8.5)$, obtained by SUPG method

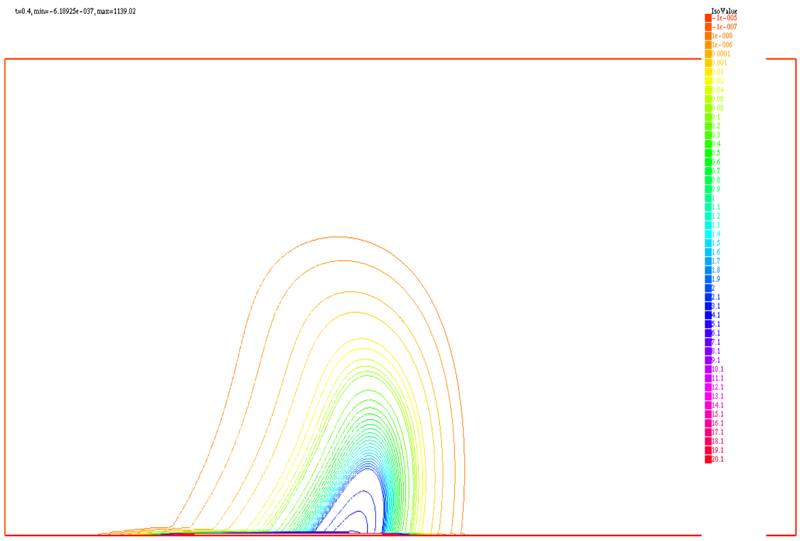

Figure 7: $c(x,t)$ for $t = 0.4$, $u = (2.1,\ 8.5)$, obtained by EAFE method

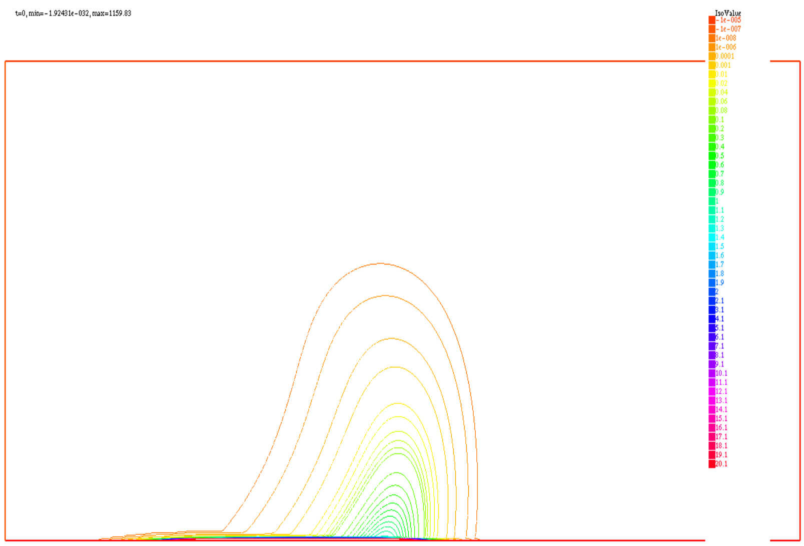

Figure 8: $c(x,t)$ for $t = 0.5$, $u = (2.1,\ 8.5)$ – moment of control switching. Obtained by EAFE method

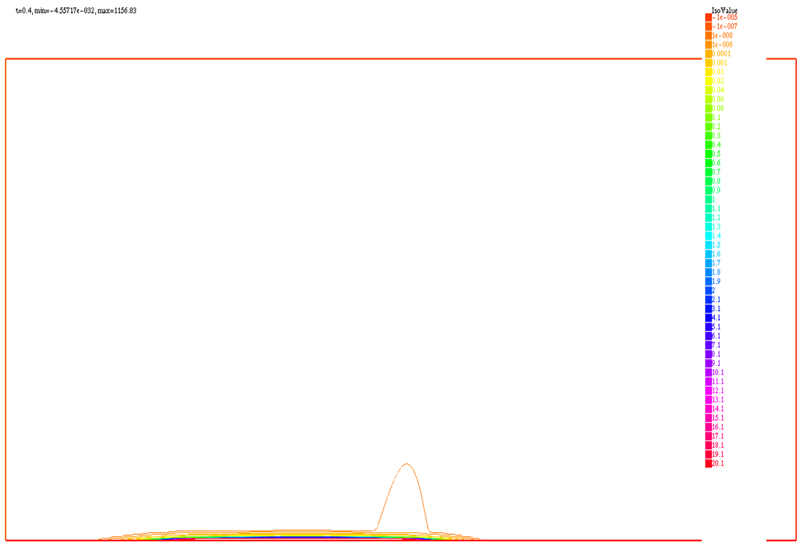

Figure 9: $c(x,t)$ for $t = 0.9$, $u = (1.6,\ 8.5)$, obtained by EAFE method

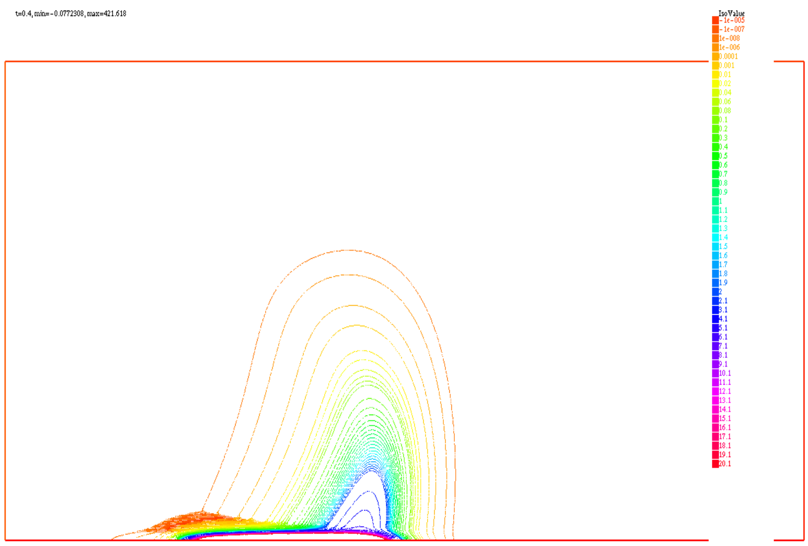

Figure 10: $c(x,t)$ for $t = 0.4$, $u = (2.1,\ 8.5)$, obtained by SUPG method

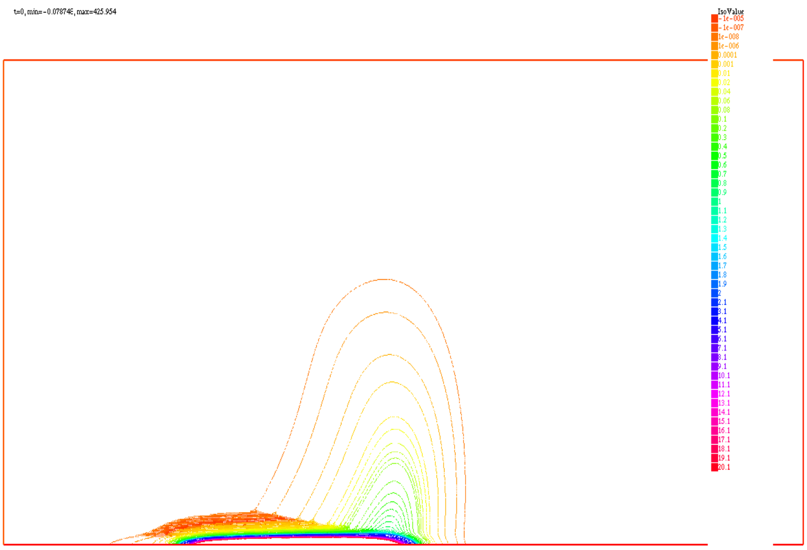

Figure 11: $c(x,t)$ for $t = 0.5$, $u = (2.1,\ 8.5)$ – moment of control switching. Obtained by SUPG method

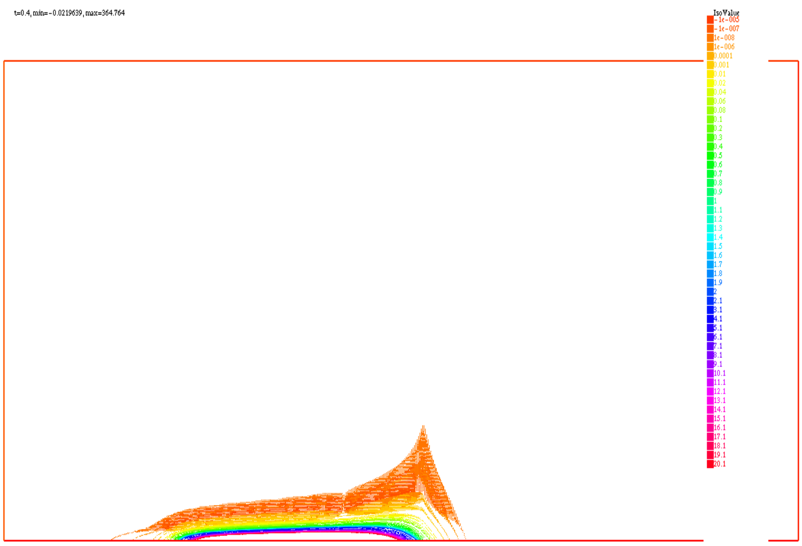

Figure 12: $c(x,t)$ for $t = 0.9$, $u = (1.6,\ 8.5)$, obtained by SUPG method. Oscillations are formed around the boundary layer.

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