Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

Algebraic multilevel preconditioning in $H(\Omega, curl)$

S. Tomar

RICAM-Report 2012-27

ALGEBRAIC MULTILEVEL PRECONDITIONING IN $H(\Omega, \text{curl})$

S. K. TOMAR

ABSTRACT. An algebraic multilevel iteration method for solving system of linear algebraic equations arising in $H(\Omega, \text{curl})$ space is presented. The algorithm is developed for the discrete problem obtained by using the space of lowest order edge elements. The theoretical analysis of the method is based only on some algebraic sequences and generalized eigenvalues of local (element-wise) problems. Explicit recursion formulae are derived to compute the element matrices and the constant γ (which measures the quality of the space splitting) at any given level. It is proved that the proposed method is robust with respect to the problem parameters, and is of optimal order complexity. Supporting numerical results, including the case when the parameters have jumps, are also presented.

1. INTRODUCTION

Consider the finite element discretization of variational problems related to the bilinear form

$$\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) := \alpha(\boldsymbol{u}, \boldsymbol{v}) + \beta(\text{curl } \boldsymbol{u}, \text{curl } \boldsymbol{v}), \quad \alpha, \beta \in \mathbb{R}^+, \tag{1}$$

defined on the two-dimensional Hilbert space

$$H(\Omega, \text{curl}) := \{\boldsymbol{v} \in (L^2(\Omega))^2 : \text{curl } \boldsymbol{v} \in L^2(\Omega)\}. \tag{2}$$

Here $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain, curl $\boldsymbol{v} = \partial_x v_2 - \partial_y v_1$ is the scalar curl of a two-dimensional vector $\boldsymbol{v} = [v_1, v_2]^T$, and $(\cdot, \cdot)$ denotes the inner-product in $L^2(\Omega)$. For $\alpha = \beta = 1$, the bilinear form (1) is precisely the inner-product in $H(\Omega, \text{curl})$.

Associated with the inner-product $\mathcal{H}$, there exists a linear operator $\boldsymbol{A} := \alpha \boldsymbol{I} + \beta \mathbf{curl}$ curl, which is determined by the relation

$$(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H(\Omega, \text{curl}), \tag{3}$$

where the vector curl of a scalar function w is defined as $\mathbf{curl}\ w = [\partial_y w, -\partial_x w]^T$. Given a finite element space $\mathcal{V}_h$ of $H(\Omega, \text{curl})$, the symmetric and positive-definite (SPD) operator $A_h : \mathcal{V}_h \to \mathcal{V}_h$, which is the discretization of the operator $\boldsymbol{A}$ together with natural boundary conditions, is determined by

$$(A_h u_h, v_h) = \mathcal{H}(u_h, v_h), \quad \forall v_h \in \mathcal{V}_h. \tag{4}$$

The operator equation $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$, for $\boldsymbol{f} \in (L^2(\Omega))^2$, then leads to the following discrete problem

$$A_h u_h = f_h, \tag{5}$$

which is uniquely solvable. Such problems frequently occur in various contexts in electromagnetism, e.g., low-frequency time-harmonic Maxwell equations [24], or some formulations of the (Navier -) Stokes equations [13]. Therefore, developing fast solvers for large system of equations (5) is of significant importance.

Preconditioning methods for such linear systems within the framework of domain decomposition (overlapping Schwarz), multigrid and auxiliary space methods have been proposed by several authors. The first results for multigrid in $H(\Omega, \text{curl})$ (within the framework of overlapping Schwarz methods) were obtained by Hiptmair in [14]. A unified treatment of multigrid methods for $H(\Omega, \text{curl})$ and $H(\Omega, \text{div})$ was presented by Hiptmair and Toselli in [15]. However, the condition number estimates of their preconditioned system were not robust with respect to the parameters α and β. Arnold et al. [1] employed the multigrid framework by developing necessary estimates for mixed finite element methods (FEM) based on discretizations of $H(\Omega, \text{div})$ and $H(\Omega, \text{curl})$, and thereby obtained parameter independent condition number estimates of the preconditioned system. Pasciak and Zhao studied the overlapping Schwarz

Date: First version: November 28, 2012, Second version: January 18, 2013.

1991 *Mathematics Subject Classification.* 65N30, 65N22, 65N55.

Key words and phrases. Algebraic multilevel iteration method, lowest-order Nedelec space, optimal order complexity, Maxwell's equation, $H(\Omega, \text{curl})$ space.

methods for $H(\Omega, \mathrm{curl})$ in polyhedral domains in [28], and Reitzinger and Schoeberl studied algebraic multigrid methods for edge elements in [29]. Auxiliary space preconditioning, proposed by Xu in [33], was studied for $H_0(\Omega, \mathrm{curl})$ (the space $H(\Omega, \mathrm{curl})$ with zero tangential trace) by Hiptmair et al. [16]. Nodal auxiliary space preconditioning in $H(\Omega, \mathrm{curl})$ and $H(\Omega, \mathrm{div})$ was studied by Hiptmair and Xu in [17], and the proposed preconditioner was robust with respect to the parameters α and β.

In this paper we develop optimal order iterative solver for the solution of the linear system (5) based on algebraic multilevel iterative (AMLI) methods. AMLI methods were introduced by Axelsson and Vassilevski in a series of papers [3, 4, 5, 6]. The AMLI methods, which are recursive extensions of two-level multigrid methods for FEM [2], have been extensively analyzed in the context of conforming and non-conforming FEM (including discontinuous Galerkin methods), see [8, 12, 18, 20, 21, 22]. For a detailed systematic exposition of AMLI methods, see the monographs [19, 31]. To reduce the overall complexity of AMLI methods (to achieve optimal computational complexity), various stabilization techniques can be used. In the original work [3, 4], the stabilization was achieved by employing properly shifted and scaled Chebyshev polynomials. This approach requires the computation of polynomial coefficients which depends on the bounds of the eigenvalues of the preconditioned system. Alternatively, some inner iterations at certain (or all) coarse levels can be used to stabilize the outer iterations, which lead to parameter-free AMLI methods [5, 6, 18, 25, 26, 27]. These methods utilize a sequence of coarse-grid problems that are obtained from repeated application of a natural (and simple) hierarchical basis transformation, which is computationally advantageous. Moreover, the underlying technique of these methods often requires only a few minor adjustments (mainly two-level hierarchical basis transformation) even if the underlying problem changes significantly.

Our analysis is based only on some algebraic sequences and the generalized eigenvalues of local (element-wise) problems. We derive explicit recursion formulae to compute the element matrices and the constant γ (which measures the quality of the space splitting) at any given level. The method is shown to be robust with respect to the parameters, i.e., the results hold uniformly for $0 < \alpha, \beta < \infty$.

The remainder of this paper is organized as follows. In Section 2 we briefly discuss the finite element discretization of the model problem (1) using the lowest-order Nedelec space. Section 3 starts with a brief description of the AMLI procedure (in Section 3.1), and is followed by the construction of the hierarchical splitting of the lowest-order Nedelec space (in Section 3.2). In Section 3.3 a local two- and multi-level analysis is then presented and the main result is proved. Finally, in Section 4 we present numerical experiments. These include the cases with known analytical solution ($\alpha = \beta = 1$), fixing one of the parameters α or β and varying other from 10^{-6} to 10^{6}, and jumping coefficients. The conclusions are drawn in Section 5.

2. Finite element discretization using Nedelec elements

We consider the tessellation of $\Omega \subset \mathbb{R}^2$ using rectangular elements, and choose the reference element $\hat{K}$ as $[-1,1] \times [-1,1]$. Let $P_{r_1,r_2}(\hat{K})$ denote the space of polynomials of degree $\leqslant r_1$ in x and $\leqslant r_2$ in y. Also, let $P_r(\partial\hat{K})$ denote the space of polynomials of degree $\leqslant r$ on $\partial\hat{K}$. For the construction of $\mathcal{V}_h$, we use the space of lowest-order edge elements (Nedelec space of first kind), which is denoted by N^0. The space $\mathrm{N}^0(\hat{K})$ is defined as

$$\mathrm{N}^0(\hat{K}) = P_{0,1}(\hat{K}) \times P_{1,0}(\hat{K}) = \left\{ \boldsymbol{v}(\hat{x},\hat{y}) = \begin{bmatrix} v_1 + v_2\hat{y} \\ v_3 + v_4\hat{x} \end{bmatrix} \right\}. \tag{6}$$

Thus, the local basis for N^0 has dimension 4. Moreover, for $\boldsymbol{v}_0 \in \mathrm{N}^0(\hat{K})$ we have

$$\mathrm{curl}\, \boldsymbol{v}_0 \in P_{0,0}\,, \quad \boldsymbol{v}_0 \cdot \boldsymbol{t}|_{\partial\hat{K}} \in P_0(\partial\hat{K}), \tag{7}$$

where $\boldsymbol{t}$ denotes the unit tangential vector to the element boundaries. For further details the reader is referred to, e.g., [24].

Now let $F : \hat{K} \to \mathbb{R}^2$ be a diffeomorphism of the reference element $\hat{K}$ onto a physical element K, i.e., $K = F(\hat{K})$. By $\mathcal{J}$ we denote the Jacobian matrix of the mapping, and by $\mathcal{J}_D$ its determinant, which are defined as

$$\mathcal{J} = \begin{pmatrix} \partial_{\hat{x}} x & \partial_{\hat{y}} x \\ \partial_{\hat{x}} y & \partial_{\hat{y}} y \end{pmatrix}, \quad \mathcal{J}_D = |\mathrm{det}\mathcal{J}| = \partial_{\hat{x}} x\, \partial_{\hat{y}} y - \partial_{\hat{y}} x\, \partial_{\hat{x}} y > 0.$$

Then we have the following transformation relations:

$$\boldsymbol{w} = \mathcal{J}^{-T}\hat{\boldsymbol{w}}; \quad \mathrm{curl}\,\boldsymbol{w} = \mathcal{J}_D^{-1}\mathcal{J}\,\mathrm{curl}\,\hat{\boldsymbol{w}}, \quad \forall \boldsymbol{w} \in H(K, \mathrm{curl}), \hat{\boldsymbol{w}} \in H(\hat{K}, \mathrm{curl}). \tag{8}$$

The vector transformation $\hat{\boldsymbol{w}} \to \mathcal{J}^{-T}\hat{\boldsymbol{w}}$ is called the covariant transformation, and $\hat{\boldsymbol{w}} \to \mathcal{J}_D^{-1}\mathcal{J}\hat{\boldsymbol{w}}$ is the well known Piola transformation.

We denote the element matrix for $\int_K \boldsymbol{u}\cdot\boldsymbol{v}$ by L_K, and for $\int_K \mathrm{curl}\,\boldsymbol{u}\,\mathrm{curl}\,\boldsymbol{v}$ by C_K. For the N^0 space based on uniform mesh composed of square elements, the element matrices L_K and C_K have the following structure

$$L_K = \frac{1}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad C_K = \frac{1}{h^2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{9}$$

The overall element matrix A_K, which is defined as $A_K = \alpha C_K + \beta B_K$, is thus given by

$$A_K = \frac{1}{6h^2}\begin{bmatrix} 2\alpha h^2 + 6\beta & \alpha h^2 - 6\beta & -6\beta & 6\beta \\ \alpha h^2 - 6\beta & 2\alpha h^2 + 6\beta & 6\beta & -6\beta \\ -6\beta & 6\beta & 2\alpha h^2 + 6\beta & \alpha h^2 - 6\beta \\ 6\beta & -6\beta & \alpha h^2 - 6\beta & 2\alpha h^2 + 6\beta \end{bmatrix}. \tag{10}$$

Letting $e = \kappa h^2$, with $\kappa = \alpha/\beta$, the element matrix can be written as

$$A_K = \frac{\beta}{6h^2}\begin{bmatrix} 2e + 6 & e - 6 & -6 & 6 \\ e - 6 & 2e + 6 & 6 & -6 \\ -6 & 6 & 2e + 6 & e - 6 \\ 6 & -6 & e - 6 & 2e + 6 \end{bmatrix}. \tag{11}$$

Clearly, for all $\alpha, \beta \in \mathbb{R}^+$, and thus $\kappa \in \mathbb{R}^+$, we have $e > 0$. Note that for fixed κ, and $h \to 0$, the element matrix A_K is dominated by the matrix C_K (which has a non-zero kernel), whereas for moderate values of h it is a regular matrix. The near-nullspace of the matrix A_K is given by the nullspace of the matrix C_K, which is associated with the local bilinear form $C_K(\boldsymbol{u}, \boldsymbol{v}) := (\mathrm{curl}\,\boldsymbol{u}, \mathrm{curl}\,\boldsymbol{v})_K$. As we shall see in the analysis, the proposed method is of optimal order for all $0 < \alpha, \beta < \infty$.

Lemma 2.1. (Near-nullspace of matrix A_K). *The element matrix A_K* (10) *is symmetric positive definite (SPD). Moreover, the nullspace of the matrix C_K for a general element K with nodal coordinates (x_i, y_i), $i \in \{1, 2, 3, 4\}$ is given by*

$$\ker(C_K) = \mathrm{span}\{(1, 1, 0, 0)^T, (0, 0, 1, 1)^T, (x_1, x_2, y_3, y_4)^T\}. \tag{12}$$

Furthermore, in case of a uniform mesh composed of square N^0 elements, the matrix C_K is same for each element K and its nullspace is given by

$$\ker(C_K) = \mathrm{span}\{(1, 1, 0, 0)^T, (0, 0, 1, 1)^T, (-1, 0, 0, 1)^T\}.$$

Proof. Since the coefficients α and β in (10) are positive, it follows from equation (5) that A_K is SPD for a general element K. Moreover, it can easily be seen that there exist positive constants c_1 and c_2 such that the following inequalities hold:

$$c_1\,\zeta(\boldsymbol{w}, \boldsymbol{w}) \leqslant \mathbf{w}^T C_K \mathbf{w} \leqslant c_2\,\zeta(\boldsymbol{w}, \boldsymbol{w}). \tag{13}$$

Here $\boldsymbol{w}$ is a function that is linear in one component and constant in the other component, and the function $\zeta(\cdot, \cdot)$ is defined by

$$\zeta(\boldsymbol{w}, \boldsymbol{w}) := \int_K (\partial_x w_2 - \partial_y w_1)^2 dK.$$

Thus (for the particular choice of $\boldsymbol{w}$), we get $C_K\mathbf{w} = 0$ if and only if $\boldsymbol{w} = (\alpha_1 + \alpha_3 y, \alpha_2 + \alpha_3 x)^T$, for all constants $\alpha_i, i = 1, 2, 3$. By setting one of these constants to 1 and the other constants to 0, we obtain the three linear independent vectors $(x_1, x_2, y_3, y_4)^T$, $(1, 1, 0, 0)^T$, and $(0, 0, 1, 1)^T$, that span the kernel of C_K. The remaining part related to uniform mesh is trivial. Note that, in case of a uniform mesh composed of square N^0 elements, since the vector $(1, -1, -1, 1)^T$ is orthogonal to the kernel of C_K, it is clear that the rank-one matrix C_K is of the form $c \cdot (1, -1, -1, 1)^T \cdot (1, -1, -1, 1)$, for some constant c. □

Remark 2.2. When using the lowest order Nedelec elements, the matrix C_K is always of rank one. In the global assembly this yields a matrix C whose rank equals the number of elements in the mesh. That is, the kernel of the global matrix C has dimension $\dim(\ker(C)) = n_E - n_K$, where n_E denotes the number of faces and n_K the number of elements in the finite element mesh. Thereby, the dimension of the kernel is slightly more than half of the total number of degrees of freedom.

3. Algebraic multilevel iteration

For the solution of the linear system arising from (5), we describe and analyze the AMLI method in the remainder of this section. Our presentation follows Reference [22].

3.1. **The AMLI procedure.** In what follows we will denote by $M^{(\ell)}$ a preconditioner for a finite element (stiffness) matrix $A^{(\ell)}$ corresponding to a ℓ times refined mesh ($0 \leqslant \ell \leqslant L$). We will also make use of the corresponding ℓ^{th} level hierarchical matrix $\hat{A}^{(\ell)}$, which is related to $A^{(\ell)}$ via a two-level hierarchical basis (HB) transformation $J^{(\ell)}$, i.e.,

$$\hat{A}^{(\ell)} = J^{(\ell)} A^{(\ell)} (J^{(\ell)})^T. \tag{14}$$

The transformation matrix $J^{(\ell)}$ specifies the space splitting, which will be described in detail in Section 3.2. By $A_{ij}^{(\ell)}$ and $\hat{A}_{ij}^{(\ell)}$, $1 \leqslant i, j \leqslant 2$, we denote the blocks of $A^{(\ell)}$ and $\hat{A}^{(\ell)}$ that correspond to the fine-coarse partitioning of degrees of freedom (DOF) where the DOF associated with the coarse mesh are numbered last.

The aim is to build a multilevel preconditioner $M^{(L)}$ for the coefficient matrix $A^{(L)} := A_h$ at the level of the finest mesh that has a uniformly bounded (relative) condition number

$$\varkappa({M^{(L)}}^{-1} A^{(L)}) = O(1),$$

and an optimal computational complexity, that is, linear in the number of degrees of freedom N_L at the finest mesh (grid). In order to achieve this goal hierarchical basis methods can be combined with various types of stabilization techniques.

One particular purely algebraic stabilization technique is the so-called Algebraic Multi-Level Iteration (AMLI) method, where a specially constructed matrix polynomial $p^{(\ell)}$ of degree ν_ℓ can be employed at some (or all) levels ℓ. The AMLI algorithm has been originally introduced and studied in a multiplicative form, see [3, 4].

We have the following two-level hierarchical basis representation at level ℓ

$$\hat{A}^{(\ell)} = \begin{bmatrix} \hat{A}_{11}^{(\ell)} & \hat{A}_{12}^{(\ell)} \\ \hat{A}_{21}^{(\ell)} & \hat{A}_{22}^{(\ell)} \end{bmatrix} = \begin{bmatrix} A_{11}^{(\ell)} & \hat{A}_{12}^{(\ell)} \\ \hat{A}_{21}^{(\ell)} & A^{(\ell-1)} \end{bmatrix}. \tag{15}$$

Starting at level 0 (associated with the coarsest mesh), on which a complete LU factorization of the matrix $A^{(0)}$ is performed, we define

$$M^{(0)} := A^{(0)}. \tag{16}$$

Given the preconditioner $M^{(\ell-1)}$ at level $\ell - 1$, the preconditioner $M^{(\ell)}$ at level ℓ is then defined by

$$M^{(\ell)} := L^{(\ell)} U^{(\ell)}, \tag{17}$$

where

$$L^{(\ell)} := \begin{bmatrix} C_{11}^{(\ell)} & 0 \\ \hat{A}_{21}^{(\ell)} & C_{22}^{(\ell)} \end{bmatrix}, \quad U^{(\ell)} := \begin{bmatrix} I & {C_{11}^{(\ell)}}^{-1} \hat{A}_{12}^{(\ell)} \\ 0 & I \end{bmatrix}. \tag{18}$$

Here $C_{11}^{(\ell)}$ is a preconditioner for the pivot block $A_{11}^{(\ell)}$, and

$$C_{22}^{(\ell)} := A^{(\ell-1)} \left(I - p^{(\ell)} ({M^{(\ell-1)}}^{-1} A^{(\ell-1)}) \right)^{-1} \tag{19}$$

is an approximation to the Schur complement $S = A^{(\ell-1)} - \hat{A}_{21}^{(\ell)} {C_{11}^{(\ell)}}^{-1} \hat{A}_{12}^{(\ell)}$, where $A^{(\ell-1)} = \hat{A}_{22}^{(\ell)}$ is the stiffness matrix at the coarse level $\ell - 1$, and $p^{(\ell)}$ is a certain stabilization polynomial of degree ν_ℓ satisfying the condition

$$0 \leqslant p^{(\ell)}(t) < 1, \quad 0 < t \leqslant 1, \quad p^{(\ell)}(0) = 1. \tag{20}$$

It is easily seen that (19) is equivalent to

$$C_{22}^{(\ell)^{-1}} = M^{(\ell-1)^{-1}} q^{(\ell)}(A^{(\ell-1)} M^{(\ell-1)^{-1}}), \tag{21}$$

where the polynomial $q^{(\ell)}$ is given by

$$q^{(\ell)}(x) = \frac{1 - p^{(\ell)}(x)}{x}. \tag{22}$$

We note that the multilevel preconditioner defined via (17) is getting close to a two-level method when $q^{(\ell)}(x)$ closely approximates $1/x$, in which case $C_{22}^{(\ell)^{-1}} \approx A^{(\ell-1)^{-1}}$. In order to construct an efficient multilevel method the action of $C_{22}^{(\ell)^{-1}}$ on an arbitrary vector should be much cheaper to compute (in terms of the number of arithmetic operations) than the action of $A^{(\ell-1)^{-1}}$. Optimal order solution algorithms typically require that the arithmetic work for one application of $C_{22}^{(\ell)^{-1}}$ is of the order $O(N_{\ell-1})$ where $N_{\ell-1}$ denotes the number of unknowns at level $\ell - 1$.

It is well known from the theory introduced in [3, 4] that a properly shifted and scaled Chebyshev polynomial $p^{(\ell)} := p_{\nu_\ell}$ of degree ν_ℓ can be used to stabilize the condition number of $M^{(\ell)^{-1}} \hat{A}^{(\ell)}$ (and thus obtain optimal order computational complexity). Other polynomials such as the best polynomial approximation of $1/x$ in uniform norm also qualify for stabilization, see, e.g., [23]. Alternatively, in the nonlinear AMLI method, see, e.g., [6], a few inner flexible conjugate gradient (FCG) type iterations (for the FCG algorithm see, e.g., [25]) are performed in order to improve (or freeze) the residual reduction factor of the outer FCG iteration. In general, the resulting nonlinear (variable step) multilevel preconditioning method is equally efficient, and, because its realization does not rely on any spectral bounds, is easier to implement than the linear AMLI method (based on a stabilization polynomial). For a convergence analysis of nonlinear AMLI see, e.g., [18, 26, 27, 31].

Typically, the iterative solution process is of optimal order of computational complexity if the degree $\nu_\ell = \nu$ of the matrix polynomial (or alternatively, the number of inner iterations for nonlinear AMLI) at level ℓ satisfies the optimality condition

$$1/\sqrt{(1-\gamma^2)} < \nu < \tau, \tag{23}$$

where $\tau \approx \tau_\ell = N_\ell / N_{\ell-1}$ denotes the reduction factor of the number of degrees of freedom (DOF), and γ denotes the constant in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality. The value of τ is approximately 4 in case of the sequence of non-nested N^0 spaces, which we will construct in the next subsection. For a more detailed discussion of AMLI methods, including implementation issues see, e.g., [19, 31].

Remark 3.1. The preconditioner defined in (17) is of multiplicative form. The introduction of AMLI methods was based on the multiplicative form, see [3, 4, 5, 6], and is commonly used in practice. However, it is also possible to choose the preconditioner in the additive form, which is defined as follows

$$M_A^{(\ell)} := \begin{bmatrix} C_{11}^{(\ell)} & 0 \\ 0 & C_{22}^{(\ell)} \end{bmatrix}. \tag{24}$$

In this case the optimal order of computational complexity demands that the matrix polynomial degree (or the number of inner iterations of nonlinear AMLI) satisfy the following relation

$$\sqrt{(1+\gamma)/(1-\gamma)} < \nu < \tau. \tag{25}$$

3.2. **Hierarchical splitting of the lowest order Nedelec space.** The AMLI methods we are considering here, for the solution of (5), are based on a proper splitting of the N^0 subspace of $H(\Omega, \mathrm{curl})$. The particular two-level HB transformation that induces this splitting was introduced in the context of linear nonconforming (Crouzeix-Raviart) elements in [8]. It was later studied for quadrilateral rotated bilinear (Rannacher-Turek) type elements in [12]. In the following we will briefly describe this transformation and study the related splitting of the lowest order Nedelec space.

Consider two consecutive discretizations $\mathcal{T}_H$ (coarse level) and $\mathcal{T}_h$ (fine level). Figure 1 illustrates a macro-element G (at fine level) obtained from a coarse element by one regular mesh-refinement step. We see that in this case the corresponding local (and thus the global) finite element spaces $\mathcal{V}_H$ and $\mathcal{V}_h$ are

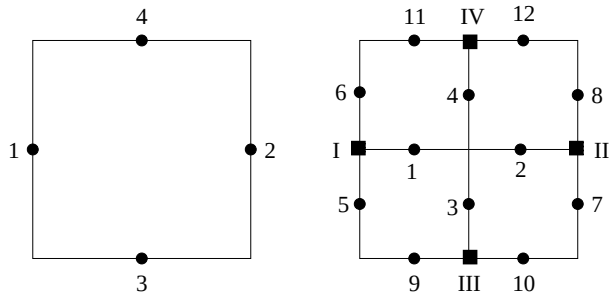

FIGURE 1. Macro-element obtained after one regular mesh-refinement step

not nested. Let $\varphi_G = \{\phi_i(x,y)\}_{i=1}^{12}$ be the macro-element vector of the nodal basis functions, and A_G be the macro-element stiffness matrix corresponding to $G \in \mathcal{T} = \mathcal{T}_h$. The global stiffness matrix A_h can be written as

$$A_h = \sum_{G\in\mathcal{T}} R_G^T A_G R_G,$$

where R_G denotes the natural inclusion (canonical injection) of the 12×12 matrix A_G for all G in $\mathcal{T}$. Using the local numbering of DOF, as shown in Figure 1 (right picture), a macro-element level (local) transformation matrix J_G is constructed based on differences and aggregates of each pair of basis functions ϕ_i and ϕ_j that correspond to a macro element edge, i.e.,

$$J_G = \frac{1}{2}\begin{bmatrix} 2 & & & & & & & & & & & \\ & 2 & & & & & & & & & & \\ & & 2 & & & & & & & & & \\ & & & 2 & & & & & & & & \\ & & & & 1 & -1 & & & & & & \\ & & & & & & 1 & -1 & & & & \\ & & & & & & & & 1 & -1 & & \\ & & & & & & & & & & 1 & -1 \\ & & & & 1 & 1 & & & & & & \\ & & & & & & 1 & 1 & & & & \\ & & & & & & & & 1 & 1 & & \\ & & & & & & & & & & 1 & 1 \end{bmatrix}, \tag{26}$$

cf. [12]. This transformation defines a two-level hierarchical basis $\hat{\varphi}_G$ locally, namely, $\hat{\varphi}_G = J_G\varphi_G$. Then the hierarchical two-level macro-element matrix is given by

$$\hat{A}_G = J_G A_G J_G^T,$$

and the related global two-level matrix can be obtained via assembling, i.e., $\hat{A}_h = \sum_{G\in\mathcal{T}} R_G^T \hat{A}_G R_G$. Alternatively, one can compute the matrix $\hat{A}_h$ via the triple matrix product

$$\hat{A}_h = J A_h J^T, \tag{27}$$

where the global transformation matrix J is induced by the local transformations, i.e.,

$$J|_G = J_G, \quad \forall G \in \mathcal{T}.$$

In other words, global and local transformations are compatible in the sense that restricting J to the DOF of any macro-element G we obtain J_G. Now, if we number those DOF that correspond to interior nodes of the macro elements first, the global two-level stiffness matrix $\hat{A}_h$ has the 2×2 block structure

$$\hat{A}_h = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \tag{28}$$

where $\hat{A}_{11}$ corresponds to the "interior unknowns". We follow the "first reduce" (FR) approach, see e.g., [8, 12], where these interior unknowns are first eliminated *exactly*. This static condensation step can be

written in the form

$$\hat{A}_h = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & B \end{bmatrix} \begin{bmatrix} I_1 & \hat{A}_{11}^{-1}\hat{A}_{12} \\ 0 & I_2 \end{bmatrix}, \tag{29}$$

with the Schur complement $B = \hat{A}_{22} - \hat{A}_{21}\hat{A}_{11}^{-1}\hat{A}_{12}$. Next, the matrix B is partitioned into 2×2 blocks, i.e.,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{30}$$

where each row (and each column) of B_{11} corresponds to a difference of two basis functions associated with one macro-element edge (face) and each row (and each column) of B_{22} corresponds to an aggregate of such a pair of basis functions. The matrix B_{22} at level ℓ then defines the coarse-grid matrix $A^{(\ell-1)}$ in the AMLI hierarchy, cf. (15). This algorithm can be applied recursively on each level $\ell = L, L-1, \ldots, 1$. The resulting algorithm is then of optimal computational complexity, see e.g., [22, Remark 3.1].

3.3. **Local analysis.** In the two-level framework we denote by $\mathcal{V}_1$ and $\mathcal{V}_2$ the subspaces of the finite element space $\mathcal{V}_h$. The space $\mathcal{V}_2$ is spanned by the coarse-space basis functions (aggregates) and $\mathcal{V}_1$ is the complement of $\mathcal{V}_2$ in $\mathcal{V}_h$, i.e., $\mathcal{V}_h$ is a direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$:

$$\mathcal{V}_h = \mathcal{V}_1 \oplus \mathcal{V}_2. \tag{31}$$

A measure for the quality of this splitting is the constant γ in the strengthened CBS inequality, which is defined by the relation

$$\gamma = \cos(\mathcal{V}_1, \mathcal{V}_2) := \sup_{\boldsymbol{u} \in \mathcal{V}_1,\ \boldsymbol{v} \in \mathcal{V}_2} \frac{\mathcal{H}(\boldsymbol{u}, \boldsymbol{v})}{\sqrt{\mathcal{H}(\boldsymbol{u}, \boldsymbol{u})\mathcal{H}(\boldsymbol{v}, \boldsymbol{v})}}.$$

It is well known (see, e.g., [2]) that γ can be estimated locally over each macro element G, and that $\gamma = \max_G \gamma_G$, where

$$\gamma_G := \sup_{\boldsymbol{u} \in \mathcal{V}_1(G),\ \boldsymbol{v} \in \mathcal{V}_2(G)} \frac{\mathcal{H}_G(\boldsymbol{u}, \boldsymbol{v})}{\sqrt{\mathcal{H}_G(\boldsymbol{u}, \boldsymbol{u})\mathcal{H}_G(\boldsymbol{v}, \boldsymbol{v})}}.$$

The spaces $\mathcal{V}_1(G)$, $\mathcal{V}_2(G)$, and the bilinear form $\mathcal{H}_G(\boldsymbol{u}, \boldsymbol{v})$ correspond to the restriction of $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v})$, respectively, to the macro element G.

We perform this local analysis on the matrix level, where the splitting (31) is obtained via the two-level hierarchical basis transformation described in Section 3.2, and the space $\mathcal{V}_h$ corresponds to the choice of lowest order Nedelec elements. In this setting the upper left block of $\hat{A}_h$ is block-diagonal (with diagonal blocks of $\hat{A}_{11}$ of size 4×4, which can be associated with the interior nodes $\{1, 2, 3, 4\}$ in the right picture of Figure 1). Therefore, we first compute the local Schur complements arising from static condensation of the interior DOF and obtain hereby the (8×8) matrices B_G. Next we split each matrix B_G as

$$B_G = \begin{bmatrix} B_{G,11} & B_{G,12} \\ B_{G,21} & B_{G,22} \end{bmatrix} \begin{matrix} \} \text{ differences} \\ \} \text{ aggregates} \end{matrix},$$

written again in two-by-two block form (with blocks of size 4×4). We have thus reduced the problem of estimating the CBS constant of the splitting (31) to a small-sized local problem that involves the matrix B_G. Following the general theory, see [2, 11], to estimate the CBS constant γ, it suffices to compute the minimal eigenvalue of the generalized eigenproblem

$$S_G \mathbf{v}_G = \lambda_{G,\min} B_{G,22} \mathbf{v}_G, \quad \forall \mathbf{v}_G, \tag{32}$$

where $S_G = B_{G,22} - B_{G,21}B_{G,11}^{-1}B_{G,12}$. The CBS constant γ can then be estimated as follows

$$\gamma^2 \leqslant \max_{G \in \mathcal{T}} \gamma_G^2 = \max_{G \in \mathcal{T}} (1 - \lambda_{G,\min}). \tag{33}$$

Note that the matrix $B_{G,11}$ is a well conditioned matrix, see Figure 2, and therefore, it can be inverted cheaply, either by an iterative process or by, for example, an incomplete LU factorization [30], which is denoted by B_{11}^i in Figure 2.

We now first prove two auxiliary (stand-alone) results on algebraic sequences, which we will use to bound the CBS constant γ.

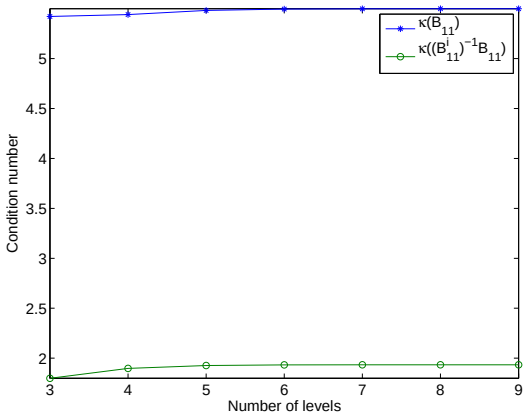

FIGURE 2. Condition number of the matrix $B_{G,11}$

Lemma 3.2. *For all $e > 0$, consider the coupled sequences*

$$b_0 = e - 6, \qquad a_0 = 2e + 6 = 2(b_0 + 9), \tag{34a}$$

$$b_{\ell+1} = -b_\ell^2/a_\ell, \qquad a_{\ell+1} = 2a_\ell + b_{\ell+1}, \quad \ell = 0, 1, 2, \ldots. \tag{34b}$$

Let $r_\ell = b_\ell/a_\ell$. Then we have

$$b_{\ell+1}/a_\ell = -r_\ell^2, \quad a_{\ell+1}/a_\ell = 2 - r_\ell^2, \quad r_{\ell+1} = -r_\ell^2/(2 - r_\ell^2). \tag{35}$$

Moreover, the following bound holds for all $\ell = 0, 1, 2, \ldots$

$$a_\ell > \ldots a_1 > a_0 > 6, \tag{36a}$$

$$0 \leqslant r_\ell^2 \leqslant \ldots \leqslant r_1^2 \leqslant r_0^2 < 1. \tag{36b}$$

Proof. Using the definition of r_ℓ in (34b), we get $b_{\ell+1}/a_\ell = -r_\ell^2$, and thus $a_{\ell+1}/a_\ell = 2 - r_\ell^2$. The last relation of (35) then immediately follows. From (34b) and (35) we also note the following relations

$$a_{\ell+1} - b_{\ell+1} = 2a_\ell, \quad a_{\ell+1} + b_{\ell+1} = \frac{2}{a_\ell}(a_\ell^2 - b_\ell^2) = 2a_\ell(1 - r_\ell^2), \quad \frac{a_{\ell+1} + b_{\ell+1}}{a_{\ell+1} - b_{\ell+1}} = 1 - r_\ell^2. \tag{37}$$

Clearly, for $e > 0$, we have $a_0 > 6$, and since $r_0 = b_0/a_0 = (e - 6)/(2e + 6)$, it is easy to see that $-1 < r_0 < 1/2$. The latter also implies that $0 \leqslant r_0^2 < 1$. We now prove the remaining bounds using induction.

$\ell = 0$. Since $a_1/a_0 = 2 - r_0^2 > 1$, we have $a_1 > a_0 > 6$. Moreover, $r_1 = -r_0^2/(2 - r_0^2)$. This implies that $-1 < r_1 \leqslant 0$, and thus $0 \leqslant r_1^2 < 1$. Furthermore, when $r_0 \neq 0$, we have

$$r_1^2 = \left(\frac{-r_0^2}{2 - r_0^2}\right)^2 \Rightarrow \frac{r_1^2}{r_0^2} = \frac{r_0^2}{(2 - r_0^2)^2} < 1.$$

And, since $r_1 = 0$ if $r_0 = 0$, we have $r_1^2 \leqslant r_0^2 < 1$.

$\ell = n$. Assume that the relations (36) hold for $\ell = n$. Since $a_{n+1}/a_n = 2 - r_n^2 > 1$, we have $a_{n+1} > a_n > 6$. Moreover, $r_{n+1} = -r_n^2/(2 - r_n^2)$. This implies that $-1 < r_{n+1} \leqslant 0$, and thus $0 \leqslant r_{n+1}^2 < 1$. Also, when $r_n \neq 0$, we have

$$r_{n+1}^2 = \left(\frac{-r_n^2}{2 - r_n^2}\right)^2 \Rightarrow \frac{r_{n+1}^2}{r_n^2} = \frac{r_n^2}{(2 - r_n^2)^2} < 1.$$

And, since $r_{n+1} = 0$ if $r_n = 0$, we have $r_{n+1}^2 \leqslant r_n^2 < 1$.

This concludes the proof. □

From Lemma 3.2, we also note the following bounds

$$-1 < r_0 < 1/2, \text{ and } -1 < r_\ell \leqslant 0 \quad \forall\, \ell = 1, 2, \ldots. \tag{38}$$

Lemma 3.3. *Let $e > 0$ and the sequences a_ℓ and b_ℓ be as defined in Lemma 3.2. Then for*

$$c_\ell^2 = \frac{36(a_\ell + b_\ell)}{(a_\ell^2 - 36)(a_\ell - b_\ell)}, \tag{39}$$

the following bounds hold for all $\ell = 0, 1, 2, \ldots$

$$c_\ell^2 < c_{\ell-1}^2 < \ldots < c_1^2 < c_0^2 < 3/8. \tag{40}$$

Proof. From $a_0 = 2e + 6$ and $b_0 = e - 6$, we have $a_0 - b_0 = e + 12$, $a_0 + b_0 = 3e$, $a_0 - 6 = 2e$, and $a_0 + 6 = 2(e + 6)$. Substituting these relations in the definition of c_0^2, we get

$$c_0^2 = \frac{27}{(e + 6)(e + 12)} < 3/8. \tag{41}$$

Now

$$c_1^2 - c_0^2 = \frac{36\left((a_1 + b_1)(a_0^2 - 36)(a_0 - b_0) - (a_0 + b_0)(a_1^2 - 36)(a_1 - b_1)\right)}{(a_1^2 - 36)(a_1 - b_1)(a_0^2 - 36)(a_0 - b_0)}.$$

Substituting the values of a_0, a_1, b_0 and b_1, and after some lengthy, but simple calculations, we find that

$$c_1^2 - c_0^2 = \frac{108e\left(-9e^2(312 + 80e + 5e^2)\right)}{(e + 3)(a_1^2 - 36)(a_1 - b_1)(a_0^2 - 36)(a_0 - b_0)}.$$

Since the denominator is a positive quantity, we get $c_1^2 - c_0^2 < 0$, and thus

$$c_1^2 < 3/8. \tag{42}$$

For remaining bounds, we again use induction. Note that, using (37) we get

$$c_{\ell+1}^2 = \frac{36(a_{\ell+1} + b_{\ell+1})}{(a_{\ell+1}^2 - 36)(a_{\ell+1} - b_{\ell+1})} = \frac{36(1 - r_\ell^2)}{(a_{\ell+1}^2 - 36)}. \tag{43}$$

Therefore, to show that $c_{\ell+1}^2 < 3/8$, it suffices to show that

$$a_{\ell+1}^2 - 36 > 96(1 - r_\ell^2). \tag{44}$$

Since $c_1^2 < 3/8$, we clearly have $a_1^2 - 36 > 96(1 - r_0^2)$. Now assume that the relation (44) holds for $\ell = n - 1$, i.e.,

$$a_n^2 - 36 > 96(1 - r_{n-1}^2). \tag{45}$$

Multiplying (45) by $(2 - r_n^2)^2$ and subtracting 36 from both sides we get

$$\begin{aligned} &(2 - r_n^2)^2 a_n^2 - 36 > 36(2 - r_n^2)^2 + 96(1 - r_{n-1}^2)(2 - r_n^2)^2 - 36 \\ \Rightarrow \quad &a_{n+1}^2 - 36 > 96\left((2 - r_n^2)^2(11/8 - r_{n-1}^2) - 3/8\right). \end{aligned}$$

We need to show that $(2 - r_n^2)^2(11/8 - r_{n-1}^2) - 3/8 > 1 - r_n^2$, i.e.,

$$g_n := (2 - r_n^2)^2(11/8 - r_{n-1}^2) + r_n^2 - 11/8 > 0. \tag{46}$$

From the recurrence relation on r_n from (35), we have

$$r_n^2 = \frac{r_{n-1}^4}{(2 - r_{n-1}^2)^2}, \quad 2 - r_n^2 = \frac{(r_{n-1}^4 - 8r_{n-1}^2 + 8)}{(2 - r_{n-1}^2)^2}.$$

Substituting these relations in g_n, and after some lengthy calculations we obtain

$$g_n = \frac{(1 - r_{n-1}^2)^2}{(2 - r_{n-1}^2)^4}(-r_{n-1}^6 + 15r_{n-1}^4 - 64r_{n-1}^2 + 66). \tag{47}$$

Now for $r_{n-1}^2 \in [0, 1)$, we have

$$1 - r_{n-1}^2 > 0, \quad 2 - r_{n-1}^2 > 0, \quad 66 - 64r_{n-1}^2 > 0, \quad 15r_{n-1}^4 - r_{n-1}^6 \geqslant 0,$$

which proves that $g_n > 0$, and that $a_{n+1}^2 - 36 > 96(1 - r_n^2)$. Therefore, the inequality (44) holds for all $\ell = 0, 1, \ldots$.

To prove the monotonicity of c_ℓ^2, we show that

$$f_\ell := c_{\ell+1}^2/c_\ell^2 < 1. \tag{48}$$

Using (43) we get

$$f_\ell = \frac{(1-r_\ell^2)(a_\ell^2-36)}{(1-r_{\ell-1}^2)(a_{\ell+1}^2-36)}.$$

Multiplying numerator and denominator by $(2-r_\ell^2)^2$, we obtain

$$\begin{aligned} f_\ell &= \frac{(1-r_\ell^2)\left((2-r_\ell^2)^2a_\ell^2-36(2-r_\ell^2)^2\right)}{(1-r_{\ell-1}^2)(a_{\ell+1}^2-36)(2-r_\ell^2)^2} \\ &= \frac{(1-r_\ell^2)}{(1-r_{\ell-1}^2)}\frac{\left(a_{\ell+1}^2-36+36(1-(2-r_\ell^2)^2)\right)}{(a_{\ell+1}^2-36)(2-r_\ell^2)^2} \\ &= \frac{(1-r_\ell^2)}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2}+\frac{36(1-r_\ell^2)(1-(2-r_\ell^2)^2)}{(1-r_{\ell-1}^2)(a_{\ell+1}^2-36)(2-r_\ell^2)^2}. \end{aligned}$$

Now since $c_{\ell+1}^2 < 3/8$, we have $(1-r_\ell^2)/(a_{\ell+1}^2-36) < 1/96$ from (44). Therefore,

$$\begin{aligned} f_\ell &< \frac{(1-r_\ell^2)}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2}+\frac{36(1-(2-r_\ell^2)^2)}{96(1-r_{\ell-1}^2)(2-r_\ell^2)^2} \\ &= \frac{(1-r_\ell^2)+\frac{3}{8}(1-(2-r_\ell^2)^2)}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2} = \frac{11/8-r_\ell^2-\frac{3}{8}(2-r_\ell^2)^2}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2}. \end{aligned}$$

This gives

$$\begin{aligned} f_\ell - 1 &< \frac{11/8-r_\ell^2-\frac{3}{8}(2-r_\ell^2)^2-(1-r_{\ell-1}^2)(2-r_\ell^2)^2}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2} \\ &= \frac{11/8-r_\ell^2+(2-r_\ell^2)^2(-11/8+r_{\ell-1}^2)}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2}. \end{aligned}$$

Using (46) we therefore get

$$f_\ell - 1 < \frac{-g_\ell}{(1-r_{\ell-1}^2)(2-r_\ell^2)^2} < 0,$$

since $g_\ell > 0$, $1-r_{\ell-1}^2 > 0$, and $(2-r_\ell^2)^2 > 0$. This proves (48) and concludes the proof. □

The sequences a_ℓ, b_ℓ, r_ℓ and c_ℓ^2 are plotted in Figure 3. We are now in a position to prove the following theorem which provides a theoretical estimate that holds on all levels of recursive splitting of the N^0 subspace of $H(\Omega,\mathrm{curl})$.

Theorem 3.4. *Consider the bilinear form* (1) *where* $0 < \alpha,\beta < \infty$, *and the related discrete problem (5) on the* N^0 *subspace of* $H(\Omega,\mathrm{curl})$ *where the underlying rectangular mesh is uniform. Under these assumptions the CBS constant* γ *related to the hierarchical splitting (31) has the upper bound* $\gamma \leqslant \gamma_G < \sqrt{3/8}$, *which holds for each step of the recursive hierarchical splitting. Moreover,* $\gamma^{(L-\ell)}$ *is monotonically strictly decreasing and has an upper bound of* $\sqrt{3/8}$ *for all* $\ell = 0,1,\ldots,L$, *i.e.,*

$$\gamma^{(0)} < \gamma^{(1)} < \ldots < \gamma^{(\ell)} < \ldots < \gamma^{(L-1)} < \gamma^{(L)} < \sqrt{3/8}. \tag{49}$$

Proof. In order to prove this uniform bound for γ we study the generalized eigenproblem (32). At level L of the finest discretization the macro-element matrix $\hat{A}_G$, which is the same for all G in $\mathcal{T}_{h_L}$ for a uniform mesh, can be represented in the form

$$\hat{A}_G^{(L)} = J_G\left(\sum_{K\in G\subset\mathcal{T}_{h_\ell}} R_K^T A_K^{(L)} R_K\right) J_G^T \tag{50}$$

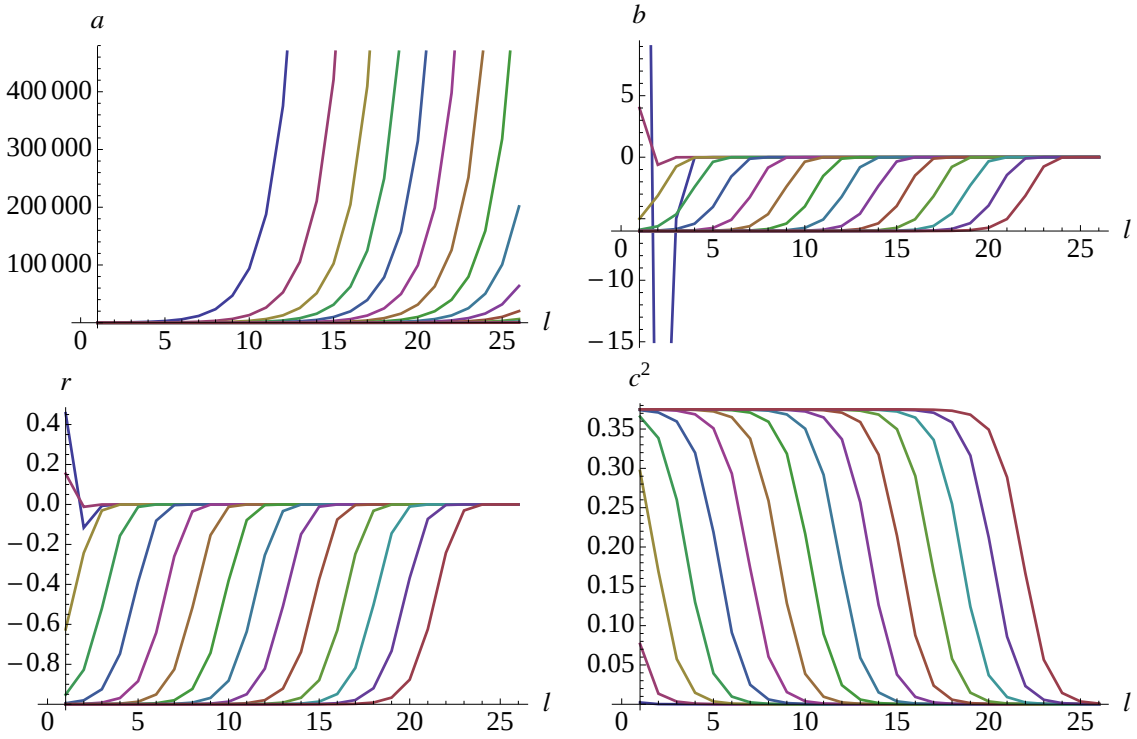

FIGURE 3. a_ℓ, b_ℓ, r_ℓ, and c_ℓ^2 for $e = 10^{m_0}$, where $m_0 = \{2, 1, 0, \ldots -11, -12\}$ (left to right)

where

$$A_K^{(L)} = \frac{\beta}{6h^2}\begin{bmatrix} a_0 & b_0 & -6 & 6 \\ b_0 & a_0 & 6 & -6 \\ -6 & 6 & a_0 & b_0 \\ 6 & -6 & b_0 & a_0 \end{bmatrix}, \quad \forall K \in G,\ \forall G \subset \mathcal{T}_{h_L}, \tag{51}$$

with $a_0 = 2e + 6$, $b_0 = e - 6$, $e = \kappa h^2 > 0$, with $\kappa = \alpha/\beta$, and the local transformation matrix J_G is defined according to (26).

The lower-right 4×4 block of the matrix B_G and the Schur complement S_G for the first splitting (at level L) are to be found

(52)

$$B_{G,22}^{(L)} = \frac{\beta}{6h^2}\begin{bmatrix} p_0 & q_0 & -3/2 & 3/2 \\ q_0 & p_0 & 3/2 & -3/2 \\ -3/2 & 3/2 & p_0 & q_0 \\ 3/2 & -3/2 & q_0 & p_0 \end{bmatrix}, \quad S_G^{(L)} = \frac{\beta}{6h^2}\begin{bmatrix} s_0 & t_0 & -3/2 & 3/2 \\ t_0 & s_0 & 3/2 & -3/2 \\ -3/2 & 3/2 & s_0 & t_0 \\ 3/2 & -3/2 & t_0 & s_0 \end{bmatrix}.$$

with

$$p_0 = a_0/2 - b_0^2/4a_0, \qquad q_0 = -b_0^2/4a_0,$$
$$s_0 = \frac{108a_0 - 2a_0^3 + 72b_0 + a_0b_0^2}{144 - 4a_0^2}, \qquad t_0 = \frac{36a_0 + 72b_0 + a_0b_0^2}{144 - 4a_0^2}.$$

The generalized eigenproblem (32) has two different two-fold eigenvalues, namely $\lambda_{1,2} = 1$ and

$$\lambda_{3,4} = \frac{a_0(a_0^2 - a_0b_0 - 72)}{(a_0^2 - 36)(a_0 - b_0)},$$

which shows that

$$\left(\gamma_G^{(L)}\right)^2 \leqslant 1 - \lambda_{3,4} = \frac{36(a_0 + b_0)}{(a_0 - 36)(a_0 - b_0)}. \tag{53}$$

Note that the coefficient β does not appear in the bound for γ since the factor $\frac{\beta}{6h^2}$ appear in both the matrices of the generalized eigenproblem (32), and thus does not affect the eigenvalues.

Now in order to compute a similar bound for the second splitting (at level $L-1$) we have to use the relation $A_K^{(L-1)} := B_{G,22}^{(L)}$. In general, for the $(\ell+1)^{\text{th}}$ splitting (at level $L-\ell$) the relation

$$A_K^{(L-\ell)} := B_{G,22}^{(L-\ell+1)} \tag{54}$$

is to be used in the assembly of $\hat{A}_G^{L-\ell}$, i.e.,

$$\hat{A}_G^{(L-\ell)} = J_G \left(\sum_{K \in G \subset \mathcal{T}_{h_{L-\ell}}} R_K^T A_K^{(L-\ell)} R_K \right) J_G^T. \tag{55}$$

Repeating the computations, we find that the relation (55) holds for all levels $\ell = 1, 2, \ldots, L-1, L$, and the element stiffness matrix $A_K^{L-\ell}$ (after ℓ coarsening steps) is given by

$$A_K^{(L-\ell)} = \frac{\beta}{6(2^\ell h)^2} \begin{bmatrix} a_\ell & b_\ell & -6 & 6 \\ b_\ell & a_\ell & 6 & -6 \\ -6 & 6 & a_\ell & b_\ell \\ 6 & -6 & b_\ell & a_\ell \end{bmatrix}, \quad \forall K \in G,\ \forall G \subset \mathcal{T}_{h_{L-\ell}}, \tag{56}$$

where the sequences a_ℓ and b_ℓ are defined in (34). Thus, the bound for γ_G at level $L-\ell$ reads

$$\left(\gamma_G^{(L-\ell)}\right)^2 = \frac{36(a_\ell + b_\ell)}{(a_\ell^2 - 36)(a_\ell - b_\ell)}. \tag{57}$$

The result (49) then follows by taking $\gamma_G^{L-\ell} = c_\ell$ in Lemma 3.3. □

Remark 3.5. The curves in the lower-right part of Figure 3 show the behavior of γ_G^2 (defined by (57)). We observe that γ_G^2 approaches zero when the splitting is applied many times (increasing ℓ from left to right), which means that the two subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ in (31) become increasingly orthogonal to each other as the recursion proceeds. Therefore, on (very) coarse levels, the upper bound $3/8$ for γ_G^2, and thus for γ, is quite pessimistic.

4. Numerical results

All the numerical experiments presented in this section are performed using Matlab R2012a on an HP Z400 workstation with 6 core 3.33GHz CPU and 12 GB RAM. For all the numerical experiments, we consider a mesh of square elements of size $h = 1/8, 1/64, \ldots, 1/2048$ (i.e., up to $8,392,704$ DOF for the finest level). We use a direct solver on the coarsest grid that consists of 2×2 elements. Hence, the multilevel procedure is based on 1 to 9 levels of regular mesh refinement (resulting in a k-level method, $k = 3, \ldots, 11$). The initial guess is chosen as a zero vector, and the stopping criteria is chosen as

$$\left(\|r^{(n_{\text{it}})}\| / \|r^{(0)}\| \right) \leqslant 10^{-8},$$

where n_{it} is the number of iterations reported in the tables. The average residual reduction factor ρ, which is also reported in the tables, is defined as

$$\rho := \left(\|r^{(n_{\text{it}})}\| / \|r^{(0)}\| \right)^{\frac{1}{n_{\text{it}}}}.$$

Example 4.1. *Consider the model problem* (1) *in a unit square, and fix the coefficients* $\alpha = \beta = 1$. *The problem data is chosen such that the exact solution is given by* $\boldsymbol{u} = (\pi \sin \pi x \cos \pi y, -\pi \cos \pi x \sin \pi y)^T$.

For the W-cycle method, we chose two-types of stabilization polynomials $q^{(\ell)}$. One is based on Chebyshev polynomials (see, e.g., [4, 19, 31], denoted in the tables by T), for which the polynomial $q^{(\ell)}$ is defined as $2/(s-b) - t/(s-b)^2$, where $s = \sqrt{1+b+b^2-\gamma^2}$, and b is some constant estimating the upper bound of the condition number of preconditioned B_{11} block. The other one is based on the polynomial of best uniform approximation to $1/x$ (see, e.g., [23], denoted in the tables by X), for which the polynomial $q^{(\ell)}$ is defined as $(2-\gamma^2)/(1-\gamma^2) - t/(1-\gamma^2)$. The results for the V-cycle and W-cycle multiplicative AMLI method are presented in Table 1. The second column confirms the error convergence behavior. We see that for decreasing h the growth in the iteration number for V-cycle is moderate (as expected), whereas both the W-cycle versions (T and X) exhibit h-independence. Moreover, the total time (factorization and solver) reported in eighth and eleventh columns also confirms that both the versions of W-cycle are of practical optimal complexity (slight increase in time may be attributed to

the implementation issues). We note that the X version W-cycle gives slightly better results than the T version W-cycle.

TABLE 1. Convergence results for multiplicative AMLI, $\alpha = \beta = 1, \chi = \boldsymbol{u} - \boldsymbol{u}_h$

		V-cycle			W-cycle (T)			W-cycle (X)		
$1/h$	$\|\mathrm{curl}\,\chi\|_{L^2(\Omega)}$	n_{it}	ρ	t_{sec}	n_{it}	ρ	t_{sec}	n_{it}	ρ	t_{sec}
8	0.15946423	7	0.049	0.00	7	0.049	0.00	7	0.049	0.00
16	0.08005229	8	0.094	0.01	8	0.083	0.01	8	0.094	0.01
32	0.04006629	10	0.143	0.01	9	0.104	0.01	8	0.097	0.01
64	0.02003817	11	0.174	0.04	9	0.105	0.05	8	0.100	0.04
128	0.01001971	12	0.201	0.14	9	0.108	0.16	8	0.095	0.14
256	0.00500993	13	0.224	0.54	9	0.109	0.55	8	0.088	0.51
512	0.00250498	14	0.246	2.41	9	0.110	2.22	8	0.083	2.09
1024	0.00125249	14	0.267	10.74	9	0.110	9.35	8	0.078	8.99
2048	0.00062624	16	0.313	49.79	9	0.110	40.29	8	0.073	38.74

We now test the AMLI method with additive preconditioning. The results for the V-cycle and both the W-cycle additive AMLI methods are presented in Table 2. We also present the results for nonlinear variant of AMLI method, see, e.g., [5, 6, 18, 19, 25, 26, 27], in the last three columns (denoted in the tables by N, W-cycle referring to two inner iterations). Surprisingly, in the additive form, the T version W-cycle gives much better results than the X version W-cycle, where the latter appears to be stabilizing only towards the very fine mesh (many recursive levels). This can be attributed to the fact that for the additive preconditioning, for the choice of $\gamma = \sqrt{3/8}$, we require that $\nu > \sqrt{(1+\gamma)/(1-\gamma)} > 2$. The results of nonlinear W-cycle further improve the results of T version W-cycle (linear). Since the nonlinear W-cycle AMLI method gives the best results (and is free from parameters b and γ), in the remaining numerical experiments we will only present the results from nonlinear W-cycle AMLI method.

TABLE 2. Convergence results for additive AMLI, $\alpha = \beta = 1$

	V-cycle			W-cycle (T)			W-cycle (X)			W-cycle (N)		
$1/h$	n_{it}	ρ	t_{sec}	n_{it}	ρ	t_{sec}	n_{it}	ρ	t_{sec}	n_{it}	ρ	t_{sec}
8	10	0.153	0.00	10	0.153	0.00	10	0.153	0.00	10	0.153	0.00
16	17	0.300	0.01	17	0.299	0.01	17	0.299	0.01	12	0.208	0.01
32	20	0.391	0.02	19	0.346	0.03	23	0.446	0.03	12	0.209	0.03
64	25	0.472	0.06	19	0.372	0.08	31	0.550	0.13	12	0.197	0.08
128	30	0.538	0.21	21	0.386	0.26	44	0.653	0.47	11	0.179	0.23
256	34	0.575	0.87	19	0.377	0.79	56	0.712	1.82	11	0.167	0.76
512	39	0.617	3.99	19	0.361	3.04	60	0.735	6.85	9	0.127	2.61
1024	44	0.657	19.20	19	0.362	12.46	65	0.751	28.90	9	0.117	10.65
2048	50	0.685	91.72	19	0.371	52.99	65	0.752	121.49	8	0.098	43.03

Example 4.2. *Consider the model problem* (1) *in a unit square, fix the coefficient* $\beta = 1$ *and take* $\alpha = 10^{m_0}$ *for* $m_0 = \{-6, -3, 0, 3, 6\}$. *The right hand side (RHS) vector is all ones.*

The results for the multiplicative AMLI method for varying α are presented in Table 3 for V- and nonlinear W-cycle. We see that the V-cycle shows some effect of α, with a moderate growth in the number of iterations for decreasing h, however, the nonlinear W-cycle is independent of h, and is fully robust with respect to α. Note that towards very large values of α, the system matrix is well-conditioned, and the hierarchical splitting approaches orthogonal decomposition, therefore, the V-cycle method also exhibits optimal order complexity.

Example 4.3. *Consider the model problem* (1) *in a unit square, fix the coefficient* $\alpha = 1$ *and take* $\beta = 10^{m_0}$ *for* $m_0 = \{-6, -3, 0, 3, 6\}$. *The RHS vector is all ones.*

TABLE 3. Convergence results for multiplicative AMLI, $\beta = 1$

	n_{it}									
$\alpha \rightarrow$	10^{-6}		10^{-3}		10^{0}		10^{3}		10^{6}	
$1/h$	V	W	V	W	V	W	V	W	V	W
8	9	9	9	9	9	9	4	4	2	2
16	12	10	12	10	12	10	7	6	2	2
32	15	10	15	10	14	10	9	8	2	2
64	17	10	17	10	16	10	11	9	2	2
128	20	9	20	9	17	9	12	9	3	3
256	22	9	22	9	18	9	14	9	4	4
512	26	9	26	9	21	9	16	9	6	6
1024	28	9	28	9	23	9	17	9	8	8
2048	28	9	31	8	25	8	20	8	10	8

TABLE 4. Convergence results for multiplicative AMLI, $\alpha = 1$

	n_{it}									
$\beta \rightarrow$	10^{-6}		10^{-3}		10^{0}		10^{3}		10^{6}	
$1/h$	V	W	V	W	V	W	V	W	V	W
8	2	2	4	4	9	9	9	9	9	9
16	2	2	7	6	12	10	12	10	12	10
32	2	2	9	8	14	10	15	10	15	10
64	2	2	11	9	16	10	17	10	17	10
128	3	3	12	9	17	9	20	9	20	9
256	4	4	14	9	18	9	22	9	22	9
512	6	6	16	9	21	9	26	9	26	9
1024	8	8	17	9	23	9	28	9	28	9
2048	10	8	20	8	25	8	31	8	28	9

The results for the multiplicative AMLI method for varying β are presented in Table 4 for V- and W-cycles. The results are qualitatively the same as in Table 3 for varying α, with the parameter value reversing the behavior of the solver.

Example 4.4. *Consider the model problem* (1) *in a unit square, and fix the coefficient* $\beta = 1$. *The coefficient* α *is chosen as* 1 *in* $[0, 0.5]^2 \bigcup (0.5, 1]^2$ *and* κ *elsewhere, where* $\kappa = 10^{m_0}$, *and* $m_0 = \{-6, -4, -2, 0\}$. *The RHS vector is all ones.*

Finally, the results for the multiplicative AMLI method for the case with jump in the coefficients, which are presented in Table 5 for V- and nonlinear W-cycles, show robustness with respect to the jump in the coefficients.

5. CONCLUSION

We have presented an optimal order AMLI method for problems in the $H(\Omega, \text{curl})$ space. The main result of our local analysis (Theorem 3.4) shows that a second order stabilization polynomial (or two inner iterations in nonlinear method), i.e., a $W-$cycle, is sufficient to stabilize the AMLI process. We derived explicit bounds for the multilevel behavior of γ that are robust with respect to the coefficients in the $H(\Omega, \text{curl})$ model problem. The presented numerical results (including jumping coefficients case) confirm the robustness and efficiency of the proposed method.

Acknowledgements. The author is very grateful to Dr. Johannes Kraus (RICAM, Linz) for insightful discussions on AMLI methods.

REFERENCES

[1] Arnold DN, Falk RS, Winther R. Multigrid in $H(\text{div})$ and $H(\text{curl})$. *Numer. Math.*, 2000; **85**:197–217.

TABLE 5. Convergence results for multiplicative AMLI with jump in the coefficients, $\beta = 1$

	n_{it}							
$\kappa \rightarrow$	10^0		10^{-2}		10^{-4}		10^{-6}	
$1/h$	V	W	V	W	V	W	V	W
8	9	9	10	10	10	10	10	10
16	12	10	12	11	13	11	13	11
32	14	10	15	11	15	11	16	11
64	16	10	17	11	18	11	19	11
128	17	9	20	11	20	11	21	11
256	18	9	22	10	22	11	24	11
512	21	9	23	10	26	11	26	11
1024	23	9	26	10	28	11	28	11
2048	25	8	28	10	32	11	32	11

[2] Axelsson O, Gustafsson I. Preconditioning and two-level multigrid methods of arbitrary degree of approximations. *Math. Comp.*, 1983; **40**:219–242.
[3] Axelsson O, Vassilevski PS. Algebraic multilevel preconditioning methods I. *Numer. Math.*, 1989; **56**:157–177.
[4] Axelsson O, Vassilevski PS. Algebraic multilevel preconditioning methods II. *SIAM J. Numer. Anal.*, 1990; **27**:1569–1590.
[5] Axelsson O, Vassilevski PS. A black box generalized conjugate gradient solver with inner iterations and variable-step preconditioning. *SIAM J. Matrix Anal. Appl.*, 1991; **12**(4):625–644.
[6] Axelsson O, Vassilevski P. Variable-step multilevel preconditioning methods, I: self-adjoint and positive definite elliptic problems. *Numer. Lin. Alg. Appl.*, 1994; **1**:75–101.
[7] Bank R, Dupont T. An optimal order process for solving finite element equations. *Math. Comp.*, 1981; **36**:427–458.
[8] Blaheta R, Margenov S, Neytcheva M. Robust optimal multilevel preconditioners for non-conforming finite element systems. *Numer. Lin. Alg. Appl.*, 2005; **12**(5-6):495–514.
[9] Bochev P.B., Garasi C.J., Hu J.J., Robinson A.C., Tuminaro R.S. An improved algebraic multigrid method for solving Maxwell's equations. *SIAM J. Sci. Comput.*, 2003; **25**:623–642.
[10] Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, Berlin, 1991.
[11] Eijkhout V, Vassilevski PS. The role of the strengthened Cauchy-Bunyakowski-Schwarz inequality in multilevel methods. *SIAM Review*, 1991; **33**:405–419.
[12] Georgiev I, Kraus J, Margenov S. Multilevel preconditioning of rotated bilinear non-conforming FEM problems. *Comput. Math. Appl.*, 2008; **55**:2280–2294.
[13] Girault V., Raviart P.A. Finite Element Methods for Navier-Stokes Equations. *Springer-Verlag*, 1986.
[14] Hiptmair R. Multigrid method for Maxwell's equations. *SIAM J. Numer. Anal.*, 1998; **36**(1):204–225.
[15] Hiptmair R, Toselli A. Overlapping Schwarz methods for vector-valued elliptic problems in three dimensions. In *Parallel solution of PDEs*, IMA Volumes in Mathematics and its Applications, Springer-Verlag, Berlin, 1998.
[16] Hiptmair R, Widmer G, Zou J. Auxiliary space preconditioning in $H_0(\mathbf{curl}\,, \Omega)$. *Numer. Math.*, 2006; **103**:435–459.
[17] Hiptmair R, Xu J. Nodal auxiliary space preconditioning in $H(\Omega, \mathrm{curl})$ and $H(\Omega, \mathrm{div})$ spaces. *SIAM J. Numer. Anal.*, 2007; **45**(6): 2483–2509.
[18] Kraus J. An algebraic preconditioning method for M-matrices: linear versus nonlinear multilevel iteration. *Numer. Lin. Alg. Appl.*, 2002; **9**:599–618.
[19] Kraus J, Margenov S. *Robust Algebraic Multilevel Methods and Algorithms*. *Radon Series on Computational and Applied Mathematics*, **5**, de Gruyter, Berlin, New York, 2009. ISBN 978-3-11-019365-7.
[20] Kraus J, Tomar SK. Multilevel preconditioning of two-dimensional elliptic problems discretized by a class of discontinuous Galerkin methods. *SIAM J. Sci. Comput.*, 2008; **30**:684–786.
[21] Kraus J, Tomar SK. A multilevel method for discontinuous Galerkin approximation of three-dimensional anisotropic elliptic problems. *Numer. Linear Algebra Appl.*, 2008; **15**(5):417–438.
[22] Kraus J, Tomar SK. Algebraic multilevel iteration method for lowest order Raviart-Thomas space and applications. *Int. J. Numer. Meth. Engng*, 2011; **86**:1175–1196.
[23] Kraus J, Vassilevski P, Zikatanov, L. Polynomial of best uniform approximation to $1/x$ and smoothing in two-level methods. *Comput. Methods Appl. Math.*, 2012; **12**(4): 448–468.
[24] Monk P. Finite Element Methods for Maxwell's Equations. *Oxford University Press*, 2003.
[25] Notay Y. Flexible conjugate gradients. *SIAM J. Sci. Comput.*, 2000; **22**(4):1444–1460.
[26] Notay Y. Robust parameter-free algebraic multilevel preconditioning. *Numer. Lin. Alg. Appl.*, 2002; **9**:409–428.
[27] Notay Y, Vassilevski PS. Recursive Krylov-based multigrid cycles. *Numer. Linear Algebra Appl.*, 2008; **15**(5):473–487.
[28] Pasciak JE, Zhao J. Overlapping Schwarz methods in $H(\Omega, \mathrm{curl})$ on polyhedral domains. *J. Numer. Math.*, 2002; **10**(3):221–234.
[29] Reitzinger S, Schoeberl J. Algebraic multigrid for edge elements. *Numer. Linear Algebra Appl.*, 2002; **9**:223–238.
[30] Saad Y. *Iterative Methods for Sparse Linear Systems.* PWS Publishing Company, Boston, 1996.
[31] Vassilevski PS. Multilevel block factorization preconditioners. *Springer*, New York, 2008.
[32] Xu J. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 1992; **34**(4): 581–613.

[33] Xu J. The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids. *Computing*, 1996; **56**: 215–235.

Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences,, Altenbergerstrasse 69, 4040 Linz, Austria

E-mail address: satyendra.tomar@ricam.oeaw.ac.at