

# **Condition number estimates for matrices arising in the isogeometric discretizations**

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# Condition number estimates for matrices arising in the isogeometric discretizations

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## Abstract

We derive the bounds for the extremal eigenvalues and the spectral condition number of matrices for isogeometric discretizations of elliptic partial differential equations in  $\Omega \in \mathbb{R}^d$ ,  $d = 2, 3$ . For the  $h$ -refinement, the condition number of the stiffness matrix is bounded above and below by a constant times  $h^{-2}$ , and the condition number of the mass matrix is uniformly bounded. For the  $p$ -refinement, the condition number is bounded above by  $p^{2d}A^{pd}$  and  $p^{2(d-1)}A^{pd}$  for the stiffness matrix and the mass matrix respectively. Numerical results supporting the theoretical estimates are presented. Some numerical results on the condition number for varying smoothness of the basis functions are also discussed.

*Keywords:* B-Splines, NURBS, Galerkin formulation, Isogeometric method, Stiffness matrix, Mass matrix, Eigenvalues, Condition number,  $h$ - $p$ - $r$ -refinement

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## 1. Introduction

Isogeometric analysis introduced by Hughes et al. in 2005 [27], has generated a significant research interest worldwide. Many research papers have appeared where the isogeometric analysis has been used for a wide variety of problems. Most of the research activities in isogeometric analysis have focused on the use of Non-Uniform Rational B-Spline (NURBS) basis functions, e.g. [1, 5, 10, 27]. In isogeometric analysis the computational geometry (e.g. circle) is represented exactly from the information and the basis functions given by Computer Aided Design (CAD), a main advantage over traditional FEM, where the basis functions and the computational geometry (i.e. the mesh) are defined using piecewise polynomials. Isogeometric analysis is not restricted to NURBS basis functions, other type of basis functions, e.g. T-Splines, generalized B-Splines, sub-D, are also being used by researchers. It has been argued in [10] that NURBS based isogeometric method leads to qualitatively more accurate results than standard polynomials based finite element method. Another noteworthy advantage of isogeometric method over classical finite element method is the higher continuity of the solution. It is a difficult and cumbersome task to achieve even  $C^1$  inter-element continuity in finite element method, whereas isogeometric method offers up to  $C^{p-m}$  continuity, where  $p$  denotes the degree of the basis functions and  $m$  denotes the knot-multiplicity. Summerily, isogeometric analysis provides a powerful tool to compute highly continuous numerical solution of PDEs arising in engineering sciences.

Since the introduction of isogeometric analysis, most of its progress has been focused on the applications and discretization properties. Nevertheless, when dealing with large problems,

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the cost of solving the linear system of equations arising from the isogeometric discretization becomes an important issue. Clearly, the discretization matrix  $A$  gets denser with increasing polynomial degree  $p$ . Therefore, the cost of a direct solver, particularly for large problems, becomes prohibitively expensive. The most practical way to solve them is to resort to an iterative method. Since the convergence rate of such methods is strongly affected by the condition number of the system matrix  $A$ , it is important to assess this quantity as a function of the mesh size  $h$  for the  $h$ -refinement, or as a function of the degree  $p$  for the  $p$ -refinement. Note that in the  $p$ -refinement, improved approximate solutions are sought by increasing  $p$  while the mesh of the domain, and thus the maximum quadrilateral diameter  $h$ , is held fixed, whereas in the  $h$ -refinement, improved approximations are obtained by refining the mesh, and thus reducing  $h$ , while  $p$  is held fixed. In this paper we consider both the cases, i.e. the  $h$ -refinement and the  $p$ -refinement. Our main results provide upper and lower bounds for the condition number of the stiffness matrix and the mass matrix for the  $h$ -refinement, and upper bounds for the condition number of the stiffness matrix and the mass matrix for the  $p$ -refinement.

It is well known fact that for the  $h$ -refinement, when applied to second order elliptic problems on a regular mesh, the condition number of the finite element stiffness matrix scales as  $h^{-2}$ , and the condition number of the mass matrix is bounded uniformly, independent of  $h$ , see e.g. [3, 8]. This is true for a great variety of elements and independent of the dimension of the problem domain. Our results are in agreement with the fact that for  $h$ -refinement whatever may be the underlying basis functions from the respective function space, the above bounds remain same [2]. These results are useful in many theoretical analysis that relate to the  $h$ -refinement. For example, in convergence analysis of multigrid methods, these results are one of the key elements in deriving convergence factors, for finite element analysis, see e.g. [7, 22, 40], and for isogeometric analysis, see [21].

It is known that the order of the approximation error of the numerical solution depends on the choice of the finite dimensional subspace, and not on the choice of its basis. Therefore, when working with finite element method or isogeometric method for elliptic problems, one should think in terms of function spaces rather than on the choice of particular basis functions. Nevertheless, the choice of the basis functions affects the condition number of the stiffness matrix and the mass matrix, which influences the performance of iterative solvers. To the best of our knowledge, there is no general theory to characterize the extremal eigenvalues or the condition number based on a set of general polynomial basis functions, see e.g. [4, 32, 34, 33]. Unlike the  $h$ -refinement, the condition number heavily depends on the choice of basis functions for the  $p$ -refinement. For different choices of basis functions the condition number may grow algebraically or exponentially. Olsen and Douglas, Jr. [36] estimated the condition number bounds of finite element matrices for tensor product elements with two choices of basis functions. For the first choice, Lagrange elements, it is proved that the condition number grows exponentially in  $p$ , whereas for the second choice, hierarchical basis functions based on Chebychev polynomials, the condition number grows rapidly but only algebraically in  $p$ . Similar results on the condition number bounds can be found in, e.g., [19, 26, 31]. Due to the larger support of NURBS basis functions, the band of the stiffness matrix corresponding to the NURBS-based isogeometric method is less sparse than the one arising from finite element procedures. Therefore a larger condition number is expected. Our results for the  $p$ -refinement show that the condition number of system matrices in isogeometric method grows exponentially.

The remainder of the article is organized as follows. In Sect. 2 we briefly describe the model problem and its discretization. Notations and definitions of B-Splines and NURBS are given in Sect. 3. The condition number of B-Spline basis functions is also discussed here. In Sect. 4 we give the bounds for the eigenvalues and the condition number of the stiffness matrix and the mass matrix arising in isogeometric discretizations. Both the cases, i.e. the  $h$ -refinement and

the  $p$ -refinement, are discussed here. Numerical results, which support theoretical estimates, are then reported in Sect. 5. Finally, some conclusions are drawn in Sect. 6.

## 2. Model problem and its discretization

Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be an open, bounded and simply connected Lipschitz domain with Dirichlet boundary  $\partial\Omega$ . We consider the Poisson equation,

$$\Delta u = -f \quad \text{in } \Omega, \quad (1a)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (1b)$$

where  $f : \Omega \rightarrow \mathbb{R}$  is given. The aim is to find  $u : (\Omega \cup \partial\Omega) \rightarrow \mathbb{R}$  which satisfies (1). We consider Galerkin's formulation of the problem which is commonly used in isogeometric analysis. Since we are interested in the study of the condition number, therefore we shall not go into the details of the solution properties, and restrict our-selves to the study of the condition number of resulting system matrices.

Isogeometric analysis has the same theoretical foundation as finite element analysis, namely the variational form of a partial differential equation. For this we define the function space, denoted by  $\mathcal{S}$ , as all the functions which have square integrable derivatives and also satisfy  $u|_{\partial\Omega} = 0$ , i.e.

$$\mathcal{S} = \{u : u \in H^1(\Omega), u|_{\partial\Omega} = 0\}, \quad (2)$$

where  $H^1(\Omega) = \{u : D^\alpha u \in L^2(\Omega), |\alpha| \leq 1\}$  is the Sobolev space,  $\alpha \in \mathbb{N}^d$  is a multi-index,  $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_d^{\alpha_d}$  and  $D_i^j = \frac{\partial^j}{\partial x_i^j}$ .

We now write the variational formulation of the model problem by multiplying it by an arbitrary function  $v \in \mathcal{S}$  and integrating by parts. For a given  $f$ : Find  $u \in \mathcal{S}$  such that for all  $v \in \mathcal{S}$

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} f v \, d\Omega.$$

We may rewrite above as

$$a(u, v) = L(v), \quad (3)$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega, \quad \text{and} \quad L(v) = \int_{\Omega} f v \, d\Omega.$$

It is clear that  $a(\cdot, \cdot)$  is bilinear, continuous and coercive on  $\mathcal{S}$ , and  $L(\cdot)$  is linear form associated to the original equation.

Let  $\mathcal{S}^h$  be the finite dimensional approximations of  $\mathcal{S}$ , i.e.  $\mathcal{S}^h \subset \mathcal{S}$ . The Galerkin form of the problem is: Find  $u^h \in \mathcal{S}^h$  such that for all  $v^h \in \mathcal{S}^h$

$$a(u^h, v^h) = L(v^h). \quad (4)$$

It is well known that (4) is a well-posed problem and has a unique solution.

By approximating  $u_h$  and  $v_h$  using splines (which will be discussed in the next section) basis functions  $N_i$ ,  $i = 1, 2, \dots, n_h$ , where  $n_h = \mathcal{O}(h^{-2})$ , the variational formulation (4) is transformed in to a set of linear algebraic equations

$$A\mathbf{u} = \mathbf{f}, \quad (5)$$

where  $A$  denotes the stiffness matrix obtained from the bilinear form  $a(\cdot, \cdot)$ , i.e.

$$A = (a_{i,j}) = (a(N_i, N_j)), \quad i, j = 1, 2, 3, \dots, n_h,$$

$\mathbf{u}$  denotes the vector of unknown degrees of freedom (DOF), and  $\mathbf{f}$  denotes the right hand side (RHS) vector from the known data of the problem. It is clear that  $A$  is a real symmetric positive definite matrix.

### 3. Splines and their condition number bounds

Non-uniform rational B-Splines are commonly used in isogeometric analysis. Since NURBS are built from B-Splines, a discussion of B-Splines is a natural starting point. We give a brief description of B-Splines and their properties. This is followed by the introduction of NURBS, and the bounds on the condition number of these basis functions are discussed next.

#### 3.1. B-Splines and NURBS

**Definition 1.** Let  $\Xi_1 = \{\xi_i : i = 1, \dots, n+p+1\}$  be a non-decreasing sequence of real numbers called the knot vector, where  $\xi_i$  is the  $i^{\text{th}}$  knot,  $p$  is the polynomial degree, and  $n$  is the number of basis function. With a knot vector in hand, the B-Spline basis functions denoted by  $N_i^p(\xi)$  are (recursively) defined starting with a piecewise constant ( $p = 0$ )

$$N_i^0(\xi) = \begin{cases} 1 & \text{if } \xi \in [\xi_i, \xi_{i+1}), \\ 0 & \text{otherwise,} \end{cases} \quad (6a)$$

$$N_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi), \quad (6b)$$

where  $0 \leq i \leq n, p \geq 1$  and  $\frac{0}{0}$  is considered as zero.

The above is usually referred as the *Cox-de Boor recursion formula*, see e.g. [14]. For a B-Spline basis function of degree  $p$ , an interior knot can be repeated at most  $p$  times, and the boundary knots can be repeated at most  $p + 1$  times. A knot vector for which the two boundary knots are repeated  $p + 1$  times is said to be open. In this case, the basis functions are interpolatory at the first and the last knot. Important properties of the B-Spline basis functions include nonnegativity, partition of unity, local support and  $C^{p-k}$ -continuity.

**Definition 2.** A B-Spline curve  $C(\xi)$ , is defined by

$$C(\xi) = \sum_{i=1}^n P_i N_i^p(\xi) \quad (7)$$

where  $\{P_i : i = 1, \dots, n\}$  are the control points and  $N_i^p$  are B-Spline basis functions defined in (6).

The previous definitions are easily generalized to the higher dimensional cases by means of tensor product. Using tensor product of one-dimensional B-Spline functions, a B-Spline surface  $S(\xi, \eta)$  is defined as follows:

$$S(\xi, \eta) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} N_{i,j}^{p_1,p_2}(\xi, \eta) P_{i,j}, \quad (8)$$

where  $P_{i,j}, i = 1, 2, \dots, n_1, j = 1, 2, \dots, n_2$ , denote the control points,  $N_{i,j}^{p_1,p_2}$  is the tensor product of B-Spline basis functions  $N_i^{p_1}$  and  $N_j^{p_2}$ , and  $\Xi_1 = \{\xi_1, \xi_2, \dots, \xi_{n_1+p_1+1}\}$  and  $\Xi_2 = \{\eta_1, \eta_2, \dots, \eta_{n_2+p_2+1}\}$  are the corresponding knot vectors. Similarly B-Spline solids can be defined.

It is known that polynomials can not exactly describe frequently encountered shapes in engineering, particularly the conic family, e.g. circle. While B-Splines (polynomials) are flexible and have many nice properties for curve design, they are also incapable of representing such curves exactly. Such limitations are overcome by NURBS functions which can be used to exactly represent a wide array of objects. Rational representation of conics originates from projective geometry. The “coordinates” in the additional dimension are called weights, which we shall denote by  $w$ . Furthermore, let  $\{P_i^w\}$  be a set of control points for a projective B-Spline curve in  $\mathbb{R}^3$ . For the desired NURBS curve in  $\mathbb{R}^2$ , the weights and the control points are derived by the relations

$$w_i = (P_i^w)_3, \quad (P_i)_d = (P_i^w)/w_i, \quad d = 1, 2, \quad (9)$$

where  $w_i$  is called the  $i^{\text{th}}$  weight and  $(P_i)_d$  is the  $d^{\text{th}}$ -dimension component of the vector  $P_i$ . The weight function  $w(\xi)$  is defined as

$$w(\xi) = \sum_{i=1}^n N_i^p(\xi) w_i. \quad (10)$$

Then, the NURBS basis functions and curve are defined by

$$R_i^p(\xi) = \frac{N_i^p(\xi) w_i}{w(\xi)}, \quad C(\xi) = \sum_{i=1}^n R_i^p(\xi) P_i. \quad (11)$$

The NURBS surfaces are analogously defined as follows

$$S(\xi, \eta) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} R_{i,j}^{p_1,p_2}(\xi, \eta) P_{i,j}, \quad (12)$$

where  $R_{i,j}^{p_1,p_2}$  is the tensor product of NURBS basis functions  $R_i^{p_1}$  and  $R_j^{p_2}$ . NURBS functions also satisfy the properties of B-Spline functions. For a detailed exposition see, e.g. [38, 39, 43].

### 3.2. Derivatives of Splines

Derivatives of splines, see e.g. [20], and their conditioning are very important for the estimation of the condition number of the stiffness matrix. The recursive definition of B-Spline functions allow us to seek the relationship between the derivative of a B-Spline basis function and lower degree basis function.

**Definition 3.** *The derivative of  $i^{\text{th}}$  B-Spline basis function defined in (6), is given by*

$$\frac{d}{d\xi} N_i^p(\xi) = \frac{p}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) - \frac{p}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi). \quad (13)$$

By repeating differentiation of (13) we get the general formula for any order derivative. Since we are interested in the first derivative only, we ignore further details.

The derivatives of rational functions will clearly depend on the derivatives of their non-rational counterpart. Definition 3 can be generalized for NURBS as follows.

**Definition 4.** *The derivative of  $i^{\text{th}}$  NURBS basis function is given by*

$$\frac{d}{d\xi} R_i^p(\xi) = w_i \frac{w(\xi) \frac{d}{d\xi} N_i^p(\xi) - \frac{d}{d\xi} w(\xi) N_i^p(\xi)}{(w(\xi))^2}. \quad (14)$$

where  $w_i$  and  $w(\xi)$  are defined in (9) and (10), respectively.

### 3.3. Condition number of B-Splines

To bound the condition number of matrices resulting from isogeometric discretizations, first we need to know the bounds on spline basis functions in  $L_s$ -norm<sup>1</sup>, where  $s \in [1, \infty]$ , which is briefly discussed in this section. We estimate the size of the coefficients of a polynomial of degree  $p$  in two dimensions when it is represented using the tensor product structure of B-Spline basis functions. The condition number of a basis can be defined as follows.

**Definition 5.** A basis  $\{N_i\}$  of a normed linear space is said to be stable with respect to a vector norm if there are constants  $K_1$  and  $K_2$  such that for all coefficients  $\{v_i\}$  the following relation holds

$$K_1^{-1} \|\{v_i\}\| \leq \left\| \sum_i v_i N_i \right\| \leq K_2 \|\{v_i\}\|. \quad (15)$$

The number  $\kappa = K_1 K_2$ , with  $K_1$  and  $K_2$  as small as possible, is called the condition number of  $\{N_i\}$  with respect to  $\|\cdot\|$ . Note that we use the symbols  $\|\cdot\|$  and  $\|\{\cdot\}\|$  for the norms in the vector space and the vector norm, respectively.

Such condition numbers give an upper bound for magnification of error in the coefficients to the function values. Indeed, if  $f = \sum_i f_i N_i \neq 0$  and  $g = \sum_i g_i N_i$ , then it follows immediately from (15) that

$$\frac{\|f - g\|}{\|f\|} \leq \kappa \frac{\|\{f_i - g_i\}\|}{\|\{f_i\}\|}.$$

More details on the approximation properties and the stability of B-Splines can be found in, e.g., [23, 24, 25, 29, 30, 35, 37]. We shall use these estimates on  $\kappa$  to estimate the bounds on the condition number of the stiffness matrix and the mass matrix.

It is of central importance for working with B-Spline basis functions that its condition number is bounded independently of the underlying knot sequence. That is, the condition number of B-Splines does not depend on the multiplicity of the knots of knot vector. This fact was proved by de Boor in 1968 for the sup-norm and in 1973 for any  $L_s$ -norm, see [11, 12, 13, 16]. In [12] he gave the direct estimate that the worst condition number of a B-Spline of degree  $p$  with respect to  $s$ -norm is bounded above by  $p9^p$ , and conjectured that the real value of  $\kappa$  grows like  $2^p$ , which is seen far better than the direct estimate:

$$\kappa < p9^p \quad (\text{direct estimate}), \quad (16a)$$

$$\kappa \sim 2^p \quad (\text{conjecture}). \quad (16b)$$

In [15], de Boor discussed that the exact condition number of B-Spline basis may be hard to determine. Scherer and Shadrin proved that the upper bound of the condition number of a B-Spline of degree  $p$  with respect to  $s$ -norm is bounded by  $p^{\frac{1}{2}}4^p$ , see [41], i.e.

$$\kappa < p^{\frac{1}{2}}4^p. \quad (17)$$

This improves de Boor's estimate  $\kappa < p9^p$  and stands closer to his conjecture that  $\kappa \sim 2^p$ . Later, in [42], Scherer and Shadrin proved the following result.

**Lemma 6.** For all  $p$  and all  $s \in [1, \infty]$ ,

$$\kappa < p2^p. \quad (18)$$

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<sup>1</sup>Note that we use the subscript 's' to represent the norm in  $L_s$ -space because we reserve 'p' to represent the degree of the spline basis functions.

The above lemma confirms the de Boor's conjecture up to a polynomial factor. Further possible approaches by which the polynomial factor could be removed are also discussed in [42].

The above one-dimensional B-Spline condition number can be easily generalized to  $d$ -dimensions. For the tensor product B-Spline basis of degree  $p$  in  $d$ -dimensions, using (18) one obtains the condition number estimate as follows

$$\kappa < (p2^p)^d. \quad (19)$$

#### 4. Estimates of condition number

This section is devoted to the estimates for the condition number of the stiffness matrix and the mass matrix obtained from isogeometric discretization. We study the condition number of the stiffness matrix and the mass matrix with respect to  $h$ -refinement and  $p$ -refinement. For  $h$ -refinement, upper and lower bounds for the extremal eigenvalues and the condition number are given, whereas for  $p$ -refinement we prove upper and lower bounds for the maximum eigenvalue, lower bounds for the minimum eigenvalue, and upper bounds for the condition number. The constant  $C$  will be used often in this section for the generic constant that may take different values at different occasions, and is independent of  $h$  and  $p$  in the analysis with respect to  $h$ -refinement and  $p$ -refinement, respectively.

##### 4.1. Stiffness matrix

###### 4.1.1. $h$ -refinement

For simplicity, we begin with a two-dimensional domain. Let  $\Omega := (0, 1)^2$  be an open parametric domain which we will refer as a *patch*. Assume that two open knot vectors  $\Xi_1 := \{0 = \xi_1, \xi_2, \xi_3, \dots, \xi_{m_1} = 1\}$  and  $\Xi_2 := \{0 = \eta_1, \eta_2, \eta_3, \dots, \eta_{m_2} = 1\}$  are given. Associated with  $\Xi_1$  and  $\Xi_2$ , we partition the patch  $\Omega$  into a mesh

$$\mathcal{Q}_h := \{Q = (\xi_i, \xi_{i+1}) \otimes (\eta_j, \eta_{j+1}), i = p_1 + 1, 2, \dots, m_1 - p_1 - 1, j = p_2 + 1, 2, \dots, m_2 - p_2 - 1\},$$

where  $Q$  is a two-dimensional open knot-span whose diameter is denoted by  $h_Q$ . We consider a family of quasi-uniform meshes  $\{\mathcal{Q}_h\}_h$  on  $\Omega$ , where  $h = \max\{h_Q | Q \in \mathcal{Q}_h\}$  denotes the family index, see [5]. Furthermore, let  $\mathcal{S}_h$  denote the B-spline space associated with the mesh  $\mathcal{Q}_h$ . Given two adjacent elements  $Q_1$  and  $Q_2$ , by  $m_{Q_1 Q_2}$  we denote the number of continuous derivatives across their common face  $\partial Q_1 \cap \partial Q_2$ . In the analysis, we will use the following Sobolev space of order  $m \in \mathbb{N}$

$$\begin{aligned} \mathcal{H}^m(\Omega) := & \left\{ v \in L^2(\Omega) \text{ such that } v|_Q \in H^m(Q), \forall Q \in \mathcal{Q}_h, \text{ and} \right. \\ & \nabla^i(v|_{Q_1}) = \nabla^i(v|_{Q_2}) \text{ on } \partial Q_1 \cap \partial Q_2, \\ & \left. \forall i \in \mathbb{N} \text{ with } 0 \leq i \leq \min\{m_{Q_1 Q_2}, m - 1\}, \forall Q_1, Q_2 \text{ with } \partial Q_1 \cap \partial Q_2 \neq \emptyset \right\}, \end{aligned} \quad (20)$$

where  $\nabla^i$  has the usual meaning of  $i^{\text{th}}$ -order partial derivative, and  $H^m$  is the usual Sobolev space of order  $m$ . The space  $\mathcal{H}^m$  is equipped with the following semi-norms and norm

$$|v|_{\mathcal{H}^i(\Omega)}^2 := \sum_{Q \in \mathcal{Q}_h} |v|_{H^i(Q)}^2, \quad 0 \leq i \leq m, \quad \|v\|_{\mathcal{H}^m(\Omega)}^2 := \sum_{i=0}^m |v|_{\mathcal{H}^i(\Omega)}^2.$$

On a regular mesh of size  $h$ , the condition number of the finite element equations for a second-order elliptic boundary value problem can be obtained using *inverse estimates*, see the classical texts e.g. [2, 7, 8, 9]. Therefore, the similar inverse estimates are of interest in isogeometric framework using NURBS basis functions. To keep the article self-contained, we recall some results from [5, 43].



**Theorem 7.** Let  $\mathcal{S}_h$  be the spline space consisting of piecewise polynomials of degree  $p$  associated with uniform partitions. Then there exists a constant  $C = C(\text{shape})$ , such that for all  $0 \leq l \leq m$ ,

$$\|v\|_{\mathcal{H}^m(\Omega)} \leq Ch^{l-m} \|v\|_{\mathcal{H}^l(\Omega)}, \quad \forall v \in \mathcal{S}_h. \quad (21)$$

The proof of the above theorem, for a particular case  $m = 2$  and  $l = 1$ , is given in [5]. More general inverse inequalities can be easily derived following the same approach. By taking  $m = 1$  and  $l = 0$ , the following can be easily derived from (21)

$$a(v, v) = \int_{\Omega} |\nabla v|^2 \leq Ch^{-2} \|v\|^2. \quad (22)$$

Under suitable conditions the condition number related to elliptic problems in finite element analysis scales as  $h^{-2}$ , see e.g. [19, 28, 44]. We prove the similar result for the stiffness matrix arising in isogeometric discretization. To prove that we first shall prove the following result.

**Lemma 8.** There exist constants  $C_1$  and  $C_2$  independent of  $h$  (but may depend on  $p$ ), such that for all  $v = \sum_{i=1}^{n_h} v_i N_i \in \mathcal{S}_h$ , we have

$$C_1 h^2 \|\{v_i\}\|^2 \leq \left\| \sum_{i=1}^{n_h} v_i N_i \right\|^2 \leq C_2 h^2 \|\{v_i\}\|^2. \quad (23)$$

*Proof.* We only consider the non-trivial case, i.e. there exists some  $i$  for which  $v_i \neq 0$ . For any  $Q \in \mathcal{Q}_h$ , there are  $(p+1)^2$  basis functions with non-zero support. Let  $\mathcal{I}_h^Q \equiv \{i_1^Q, i_2^Q, \dots, i_{p+1}^Q\} \times \{j_1^Q, j_2^Q, \dots, j_{p+1}^Q\} \subset \{1, 2, \dots, n_h\}$  denote the index set for the basis functions which have non-zero support in  $Q$ . Also, let  $\bar{v}_q = \max_{i \in \mathcal{I}_h^Q} |v_i|$  and  $\bar{v} = \max_{i=1,2,\dots,n_h} |v_i|$ . Now using positivity and partition of unity properties of basis functions, the right hand side inequality can be proved as follows:

$$\begin{aligned} \|v\|^2 &= \sum_{Q \in \mathcal{Q}_h} \int_Q v^2 = \sum_{Q \in \mathcal{Q}_h} \int_Q \left( \sum_{i \in \mathcal{I}_h^Q} v_i N_i \right)^2 \leq \sum_{Q \in \mathcal{Q}_h} \int_Q \left( \bar{v}_q \sum_{i \in \mathcal{I}_h^Q} N_i \right)^2 \\ &\leq \sum_{Q \in \mathcal{Q}_h} \int_Q \bar{v}_q^2 \leq \sum_{Q \in \mathcal{Q}_h} h_Q^2 \bar{v}_q^2 \leq \sum_{Q \in \mathcal{Q}_h} h_Q^2 \sum_{i \in \mathcal{I}_h^Q} v_i^2 \\ &\leq h^2 \sum_{Q \in \mathcal{Q}_h} \sum_{i \in \mathcal{I}_h^Q} v_i^2 \leq C_2 h^2 \sum_{i=1}^{n_h} v_i^2 = C_2 h^2 \|\{v_i\}\|^2. \end{aligned}$$

For the left hand side inequality, we have

$$\begin{aligned} h^2 \|\{v_i\}\|^2 &= h^2 \sum_{i=1}^{n_h} v_i^2 \leq h^2 \sum_{i=1}^{n_h} \bar{v}^2 = h^2 n_h \bar{v}^2 \leq h^2 \left( \frac{C}{h} \right)^2 \bar{v}^2 = C^2 \bar{v}^2 \\ &= C^2 \|\{v_i\}\|_{L^\infty}^2 \leq C^2 K_1^2 \|v\|_{L^\infty}^2 \quad \left( \text{using (15), } K_1^{-1} \|\{v_i\}\|_{L^\infty} \leq \left\| \sum v_i N_i \right\|_{L^\infty} \right) \\ &\leq C^2 K_1^2 \|v\|^2. \end{aligned}$$

The result then follows by taking  $C_1 = \left( \frac{1}{C^2 K_1^2} \right)$ . □

We now turn to the problem of obtaining bounds on the extremal eigenvalues and the condition number. The main result concerning the condition number of the stiffness matrix is the following.

**Theorem 9.** *Let  $A$  be the stiffness matrix, i.e.  $A = (a_{ij})$ , where  $a_{ij} = a(N_i, N_j) = \int_{\Omega} \nabla N_i \cdot \nabla N_j$ , then the bounds on  $\lambda_{\max}$  and  $\lambda_{\min}$  are given by*

$$k_1 \leq \lambda_{\max} \leq k_2, \quad \text{and} \quad k_3 h^2 \leq \lambda_{\min} \leq k_4 h^2,$$

where  $k_1, k_2, k_3$  and  $k_4$  are constants independent of  $h$ . Furthermore, the bounds on  $\kappa(A)$  are given by

$$c_1 h^{-2} \leq \kappa(A) \leq c_2 h^{-2},$$

where  $c_1$  and  $c_2$  are constants independent of  $h$ .

*Proof.* Let  $v = \sum_{i=1}^{n_h} v_i N_i$ . Then  $a(v, v) = \{v_i\} \cdot A\{v_i\}$ , where  $\{v_i\} = \{v_1, v_2, \dots, v_{n_h}\}$ . Using inverse estimate (22) we get

$$\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = \frac{a(v, v)}{\|\{v_i\}\|^2} \leq \frac{Ch^{-2}\|v\|^2}{\|\{v_i\}\|^2} =: \bar{G}.$$

Let  $\bar{G}$  be the supremum of  $\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2}$ . Using (23) we can get the following upper and lower bounds on  $\bar{G}$

$$\begin{aligned} \bar{G} &= \frac{Ch^{-2}\|v\|^2}{\|\{v_i\}\|^2} \geq \frac{Ch^{-2}C_1 h^2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = CC_1 = k_1 \\ &\Rightarrow \sup \sup \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = k_1, \\ \bar{G} &= \frac{Ch^{-2}\|v\|^2}{\|\{v_i\}\|^2} \leq \frac{Ch^{-2}C_2 h^2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = CC_2 = k_2 \\ &\Rightarrow \inf \sup \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = k_2. \end{aligned}$$

Therefore

$$k_1 \leq \sup_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \leq k_2,$$

which implies

$$k_1 \leq \lambda_{\max} \leq k_2. \tag{24}$$

On the other hand, for the bounds on  $\lambda_{\min}$ , by using coercivity of bilinear form  $a(v, v)$  we get

$$\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = \frac{a(v, v)}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|_{H^1}^2}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|^2}{\|\{v_i\}\|^2} =: \underline{G}.$$

Assume  $\underline{G}$  is the infimum of  $\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2}$ . Using again (23), we get

$$\underline{G} = \frac{\alpha_1 \|v\|^2}{\|\{v_i\}\|^2} \geq \frac{\alpha_1 C_1 h^2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \alpha_1 C_1 h^2 = k_3 h^2$$

$$\begin{aligned}
&\Rightarrow \sup \inf \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = k_3 h^2, \\
\bar{G} &= \frac{\alpha_1 \|v\|^2}{\|\{v_i\}\|^2} \leq \frac{\alpha_1 C_2 h^2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \alpha_1 C_2 h^2 = k_4 h^2 \\
&\Rightarrow \inf \inf \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = k_4 h^2,
\end{aligned}$$

which implies

$$k_3 h^2 \leq \inf_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \leq k_4 h^2.$$

Therefore, we have

$$k_3 h^2 \leq \lambda_{\min} \leq k_4 h^2. \quad (25)$$

The condition number of the stiffness matrix is given by

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}}, \text{ where } \lambda_{\max} = \max_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2}, \text{ and } \lambda_{\min} = \min_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2}.$$

From (24) and (25), we get

$$c_1 h^{-2} \leq \kappa(A) \leq c_2 h^{-2}, \quad (26)$$

which concludes the proof.  $\square$

Table 1:  $\lambda_{\max}$  and  $\lambda_{\min}$  for  $\kappa(A)$

$h^{-1}$		2	4	8	16	32	64
$p = 2$							
$C^0$	$\lambda_{\max}$	2.1726	2.5607	2.6436	2.6612	2.6653	2.6663
	$\lambda_{\min}$	0.2929	0.2008	0.0726	0.0190	0.0048	0.0012
$C^1$	$\lambda_{\max}$	1.4222	1.4238	1.4896	1.4951	1.4991	1.4997
	$\lambda_{\min}$	0.3556	0.3556	0.2855	0.0756	0.0192	0.0048
$p = 3$							
$C^0$	$\lambda_{\max}$	2.1297	2.2415	2.2844	2.2961	2.2992	2.2999
	$\lambda_{\min}$	0.0284	0.0210	0.0190	0.0085	0.0021	0.0005
$C^1$	$\lambda_{\max}$	0.8962	1.1705	1.1910	1.2078	1.2129	1.2142
	$\lambda_{\min}$	0.0386	0.0386	0.0386	0.0191	0.0048	0.0012
$C^2$	$\lambda_{\max}$	1.0384	1.3698	1.5247	1.5627	1.5720	1.5743
	$\lambda_{\min}$	0.0336	0.0464	0.0522	0.0547	0.0191	0.0048
$p = 4$							
$C^0$	$\lambda_{\max}$	2.1002	2.1105	2.1174	2.1195	2.1200	2.1202
	$\lambda_{\min}$	0.0024	0.0019	0.0018	0.0017	0.0012	0.0003
$C^1$	$\lambda_{\max}$	0.8752	1.0840	1.1452	1.1606	1.1644	1.1654
	$\lambda_{\min}$	0.0030	0.0030	0.0030	0.0030	0.0021	0.0005
$C^2$	$\lambda_{\max}$	0.6780	0.9178	0.9847	1.0059	1.0118	1.0133
	$\lambda_{\min}$	0.0040	0.0048	0.0051	0.0052	0.0047	0.0012
$C^3$	$\lambda_{\max}$	0.9369	1.3334	1.7182	1.8111	1.8311	1.8357
	$\lambda_{\min}$	0.0028	0.0050	0.0072	0.0081	0.0085	0.0048

In Table 1 the extremal eigenvalues of the stiffness matrix, using basis functions with continuity from  $C^0$  to  $C^{p-1}$ , are given. The extremal eigenvalues satisfy the theoretical estimates given above, i.e. the maximum eigenvalues are independent of  $h$ , and the minimum eigenvalues asymptotically scale as  $h^{-2}$ .

#### 4.1.2. $p$ -refinement

In this section we estimate the upper bound for the condition number as a function of  $p$ . Without loss of generality, we assume a single element mesh, i.e.,  $Q = \Omega = (0, 1)^2$ . We denote  $\mathcal{S}_p$  the tensor product space of spline functions of degree  $p$ . We set next some basic technical lemmas which will be needed later on. The following lemma is well known generalization of a theorem of Markov due to Hill, Szechuan and Tamarkin, see [6, 36].

**Lemma 10** (Schmidt's inequality). *There exists a constant  $C$  (independent of  $p$ ) such that for any polynomial  $f(x)$  of degree  $p$  we have*

$$\int_{-1}^1 (f'(x))^2 dx \leq Cp^4 \int_{-1}^1 (f(x))^2 dx. \quad (27)$$

No such constant  $C$  exists so that (27) holds for all  $f(x)$  with the exponent smaller than 4.

Let  $I = (-1, 1)$ . Then using (27), we have

$$\int_I \left( \frac{dN_p(\xi)}{d\xi} \right)^2 d\xi \leq Cp^4 \int_I (N_p(\xi))^2 d\xi. \quad (28)$$

Now using (28), we get the following

$$\begin{aligned} \int_{\Omega} \nabla N_p(\xi, \eta) \cdot \nabla N_p(\xi, \eta) d\xi d\eta &= \int_I \int_I \left[ \left( \frac{\partial N_p(\xi, \eta)}{\partial \xi} \right)^2 + \left( \frac{\partial N_p(\xi, \eta)}{\partial \eta} \right)^2 \right] d\xi d\eta \\ &\leq Cp^4 \int_{\Omega} (N_p(\xi, \eta))^2 d\xi d\eta. \end{aligned} \quad (29)$$

From this we can have a similar result like in Lemma 8 for the  $p$ -refinement. Moreover, the following estimate directly follows from Schmidt's inequality and (29)

$$a(v, v) = \int_{\Omega} |\nabla v|^2 \leq Cp^4 \|v\|^2. \quad (30)$$

**Lemma 11.** *There exist constants  $C_1$  and  $C_2$  (independent of  $p$ ), such that for all*

$v = \sum_{i=1}^{n_p} v_i N_i \in \mathcal{S}_p$ , we have

$$\frac{C_1}{(p^2 4^p)^2} \|\{v_i\}\|^2 \leq \left\| \sum_{i=1}^{n_p} v_i N_i \right\|^2 \leq C_2 \|\{v_i\}\|^2, \quad (31)$$

*Proof.* From the stability of B-Splines there exists a constant  $\gamma$ , which depends on the degree  $p$ , such that

$$\left\| \sum_{i=1}^{n_p} v_i N_i \right\| \leq \|\{v_i\}\| \leq \gamma \left\| \sum_{i=1}^{n_p} v_i N_i \right\|, \quad (32)$$

where  $\gamma = p^2 4^p$ , following (19). In the estimate (31) the right hand side inequality follows easily from nonnegativity and partition of unity properties of basis functions, and the left hand side inequality follows from (32).  $\square$

Now we prove the following result, analogous to Theorem 9, for the  $p$ -refinement of isogeometric discretization.

**Theorem 12.** *Let  $\{N_i\}$  be a set of basis function of  $\mathcal{S}_p$  on a unit square. Then the following upper bound on  $\kappa(A)$  holds*

$$\kappa(A) \leq Cp^8 16^p.$$

*Proof.* We prove this theorem following the same approach as for the  $h$ -refinement estimates.

Let  $v = \sum_{i=1}^{n_p} v_i N_i$ , where  $\{v_i\} = \{v_1, v_2, \dots, v_{n_p}\}$ . Now using (30) and (31), we get

$$\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = \frac{a(v, v)}{\|\{v_i\}\|^2} \leq \frac{Cp^4 \|v\|^2}{\|\{v_i\}\|^2} \leq \frac{Cp^4 C_2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = CC_2 p^4 = Cp^4,$$

which implies that

$$\lambda_{\max} = \max_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \leq Cp^4. \quad (33)$$

To prove the lower bound for  $\lambda_{\min}$  we use (31) and coercivity of bilinear form,

$$\begin{aligned} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} &= \frac{a(v, v)}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|_{H^1}^2}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|^2}{\|\{v_i\}\|^2} \\ &\geq \frac{\alpha \frac{C_1}{(p^2 4^p)^2} \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \frac{\alpha C_1}{(p^2 4^p)^2} = \frac{C}{(p^4 16^p)}, \end{aligned}$$

which implies that

$$\lambda_{\min} = \min_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \geq \frac{C}{(p^4 16^p)}. \quad (34)$$

From (33) and (34), we get the desired result, i.e.

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{Cp^4}{\left(\frac{C}{(p^4 16^p)}\right)} \leq C(p^8 16^p).$$

□

**Remark 13.** *The above result can be easily generalized for higher dimensions. The bound for condition number of the stiffness matrix for  $d$ -dimensional problem is given by  $(p^{4+2d} 4^{pd})$ .*

Though, in the above theorem we proved upper bound on the maximum eigenvalue of the stiffness matrix using B-Spline basis functions, this is independent of the choice of the basis functions (holds for all kind of basis functions irrespective to their nature). However, from numerical experiments using B-Spline basis functions, we observe that  $\lambda_{\max}$  is uniformly bounded and is independent of  $p$ . This motivates us for further investigations.

The lower bound on the minimum eigenvalues depends on the stability of the B-Spline basis functions, which cannot be improved further (specially beyond the de Boor's conjecture). On the other hand the upper bound on the maximum eigenvalue directly depends on the upper bound of bilinear form  $a(v, v)$ . Therefore, we shall improve the bound for  $a(v, v)$  given in (30). From the derivative of a B-Spline basis function given in (13) we can obtain a new upper bound on the maximum eigenvalue of the stiffness matrix, which is  $\lambda_{\max} \leq Cp^2$ , see Appendix.

However, this bound still is not independent of  $p$ . To further improve this bound, we proceed as follows.

We have the B-Spline basis functions of degree  $p$  in one variable  $\xi$  on a unit length interval

$$N_{i,\xi}^p = (-1)^i \binom{p}{i} (\xi - 1)^{p-i} \xi^i, \quad i = 0, 1, 2, \dots, p.$$

Similarly in variable  $\eta$ , we have

$$N_{j,\eta}^p = (-1)^j \binom{p}{j} (\eta - 1)^{p-j} \eta^j, \quad j = 0, 1, 2, \dots, p.$$

B-Spline basis functions in two variables on a unit square element is simply given by the tensor product:

$$N_{i,j,\xi,\eta}^{p,p} = (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j}, \quad i, j = 0, 1, 2, \dots, p.$$

We prove the following lemma for the diagonal entries of the stiffness matrix.

**Lemma 14.** *There exists a constant  $C$  independent of  $p$ , such that*

$$(A_{(i,j),(i,j)}) = a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) = \int_0^1 \int_0^1 \nabla N_{i,j,\xi,\eta}^{p,p} \cdot \nabla N_{i,j,\xi,\eta}^{p,p} d\xi d\eta \leq C. \quad (35)$$

*Proof.* For all  $i, j = 0, 1, 2, \dots, p$  we have

$$\begin{aligned} a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) &= \int_0^1 \int_0^1 \nabla N_{i,j,\xi,\eta}^{p,p} \cdot \nabla N_{i,j,\xi,\eta}^{p,p} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left\{ \left( \frac{\partial}{\partial \xi} N_{i,j,\xi,\eta}^{p,p} \right)^2 + \left( \frac{\partial}{\partial \eta} N_{i,j,\xi,\eta}^{p,p} \right)^2 \right\} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left[ \left\{ \frac{\partial}{\partial \xi} \left( (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} \right) \right\}^2 + \right. \\ &\quad \left. \left\{ \frac{\partial}{\partial \eta} \left( (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} \right) \right\}^2 \right] d\xi d\eta \\ &= \int_0^1 \int_0^1 \left[ \left\{ \binom{p}{i} \binom{p}{j} \eta^j (\eta - 1)^{p-j} \frac{\partial}{\partial \xi} (\xi^i (\xi - 1)^{p-i}) \right\}^2 + \right. \\ &\quad \left. \left\{ \binom{p}{i} \binom{p}{j} \xi^i (\xi - 1)^{p-i} \frac{\partial}{\partial \eta} (\eta^j (\eta - 1)^{p-j}) \right\}^2 \right] d\xi d\eta \\ &= \int_0^1 \int_0^1 \left[ \left\{ \binom{p}{i} \binom{p}{j} \eta^j (\eta - 1)^{p-j} (i \xi^{i-1} (\xi - 1)^{p-i} + (p-i) \xi^i (\xi - 1)^{p-i-1}) \right\}^2 + \right. \\ &\quad \left. \left\{ \binom{p}{i} \binom{p}{j} \xi^i (\xi - 1)^{p-i} (j \eta^{j-1} (\eta - 1)^{p-j} + (p-j) \eta^j (\eta - 1)^{p-j-1}) \right\}^2 \right] d\xi d\eta \\ &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left[ \{ i \xi^{i-1} \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} + \right. \\ &\quad \left. (p-i) \xi^i \eta^j (\xi - 1)^{p-i-1} (\eta - 1)^{p-j} \}^2 \right. \\ &\quad \left. + \{ j \xi^i \eta^{j-1} (\xi - 1)^{p-i} (\eta - 1)^{p-j} + (p-j) \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j-1} \}^2 \right] d\xi d\eta \\ &= I + II, \end{aligned}$$

where

$$\begin{aligned}
I &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left\{ i \xi^{i-1} \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} + \right. \\
&\quad \left. (p-i) \xi^i \eta^j (\xi - 1)^{p-i-1} (\eta - 1)^{p-j} \right\}^2 d\xi d\eta, \\
II &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left\{ j \xi^i \eta^{j-1} (\xi - 1)^{p-i} (\eta - 1)^{p-j} + \right. \\
&\quad \left. (p-j) \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j-1} \right\}^2 d\xi d\eta.
\end{aligned}$$

Now,

$$\begin{aligned}
I &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left\{ i \xi^{i-1} \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} + \right. \\
&\quad \left. (p-i) \xi^i \eta^j (\xi - 1)^{p-i-1} (\eta - 1)^{p-j} \right\}^2 d\xi d\eta \\
&= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 (i^2 \xi^{2(i-1)} \eta^{2j} (\xi - 1)^{2(p-i)} (\eta - 1)^{2(p-j)}) d\xi d\eta + \\
&\quad \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 ((p-i)^2 \xi^{2i} \eta^{2j} (\xi - 1)^{2(p-i-1)} (\eta - 1)^{2(p-j)}) d\xi d\eta + \\
&\quad \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 (2i(p-i) \xi^{2i-1} \eta^{2j} (\xi - 1)^{2p-2i-1} (\eta - 1)^{2(p-j)}) d\xi d\eta \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 (i^2 \xi^{2(i-1)} \eta^{2j} (\xi - 1)^{2(p-i)} (\eta - 1)^{2(p-j)}) d\xi d\eta \\
&= \binom{p}{i}^2 \binom{p}{j}^2 i^2 \underbrace{\left( \int_0^1 (\xi^{2(i-1)} (\xi - 1)^{2(p-i)}) d\xi \right)}_{=: I_{11}} \underbrace{\left( \int_0^1 (\eta^{2j} (\eta - 1)^{2(p-j)}) d\eta \right)}_{=: I_{12}} \\
&= \binom{p}{i}^2 \binom{p}{j}^2 i^2 I_{11} I_{12}, \\
I_2 &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 ((p-i)^2 \xi^{2i} \eta^{2j} (\xi - 1)^{2(p-i-1)} (\eta - 1)^{2(p-j)}) d\xi d\eta \\
&= \binom{p}{i}^2 \binom{p}{j}^2 (p-i)^2 \underbrace{\left( \int_0^1 (\xi^{2i} (\xi - 1)^{2(p-i-1)}) d\xi \right)}_{=: I_{21}} \underbrace{\left( \int_0^1 (\eta^{2j} (\eta - 1)^{2(p-j)}) d\eta \right)}_{=: I_{22}} \\
&= \binom{p}{i}^2 \binom{p}{j}^2 (p-i)^2 I_{21} I_{22}, \text{ and} \\
I_3 &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 (2i(p-i) \xi^{2i-1} \eta^{2j} (\xi - 1)^{2p-2i-1} (\eta - 1)^{2(p-j)}) d\xi d\eta \\
&= \binom{p}{i}^2 \binom{p}{j}^2 2i(p-i) \underbrace{\left( \int_0^1 (\xi^{2i-1} (\xi - 1)^{2p-2i-1}) d\xi \right)}_{=: I_{31}} \underbrace{\left( \int_0^1 (\eta^{2j} (\eta - 1)^{2(p-j)}) d\eta \right)}_{=: I_{32}} \\
&= \binom{p}{i}^2 \binom{p}{j}^2 2i(p-i) I_{31} I_{32}.
\end{aligned}$$

We now compute the above integrals.

Case  $I_1$ : Clearly, for  $i = 0$ ,  $I_1 = 0$  and for  $i = 1, 2, 3, \dots, p$ ,

$$I_{11} = \int_0^1 (\xi^{2(i-1)}(\xi - 1)^{2(p-i)}) d\xi = (-1)^{2(p-i)} \int_0^1 \xi^{(2i-1)-1}(1 - \xi)^{(2p-2i+1)-1} d\xi,$$

using the integration formula for beta function, i.e.  $\int_0^1 t^{n-1}(1-t)^{m-1} dt = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$ , we get

$$\begin{aligned} I_{11} &= \frac{\Gamma(2i-1)\Gamma(2p-2i+1)}{\Gamma(2p)} = \frac{(2i-2)!(2p-2i)!}{(2p-1)!} = \frac{(2i)!(2p-2i)!}{(2p-1)!(4i^2-2i)}, \\ I_{12} &= \int_0^1 (\eta^{2j}(\eta-1)^{2(p-j)}) d\eta = (-1)^{2(p-j)} \int_0^1 \eta^{(2j+1)-1}(1-\eta)^{(2p-2j+1)-1} d\eta \\ &= \frac{\Gamma(2j+1)\Gamma(2p-2j+1)}{\Gamma(2p+2)} = \frac{(2j)!(2p-2j)!}{(2p+1)!}, \end{aligned}$$

which implies

$$\begin{aligned} I_1 &= \binom{p}{i}^2 \binom{p}{j}^2 i^2 \frac{(2i)!(2p-2i)!}{(2p-1)!(4i^2-2i)} \frac{(2j)!(2p-2j)!}{(2p+1)!} \\ &\leq \frac{1}{2} \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!} \left( \text{since } \frac{i^2}{4i^2-2i} \leq \frac{1}{2} \right). \end{aligned}$$

Case  $I_2$ : For  $i = p$ , we get  $I_2 = 0$  and for  $i = 0, 1, 2, \dots, (p-1)$ ,

$$\begin{aligned} I_{21} &= \int_0^1 (\xi^{2i}(\xi-1)^{2(p-i-1)}) d\xi = (-1)^{2(p-i-1)} \int_0^1 \xi^{(2i+1)-1}(1-\xi)^{(2p-2i-1)-1} d\xi \\ &= \frac{\Gamma(2i+1)\Gamma(2p-2i-1)}{\Gamma(2p)} = \frac{(2i)!(2p-2i-2)!}{(2p-1)!} = \frac{(2i)!(2p-2i)!}{(2p-1)!(4(p-i)^2-2(p-i))}, \\ I_{22} &= I_{12} = \frac{(2j)!(2p-2j)!}{(2p+1)!}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} I_2 &= \binom{p}{i}^2 \binom{p}{j}^2 (p-i)^2 \frac{(2i)!(2p-2i)!}{(2p-1)!(4(p-i)^2-2(p-i))} \frac{(2j)!(2p-2j)!}{(2p+1)!} \\ &\leq \frac{1}{2} \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!} \left( \text{since } \frac{(p-i)^2}{4(p-i)^2-2(p-i)} \leq \frac{1}{2} \right). \end{aligned}$$

Case  $I_3$ : Clearly,  $I_3 = 0$  for  $i = 0$  and  $i = p$ , and for  $i = 1, 2, 3, \dots, (p-1)$ ,

$$\begin{aligned} I_{31} &= \int_0^1 (\xi^{2i-1}(\xi-1)^{2p-2i-1}) d\xi = (-1)^{2(p-i)-1} \int_0^1 \xi^{(2i)-1}(\xi-1)^{(2p-2i)-1} d\xi \\ &= -\frac{\Gamma(2i)\Gamma(2p-2i)}{\Gamma(2p)} = -\frac{(2i-1)!(2p-2i-1)!}{(2p-1)!} = -\frac{(2i)!(2p-2i)!}{(2p-1)!(4i(p-i))}, \\ I_{32} &= I_{12} = \frac{(2j)!(2p-2j)!}{(2p+1)!}, \end{aligned}$$



which implies

$$\begin{aligned} I_3 &= -\binom{p}{i}^2 \binom{p}{j}^2 2i(p-i) \frac{(2i)!(2p-2i)!}{(2p-1)!(4i(p-i))} \frac{(2j)!(2p-2j)!}{(2p+1)!} \\ &= -\frac{1}{2} \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!}. \end{aligned}$$

Now, for all  $i = 0, 1, 2, \dots, p$ ,

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= \begin{cases} I_2, & \text{if } i = 0, \\ I_1 + I_2 + I_3, & \text{if } i = 1, 2, \dots, p-1, \\ I_1, & \text{if } i = p, \end{cases} \\ &\leq \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!} \\ &= \left\{ \binom{p}{i}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \right\} \left\{ \binom{p}{j}^2 \frac{(2j)!(2p-2j)!}{(2p+1)!} \right\} \\ &= \frac{(2p)}{(2p+1)} \left\{ \binom{p}{i}^2 \frac{(2i)!(2p-2i)!}{(2p)!} \right\} \left\{ \binom{p}{j}^2 \frac{(2j)!(2p-2j)!}{(2p)!} \right\} \\ &\leq \left\{ \binom{p}{i}^2 \frac{(2i)!(2p-2i)!}{(2p)!} \right\} \left\{ \binom{p}{j}^2 \frac{(2j)!(2p-2j)!}{(2p)!} \right\} = I_a I_b, \text{ where} \\ I_a &= \binom{p}{i}^2 \frac{(2i)!(2p-2i)!}{(2p)!} = \frac{p!p!}{i!i!(p-i)!(p-i)!} \frac{(2i)!(2p-2i)!}{(2p)!}, \\ I_b &= \binom{p}{j}^2 \frac{(2j)!(2p-2j)!}{(2p)!} = \frac{p!p!}{j!j!(p-j)!(p-j)!} \frac{(2j)!(2p-2j)!}{(2p)!}. \end{aligned}$$

Further, we seek the upper bound for  $I_a$ . We prove that  $I_a \leq C$  by induction on  $p$ , where  $C$  is a constant independent of  $p$ . For  $p = 1$ , we have  $I_a = 1$  for all  $i = 0, 1$ , i.e. the result holds for the base case. Assume that the result holds for  $p = m$  and for all  $i = 0, 1, 2, \dots, m$ , i.e.

$$\frac{m!m!}{i!i!(m-i)!(m-i)!} \frac{(2i)!(2m-2i)!}{(2m)!} \leq C. \quad (36)$$

Now we shall show that the result holds for  $p = m + 1$  and for all  $i = 0, 1, 2, \dots, m + 1$ . We have

$$\begin{aligned} &\frac{(m+1)!(m+1)!}{i!i!(m+1-i)!(m+1-i)!} \frac{(2i)!(2(m+1)-2i)!}{(2(m+1))!} \\ &= \begin{cases} \frac{(m+1)m!(m+1)m!}{i!i!(m+1-i)(m-i)!(m+1-i)(m-i)!} \frac{(2i)!(2(m-i+1))(2(m-i)+1)(2m-2i)!}{(2(m+1))(2m+1)(2m)!}, & \text{if } i = 0, 1, 2, \dots, m, \\ \frac{(m+1)!(m+1)!}{(m+1)!(m+1)!(0)!(0)!} \frac{(2(m+1))!(0)!}{(2(m+1))!}, & \text{if } i = m+1, \end{cases} \\ &= \begin{cases} \left( \frac{m^2+2m+1}{4m^2+6m+2} \right) \left( \frac{4(m-i)^2+6(m-i)+2}{(m-i)^2+2(m-i)+1} \right) \left\{ \frac{m!m!}{i!i!(m-i)!(m-i)!} \frac{(2i)!(2m-2i)!}{(2m)!} \right\}, & \text{if } i = 0, 1, 2, \dots, m, \\ 1, & \text{if } i = m+1. \end{cases} \end{aligned}$$

Using (36) and since  $\left(\frac{m^2 + 2m + 1}{4m^2 + 6m + 2}\right) \left(\frac{4(m-i)^2 + 6(m-i) + 2}{(m-i)^2 + 2(m-i) + 1}\right) \leq 1$ , we get for all  $i = 0, 1, 2, \dots, m + 1$ ,

$$\frac{(m+1)!(m+1)!}{i!i!(m+1-i)!(m+1-i)!} \frac{(2i)!(2(m+1)-2i)!}{(2(m+1))!} \leq C.$$

We now have  $I_a \leq C$ , where  $C$  is a constant independent of  $p$ . Similarly we can obtain that  $I_b \leq C$ . Therefore

$$I = I_a I_b \leq C.$$

Proceeding in the same way for  $II$ , we can prove that

$$II \leq C.$$

Therefore,

$$a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) = I + II \leq C,$$

which concludes the proof.  $\square$

Thus, we have proved that  $a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})$  is bounded by a constant independent of  $p$ . Since the upper bound of the diagonal entries is the upper bound of all the entries of the stiffness matrix, the maximum entry of the stiffness matrix is bounded by a constant independent of  $p$ , i.e.

$$a(N_{i,j,\xi,\eta}^{p,q}, N_{k,l,\xi,\eta}^{p,q}) \leq C. \quad (37)$$

Similarly, we can prove for three dimensional problem that

$$a(N_{i,j,k,\xi,\eta,\zeta}^{p,q,r}, N_{l,m,n,\xi,\eta,\zeta}^{p,q,r}) \leq C. \quad (38)$$

Using (37) and (38) we can give the following result.

**Lemma 15.** *The maximum eigenvalue of the stiffness matrix  $A$  can be bounded below by a constant  $C$  (independent of  $p$ ), i.e.*

$$\lambda_{\max}(A) \geq C.$$

*Proof.* We prove this by using the basics of matrix norms. The max-norm of a matrix is the element-wise norm defined by

$$\|A\|_{\max} = \max\{|a_{ij}|\}.$$

From (35), we have

$$\max\{|a_{ij}|\} = C,$$

where  $C$  is independent of  $p$ . By the equivalence of norms we have

$$\|A\|_2 \geq \|A\|_{\max} = C,$$

which implies

$$\lambda_{\max}(A) \geq C. \quad \square$$

To bound  $\lambda_{\max}$  from above we bound the spectral norm by the  $\ell_1$ -norm in the following lemma.

**Lemma 16.** *For any fixed  $k$  and  $l$  such that  $0 \leq k, l \leq p$  and for any  $0 \leq i, j \leq p$  we have*

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{k,l,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where  $C$  is a constant independent of  $p$ .

*Proof.* We have

$$N_{0,0,\xi,\eta}^{p,p} = (1 - \xi)^p(1 - \eta)^p.$$

We first prove

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where  $C$  is a constant independent of  $p$ . We have

$$\begin{aligned} & a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) \\ &= \int_0^1 \int_0^1 \nabla N_{0,0,\xi,\eta}^{p,p} \cdot \nabla N_{i,j,\xi,\eta}^{p,p} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left\{ \left( \frac{\partial}{\partial \xi} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \xi} N_{i,j,\xi,\eta}^{p,p} \right) + \left( \frac{\partial}{\partial \eta} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \eta} N_{i,j,\xi,\eta}^{p,p} \right) \right\} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left( \frac{\partial}{\partial \xi} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \xi} N_{i,j,\xi,\eta}^{p,p} \right) d\xi d\eta + \int_0^1 \int_0^1 \left( \frac{\partial}{\partial \eta} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \eta} N_{i,j,\xi,\eta}^{p,p} \right) d\xi d\eta \\ &= I + II, \end{aligned}$$

where

$$I = \int_0^1 \int_0^1 \left( \frac{\partial}{\partial \xi} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \xi} N_{i,j,\xi,\eta}^{p,p} \right) d\xi d\eta,$$

and

$$II = \int_0^1 \int_0^1 \left( \frac{\partial}{\partial \eta} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \eta} N_{i,j,\xi,\eta}^{p,p} \right) d\xi d\eta.$$

Now

$$\begin{aligned} I &= \int_0^1 \int_0^1 (-p(1 - \xi)^{p-1}(1 - \eta)^p) \\ &\quad \left( \binom{p}{i} \binom{p}{j} \eta^j (1 - \eta)^{p-j} (i\xi^{i-1}(1 - \xi)^{p-i} - (p - i)\xi^i(1 - \xi)^{p-i-1}) \right) d\xi d\eta \\ &= -p \binom{p}{i} \binom{p}{j} \left( \int_0^1 \eta^j (1 - \eta)^{2p-j} d\eta \right) \\ &\quad \left( \int_0^1 i\xi^{i-1}(1 - \xi)^{2p-i-1} d\xi - \int_0^1 (p - i)\xi^i(1 - \xi)^{2p-i-2} d\xi \right) \\ &= -p \binom{p}{i} \binom{p}{j} (I_1) (I_2 - I_3), \end{aligned}$$

where

$$I_1 = \int_0^1 \eta^j (1 - \eta)^{2p-j} d\eta, \quad I_2 = \int_0^1 i\xi^{i-1}(1 - \xi)^{2p-i-1} d\xi,$$

and

$$I_3 = \int_0^1 (p - i)\xi^i(1 - \xi)^{2p-i-2} d\xi.$$

For  $i = 0, 1, 2, \dots, p$  we have

$$\begin{aligned} I_1 &= \int_0^1 \eta^j (1 - \eta)^{2p-j} d\eta = \frac{\Gamma(j+1)\Gamma(2p-j+1)}{\Gamma(2p+2)} \\ &= \frac{(j)!(2p-j)!}{(2p+1)!} = \frac{1}{(2p+1)} \frac{(j)!(2p-j)!}{(2p)!}. \end{aligned}$$

Clearly,  $I_2 = 0$  if  $i = 0$ . For  $i = 1, 2, \dots, p$

$$\begin{aligned} I_2 &= \int_0^1 i\xi^{i-1}(1-\xi)^{2p-i-1} d\xi = i \frac{\Gamma(i)\Gamma(2p-i)}{\Gamma(2p)} \\ &= i \frac{(i-1)!(2p-i-1)!}{(2p-1)!} = \frac{2p}{(2p-i)} \frac{(i)!(2p-i)!}{(2p)!}. \end{aligned}$$

For  $i = p$  we get  $I_3 = 0$  and for  $i = 0, 1, \dots, p-1$

$$\begin{aligned} I_3 &= \int_0^1 (p-i)\xi^i(1-\xi)^{2p-i-2} d\xi \\ &= (p-i) \frac{\Gamma(i+1)\Gamma(2p-i-1)}{\Gamma(2p)} \\ &= (p-i) \frac{(i)!(2p-i-2)!}{(2p-1)!} = \frac{(p-i)(2p)}{(2p-i)(2p-i-1)} \frac{(i)!(2p-i)!}{(2p)!}. \end{aligned}$$

Therefore we have

$$I = \begin{cases} \binom{p}{j} \frac{p^2}{(4p^2-1)} \frac{(j)!(2p-j)!}{(2p)!}, & \text{if } i = 0, \\ -p \binom{p}{i} \binom{p}{j} \frac{2p}{(2p+1)} \frac{(i)!(2p-i)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!} \frac{1}{(2p-i)} \left(1 - \frac{(p-i)}{(2p-i-1)}\right), & \text{if } i = 1, 2, \dots, p-1, \\ -\binom{p}{j} \frac{2p}{(2p+1)} \frac{(p)!(p)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!}, & \text{if } i = p. \end{cases}$$

Similar expression can be easily obtained for  $II$ . We are interested to calculate the sum

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})|.$$

For  $i = 0$ , we have

$$\begin{aligned} \sum_{j=0}^p |I| &= \sum_{j=0}^p \binom{p}{j} \frac{p^2}{(4p^2-1)} \frac{(j)!(2p-j)!}{(2p)!} \\ &< \frac{1}{3} \sum_{j=0}^p \binom{p}{j} \frac{(j)!(2p-j)!}{(2p)!} \left( \text{since } \frac{p^2}{(4p^2-1)} \leq \frac{1}{3} \right) \\ &= \frac{1}{3} \sum_{j=0}^p \frac{(p)!(2p-j)!}{(2p)!(p-j)!} < \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{p!p!}{(2p)!} \right) < 1. \end{aligned}$$

For  $i = 1, 2, \dots, p-1$ , we have

$$\begin{aligned} &\sum_{i=1}^{p-1} \sum_{j=0}^p |I| \\ &= \sum_{i=1}^{p-1} \sum_{j=0}^p \binom{p}{i} \binom{p}{j} \frac{2p^2}{(2p+1)} \frac{(i)!(2p-i)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!} \frac{1}{(2p-i)} \left( 1 - \frac{(p-i)}{(2p-i-1)} \right) \end{aligned}$$

$$\begin{aligned}
&< \sum_{i=1}^{p-1} \frac{p!(2p-i)!}{(2p)!(p-i)!} \frac{p(p-1)}{(2p-i)(2p-i-1)} \sum_{j=0}^p \frac{p!(2p-j)!}{(2p)!(p-j)!} \left( \text{since } \frac{2p}{(2p+1)} \leq 1 \right) \\
&= \sum_{i=1}^{p-1} \frac{p!(2p-i)!}{(2p)!(p-i)!} \frac{p(p-1)}{(2p-i)(2p-i-1)} \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{p!p!}{(2p)!} \right) \\
&< \sum_{i=1}^{p-1} \frac{p!(2p-i)!}{(2p)!(p-i)!} \frac{2p(p-1)}{(2p-i)(2p-i-1)} \left( \text{since } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{p!p!}{(2p)!} < 2 \right) \\
&= \sum_{i=1}^{p-1} \frac{p-1}{2p-1} \frac{p!(2p-i-2)!}{(2p-2)!(p-i)!} < \frac{1}{2} \sum_{i=1}^{p-1} \frac{p!(2p-i-2)!}{(2p-2)!(p-i)!} \\
&< \frac{1}{2} \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{p!(p-1)!}{(2p-2)!} \right) < 1.
\end{aligned}$$

For  $i = p$ , we have

$$\begin{aligned}
\sum_{j=0}^p |I| &= \sum_{j=0}^p \binom{p}{j} \frac{2p}{(2p+1)} \frac{(p)!(p)!(j)!(2p-j)!}{(2p)!(2p)!} < \sum_{j=0}^p \frac{p!p!(2p-j)!}{(2p)!(2p)!(p-j)!} \\
&< \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{(p!)^4}{((2p)!)^2} \right) < 1.
\end{aligned}$$

Therefore, we have

$$\sum_{i=0}^p \sum_{j=0}^p |I| < C, \tag{39}$$

where  $C$  is independent of  $p$ . Similarly, we can get

$$\sum_{i=0}^p \sum_{j=0}^p |II| < C. \tag{40}$$

Therefore, from (39) and (40) we get

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C, \tag{41}$$

where  $C$  is a constant independent of  $p$ .

The above gives us the absolute row-sum for the first row of the stiffness matrix. Since on a uniform mesh the absolute rowsum for all rows of the stiffness matrix are of the same order upto a constant, we get the desired result, i.e.

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{k,l,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

for any fixed  $k$  and  $l$  such that  $0 \leq k, l \leq p$  and for any  $0 \leq i, j \leq p$ , where  $C$  is a constant independent of  $p$ .  $\square$

Similar results can be obtained for higher dimensions. The next lemma, which gives an upper bound for the maximum eigenvalue, is a direct consequence of the above lemma.

**Lemma 17.** *The maximum eigenvalue of the stiffness matrix  $A$  can be bounded above by a constant  $C$ , independent of  $p$ , i.e.*

$$\lambda_{\max}(A) \leq C.$$

*Proof.* We have

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{k,l,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where  $C$  is a constant independent of  $p$ , which implies  $\|A\|_1 \leq C$ . We use the following inequality between matrix norms

$$\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}.$$

Since  $A$  is symmetric matrix, we have  $\|A\|_1 = \|A\|_\infty$ , which implies

$$\|A\|_2 \leq \|A\|_1 \leq C.$$

Therefore,

$$\lambda_{\max}(A) \leq C.$$

□

From Lemma 15 and Lemma 17, we have the following result.

**Lemma 18.**  $\lambda_{\max}(A) = C$ , where  $C$  is a constant independent of  $p$ .

**Corollary 19.** For two dimensional problem the improved upper bound for the condition number of the stiffness matrix  $A$  is given as follows

$$\kappa(A) \leq C(p^2 4^p)^2 = Cp^4 16^p. \quad (42)$$

This result can be easily generalized for higher dimensions, the bound for  $d$ -dimensional problem is given as follows

$$\kappa(A) \leq p^{2d} 4^{pd}. \quad (43)$$

**Remark 20.** We have used the condition number of B-Splines  $\kappa \sim p^{2p}$  in reaching the above estimates. If we use the de Boor's conjecture (the condition number of B-Splines  $\kappa \sim 2^p$ ) instead, then the upper bound of the stiffness matrix can be further improved and given by

$$\kappa(A) \leq 4^{pd}. \quad (44)$$

## 4.2. Mass matrix

### 4.2.1. $h$ -refinement

We now estimate the condition number of the mass matrix. Let  $M$  be the mass matrix, i.e.  $M = (m_{ij})$ , where

$$m_{ij} = (N_i, N_j) = \int_{\Omega} N_i N_j \quad i, j = 1, 2, \dots, n_h.$$

The following lemma gives estimates on the maximum and minimum eigenvalues of the mass matrix with respect to  $h$ .

**Lemma 21.** For the extremal eigenvalues of the mass matrix  $M = (m_{ij}) = (N_i, N_j)$ , we have the following estimates

$$C_1 h^2 \leq \lambda_{\min} \leq \lambda_{\max} \leq C_2 h^2,$$

where  $C_1, C_2$  are constants independent of  $h$ . Furthermore

$$c_1 \leq \kappa(M) \leq c_2,$$

where  $c_1, c_2$  are constants independent of  $h$ .

*Proof.* By recalling the result from (23), we can bound both the extremal eigenvalues of the mass matrix. For the minimum eigenvalue, we have

$$\frac{\{v_i\} \cdot M\{v_i\}}{\|\{v_i\}\|^2} = \frac{(v, v)}{\|\{v_i\}\|^2} \geq \frac{C_1 h^2 \|\{v\}\|^2}{\|\{v_i\}\|^2} = C_1 h^2.$$

On the other hand, for the maximum eigenvalue we have

$$\frac{\{v_i\} \cdot M\{v_i\}}{\|\{v_i\}\|^2} = \frac{(v, v)}{\|\{v_i\}\|^2} \leq \frac{C_2 h^2 \|\{v\}\|^2}{\|\{v_i\}\|^2} = C_2 h^2.$$

Therefore we have

$$C_1 h^2 \leq \lambda_{\min} \leq \lambda_{\max} \leq C_2 h^2,$$

which implies

$$c_1 \leq \kappa(M) \leq c_2.$$

□

#### 4.2.2. $p$ -refinement

In this section we estimate the bounds on the extremal eigenvalues and the condition number of the mass matrices for  $p$ -refinement. We first prove the following lemma

**Lemma 22.** *The mass matrix is a positive matrix, and all the entries of the mass matrix are bounded above by  $\frac{C}{(2p+1)^2}$ , where  $C$  is a constant independent of  $p$ .*

*Proof.* We have

$$\begin{aligned} (M_{(i,j),(k,l)}) &= (N_{i,j,\xi,\eta}^{p,p}, N_{k,l,\xi,\eta}^{p,p}) = \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} \cdot N_{k,l,\xi,\eta}^{p,p} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left( (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi-1)^{p-i} (\eta-1)^{p-j} \right) \\ &\quad \left( (-1)^{k+l} \binom{p}{k} \binom{p}{l} \xi^k \eta^l (\xi-1)^{p-k} (\eta-1)^{p-l} \right) d\xi d\eta \\ &= \left( (-1)^{i+j+k+l} \binom{p}{i} \binom{p}{j} \binom{p}{k} \binom{p}{l} \right) \\ &\quad \int_0^1 \int_0^1 (\xi^{i+k} \eta^{j+l} (\xi-1)^{2p-i-k} (\eta-1)^{2p-j-l}) d\xi d\eta \\ &= \left( (-1)^{i+j+k+l} (-1)^{2p-i-k} (-1)^{2p-j-l} \binom{p}{i} \binom{p}{j} \binom{p}{k} \binom{p}{l} \right) \\ &\quad \left( \int_0^1 \xi^{i+k} (1-\xi)^{2p-i-k} d\xi \right) \left( \int_0^1 \eta^{j+l} (1-\eta)^{2p-j-l} d\eta \right) \\ &= \binom{p}{i} \binom{p}{j} \binom{p}{k} \binom{p}{l} \left( \int_0^1 \xi^{(i+k+1)-1} (1-\xi)^{(2p-i-k+1)-1} d\xi \right) \\ &\quad \left( \int_0^1 \eta^{(j+l+1)-1} (1-\eta)^{(2p-j-l+1)-1} d\eta \right) \\ &= (I) (II), \end{aligned}$$

where

$$\begin{aligned}
I &= \binom{p}{i} \binom{p}{k} \left( \int_0^1 \xi^{(i+k+1)-1} (1-\xi)^{(2p-i-k+1)-1} d\xi d\eta \right) \\
&= \binom{p}{i} \binom{p}{k} \frac{\Gamma(i+k+1)\Gamma(2p-i-k+1)}{\Gamma(2p+2)} \\
&= \frac{p!p!}{i!k!(p-i)!(p-k)!} \frac{(i+k)!(2p-i-k)!}{(2p+1)!} \\
&= \frac{1}{2p+1} \underbrace{\left\{ \frac{p!p!}{i!k!(p-i)!(p-k)!} \frac{(i+k)!(2p-i-k)!}{(2p)!} \right\}}_{I_1} = \frac{1}{2p+1} I_1,
\end{aligned}$$

and

$$\begin{aligned}
II &= \binom{p}{j} \binom{p}{l} \left( \int_0^1 \eta^{(j+l+1)-1} (1-\eta)^{(2p-j-l+1)-1} d\xi d\eta \right) \\
&= \binom{p}{j} \binom{p}{l} \frac{\Gamma(j+l+1)\Gamma(2p-j-l+1)}{\Gamma(2p+2)} \\
&= \frac{p!p!}{j!l!(p-j)!(p-l)!} \frac{(j+l)!(2p-j-l)!}{(2p+1)!} \\
&= \frac{1}{2p+1} \underbrace{\left\{ \frac{p!p!}{j!l!(p-j)!(p-l)!} \frac{(j+l)!(2p-j-l)!}{(2p)!} \right\}}_{II_1} = \frac{1}{2p+1} II_1.
\end{aligned}$$

Now, by induction on  $p$ , we can easily obtain that (as we proved in Lemma 14),

$$I_1 = \left\{ \frac{p!p!}{i!k!(p-i)!(p-k)!} \frac{(i+k)!(2p-i-k)!}{(2p)!} \right\} \leq C.$$

Similarly,  $II_1 \leq C$ . Therefore

$$(M_{(i,j),(k,l)}) \leq \frac{C}{(2p+1)^2}. \quad (45)$$

It is also clear that for all  $p \geq 1$  and  $i, k = 0, 1, 2, \dots, p$ ,  $I_1 > 0$ , and  $II_1 > 0$ . This implies that the mass matrix  $(M_{(i,j),(k,l)})$  is a positive matrix.  $\square$

**Lemma 23.** *The maximum eigenvalue of the mass matrix  $M$  can be bounded below as follows*

$$\lambda_{\max}(M) \geq \frac{C}{(2p+1)^2}.$$

*Proof.* Following the proof of Lemma 15 and (45), we get the desired result.  $\square$

To bound  $\lambda_{\max}$  from above we bound the spectral norm by the  $\ell_1$ -norm of the mass matrix. In the following lemma we first compute the  $\ell_1$ -norm of the mass matrix.

**Lemma 24.** *For the mass matrix  $M$  on a unit square element, we have*

$$\|M\|_1 = \frac{1}{(p+1)^2}.$$



*Proof.* We have

$$\begin{aligned}
\|M\|_1 &= \max_{i,j} \sum_{k,l} \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} \cdot N_{k,l,\xi,\eta}^{p,p} d\xi d\eta = \max_{i,j} \sum_{k,l} (N_{i,j,\xi,\eta}^{p,p}, N_{k,l,\xi,\eta}^{p,p}) \\
&= \max_{i,j} (N_{i,j,\xi,\eta}^{p,p}, \sum_{k,l} N_{k,l,\xi,\eta}^{p,p}) = \max_{i,j} (N_{i,j,\xi,\eta}^{p,p}, 1) \quad \left( \text{since } \sum_{k,l} N_{k,l,\xi,\eta}^{p,p} = 1 \right) \\
&= \max_{i,j} \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} d\xi d\eta.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} d\xi d\eta &= \int_0^1 \int_0^1 (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} d\xi d\eta \\
&= \binom{p}{i} \binom{p}{j} \int_0^1 \int_0^1 (-1)^{i+j+p-i+p-j} \xi^i \eta^j (1 - \xi)^{p-i} (1 - \eta)^{p-j} d\xi d\eta \\
&= \binom{p}{i} \binom{p}{j} \left( \int_0^1 \xi^{(i+1)-1} (1 - \xi)^{(p-i+1)-1} d\xi \right) \left( \int_0^1 \eta^{(j+1)-1} (1 - \eta)^{(p-j+1)-1} d\eta \right) \\
&= \binom{p}{i} \binom{p}{j} \left( \frac{\Gamma(i+1)\Gamma(p-i+1)}{\Gamma(p+2)} \right) \left( \frac{\Gamma(j+1)\Gamma(p-j+1)}{\Gamma(p+2)} \right) \\
&= \frac{p!}{i!(p-i)!} \frac{p!}{j!(p-j)!} \frac{i!(p-i)!}{(p+1)!} \frac{j!(p-j)!}{(p+1)!} = \frac{1}{(p+1)^2}.
\end{aligned}$$

The above implies

$$\max_{i,j} \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} d\xi d\eta = \frac{1}{(p+1)^2},$$

which concludes the proof.  $\square$

The symmetry of  $M$  implies

$$\|M\|_\infty = \|M\|_1 = \frac{1}{(p+1)^2}. \tag{46}$$

**Lemma 25.** *The maximum eigenvalue of the mass matrix  $M$  can be bounded above as follows*

$$\lambda_{\max}(M) \leq \frac{1}{(p+1)^2}.$$

*Proof.* We have the following inequality for matrix norms

$$\|M\|_2^2 \leq \|M\|_1 \|M\|_\infty.$$

Using Lemma 24 and (46), we get the bound on the spectral norm of  $M$

$$\|M\|_2 \leq \frac{1}{(p+1)^2},$$

which gives the desired result.  $\square$

**Remark 26.** *In fact, we get  $\lambda_{\max}(M) = \frac{1}{(p+1)^2}$  by Lemma 22 and by [45, Lemma 2.5].*

**Lemma 27.** *There exists a constant  $C$ , independent of  $p$ , such that the minimum eigenvalue of the mass matrix  $M$  can be bounded below as follows*

$$\lambda_{\min}(M) \geq \frac{C}{p^4 16^p}.$$

*Proof.* To bound the minimum eigenvalue from below we use the left hand side inequality of (31). We have

$$\frac{\{v_i\} \cdot M\{v_i\}}{\|\{v_i\}\|^2} = \frac{(v, v)}{\|\{v_i\}\|^2} \geq \frac{\frac{C}{p^4 16^p} \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \frac{C}{p^4 16^p}.$$

Therefore,  $\lambda_{\min}(M) \geq \frac{C}{p^4 16^p}$ , where  $C$  is a constant, independent of  $p$ .  $\square$

The following lemma gives us the upper bound for the condition number of the mass matrix.

**Lemma 28.** *The condition number of the mass matrix  $M$  is bounded above by*

$$\kappa(M) \leq Cp^2 16^p,$$

where  $C$  is a constant, independent of  $p$ .

*Proof.* From Lemma 25 and Lemma 27 we have

$$\frac{C}{p^4 16^p} \leq \lambda_{\min} \leq \lambda_{\max} \leq \frac{1}{(p+1)^2},$$

which gives the desired result.  $\square$

**Remark 29.** *The above bound can be easily generalized for  $d$ -dimensional problem, and is given as follows*

$$\kappa(M) \leq p^{2(d-1)} 4^{pd}. \quad (47)$$

Following Remark 20, by using the de Boor's conjecture, the upper bound for the condition number of the mass matrix can be further improved and given as follows

$$\kappa(M) \leq p^{-2} 4^{pd}. \quad (48)$$

**Remark 30.** *We have done all the analysis on the parametric domain  $(0, 1)^2$ . To get the results for physical domain we can define an invertible NURBS geometrical map from parametric domain to physical domain, and with suitable transformations we get the results for physical domain. For details we refer the article by Bazilevs et al. [5].*

## 5. Numerical results

In this section we provide the numerical results. Apart from the  $h$ -refinement and the  $p$ -refinement, for which we have established theoretical results, we also provide numerical results for the  $r$ -refinement, where we have a possibility to vary the continuity of the basis functions from  $C^0$  to  $C^{p-1}$ . As we shall see, however, the difference between the condition numbers for  $C^0$  and  $C^{p-1}$  continuous basis functions is hardly of order  $p$ , and the results are dominated by the exponent term  $4^{pd}$ .

The numerical discretizations are performed using the Matlab toolbox GeoPDEs [17, 18].

Table 2: Condition number of the stiffness matrix  $A$ 

$p \backslash h^{-1}$	2	4	8	16	32	64
$C^0$ inter element continuity						
2	7.41	12.75	36.40	140.00	554.99	2215.03
3	75.11	106.56	120.34	269.99	1075.42	4297.18
4	881.40	1099.74	1189.14	1214.75	1761.85	7041.29
5	11093.95	12951.15	13680.12	13886.00	13939.10	13952.48
$C^{p-1}$ inter element continuity						
2	4.00	4.00	5.22	19.77	78.14	311.58
3	30.93	29.51	29.19	28.56	82.10	327.21
4	339.92	269.23	240.03	222.55	215.00	381.73
5	4177.20	3220.60	2148.25	1812.58	1700.63	1688.11

### 5.1. $h$ -refinement

For the  $h$ -refinement, the condition number of the stiffness matrix is shown in Table 2 for  $C^0$  and  $C^{p-1}$  continuous basis functions. Numerical results are provided from  $p = 2$  to  $p = 5$ . The results show a different behavior than the classical finite element method for higher  $p$ . In classical finite element method, the condition number of the stiffness matrix is of order  $h^{-2}$  even for a coarse mesh-size, but in isogeometric discretizations, for higher  $p$  the condition number does not appear to be of order  $h^{-2}$  for a coarse mesh-size. This is due to the stability of B-Splines. The condition number of B-Splines heavily depends on polynomial degree (as explained in Sec. 3), and scales as  $(p2^p)^d$ . This factor  $(p2^p)^d$  dominates the factor  $h^{-2}$  for coarse meshes. However, the numerical results support the theoretical findings asymptotically (for reasonably refined meshes) for any polynomial degree.

In Table 3, we present the condition number of the mass matrix for  $C^0$  and  $C^{p-1}$  continuous basis functions. We see that the condition number is bounded uniformly by a constant independent of  $h$ , which confirms the theoretical estimates.

Table 3: Condition number of the mass matrix  $M$ 

$p \backslash h^{-1}$	2	4	8	16	32
$C^0$ inter element continuity					
2	181.43	203.92	208.18	208.50	208.95
3	2295.08	2560.70	2629.67	2641.57	2641.85
4	29390.74	32123.43	32891.28	33071.24	33103.92
5	387089.92	414957.07	422941.81	424969.15	425438.49
$C^{p-1}$ inter element continuity					
2	89.679	109.68	108.51	109.85	111.29
3	915.558	799.941	737.379	708.010	715.89
4	11773.17	6795.46	5381.96	4762.53	4750.07
5	163371.70	77448.11	42580.04	33560.40	32587.27

Table 4:  $\lambda_{\max}$ ,  $\lambda_{\min}$ , and  $\kappa(A)$  for  $p = 2$  to  $p = 15$  on one element

$p$	$\lambda_{\max}$	$\lambda_{\min}$	$\kappa(A)$	$\frac{\kappa_p(A)}{\kappa_{p-1}(A)}$
2	3.5e-01	3.5e-01	1.0e+00	–
3	4.5e-01	3.8e-02	1.1e+01	11.67
4	4.1e-01	2.9e-03	1.3e+02	11.82
5	3.5e-01	2.1e-04	1.6e+03	11.80
6	3.3e-01	1.5e-05	2.1e+04	13.31
7	3.3e-01	1.1e-06	2.9e+05	13.77
8	3.1e-01	7.8e-08	4.0e+06	13.71
9	3.0e-01	5.4e-09	5.6e+07	13.88
10	3.0e-01	3.7e-10	8.1e+08	14.33
11	3.0e-01	2.5e-11	1.1e+10	14.58
12	3.0e-01	1.7e-12	1.7e+11	14.63
13	3.0e-01	1.1e-13	2.5e+12	14.70
14	2.9e-01	8.0e-15	3.7e+13	14.77
15	2.9e-01	5.0e-16	5.5e+14	14.62

## 5.2. $p$ -refinement

We perform numerical experiments to obtain the maximum and minimum eigenvalues, and the condition number of the stiffness matrix and the mass matrix. The eigenvalues and the condition number are obtained on a single unit square element.

In Table 4, we present the extremal eigenvalues and the condition number of the stiffness matrix for  $p = 2$  to  $p = 15$  (for higher  $p$  roundoff errors start contaminating the results). We observe that the maximum eigenvalue scales as a constant independent of  $p$ , and the minimum eigenvalue is bounded from below by the bound given in Theorem 12, i.e.  $\lambda_{\min} \geq C/(p^4 16^p)$ , and the ratio  $\kappa_p(A)/\kappa_{p-1}(A)$  is bounded by 16. In Fig. 1, we plot these results, which confirm the behavior of condition number according to the theoretical estimates.

The extremal eigenvalues and the condition number of the mass matrix for increasing  $p$  are presented in Table 5. In Fig. 2, we plot extremal eigenvalues and the condition number of the

Table 5:  $\lambda_{\max}$ ,  $\lambda_{\min}$ , and  $\kappa(M)$  for  $p = 2$  to  $p = 10$  on one element

$p$	$\lambda_{\max}$	$\lambda_{\min}$	$\kappa(M)$	$\frac{\kappa_p(M)}{\kappa_{p-1}(M)}$
2	1.1e-01	1.1e-03	1.0e+02	–
3	6.2e-02	5.1e-05	1.2e+03	12.25
4	4.0e-02	2.5e-06	1.5e+04	12.96
5	2.7e-02	1.3e-07	2.1e+05	13.44
6	2.0e-02	6.9e-09	2.9e+06	13.80
7	1.5e-02	3.7e-10	4.1e+07	14.06
8	1.2e-02	2.0e-11	5.9e+08	14.27
9	1.0e-02	1.1e-12	8.5e+09	14.44
10	8.2e-03	6.6e-14	1.2e+11	14.58

mass matrix against polynomial degree. We perform numerical results for  $p = 2$  to  $p = 10$  (we stop at  $p = 10$  due to roundoff errors). Numerical results confirm the theoretical estimates given in Lemma 25, Lemma 27, and Lemma 28.

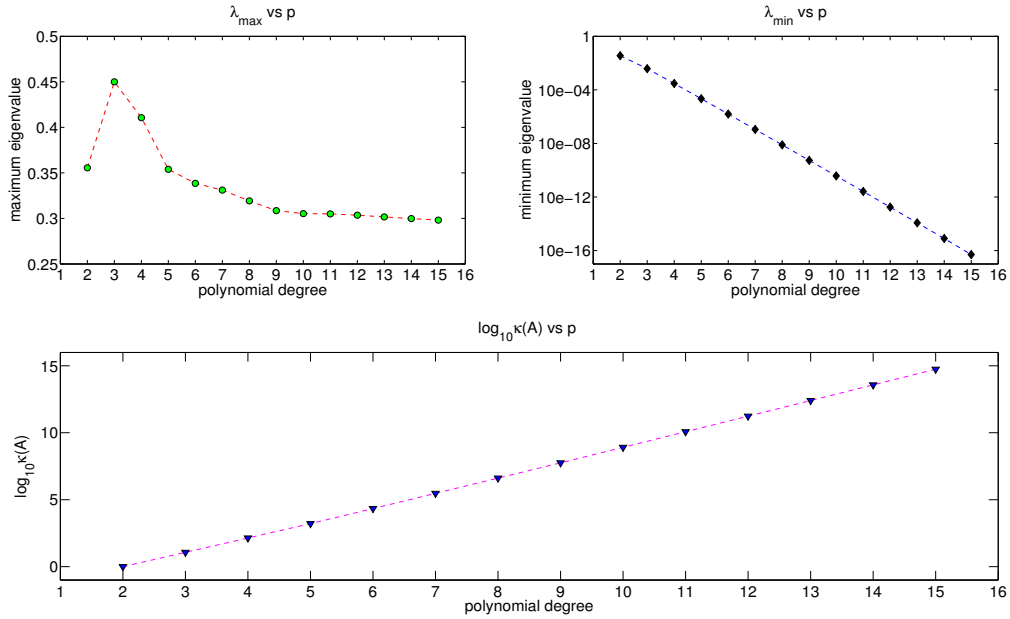


Figure 1: The maximum eigenvalue, the minimum eigenvalue, and the condition number of the stiffness matrix is shown on a square element for increasing  $p$ . In the top-left graph we plot the maximum eigenvalue versus polynomial degree, in the top-right the graph is given for the minimum eigenvalue versus polynomial degree, and at bottom the condition number  $\kappa(A)$  is plotted against polynomial degree.

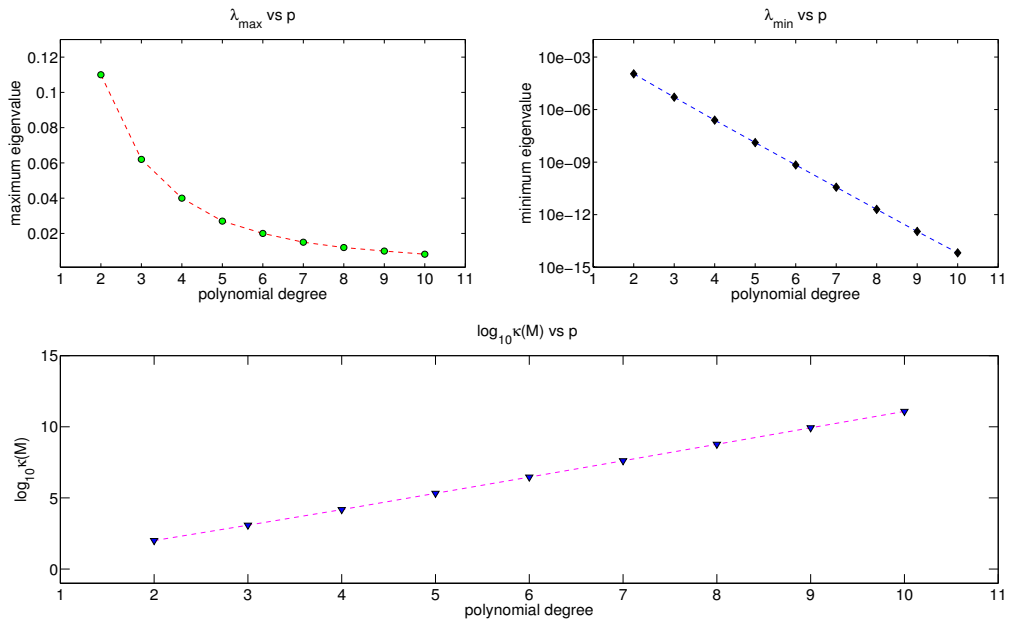


Figure 2: The graphs of the extremal eigenvalues of the mass matrix  $M$  against polynomial degree are given in the top left and top right. At bottom the condition number of  $M$  is plotted against polynomial degree.

### 5.3. $r$ -refinement

In this section we study how the condition number behaves with increasing smoothness of basis functions, i.e., with respect to the  $r$ -refinement. We numerically compute to get the condition number of the stiffness matrix and the mass matrix for increasing smoothness for both, the  $h$ -refinement and the  $p$ -refinement. For  $h$ -refinement and reasonably refined mesh size, we obtain the best (minimum) and the worst (maximum) condition number for  $C^{p-1}$  and  $C^0$ -continuous basis functions, respectively. Though, one would expect the same for  $p$ -refinement, however, from numerical tests we see that the best condition number is obtained for  $C^{\lfloor \frac{p}{2} \rfloor}$  or  $C^{\lceil \frac{p}{2} \rceil}$  continuous basis functions. Here,  $C^{\lfloor \frac{p}{2} \rfloor}$  and  $C^{\lceil \frac{p}{2} \rceil}$  denote the nearest integer value from above and below, respectively.

#### 5.3.1. $h$ -refinement

In Fig. 3 and Fig. 4, we plot the extremal eigenvalues of the stiffness matrix and the mass matrix, respectively for increasing smoothness of basis functions with respect to the  $h$ -refinement. In Table 6-9, we present the  $\kappa(A)$  for varying smoothness of basis functions for  $p = 2$  to  $p = 5$ . In Table 10 the condition number of mass matrix is shown for  $p = 5$ . The following observation can be made from numerical results.

- The condition number of the stiffness matrix is the best for  $C^{\lfloor \frac{p}{2} \rfloor}$  or  $C^{\lceil \frac{p}{2} \rceil}$  for coarse meshes. However, asymptotically the condition number is the best and worst for  $C^{p-1}$  and  $C^0$ -continuous basis functions, respectively. See Table 6-9.
- The maximum eigenvalue of the stiffness matrix is of order one for all cases of smoothness of the basis functions. The minimum eigenvalue of the stiffness matrix keeps moving away from zero with increasing smoothness. See Fig. 3.
- For the mass matrix, the maximum and minimum eigenvalues uniformly keep moving away from zero with increasing smoothness of basis functions, i.e. the condition number is the best and the worst for  $C^{p-1}$  and  $C^0$  continuous basis functions, respectively. See Fig. 4 and Table 10.

#### 5.3.2. $p$ -refinement

For  $p$ -refinement, we perform numerical computations on a  $2 \times 2$  mesh (because there is no varying regularity for one element). In Table 11 and Table 12, we present the condition number of the stiffness matrix and the mass matrix, respectively for  $p = 2$  to  $p = 10$  with increasing continuity from  $C^0$  to  $C^{p-1}$ . In Fig. 5 and Fig. 6, we plot the condition number of the stiffness matrix and the mass matrix, respectively with varying continuity of basis functions with all possible cases (from minimum  $C^0$  to maximum  $C^{p-1}$ ). We have the following observations from numerical results.

Table 6:  $\kappa(A), p = 2$

$h^{-1}$	$C^1$	$C^0$
1	1	1
2	4.00	7.42
4	4.00	12.76
8	5.22	36.41
16	19.77	140.01
32	78.14	554.99
64	311.58	2.2e+03
128	1.2e+03	8.9e+03

Table 7:  $\kappa(A), p = 3$

$h^{-1}$	$C^2$	$C^1$	$C^0$
1	11.67	11.67	11.67
2	30.93	23.23	75.11
4	29.51	30.35	106.56
8	29.19	30.88	120.34
16	28.56	63.20	269.99
32	82.10	252.23	1.1e+03
64	327.21	1.0e+03	4.3e+03
128	1.3e+03	4.0e+03	1.7e+04

Table 8:  $\kappa(A), p = 4$ 

$h^{-1}$	$C^3$	$C^2$	$C^1$	$C^0$
1	137.88	137.88	137.88	137.88
2	339.92	167.95	293.90	881.41
4	269.23	191.80	364.01	1.1e+03
8	240.03	193.02	384.55	1.2e+03
16	222.55	192.22	389.73	1.2e+03
32	215.00	213.24	545.09	1.8e+03
64	381.73	842.41	2.2e+03	7.0e+03
128	1.5e+03	3.4e+03	8.7e+03	2.8e+04

Table 9:  $\kappa(A), p = 5$ 

$h^{-1}$	$C^4$	$C^3$	$C^2$	$C^1$	$C^0$
1	1.6e+03	1.6e+03	1.6e+03	1.6e+03	1.6e+03
2	4.2e+03	1.9e+03	1.7e+03	3.8e+03	1.1e+04
4	3.2e+03	1.3e+03	1.4e+03	4.3e+03	1.3e+04
8	2.1e+03	1.0e+03	1.4e+03	4.5e+03	1.4e+04
16	1.8e+03	975.13	1.3e+03	4.5e+03	1.4e+04
32	1.7e+03	967.04	1.3e+03	4.6e+03	1.4e+04
64	1.7e+03	967.47	1.5e+03	4.6e+03	1.4e+04
128	1.8e+03	2.9e+03	5.8e+03	1.3e+04	4.4e+04

- The condition number of the stiffness matrix and the mass matrix is the worst for the  $C^0$  continuous basis functions. However, for all possible cases of smoothness the condition number of the stiffness matrix and the mass matrix is not the best for maximum smoothness, i.e  $C^{p-1}$  continuous basis functions.
- In Table 11, for  $p = 2$  to  $p = 9$ , the condition number of the stiffness matrix decreases from  $C^0$  to  $C^{\lfloor \frac{p}{2} \rfloor}$  continuous basis functions and then it increases up to  $C^{p-1}$ , and for  $p = 10$  the condition number decreases from  $C^0$  to  $C^{\frac{p}{2}-1}$ , and then increases up to  $C^{p-1}$ .
- The condition number of the mass matrix, for  $p = 2$  to  $p = 6$ , decreases from  $C^0$  to  $C^{\lceil \frac{p}{2} \rceil}$  continuous basis functions, and then it increases up to  $C^{p-1}$ . For  $p = 7$  to  $p = 10$  the condition number decreases from  $C^0$  to  $C^{\lfloor \frac{p}{2} \rfloor}$ , and then increases up to  $C^{p-1}$ .

Table 10:  $\kappa(M)$  for  $h$ -refinement with varying smoothness,  $p = 5$ 

$h^{-1}$	$C^0$	$C^1$	$C^2$	$C^3$	$C^4$
1	2.1e+05	2.1e+05	2.1e+05	2.1e+05	2.1e+05
2	3.9e+05	2.6e+05	1.8e+05	1.1e+05	1.6e+05
4	4.1e+05	2.2e+05	8.2e+04	6.0e+04	7.7e+04
8	4.2e+05	2.1e+05	7.4e+04	4.7e+04	4.3e+04
16	4.2e+05	2.1e+05	7.5e+04	4.5e+04	3.4e+04
32	4.3e+05	2.1e+05	7.5e+04	4.6e+04	3.3e+04
64	4.3e+05	2.1e+05	7.5e+04	4.6e+04	3.3e+04

- In Fig. 5, Fig. 6, Table 11 and Table 12, we see that there is a difference of a factor about  $p$  between the best and the worst condition numbers, which is negligible as the exponent term  $4^{pd}$  is the dominating factor for higher  $p$ .

## 6. Conclusions

We have provided the bounds for the minimum eigenvalue, maximum eigenvalue and the condition numbers of the stiffness matrix and the mass matrix for the Laplace operator with the  $h$ -refinement and the  $p$ -refinement of the isogeometric discretizations based on B-Spline (NURBS) basis functions. We proved that in the  $h$ -refinement, like finite element method, the condition number of the stiffness matrix scales as  $h^{-2}$ , and for the mass matrix it scales as constant independent of  $h$ . For  $p$ -refinement, we have shown that the condition number of the stiffness matrix and the mass matrix grows exponentially in  $p$ .

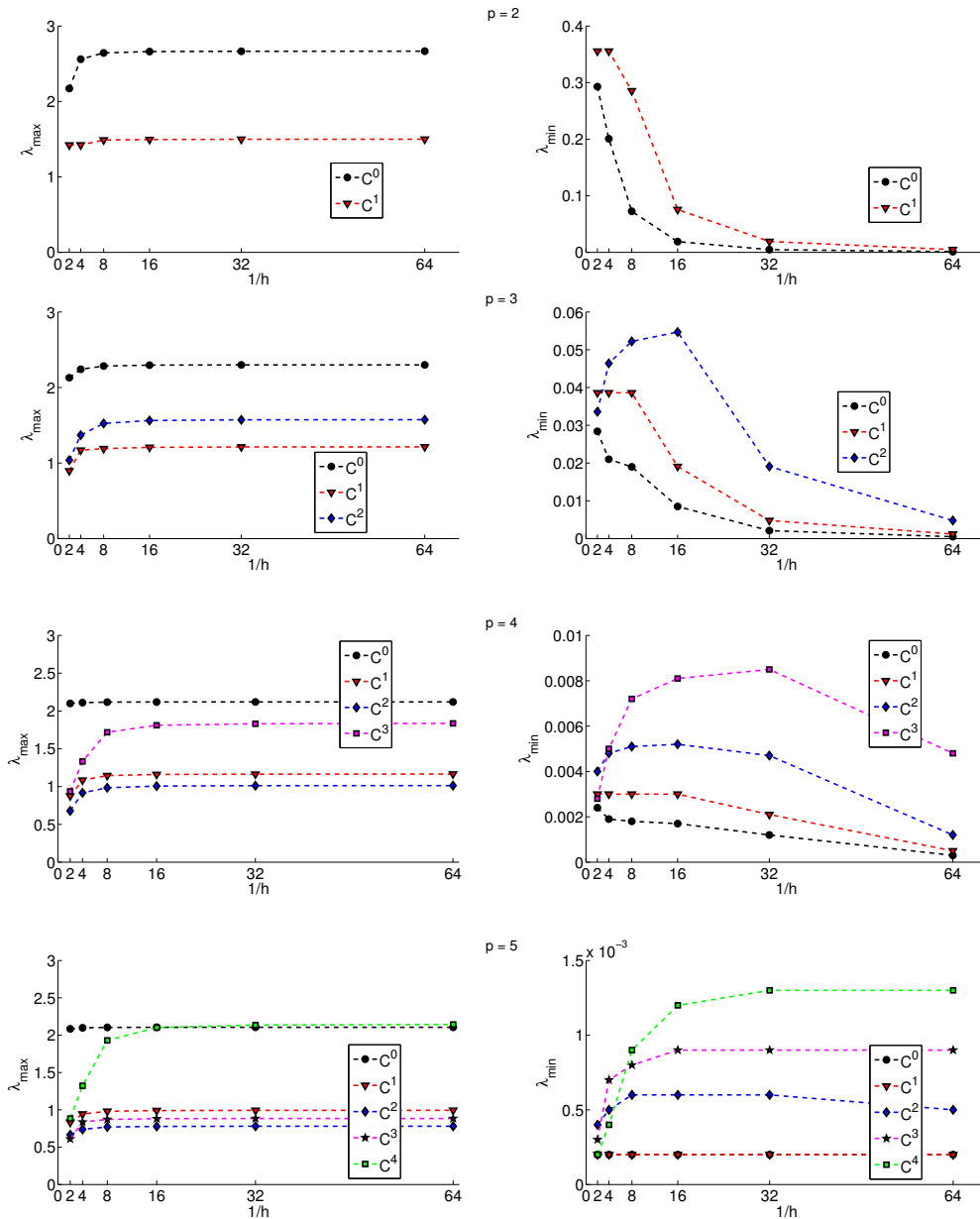


Figure 3: The extremal eigenvalues of the stiffness matrix  $A$  w.r.t. the  $h$ -refinement and varying smoothness.



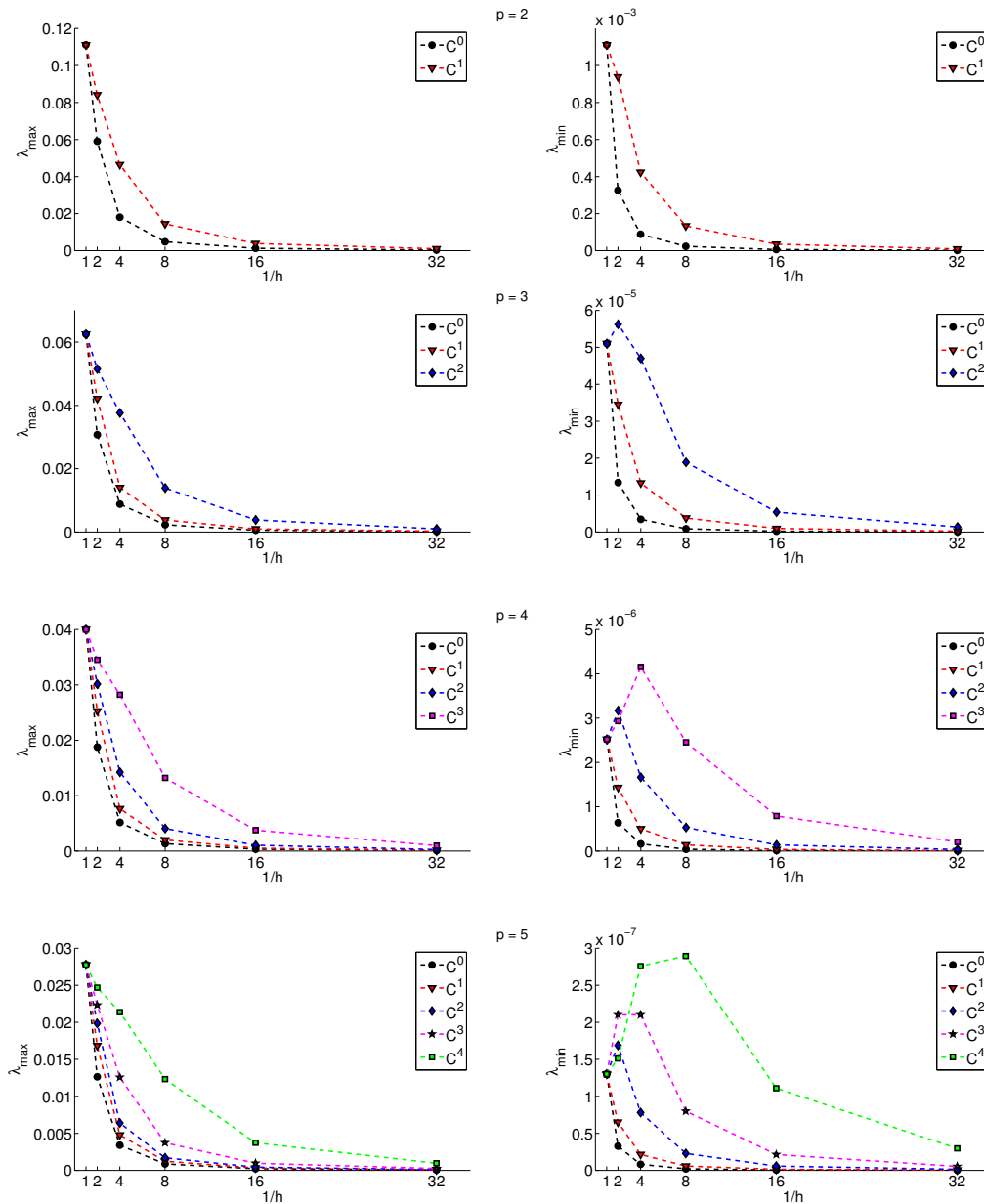


Figure 4: The extremal eigenvalues of the stiffness matrix  $M$  w.r.t. the  $h$ -refinement and varying smoothness.

For  $p$ -refinement, the estimates for the maximum eigenvalues of the stiffness matrix and the mass matrix are sharp and can not be improved. However, the estimates for the minimum eigenvalues of the stiffness matrix and the mass matrix depend on the stability constant of B-Splines. In reaching these estimates we have used the stability constant of B-Splines as  $p2^p$ . Using the de Boor's conjecture (the stability constant of B-Splines given by  $2^p$ , which is the best known bound to our knowledge), these estimates can be further improved according to Remarks 20 and 29. Unfortunately, a sharp estimate for the stability constant is not known, and therefore, a sharp estimate for the minimum eigenvalue is hard to determine. Furthermore, the effect of continuity of basis functions is negligible on the condition number because the difference between the best and the worst condition number is about a factor of  $p$ , which is highly dominated by the exponent term  $4^{pd}$ .

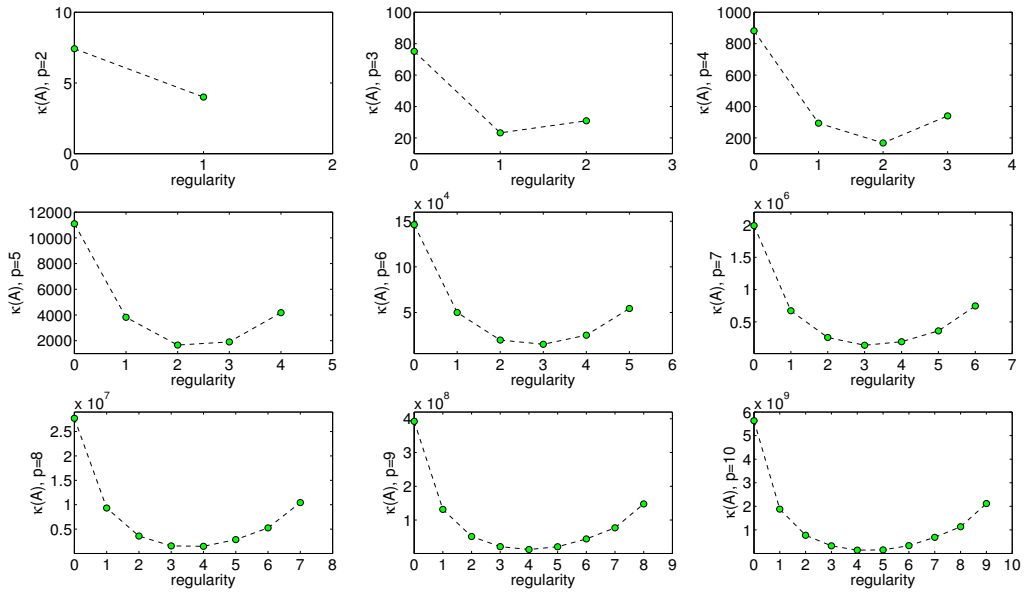


Figure 5: The behavior of the condition number of the stiffness matrix on a  $2 \times 2$  mesh with varying continuity of basis functions on interfaces. The results are shown from  $p = 2$  to  $p = 10$  and with smoothness from  $C^0$  to  $C^{p-1}$ .

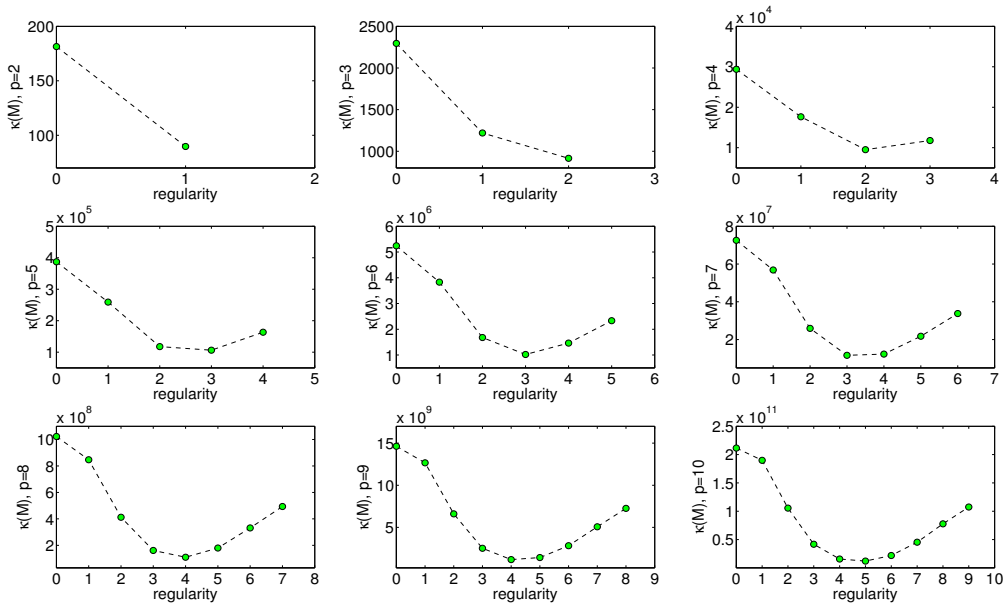


Figure 6: The condition number of the mass matrix is plotted on a  $2 \times 2$  mesh with varying continuity of basis functions on interfaces. The results are shown from  $p = 2$  to  $p = 10$  and with smoothness from  $C^0$  to  $C^{p-1}$ .

Table 11: Condition number of the stiffness matrix  $A$  for  $p = 2$  to  $p = 10$  with continuity from  $C^0$  to  $C^{p-1}$

$h^{-1} = 2$	$\kappa(A)$									
	$C^0$	$C^1$	$C^2$	$C^3$	$C^4$	$C^5$	$C^6$	$C^7$	$C^8$	$C^9$
2	7.4e+00	4.0e+00	-	-	-	-	-	-	-	-
3	7.5e+01	2.3e+01	3.1e+01	-	-	-	-	-	-	-
4	8.8e+02	2.9e+02	1.7e+02	3.4e+02	-	-	-	-	-	-
5	1.1e+04	3.8e+03	1.7e+03	1.9e+03	4.2e+03	-	-	-	-	-
6	1.5e+05	5.0e+04	2.0e+04	1.5e+04	2.5e+04	5.4e+04	-	-	-	-
7	2.0e+06	6.7e+05	2.6e+05	1.4e+05	1.9e+05	3.6e+05	7.5e+05	-	-	-
8	2.8e+07	9.3e+06	3.6e+06	1.6e+06	1.5e+06	2.8e+06	5.2e+06	1.0e+07	-	-
9	3.9e+08	1.3e+08	5.1e+07	2.1e+07	1.3e+07	2.1e+07	4.4e+07	7.7e+07	1.5e+08	-
10	5.6e+09	1.9e+09	7.7e+08	3.2e+08	1.4e+08	1.5e+08	3.3e+08	6.9e+08	1.1e+09	2.1e+09

Table 12: Condition number of the mass matrix  $M$  for  $p = 2$  to  $p = 10$  with continuity from  $C^0$  to  $C^{p-1}$

$h^{-1} = 2$	$\kappa(M)$									
	$C^0$	$C^1$	$C^2$	$C^3$	$C^4$	$C^5$	$C^6$	$C^7$	$C^8$	$C^9$
2	1.8e+02	9.0e+01	-	-	-	-	-	-	-	-
3	2.3e+03	1.2e+03	9.2e+02	-	-	-	-	-	-	-
4	2.9e+04	1.8e+04	9.5e+03	1.2e+04	-	-	-	-	-	-
5	3.9e+05	2.6e+05	1.2e+05	1.1e+05	1.6e+05	-	-	-	-	-
6	5.2e+06	3.8e+06	1.7e+06	1.0e+06	1.5e+06	2.3e+06	-	-	-	-
7	7.3e+07	5.7e+07	2.6e+07	1.2e+07	1.2e+07	2.2e+07	3.4e+07	-	-	-
8	1.0e+09	8.5e+08	4.1e+08	1.6e+08	1.1e+08	1.8e+08	3.3e+08	4.9e+08	-	-
9	1.5e+10	1.3e+10	6.6e+09	2.5e+09	1.2e+09	1.4e+09	2.8e+09	5.1e+09	7.3e+09	-
10	2.1e+11	1.9e+11	1.1e+11	4.2e+10	1.6e+10	1.2e+10	2.2e+10	4.5e+10	7.8e+10	1.1e+11

## Appendix A.

**Lemma 31.** *There exists a constant  $C$  independent of  $p$ , such that for all  $v = \sum_{i=1}^{n_p} v_i N_i \in \mathcal{S}_p$ , we have*

$$a(v, v) = \int_{\Omega} |\nabla v|^2 \leq Cp^2. \quad (\text{A.1})$$

*Proof.* From the derivative of a B-Spline basis function given in (13), we have

$$\frac{d}{d\xi} N_i^p(\xi) = \frac{p}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) - \frac{p}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi).$$

On a single element mesh we have  $\xi_{i+p} - \xi_i = \xi_{i+p+1} - \xi_{i+1} = 1$ , which implies

$$\begin{aligned} \frac{d}{d\xi} N_i^p(\xi) &= p (N_i^{p-1}(\xi) - N_{i+1}^{p-1}(\xi)) \\ \Rightarrow \left( \frac{d}{d\xi} N_i^p(\xi) \right)^2 &= p^2 (N_i^{p-1}(\xi) - N_{i+1}^{p-1}(\xi))^2 \\ &\leq p^2 \left( (N_i^{p-1}(\xi))^2 + (N_{i+1}^{p-1}(\xi))^2 \right). \end{aligned}$$

Integrating over unit interval  $I$ , we get

$$\begin{aligned} \int_I \left( \frac{d}{d\xi} N_i^p(\xi) \right)^2 d\xi &\leq p^2 \left( \int_I (N_i^{p-1}(\xi))^2 d\xi + \int_I (N_{i+1}^{p-1}(\xi))^2 d\xi \right) \\ &\leq p^2 (\|N_i^{p-1}(\xi)\|^2 + \|N_{i+1}^{p-1}(\xi)\|^2) \leq Cp^2. \end{aligned} \quad (\text{A.2})$$

Now using above we get

$$\begin{aligned} \int_{\Omega} \nabla N_i^p(\xi, \eta) \cdot \nabla N_i^p(\xi, \eta) d\xi d\eta &= \int_I \int_I \left[ \left( \frac{\partial N_i^p(\xi, \eta)}{\partial \xi} \right)^2 + \left( \frac{\partial N_i^p(\xi, \eta)}{\partial \eta} \right)^2 \right] d\xi d\eta \\ &\leq Cp^2, \end{aligned}$$

which concludes the proof. □

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- [1] F. Auricchio, L. Beirao da Veiga, A. Buffa, C. Lovadina, A. Reali and G. Sangalli. A fully “locking-free” isogeometric approach for plane linear elasticity problems: A stream function formulation. *Comput. Methods Appl. Mech. Engrg.* **197**, 160-172, 2007.
- [2] O. Axelsson and V. A. Barker. *Finite Element Solution of Boundary Value Problems: Theory and Computation*. Society for Industrial and Applied Mathematics Philadelphia, PA, USA, 2001.
- [3] I. Babuska, J.E. Osborn. Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems *Mathematics of computation*, Volume 52, No. 186. 275-297, 1989.

- [4] I. Babuska, B.A. Szabo, and I.N. Katz. The  $p$ -version of the Finite Element Method. *SIAM Journal on Numerical Analysis, Volume 18, No. 3.* 515-545, 1981.
- [5] Y. Bazilevs, L. Beirao Da Veiga, J.A. Cottrell, T.J.R. Hughes and G. Sangalli. Isogeometric analysis: Approximation, Stability and error estimates for  $h$ -refined meshes. *Math. Models Methods Appl. Sci.* **16(7)**, 1031-1090, 2006.
- [6] R. Bellman. A note on an inequality of E. Schmidt. *Amer. Math. Soc.*, **50**, 734-736, 1944.
- [7] D. Braess. *Finite Elements: Theory, Fast Solvers and Applications in Solid Mechanics.* Cambridge University Press, 2007.
- [8] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer, 2008.
- [9] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North Holland Publishing Company, 1978.
- [10] J.A. Cottrell, T.J.R. Hughes and Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA.* Wiley, 2009.
- [11] C. de Boor. On Calculating with B-Splines. *Journal of Approximation Theory* **6**, pp. 50-62, 1972.
- [12] C. de Boor. On local linear functionals which vanish at all  $B$ -splines but one. *Theory of Approximation with Applications, A. G. Law and N. B. Sahney (eds.), Academic Press (New York).* 120-145, 1976.
- [13] C. de Boor. Splines as linear combinations of B-Splines, a survey. In *Approximation Theory II, G. G. Lorentz, C. K. Chui and L. L. Schumaker, (Eds.), Academic Press (New York),* pp. 1-47, 1976.
- [14] C. de Boor. *A Practical Guide to Splines.* Springer-Verlag, New York, 1978.
- [15] C. de Boor. The exact condition of the B-Spline basis may be hard to determine. *Journal of Approximation theory.* **60**, 344-359, 1990.
- [16] C. de Boor and J.W. Daniel. Splines with Nonnegative B-Spline Coefficients. *Mathematics of Computation* **28(126)**, pp. 565-568, 1974.
- [17] C. de Falco, A. Reali and R. Vazquez. GeoPDEs: A research tool for Isogeometric Analysis of PDEs. *Adv. Eng. Softw.* **42**, 1020-1034, 2011.
- [18] C. de Falco, A. Reali and R. Vazquez. GeoPDEs webpage. <http://geopdes.sourceforge.net>
- [19] A. Ern, J.-L. Guermond. Evaluation of the condition number in linear systems arising in finite element approximations. *ESAIM Math. Mod. Numer. Anal.* **40(1)**, 29-48, 2006.
- [20] M.S. Floater. Evaluation and properties of the derivative of a NURBS curve. *Mathematical Methods in CAGD, T. Lyche and L.L. Schumaker (eds.), Academic Press, Boston,* **2**, pp. 261-274, 1992.

- [21] K.P.S. Gahalaut, J.K. Kraus and S.K. Tomar. Multigrid Methods for Isogeometric Discretization. *Comput. Methods Appl. Mech. Engrg.*. Available online, doi: <http://dx.doi.org/10.1016/j.cma.2012.08.015>, 2012.
- [22] W. Hackbusch. *Iterative Solution of Large Sparse Systems of Equations*. Springer, 1994.
- [23] K. Hoellig. Multivariate Splines. *SIAM J. Numer. Anal.* **19(5)**, pp. 1013-1031, 1982.
- [24] K. Hoellig. Stability of B-Spline basis via knot insertion. *Computer Added Geometric Design*. **17**, pp. 447-450, 2000.
- [25] K. Hoellig, U. Reif and J. Wipper. Weighted Extended B-Spline Approximation of Dirichlet Problems. *SIAM J. Numer. Anal.* **39(2)**, pp. 442-462, 2002.
- [26] N. Hu, X-Z Guo, and I.N. Katz. Bounds for eigenvalues and condition numbers in the  $p$ -version of the finite element method. *Mathematics of computation*. **67/224**, 1423-1450, 1998.
- [27] T.J.R. Hughes, J.A. Cottrell and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.* **194**, 4135-4195, 2005.
- [28] C. Johnson. *Numerical solution of partial differential equations by the finite element method*. Cambridge University Press, Cambridge, New York, 1987.
- [29] T. Lyche, K. Scherer. On the sup-norm condition number of the multivariate triangular Bernstein basis. *Multivariate Approximation and Splines*, G. Nuernberger, J. W. Schmidt, and G. Walz (eds.), ISNM.125, Birkhuser Verlag, Basel. 141-151, 1997.
- [30] T. Lyche, K. Scherer. On the  $p$ -norm condition number of the multivariate triangular Bernstein basis *Journal of Computational and Applied Mathematics*. **119**, 259273, 2000.
- [31] J.F. Maitre and O. Pourquier. About the conditioning of matrices in the  $p$ -version of the finite element method for second order elliptic problems. *Journal of Computational and Applied Mathematics* **63**, 341-348, 1995.
- [32] J.F. Maitre and O. Pourquier. Condition number and diagonal preconditioning: Comparison of the  $p$ -version and the spectral element methods. *Numer. Math.* **74**, pp. 69-84, 1996.
- [33] J.M. Melenk. On condition numbers in hp-FEM with GaussLobatto-based shape functions. *Journal of Computational and Applied Mathematics. Volume 139, Issue 1*. 21-48, 2002.
- [34] J.M. Melenk and I. Babuska. The Partition of Unity Finite Element Method: Basic Theory and Applications. *Comput. Methods Appl. Mech. Engrg.* **139**, pp. 289-314, 1996.
- [35] B. Mössner and U. Reif. Stability of tensor product B-Splines on domains. *J. Approx. Theory* **154**, 1-19, 2008.
- [36] E.T. Olsen, J. Douglas, Jr. Bounds on spectral condition numbers of matrices arising in the  $p$ -version of the finite element method. *Numerische Mathematik*. **69**, 333-352, 1995.

- [37] J.M. Peña. B-Splines and Optimal Stability. *Mathematics of Computation* **66**, pp. 1555-1560, 1997.
- [38] L. Piegl and W. Tiller. *The NURBS Book (Monographs in Visual Communication)*, Second ed., Springer-Verlag, 1997.
- [39] D.F. Rogers. *An Introduction to NURBS With Historical Perspective*. Academic Press, 2001.
- [40] Y. Saad. *Iterative Methods for Sparse Linear Systems*. PWS Publishing Company, Boston, 1996.
- [41] K. Scherer, A.Y. Shadrin. New upper bound for the B-Spline basis condition number, I. *East J. Approx.* **2**, 331-342, 1996.
- [42] K. Scherer, A.Y. Shadrin. New upper bound for the B-Spline basis condition number, II. A proof of de Boor's  $2^k$ -conjecture. *Journal of Approximation theory*. **99**, 217-229, 1999.
- [43] L.L. Schumaker. *Spline Functions: Basic Theory*. Cambridge University Press, 2007.
- [44] G. Strang and G.J. Fix. *An Analysis of the Finite Element Method*. Prentice Hall, 1973.
- [45] R.S. Varga. *Matrix Iterative Analysis*. Prentice Hall, New Jersey, 1965.