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A. Korporal, G. Regensburger

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On the product of projectors and generalized inverses

Anja Korporal, Georg Regensburger[1,*]

Johann Radon Institute for
Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences,
4040 Linz, Austria

Abstract

We consider generalized inverses of linear operators on arbitrary vector spaces and study the question when their product in reverse order is again a generalized inverse. It turns out that this problem is equivalent to the question when the product of projectors is again a projector, and we discuss necessary and sufficient conditions in terms of the defining spaces. We present a new representation of the product of generalized inverses that does not require explicit knowledge of the factors. Our approach is based on implicit representations of subspaces via their orthogonals in the dual space, and we formulate a duality principle for statements about generalized inverses. For Fredholm operators, the corresponding computations reduce to finite-dimensional problems. We also illustrate our results with examples for matrices.

Keywords: Generalized inverse, Projector, Reverse order law, Fredholm Operator, Duality
2000 MSC: 15A09, 47A05

1. Introduction

Analogs of the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ for bijective operators have been studied intensively for various kinds of generalized inverses. Most articles are concerned with the matrix case, see for example [1–6], for infinite-dimensional vector spaces, usually additional topological structures like Banach or Hilbert spaces are assumed [7–9].

The validity of the reverse order law can be reduced to the question whether the product of two projectors is a projector. This problem is studied in [10, 11] for finite-dimensional vector spaces. We discuss necessary and sufficient conditions that carry over to arbitrary vector spaces and can be expressed only in terms of kernel and image of the respective operators (Section 4, 5). Moreover, we study the commutativity of two projectors, leading to sufficient conditions for the reverse order law.

Assuming the reverse order law to hold, our main result (Theorem 24) gives a representation of the product of two generalized inverses that can be computed using only kernel and image of

*Corresponding Author.
Email addresses: anja.korporal@oeaw.ac.at (Anja Korporal), georg.regensburger@oeaw.ac.at (Georg Regensburger)
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the generalized inverses of the factors. In this description, we rely on implicit representations of subspaces via their orthogonals in the dual space and avoid the computation of generalized inverses by using the associated transpose map in the dual space. In our approach, we systematically use duality results that hold in arbitrary vector spaces and a corresponding duality principle for statements about generalized inverses (Appendix A).

Our interest in generalized inverses in an infinite-dimensional setting is motivated by linear boundary problems, since their solution operators (Green's operators) are generalized inverses. In the context of singular boundary problems [12, 13], the results presented can be used to decide if the composition of two (generalized) Green's operators is again a Green's operator and which boundary problem it solves, generalizing the results for regular problems from [14, 15].

Boundary problems for ordinary differential equations correspond to Fredholm operators, for which the conditions for the reverse order law can be checked algorithmically (Section 6). Detailed examples for matrices, illustrating our results, are given in Section 7.

2. Generalized inverses

In this section, we recall basic properties of generalized inverses. For further details and proofs, we refer to [16, 7] and the references therein. Throughout this article, U, V, and W always denote vector spaces over the same field F.

Definition 1. Let $T\colon V \to W$ be linear. We call a linear map $G\colon W \to V$ an *inner inverse* of T if $TGT = T$ and an *outer inverse* of T if $GTG = G$. If G is an inner and an outer inverse of T, we call G an *algebraic generalized inverse* of T.

This terminology of generalized inverses is adopted from [16]; other sources refer to inner inverses as generalized inverses or g-inverses, whereas algebraic generalized inverses are also called reflexive generalized inverses.

Proposition 2. *Let $T\colon V \to W$ and $G\colon W \to V$ be linear. The following statements are equivalent:*

(i) G is an outer inverse of T,

(ii) GT is a projector and $\operatorname{Im} GT = \operatorname{Im} G$*,*

(iii) GT is a projector and $V = \operatorname{Im} G \dotplus \operatorname{Ker} GT$*,*

(iv) GT is a projector and $W = \operatorname{Im} T + \operatorname{Ker} G$*,*

(v) TG is a projector and $\operatorname{Ker} TG = \operatorname{Ker} G$*,*

(vi) TG is a projector and $W = \operatorname{Ker} G \dotplus \operatorname{Im} TG$*,*

(vii) TG is a projector and $\operatorname{Im} G \cap \operatorname{Ker} T = \{0\}$.

From (iii) and (vi), we see that each outer inverse gives rise to a direct sum decomposition of V and W

$$V = \operatorname{Im} G \dotplus T^{-1}(\operatorname{Ker} G) \quad \text{and} \quad W = \operatorname{Ker} G \dotplus T(\operatorname{Im} G). \tag{1}$$

Corresponding to (vii) and (vi), for given subspaces $B \leq V$ and $E \leq W$ with

$$B \cap \operatorname{Ker} T = \{0\} \quad \text{and} \quad W = E \dotplus T(B), \tag{2}$$

we can construct an outer inverse G of T with $\operatorname{Im} G = B$ and $\operatorname{Ker} G = E$ as follows; cf. [7, Cor. 8.2]. We consider the projector Q with

$$\operatorname{Im} Q = T(B), \quad \operatorname{Ker} Q = E. \tag{3}$$

The restriction $T|_B \colon B \to T(B)$ is bijective since $B \cap \operatorname{Ker} T = \{0\}$, and we can define $G = (T|_B)^{-1} Q$. One easily verifies that G is an outer inverse of T with $\operatorname{Im} G = B$ and $\operatorname{Ker} G = E$. Since by (1) we have $V = B \dotplus T^{-1}(E)$, we define the projector P in analogy to Q by

$$\operatorname{Im} P = T^{-1}(E), \quad \operatorname{Ker} P = B \tag{4}$$

Then, by definition and by Proposition 2, we have

$$GTG = G, \quad TG = Q, \quad \text{and} \quad GT = 1 - P, \tag{5}$$

and G is determined uniquely by these equations. Hence an outer inverse depends only on the choice of B and E in (2). We use the notations

$$G = \mathrm{O}(T, B, E) \quad \text{and} \quad G = \mathrm{O}(T, P, Q)$$

for P and Q as in (4) and (3).

Obviously, G is an outer inverse of T if and only if T is an inner inverse of G. Therefore, we get a result analogous to Proposition 2 by interchanging the role of T and G.

Proposition 3. *Let $T \colon V \to W$ and $G \colon W \to V$ be linear. The following statements are equivalent:*

(i) *G is an inner inverse of T,*

(ii) *TG is a projector and* $\operatorname{Im} TG = \operatorname{Im} T$,

(iii) *TG is a projector and* $W = \operatorname{Im} T \dotplus \operatorname{Ker} TG$,

(iv) *TG is a projector and* $V = \operatorname{Im} G + \operatorname{Ker} T$,

(v) *GT is a projector and* $\operatorname{Ker} GT = \operatorname{Ker} T$,

(vi) *GT is a projector and* $V = \operatorname{Ker} T \dotplus \operatorname{Im} GT$,

(vii) *GT is a projector and* $\operatorname{Im} T \cap \operatorname{Ker} G = \{0\}$.

The construction of inner inverses is not completely analogous to outer inverses, see [16, Prop. 1.3]. In particular, an inner inverse can be chosen arbitrarily on a complement of $\operatorname{Im} T$ and is therefore not uniquely determined by kernels and images of the projectors TG and GT. Nevertheless, for all complements B and E of $\operatorname{Ker} T$ and $\operatorname{Im} T$ with

$$V = \operatorname{Ker} T \dotplus B \quad \text{and} \quad W = \operatorname{Im} T \dotplus E,$$

there is an inner inverse G of T with

$$B = \operatorname{Im} GT \quad \text{and} \quad E = \operatorname{Ker} TG. \tag{6}$$

We use the notation $G \in \mathrm{I}(T, B, E)$ for such an inner inverse.

Combining the properties of inner and outer inverses gives the following characterizations of algebraic generalized inverses.

Proposition 4. *Let $T\colon V \to W$ and $G\colon W \to V$ be linear. The following statements are equivalent:*

(i) *G is an algebraic generalized inverse of T,*

(ii) *TG is a projector and $V = \operatorname{Ker} T \dotplus \operatorname{Im} G$,*

(iii) *GT is a projector and $W = \operatorname{Ker} G \dotplus \operatorname{Im} T$,*

(iv) *TG is a projector and $\operatorname{Im} TG = \operatorname{Im} T$, $\operatorname{Ker} TG = \operatorname{Ker} G$,*

(v) *GT is a projector and $\operatorname{Im} GT = \operatorname{Im} G$, $\operatorname{Ker} GT = \operatorname{Ker} T$.*

For constructing algebraic generalized inverses, we proceed similar to outer inverses, only that $B \leq V$ now has to be a complement of $\operatorname{Ker} T$, so that in this case (1) simplifies to

$$V = \operatorname{Ker} T \dotplus B \quad \text{and} \quad W = \operatorname{Im} T \dotplus E. \tag{7}$$

Since also algebraic generalized inverses are determined uniquely by the choice of respectively B and E in (7) or by the corresponding projectors P and Q with

$$\operatorname{Im} P = \operatorname{Ker} T, \quad \operatorname{Ker} P = B \quad \text{and} \quad \operatorname{Im} Q = \operatorname{Im} T, \quad \operatorname{Ker} Q = E,$$

we use the notations

$$G = \mathrm{G}(T, B, E) \quad \text{and} \quad G = \mathrm{G}(T, P, Q).$$

From the construction it is clear that

$$G = \mathrm{G}(T, P, Q) \iff T = \mathrm{G}(G, 1 - Q, 1 - P).$$

Note that a projector $P\colon V \to V$ is a special case of an algebraic generalized inverse with $P = \mathrm{G}(P, 1 - P, P)$.

Proposition 5. *Let $T_1\colon V \to W$ and $T_2\colon U \to V$ be linear with outer (resp. inner) inverses G_1 and G_2. Let $P = G_1T_1$ and $Q = T_2G_2$. Then G_2G_1 is an outer (resp. inner) inverse of T_1T_2 if and only if QP (resp. PQ) is a projector.*

Proof. Let G_2G_1 be an outer inverse of T_1T_2, that is,

$$G_2G_1 = G_2G_1T_1T_2G_2G_1.$$

Multiplying with T_2 from the left and with T_1 from the right yields

$$T_2G_2G_1T_1 = T_2G_2G_1T_1T_2G_2G_1T_1,$$

thus $QP = T_2G_2G_1T_1$ is a projector. For the other direction, we multiply the previous equation with with G_2 from the left and G_1 from the right and obtain (2), using that $G_1T_1G_1 = G_1$ and $G_2T_2G_2 = G_2$. The proof for inner inverses follows by interchanging the roles of T_i and G_i. □

The previous result for inner inverses in the matrix case is mentioned in [17, Section 1.7] or [4], and proven in [18].

Corollary 6. *Let $T_1\colon V \to W$ and $T_2\colon U \to V$ be linear with algebraic generalized inverses G_1 and G_2. Let $P = G_1T_1$ and $Q = T_2G_2$. Then G_2G_1 is an algebraic generalized inverse of T_1T_2 if and only if PQ and QP are projectors.*

3. Kernel of compositions

In this section, we describe the inverse image of the composition of two linear maps using inner inverses. For projectors, kernel and image of the composition can be expressed only in terms of kernel and image of the corresponding factors.

First, we recall some elementary characterizations of projectors, which we use without further reference.

Lemma 7. *For a linear map $P\colon V \to V$, the following conditions are equivalent:*

(i) $P^2 = P$,

(ii) $P|_{\operatorname{Im} P} = 1$,

(iii) $\operatorname{Im}(1-P) = \operatorname{Ker} P$ *or* $\operatorname{Im} P = \operatorname{Ker}(1-P)$.

Proposition 8. *Let $T_1\colon V \to W$ and $T_2\colon U \to V$ be linear and G_2 an inner inverse of T_2. For a subspace $W_1 \leq W$, we have*

$$(T_1T_2)^{-1}(W_1) = G_2(T_1^{-1}(W_1) \cap \operatorname{Im} T_2) \dotplus \operatorname{Ker} T_2$$

for the inverse image of the composite. In particular, we have

$$\operatorname{Ker} T_1T_2 = G_2(\operatorname{Ker} T_1 \cap \operatorname{Im} T_2) \dotplus \operatorname{Ker} T_2.$$

Proof. Since T_2G_2 is a projector onto $\operatorname{Im} T_2$ by Proposition 3 (ii), we have

$$\begin{aligned} T_1T_2(G_2(T_1^{-1}(W_1) \cap \operatorname{Im} T_2) + \operatorname{Ker} T_2) &= T_1Q_2(T_1^{-1}(W_1) \cap \operatorname{Im} T_2) + 0 \\ &= T_1(T_1^{-1}(W_1) \cap \operatorname{Im} T_2) \leq W_1 \cap \operatorname{Im} T_1T_2 \leq W_1. \end{aligned}$$

Conversely, let $u \in (T_1T_2)^{-1}(W_1)$. Then $T_2u = v$ with $v \in T_1^{-1}(W_1)$. Since also $v \in \operatorname{Im} T_2$, we have

$$T_2(u - G_2v) = T_2u - Q_2v = T_2u - v = v - v = 0,$$

i.e. $u - G_2v \in \operatorname{Ker} T_2$. Writing $u = G_2v + u - G_2v$ yields $u \in G_2(T_1^{-1}(W_1) \cap \operatorname{Im} T_2) + \operatorname{Ker} T_2$. The sum is direct, since by Proposition 3 (vi), we have $U = \operatorname{Ker} T_2 \dotplus \operatorname{Im} G_2T_2$. □

Corollary 9. *Let $T\colon V \to W$ be linear and let $P\colon V \to V$ and $Q\colon W \to W$ be projectors. Then we have*

$$\operatorname{Ker} TQ = (\operatorname{Ker} T \cap \operatorname{Im} Q) \dotplus \operatorname{Ker} Q$$

and

$$\operatorname{Im} PT = (\operatorname{Im} T + \operatorname{Ker} P) \cap \operatorname{Im} P.$$

Proof. Applying Proposition 8 yields

$$\operatorname{Ker} TQ = Q(\operatorname{Ker} T \cap \operatorname{Im} Q) \dotplus \operatorname{Ker} Q = (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus \operatorname{Ker} Q.$$

The statement for the image follows from the Duality Principle 33. □

This result generalizes [18, Lemma 2.2], where kernel and image of a product PQ of two projectors are computed like above for the case of PQ again being a projector.

4. Products of projectors

In view of Proposition 5, we study some necessary and sufficient conditions for the product of two projectors being a projector. Throughout this section let $P, Q\colon V \to V$ denote projectors. We start with some results from [19, 10, 18] that together with Corollary 9 allow to characterize the idempotency of PQ only in terms of kernel and image of P and Q. Using the Duality Principle 33, we derive additional equivalent conditions.

The following necessary and sufficient condition for the product of P and Q to be a projector is mentioned as an exercise without proof in [19, p. 339]. In [10, Lemma 3] the same result is formulated for matrices, but the proof is valid for arbitrary vector spaces, and we repeat it for completeness.

Lemma 10. *The composition PQ is a projector if and only if*

$$\operatorname{Im} PQ \leq \operatorname{Im} Q \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q). \tag{8}$$

Proof. Let PQ be a projector and $v \in \operatorname{Im} PQ$. Since $\operatorname{Im}(1-Q) = \operatorname{Ker} Q$ and $P(1-Q)PQ = PQ - PQPQ = 0$, we have

$$v = PQv = QPQv + (1-Q)PQv \in \operatorname{Im} Q + (\operatorname{Ker} P \cap \operatorname{Ker} Q).$$

The sum is direct, since $\operatorname{Im} Q \cap \operatorname{Ker} Q = \{0\}$. Now assume (8) and let $v \in \operatorname{Im} PQ$. Then $v = q + k$ with $q \in \operatorname{Im} Q$ and $k \in \operatorname{Ker} P \cap \operatorname{Ker} Q$. Since P is a projector and $v \in \operatorname{Im} PQ \leq \operatorname{Im} P$, we have $v = Pv = Pq + Pk = Pq$. So we have $PQv = PQ(q+k) = Pq = v$, and PQ is a projector. □

Another necessary and sufficient condition for the matrix case is given in [18, Lemma 2.2]. We give the proof from [11], which carries over to arbitrary vector spaces.

Lemma 11. *The composition PQ is a projector if and only if*

$$\operatorname{Im} Q \leq \operatorname{Im} P \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q). \tag{9}$$

Proof. Let PQ be a projector and $v \in \operatorname{Im} Q$. We have

$$v = Pv + Q(1-P)v + (1-Q)(1-P)v = Pv + Q(1-P)Qv + (1-Q)(1-P)Qv,$$

since $v = Qv$. By assumption $PQ(1-P)Q = PQ - PQPQ = 0$, hence

$$Q(1-P)Qv \in \operatorname{Ker} P \cap \operatorname{Im} Q.$$

Similarly $P(1-Q)(1-P)Q = 0$, and since $\operatorname{Im}(1-Q) = \operatorname{Ker} Q$, we have

$$(1-Q)(1-P)Qv \in \operatorname{Ker} P \cap \operatorname{Ker} Q.$$

The sum is direct since $\operatorname{Im} P \cap \operatorname{Ker} P = \{0\}$ and $\operatorname{Im} Q \cap \operatorname{Ker} Q = \{0\}$. Now assume (9) and let $v \in V$. Then we have $Qv = p + k_1 + k_2$ with $p \in \operatorname{Im} P$, $k_1 \in \operatorname{Ker} P \cap \operatorname{Im} Q$, and $k_2 \in \operatorname{Ker} P \cap \operatorname{Ker} Q$, and thus

$$PQPQv = PQP(p + k_1 + k_2) = PQp = PQ(p + k_1 + k_2) = PQQv = PQv.$$

Hence PQ is a projector. □

Theorem 12. *The following statements are equivalent:*

(i) *The composition PQ is a projector,*

(ii) $\operatorname{Im} P \cap (\operatorname{Im} Q + \operatorname{Ker} P) \leq \operatorname{Im} Q \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)$*,*

(iii) $\operatorname{Im} Q \leq \operatorname{Im} P \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)$*,*

(iv) $\operatorname{Ker} Q \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \geq \operatorname{Ker} P \cap (\operatorname{Im} Q + \operatorname{Im} P)$*,*

(v) $\operatorname{Ker} P \geq \operatorname{Ker} Q \cap (\operatorname{Im} Q + \operatorname{Ker} P) \cap (\operatorname{Im} Q + \operatorname{Im} P)$.

Proof. The equivalence of (i) and (ii) is given by Lemma 10 and Corollary 9; (i) and (iii) are equivalent by Lemma 11. By the Duality Principle 33, the last two conditions follows from (ii) and (iii). □

In view of Corollary 6, we now study some sufficient conditions for PQ as well as QP being projectors. For the equivalences of the following lemma see also [19, p. 339].

Lemma 13. *We have $PQ = Q$ if and only if* $\operatorname{Im} Q \leq \operatorname{Im} P$*, and $PQ = P$ if and only if* $\operatorname{Ker} Q \leq \operatorname{Ker} P$*, Moreover, if $PQ = Q$ or $PQ = P$, then also QP is a projector.*

Proof. If $PQ = Q$, then $\operatorname{Im} Q = \operatorname{Im} PQ \leq \operatorname{Im} P$. Conversely, if $\operatorname{Im} Q \leq \operatorname{Im} P$, then $PQ = Q$, since $P|_{\operatorname{Im} P} = 1$. If $PQ = Q$, then $QPQP = Q^2P = QP$, hence also QP is a projector. The case $PQ = P$ follows from the Duality Principle 33. □

Clearly, if P and Q commute, PQ as well as QP are projectors. As before, our goal is to characterize this property only in terms of image and kernel of P and Q. Obviously, if $PQ = QP$, then $\operatorname{Ker} P,\ \operatorname{Ker} Q \leq \operatorname{Ker} PQ$. Since by Corollary 9, we have $\operatorname{Ker} PQ \leq \operatorname{Ker} P + \operatorname{Ker} Q$, and so

$$\operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q \tag{10}$$

is a necessary condition for the commutativity of P and Q. Similarly, from $PQ = QP$, we get $\operatorname{Im} PQ \leq \operatorname{Im} P \cap \operatorname{Im} Q$; and the reverse inclusion holds by Corollary 9. Thus also the condition

$$\operatorname{Im} PQ = \operatorname{Im} P \cap \operatorname{Im} Q \tag{11}$$

is necessary; compare [3, Th. 5.1.4]. In general, (10) and (11) are necessary but not sufficient for commutativity of P and Q, see [10, Ex. 1]. In [10] also a sufficient condition for the matrix case is given by (10) and (11) along with $\operatorname{rank} PQ = \operatorname{rank} QP$. We will now discuss equivalent conditions in terms of P and Q only.

The next results are possible corrected versions of [20, Ch. 2, Ex. 71], where it is claimed that PQ is the projector onto $\operatorname{Im} P \cap \operatorname{Im} Q$ along $\operatorname{Ker} P + \operatorname{Ker} Q$ if and only if

$$\operatorname{Im} Q = (\operatorname{Im} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q).$$

However, Groß and Trenkler [10] give a counterexample. We first keep the above condition and show that it characterizes projectors onto $\operatorname{Im} P \cap \operatorname{Im} Q$ (without specification of the kernel), then we characterize projectors PQ with $\operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q$.

Proposition 14. *The composition PQ is a projector with*

$$\operatorname{Im} PQ = \operatorname{Im} P \cap \operatorname{Im} Q$$

if and only if

$$\operatorname{Im} Q = (\operatorname{Im} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q). \tag{12}$$

Proof. First assume (12). Then we have

$$\operatorname{Im} PQ = P(\operatorname{Im} Q) = P(\operatorname{Im} P \cap \operatorname{Im} Q) + P(\operatorname{Ker} P \cap \operatorname{Im} Q) = \operatorname{Im} P \cap \operatorname{Im} Q,$$

and since $PQ|_{\operatorname{Im} P \cap \operatorname{Im} Q} = 1$, PQ is a projector.

Now assume that PQ is a projector onto $\operatorname{Im} P \cap \operatorname{Im} Q$. By Corollary 9, its kernel is given by $(\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus \operatorname{Ker} Q$, which means that

$$\operatorname{Ker} Q \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Im} P \cap \operatorname{Im} Q) = V = \operatorname{Ker} Q \dotplus \operatorname{Im} Q.$$

Hence Equation (12) follows from (A.3). □

Proposition 15. *The composition PQ is a projector with*

$$\operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q$$

if and only if

$$\operatorname{Ker} P = (\operatorname{Ker} P \cap \operatorname{Ker} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q). \tag{13}$$

Proof. First, let us assume (13). Then PQ is a projector by Theorem 12 (iii), since

$$\operatorname{Im} P \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q) = \operatorname{Im} P \dotplus \operatorname{Ker} P = V \geq \operatorname{Im} Q.$$

We always have $\operatorname{Ker} Q \leq \operatorname{Ker} PQ$ and also $\operatorname{Ker} PQ \leq \operatorname{Ker} P + \operatorname{Ker} Q$ by Corollary 9, so we only have to show that $\operatorname{Ker} P \leq \operatorname{Ker} PQ$. But this follows immediately from (13), since $Q|_{\operatorname{Im} Q} = 1$.

Now let PQ be a projector with $\operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q$. Again by Corollary 9, we have

$$\operatorname{Ker} P + \operatorname{Ker} Q = (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus \operatorname{Ker} Q.$$

Computing the intersection with $\operatorname{Ker} P$ yields

$$\operatorname{Ker} P = \operatorname{Ker} P \cap ((\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus \operatorname{Ker} Q) = (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q),$$

using modularity (A.2). □

By dualizing the previous propositions, it follows from the latter that PQ is a projector with $\operatorname{Im} PQ = \operatorname{Im} P \cap \operatorname{Im} Q$ if and only if

$$\operatorname{Im} Q = (\operatorname{Im} Q + \operatorname{Im} P) \cap (\operatorname{Im} Q + \operatorname{Ker} P).$$

Analogously, PQ is a projector with $\operatorname{Ker} PQ = \operatorname{Ker} P + \operatorname{Ker} Q$ if and only if

$$\operatorname{Ker} P = (\operatorname{Ker} Q + \operatorname{Ker} P) \cap (\operatorname{Im} Q + \operatorname{Ker} P)$$

by Proposition 14.

From Proposition 14 and 15 we can also derive a necessary and sufficient condition for the commutativity of P and Q, see also [19, p. 339].

Corollary 16. *We have $PQ = QP$ if and only if*

$$\operatorname{Im} Q = (\operatorname{Im} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \tag{14}$$

and

$$\operatorname{Ker} Q = (\operatorname{Im} P \cap \operatorname{Ker} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q). \tag{15}$$

Proof. Since commutativity of P and Q implies (11), the first equation follows from Proposition 14, the second follows from (10) and Proposition 15, both applied to QP.

Let us now assume (14) and (15). Since P and Q are projectors, their kernels and images impose two direct sum decomposition of V. Hence

$$\begin{aligned} V &= \operatorname{Im} P \dotplus \operatorname{Ker} P = \operatorname{Im} Q \dotplus \operatorname{Ker} Q \\ &= (\operatorname{Im} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Im} Q) \dotplus (\operatorname{Im} P \cap \operatorname{Ker} Q) \dotplus (\operatorname{Ker} P \cap \operatorname{Ker} Q), \end{aligned}$$

which by (A.3) yields the corresponding decompositions of $\operatorname{Im} P$ and $\operatorname{Ker} P$ into

$$\operatorname{Im} P = (\operatorname{Im} Q \cap \operatorname{Im} P) \dotplus (\operatorname{Ker} Q \cap \operatorname{Im} P)$$

and

$$\operatorname{Ker} P = (\operatorname{Im} Q \cap \operatorname{Ker} P) \dotplus (\operatorname{Ker} Q \cap \operatorname{Ker} P).$$

Thus by Propositions 14 and 15, PQ and QP both are projectors satisfying

$$\operatorname{Ker} PQ = \operatorname{Ker} QP = \operatorname{Ker} P + \operatorname{Ker} Q \quad \text{and} \quad \operatorname{Im} PQ = \operatorname{Im} QP = \operatorname{Im} P \cap \operatorname{Im} Q,$$

hence they are equal. □

Obviously the roles of P and Q are permutable in (14) and (15) and the conditions remain the same also for $1 - P$ resp. $1 - Q$. This gives the following generalization of [11, Th. 3] to arbitrary vector spaces. We adopt the notation $P_{I,K}$ for the projector $P\colon V \to V$ with $\operatorname{Im} P = I$ and $\operatorname{Ker} P = K$.

Corollary 17. *Let $P_1 = P_{I_1,K_1}$ and $P_2 = P_{I_2,K_2}$. The following statements are equivalent:*

(i) $P_1P_2 = P_2P_1 = P_{I_1\cap I_2, K_1+K_2}$,

(ii) $P_1(1 - P_2) = (1 - P_2)P_1 = P_{I_1\cap K_2, K_1+I_2}$,

(iii) $(1 - P_1)P_2 = P_2(1 - P_1) = P_{K_1\cap I_2, I_1+K_2}$,

(iv) $(1 - P_1)(1 - P_2) = (1 - P_2)(1 - P) = P_{K_1\cap K_2, I_1+I_2}$.

Example 18. *We give an example, where PQ is a projector, but none of the special cases of Lemma 13 or Propositions 14 and 15 is fulfilled. Let $V = \mathbb{Q}^4$. Consider the projectors*

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then one easily checks that

$$\operatorname{Ker} Q \cap \operatorname{Im} P = \operatorname{Ker} P \cap \operatorname{Im} Q = \{0\} \textit{ and } \operatorname{Ker} P \cap \operatorname{Ker} Q = L((0,0,1,0)^T).$$

Hence PQ and QP are projectors by Lemma 11.

5. Reverse order law for generalized inverses

Combining Proposition 5 and Theorem 12 gives necessary and sufficient conditions for the reverse order law for outer inverses to hold, only in terms of the defining spaces.

Theorem 19. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, B_1, E_1)$ and $G_2 = \mathrm{O}(T_2, B_2, E_2)$. The following conditions are equivalent:*

(i) G_2G_1 *is an outer inverse of* T_1T_2,

(ii) $T_2(B_2) \cap (B_1 + E_2) \leq B_1 \dotplus (E_2 \cap T_1^{-1}(E_1))$,

(iii) $B_1 \leq T_2(B_2) \dotplus (E_2 \cap B_1) \dotplus (E_2 \cap T_1^{-1}(E_1))$,

(iv) $T_1^{-1}(E_1) \dotplus (E_2 \cap B_1) \geq E_2 \cap (B_1 + T_2(B_2))$,

(v) $E_2 \geq T_1^{-1}(E_1) \cap (B_1 + E_2) \cap (B_1 + T_2(B_2))$.

Proof. Recall that $\mathrm{Im}\, G_i = B_i$ and $\mathrm{Ker}\, G_i = E_i$. Let $Q = T_2G_2$ and $P = G_1T_1$. Then P and Q are projectors, and by (1) we have

$$\mathrm{Im}\, P = B_1, \quad \mathrm{Ker}\, P = T^{-1}(E_1), \quad \mathrm{Im}\, Q = T_2(B_2), \quad \text{and} \quad \mathrm{Ker}\, Q = E_2.$$

By Proposition 5, G_2G_1 is an outer inverse if and only if QP is a projector. Applying Theorem 12 proves the claim. □

Analogously, we obtain conditions for the reverse order law for inner inverses. By (6) and Proposition 3 (ii), (v), the projectors $P = G_1T_1$ and $Q = T_2G_2$ for inner inverses $G_1 \in \mathrm{I}(T_1, B_1, E_1)$ and $G_2 \in \mathrm{I}(T_2, B_2, E_2)$ satisfy

$$\mathrm{Im}\, P = B_1, \quad \mathrm{Ker}\, P = \mathrm{Ker}\, T_1, \quad \mathrm{Im}\, Q = \mathrm{Im}\, T_2, \quad \text{and} \quad \mathrm{Ker}\, Q = E_2.$$

Hence the corresponding statement takes a simpler form.

Theorem 20. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with inner inverses $G_1 \in \mathrm{I}(T_1, B_1, E_1)$ and $G_2 \in \mathrm{I}(T_2, B_2, E_2)$. The following conditions are equivalent:*

(i) G_2G_1 *is an inner inverse of* T_1T_2,

(ii) $B_1 \cap (\mathrm{Im}\, T_2 + \mathrm{Ker}\, T_1) \leq \mathrm{Im}\, T_2 \dotplus (\mathrm{Ker}\, T_1 \cap E_2)$,

(iii) $\mathrm{Im}\, T_2 \leq B_1 \dotplus (\mathrm{Ker}\, T_1 \cap \mathrm{Im}\, T_2) \dotplus (\mathrm{Ker}\, T_1 \cap E_2)$,

(iv) $E_2 \dotplus (\mathrm{Ker}\, T_1 \cap \mathrm{Im}\, T_2) \geq \mathrm{Ker}\, T_1 \cap (\mathrm{Im}\, T_2 + B_1)$,

(v) $\mathrm{Ker}\, T_1 \geq E_2 \cap (\mathrm{Im}\, T_2 + \mathrm{Ker}\, T_1) \cap (\mathrm{Im}\, T_2 + B_1)$.

In the case of algebraic generalized inverses $G_1 = \mathrm{G}(T_1, B_1, E_1)$ and $G_2 = \mathrm{G}(T_2, B_2, E_2)$, we have

$$T_1^{-1}(E_1) = \mathrm{Ker}\, T_1 \quad \text{and} \quad T_2(B_2) = \mathrm{Im}\, T_2, \tag{16}$$

using (7). Hence we obtain necessary and sufficient conditions for the reverse order law depending only on the kernels and images of the respective operators. Moreover, note that for fixed T_1 and T_2 the conditions for inner and algebraic generalized inverseonly depend on the choice of B_1 and E_2.

The results about compositions of projectors give some sufficient conditions for the reverse order law.

Proposition 21. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with algebraic generalized inverses G_1 and G_2. Then G_2G_1 is an algebraic generalized inverse of T_1T_2 if one of the following conditions holds.*

1. $\operatorname{Im} T_2 = (\operatorname{Im} G_1 \cap \operatorname{Im} T_2) \dotplus (\operatorname{Ker} T_1 \cap \operatorname{Im} T_2)$ *and* $\operatorname{Ker} G_2 = (\operatorname{Im} G_1 \cap \operatorname{Ker} G_2) \dotplus (\operatorname{Ker} T_1 \cap \operatorname{Ker} G_2)$.

2. $\operatorname{Im} G_1 = (\operatorname{Im} T_2 \cap \operatorname{Im} G_1) \dotplus (\operatorname{Ker} G_2 \cap \operatorname{Im} G_1)$ *and* $\operatorname{Ker} T_1 = (\operatorname{Im} T_2 \cap \operatorname{Ker} T_1) \dotplus (\operatorname{Ker} G_2 \cap \operatorname{Ker} T_1)$.

3. $\operatorname{Im} G_1 \leq \operatorname{Im} T_2$ *or* $\operatorname{Im} T_2 \leq \operatorname{Im} G_1$ *or* $\operatorname{Ker} T_1 \leq \operatorname{Ker} G_2$ *or* $\operatorname{Ker} G_2 \leq \operatorname{Ker} T_1$.

Proof. Let $P = G_1T_1$ and $Q = T_2G_2$. Then the first two conditions both imply $PQ = QP$ using Corollary 16. Using Lemma 13, the conditions in 3 respectively imply $QP = P$, $PQ = Q$, $QP = Q$, and $PQ = P$. So in all cases PQ and QP are projectors, and hence G_2G_1 is an algebraic generalized inverse by Corollary 6. □

Werner [18, Thm. 3.1] proves that for matrices it is always possible to find inner inverses such that the reverse order law holds. The proof carries over to arbitrary vector spaces and can be extended to algebraic generalized inverses. The special case of Moore-Penrose inverses is treated in [4, Thm. 3.2], and explicit solutions are constructed in [21, 22]. We state the result without proof.

Proposition 22. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear. There always exist algebraic generalized inverses G_1 and G_2 of respectively T_1 and T_2 such that G_2G_1 is an algebraic generalized inverse of T_1T_2.*

We now assume that for two linear maps $T_1 \colon V \to W$ and $T_2 \colon U \to V$ with outer inverses G_1 and G_2 the reverse order law holds, and our goal is to describe the product G_2G_1. If G_2G_1 is an outer inverse, then one verifies that

$$\mathrm{O}(T_2, P_2, Q_2)\,\mathrm{O}(T_1, P_1, Q_1) = \mathrm{O}(T_1T_2, P_2 - G_2P_1T_2, T_1Q_2G_1).$$

Note that this expression involves both outer inverses G_1 and G_2.

We can also express the outer inverse of the product in terms of its kernel and image.

Lemma 23. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, B_1, E_1)$ and $G_2 = \mathrm{O}(T_2, B_2, E_2)$. Then we have*

$$\operatorname{Ker} G_2G_1 = E_1 \dotplus T_1(B_1 \cap E_2) \quad \text{and} \quad \operatorname{Im} G_2G_1 = G_2((B_1 + E_2) \cap \operatorname{Im} T_2).$$

Proof. Recall that by definition $\operatorname{Ker} G_i = E_i$ and $\operatorname{Im} G_i = B_i$. Since T_1 is an inner inverse of G_1, the equation

$$\operatorname{Ker} G_2G_1 = E_1 \dotplus T_1(B_1 \cap E_2)$$

follows directly from Proposition 8. From Proposition 31 and 8 we get

$$\begin{aligned}(\operatorname{Im} G_2G_1)^\perp = \operatorname{Ker} G_1^*G_2^* = T_2^*(\operatorname{Ker} G_1^* \cap \operatorname{Im} G_2^*) \dotplus \operatorname{Ker} G_2^* \\ = T_2^*((\operatorname{Im} G_1)^\perp \cap (\operatorname{Ker} G_2)^\perp) \dotplus (\operatorname{Im} G_2)^\perp. \quad (17)\end{aligned}$$

Taking the orthogonal and applying again Proposition 31 yields

$$\operatorname{Im} G_2G_1 = T_2^{-1}(\operatorname{Im} G_1 + \operatorname{Ker} G_2) \cap \operatorname{Im} G_2 = T_2^{-1}(B_1 + E_2) \cap B_2.$$

With Proposition 8, this simplifies to

$$\operatorname{Im} G_2G_1 = (G_2((B_1 + E_2) \cap \operatorname{Im} T_2)) \dotplus \operatorname{Ker} T_2) \cap B_2 = G_2((B_1 + E_2) \cap \operatorname{Im} T_2))$$

using modularity (A.2) and the direct sum $U = \operatorname{Ker} T_2 \dotplus B_2$. □

In view of the previous lemma the reverse order law takes the form

$$\mathrm{O}(T_2, B_2, E_2)\,\mathrm{O}(T_1, B_1, E_1) = \mathrm{O}(T_1T_2, G_2((B_1 + E_2) \cap \operatorname{Im} T_2), E_1 + T_1(B_1 \cap E_2)).$$

In [18, Thm. 2.4] a similar result for matrices can be found. Note that also this expression depends on the explicit knowledge of the outer inverse G_2.

Using an implicit description of $\operatorname{Im} G_i$, it is possible to state the reverse order law in a form that only depends on the kernels and images of the respective outer inverses. This approach is also motivated by our application to linear boundary problems, where it is natural to define solution spaces via the boundary conditions they satisfy.

In more detail, the Galois connection from Appendix A allows in particular to represent a subspace B implicitly via the orthogonally closed subspace $\mathscr{B} = B^\perp$ of the dual space. We will therefore use the notation

$$G = \mathrm{G}(T, \mathscr{B}, E)$$

for the algebraic generalized inverse with $\operatorname{Im} G = \mathscr{B}^\perp$ and $\operatorname{Ker} G = E$ as well as the analogs for inner and outer inverses.

Theorem 24. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with outer inverses $G_1 = \mathrm{O}(T_1, \mathscr{B}_1, E_1)$ and $G_2 = \mathrm{O}(T_2, \mathscr{B}_2, E_2)$. If G_2G_1 is an outer inverse of T_1T_2, we have*

$$\mathrm{O}(T_2, \mathscr{B}_2, E_2)\,\mathrm{O}(T_1, \mathscr{B}_1, E_1) = \mathrm{O}(T_1T_2, \mathscr{B}_2 \dotplus T_2^*(\mathscr{B}_1 \cap E_2^\perp), E_1 \dotplus T_1(\mathscr{B}_1^\perp \cap E_2)), \qquad (18)$$

where T_2^* denotes the transpose of T_2.

Proof. From Lemma 23 we immediately see that

$$\operatorname{Ker} G_2G_1 = E_1 \dotplus T_1(\mathscr{B}_1^\perp \cap E_2).$$

Using (17), we obtain

$$(\operatorname{Im} G_2G_1)^\perp = T_2^*((\operatorname{Im} G_1)^\perp \cap (\operatorname{Ker} G_2)^\perp) \dotplus (\operatorname{Im} G_2)^\perp = T_2^*(\mathscr{B}_1 \cap E_2^\perp) \dotplus \mathscr{B}_2,$$

and thus (18) holds. □

6. Fredholm operators

We now turn to algorithmic aspects of the previous results. As already emphazised, for arbitrary vector spaces we can express conditions for the reverse order law only in terms of the input data. Nevertheless, in general it will not be possible to compute sums and intersections of infinite-dimensional subspaces. For algorithmically checking the conditions of Theorem 19 or 20, and for computing the reverse order law in the form (18), we consider finite codimensional spaces and Fredholm operators.

Definition 25. A linear map $T\colon V \to W$ between vector spaces V and W is called a *Fredholm* operator if $\dim \operatorname{Ker} T < \infty$ and $\operatorname{codim} \operatorname{Im} T < \infty$.

For finite codimensional subspaces, we can compute the dimension of the orthogonal: For $V_1 \leq V$, there is a natural isomorphism $V_1^\perp \cong (V/V_1)^*$, so if $\operatorname{codim} V_1 < \infty$, we have

$$\operatorname{codim} V_1 = \dim V_1^\perp. \tag{19}$$

In this case, V_1 can be implicitly represented by a finite-dimensional subspace $V_1^\perp \leq V^*$. For a generalized inverse G of a Fredholm operator T, the spaces $\mathscr{B} \leq V^*$ and $E \leq W$ in the implicit representation $G = \mathrm{G}(T, \mathscr{B}, E)$ are finite dimensional: We have

$$\dim \mathscr{B} = \operatorname{codim} \mathscr{B}^\perp = \dim \operatorname{Ker} T < \infty$$

as well as

$$\dim E = \operatorname{codim} \operatorname{Im} T = \dim (\operatorname{Im} T)^\perp < \infty,$$

using (19) and

$$V = \operatorname{Ker} T \dotplus \mathscr{B}^\perp \quad \text{and} \quad W = \operatorname{Im} T \dotplus E.$$

For our application to boundary problems, this setting proves very useful, since ordinary differential equations only have finite-dimensional solution spaces and usually only finitely many boundary conditions $\beta_1, \dots, \beta_n \in V^*$ are imposed. For the Green's operator G, we then have $\operatorname{Im} G = \mathscr{B}^\perp$, where $\mathscr{B}$ is generated by $\beta_1, \dots, \beta_n$.

In the rest of this section, we discuss how to check the conditions of Theorem 19 and 20 for Fredholm operators algorithmically. We assume that for finite-dimensional subspaces, we can compute sums and intersections and check inclusions, both in vector spaces and in their duals. Furthermore, we assume that we have an implicit description of $\operatorname{Im} T_2 = \mathscr{C}_2^\perp$, where $\mathscr{C}_2$ is generated by $\gamma_1, \dots, \gamma_m \in V^*$.

In the following lemma, we show how to compute the intersection of a finite-dimensional with a finite codimensional subspace in V, respectively V^*.

Definition 26. Let $u = (u_1, \dots, u_m)^T \in V^m$ and $\beta = (\beta_1, \dots, \beta_n)^T \in (V^*)^n$. We call

$$\beta(u) = \begin{pmatrix} \beta_1(u_1) & \dots & \beta_1(u_m) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \dots & \beta_n(u_m) \end{pmatrix} \in F^{n \times m}$$

the *evaluation matrix* of β and u.

Lemma 27. *Let $U \leq V$ and $\mathscr{B} \leq V^*$ be generated respectively by $u = (u_1, \dots, u_m)$ and $\beta = (\beta_1, \dots, \beta_n)$. Let $k^1, \dots, k^r \in F^m$ be a basis of* $\operatorname{Ker} \beta(u)$*, and $\kappa^1, \dots, \kappa^s \in F^n$ a basis of* $\operatorname{Ker}(\beta(u))^T$*. Then the intersection $U \cap \mathscr{B}^\perp \leq V$ is generated by*

$$\sum_{i=1}^m k_i^1 u_i, \dots, \sum_{i=1}^m k_i^r u_i,$$

and the intersection $U^\perp \cap \mathscr{B} \leq V^*$ by

$$\sum_{i=1}^n \kappa_i^1 \beta_i, \dots, \sum_{i=1}^n \kappa_i^s \beta_i.$$

Proof. A linear combination $v = \sum_{\ell=1}^{m} c_\ell u_\ell$ is in $\mathscr{B}^\perp$ if and only if $\beta_i(v) = 0$ for all $1 \le i \le n$, i.e.,

$$\sum_{\ell=1}^{m} c_\ell \beta_i(u_\ell) = 0 \quad \text{for} \quad 1 \le i \le n.$$

Determining the coefficients c_ℓ leads to solving the linear system

$$\beta(u) \cdot (c_1, \ldots, c_m)^T = 0.$$

Analogously, for computing the intersection $U^\perp \cap \mathscr{B}$, we obtain the linear system

$$\sum_{\ell=1}^{n} d_\ell \beta_\ell(u_j) = 0 \quad \text{for} \quad 1 \le j \le m,$$

which means that we have to compute the kernel of $(\beta(u))^T$. □

Now we reformulate the conditions of Theorem 19 and 20, such that for Fredholm operators they only involve operations on finite-dimensional subspaces and intersections like in the previous lemma. For simplicity, we assume that G_1 and G_2 are algebraic generalized inverses of T_1 and T_2, such that $T_1^{-1}(E_1) = \operatorname{Ker} T_1$ and $T_2(\mathscr{B}_2^\perp) = \operatorname{Im} T_2$ by (16).

Corollary 28. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with algebraic generalized inverses $G_1 = \mathrm{G}(T_1, \mathscr{B}_1, E_1)$ and $G_2 = \mathrm{G}(T_2, \mathscr{B}_2, E_2)$. The following conditions are equivalent:*

(i) G_2G_1 *is an outer inverse of* T_1T_2*,*

(ii) $\mathscr{C}_2 + (\mathscr{B}_1 \cap E_2^\perp) \ge \mathscr{B}_1 \cap (E_2 \cap \operatorname{Ker} T_1)^\perp$*,*

(iii) $\mathscr{B}_1 \ge \mathscr{C}_2 \cap (E_2 \cap \mathscr{B}_1^\perp)^\perp \cap (E_2 \cap \operatorname{Ker} T_1)^\perp$*,*

(iv) $\operatorname{Ker} T_1 \dotplus (E_2 \cap \mathscr{B}_1^\perp) \ge E_2 \cap (\mathscr{B}_1 \cap \mathscr{C}_2)^\perp$*,*

(v) $E_2 \ge \operatorname{Ker} T_1 \cap (\mathscr{B}_1 \cap E_2^\perp)^\perp \cap (\mathscr{B}_1 \cap \mathscr{C}_2)^\perp$.

Proof. Taking the orthogonal of both sides of respectively 19 (ii), (iii) and applying Proposition 30 gives (ii) and (iii). For (iv) and (v), we can apply Proposition 30 directly to the corresponding conditions of Theorem 19. □

Similarly, we rewrite the conditions of Theorem 20, such that they can be checked algorithmically for Fredholm operators.

Corollary 29. *Let $T_1 \colon V \to W$ and $T_2 \colon U \to V$ be linear with algebraic generalized inverses $G_1 = \mathrm{G}(T_1, \mathscr{B}_1, E_1)$ and $G_2 = \mathrm{G}(T_2, \mathscr{B}_2, E_2)$. The following conditions are equivalent:*

(i) G_2G_1 *is an inner inverse of* T_1T_2*,*

(ii) $\mathscr{B}_1 \dotplus (\mathscr{C}_2 \cap (\operatorname{Ker} T_1)^\perp) \ge \mathscr{C}_2 \cap (\operatorname{Ker} T_1 \cap E_2)^\perp$*,*

(iii) $\mathscr{C}_2 \ge \mathscr{B}_1 \cap (\operatorname{Ker} T_1 \cap \mathscr{C}_2^\perp)^\perp \cap (\operatorname{Ker} T_1 \cap E_2)^\perp$*,*

(iv) $E_2 + (\operatorname{Ker} T_1 \cap \mathscr{C}_2^\perp) \ge \operatorname{Ker} T_1 \cap (\mathscr{C}_2 \cap \mathscr{B}_1)^\perp$*,*

(v) $\operatorname{Ker} T_1 \ge E_2 \cap (\mathscr{C}_2^\perp \cap \operatorname{Ker} T_1) \cap (\mathscr{C}_2 \cap \mathscr{B}_1)^\perp$.

Finally, we note that using Lemma 27, it also possible to determine constructively the implicit representation (18) of a product of generalized inverses.

7. Examples

In this section, we illustrate our results for finite-dimensional vector spaces. Consider the following linear maps $T_1\colon \mathbb{Q}^4 \to \mathbb{Q}^3$ and $T_2\colon \mathbb{Q}^3 \to \mathbb{Q}^4$ given by

$$T_1 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 3 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 2 \\ -1 & 5 & 4 \\ -1 & 5 & 4 \end{pmatrix}.$$

We first use Theorem 19 and 20 to check whether for algebraic generalized inverses $G_1 = \mathrm{G}(T_1, B_1, E_1)$ and $G_2 = \mathrm{G}(T_2, B_2, E_2)$ the composition G_2G_1 is an algebraic generalized inverse of T_1T_2. For testing the conditions, we only need to choose $B_1 = \operatorname{Im} G_1$ and $E_2 = \operatorname{Ker} G_2$, such that $B_1 \dotplus \operatorname{Ker} T_1 = \mathbb{Q}^4 = E_2 \dotplus \operatorname{Im} T_2$. We have

$$\operatorname{Ker} T_1 = \operatorname{span}((0,1,0,1)^T, (0,0,1,1)^T) \quad \text{and} \quad \operatorname{Im} T_2 = \operatorname{span}((1,0,-2,-2)^T, (0,1,1,1)^T),$$

so we may choose for example

$$B_1 = \operatorname{span}((1,0,0,0)^T, (0,1,0,0)^T), \quad E_2 = \operatorname{span}((1,0,0,0)^T, (0,0,1,0)^T).$$

Using (16), we obtain the necessary and sufficient condition for being an outer inverse

$$B_1 \leq \operatorname{Im} T_2 \dotplus (E_2 \cap B_1) \dotplus (E_2 \cap \operatorname{Ker} T_1)$$

from Theorem 19 (iii). Since $E_2 \cap \operatorname{Ker} T_1 = \{0\}$ and $E_2 \cap B_1 = \operatorname{span}((1,0,0,0)^T)$, the right hand side yields

$$\operatorname{span}((1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,1)^T) \geq B_1.$$

Thus for all algebraic generalized inverses G_1 and G_2 with $\operatorname{Im} G_1 = B_1$ and $\operatorname{Ker} G_2 = E_2$, the product G_2G_1 is an outer inverse of T_1T_2.

The corresponding condition for inner inverses by Theorem 20 (iii) is

$$\operatorname{Im} T_2 \leq B_1 \dotplus (\operatorname{Ker} T_1 \cap \operatorname{Im} T_2) \dotplus (\operatorname{Ker} T_1 \cap E_2).$$

Since $\operatorname{Ker} T_1 \cap \operatorname{Im} T_2 = \{0\}$, the right hand side yields B_1, which does not contain $\operatorname{Im} T_2$. Hence for the above choices of G_1 and G_2, the product G_2G_1 is never an inner inverse of T_1T_2.

Since G_2G_1 is an outer inverse, Theorem 24 allows to determine G_2G_1 directly without knowing the factors. Identifying the dual space with row vectors, the orthogonals of B_1 and E_2 are given by

$$B_1^\perp = \mathscr{B}_1 = \operatorname{span}((0,0,1,0), (0,0,0,1)), \quad E_2^\perp = \operatorname{span}((0,1,0,0), (0,0,0,1)),$$

so we have $\mathscr{B}_1^\perp \cap E_2 = \operatorname{span}((1,0,0,0)^T)$ and $\mathscr{B}_1 \cap E_2^\perp = \operatorname{span}((0,0,0,1))$. For explicitly computing G_2G_1, we also have to choose $B_2 = \operatorname{Im} G_2$ and $E_1 = \operatorname{Ker} G_1$. Since we have

$$\operatorname{Im} T_1 = \operatorname{span}((1,0,3)^T), (0,1,2)^T), \quad \operatorname{Ker} T_2 = \operatorname{span}((1,1,-1)^T),$$

we may choose the complements $E_1 = \operatorname{Ker} G_1$ and $B_2 = \operatorname{Im} G_2$ as

$$E_1 = \operatorname{span}((0,0,1)^T) \quad \text{and} \quad B_2 = \operatorname{span}((1,0,0)^T, (0,1,0)^T).$$

Then we can easily determine the kernel (18)

$$E = \operatorname{Ker} G_2G_1 = E_1 \dotplus T_1(\mathscr{B}_1^{\perp} \cap E_2) = \operatorname{span}((1,0,0)^T, (0,0,1)^T).$$

The image of G_2G_1 is by (18) given via the orthogonal

$$(\operatorname{Im} G_2G_1)^{\perp} = \mathscr{B}_2 \dotplus T_2^*(\mathscr{B}_1 \cap E_2^{\perp}) = \operatorname{span}((0,0,1), (-1,5,4)),$$

which means that $B = \operatorname{Im} G_2G_1 = L((5,1,0)^T)$. Therefore we can directly determine G as the unique outer inverse

$$G = \mathrm{O}(T_1T_2, B, E) = \begin{pmatrix} 0 & \frac{5}{12} & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One easily checks that G is an outer inverse of T. We can also verify our result by explicitly computing and multiplying the algebraic generalized inverses G_1 and G_2. We have

$$G_1 = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & \frac{5}{6} & 0 & \frac{-1}{6} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so indeed $G = G_2G_1$.

By Proposition 22, it is always possible to choose generalized inverses of T_1 and T_2, such that their product is a generalized inverse of T_1T_2. For example, we now take B_1 as before and change E_2 to

$$E_2 = \operatorname{span}((0,0,1,0)^T, (0,0,0,1)^T).$$

Then $\operatorname{Ker} T_1 \cap \operatorname{Im} T_2 = E_2 \cap B_1 = \{0\}$ and $\operatorname{Ker} T_1 \cap E_2 = \operatorname{span}((0,0,1,1)^T)$, and therefore

$$\operatorname{Im} T_2 \leq B_1 + (\operatorname{Ker} T_1 \cap E_2) \text{ and } B_1 \leq \operatorname{Im} T_2 + (\operatorname{Ker} T_1 \cap E_2).$$

Hence for all algebraic generalized inverses of T_1 and T_2 with $\operatorname{Im} G_1 = B_1$ and $\operatorname{Ker} G_2 = E_2$, we know that G_2G_1 is an algebraic generalized inverse of T_1T_2. In this case, we directly obtain from Lemma 23 that $\operatorname{Ker} G_2G_1 = \operatorname{Ker} G_1$ and $\operatorname{Im} G_2G_1 = \operatorname{Im} G_2$, so that we do not need to apply the transpose map in the dual space.

Appendix A. Duality

In the appendix, we summarize duality results for arbitrary vector spaces and their duals that generalize the standard duality for finite-dimensional vector spaces; see [23, Section 9.2 and 9.3] and [15] for further details. The notation should also remind of the analogous and well-known results for Hilbert spaces.

Let V and W be vector spaces over a field F and $\langle , \rangle \colon V \times W \to F$ be a bilinear map. For $V_1 \leq V$, we define the orthogonal

$$V_1^{\perp} = \{w \in W \mid \langle v, w\rangle = 0 \text{ for all } v \in V_1\} \leq W.$$

The orthogonal $W_1^{\perp}$ for $W_1 \leq W$ is defined analogously. A subspace U is called orthogonally closed if $U = U^{\perp\perp}$. It follows directly from the definition that for all subsets $X_1, X_2 \subseteq V$, we have

$$X_1 \subseteq X_2 \Rightarrow X_1^{\perp} \supseteq X_2^{\perp} \quad \text{and} \quad X_1 \subseteq X_1^{\perp\perp}. \tag{A.1}$$

The same holds for subsets of W. Let $\mathbb{P}(V)$ denote the projective geometry of V, that is, the partially ordered set (poset) of all subspaces ordered by inclusion. Then by (A.1) we have an order-reversing Galois connection between $\mathbb{P}(V)$ and $\mathbb{P}(W)$ defined by $U \mapsto U^\perp$.

We now consider the canonical bilinear form $V \times V^* \to F$ of a vector space V and its dual V^* defined by $\langle v, \beta\rangle \mapsto \beta(v)$. Then every subspace $U \leq V$ is orthogonally closed with respect to the canonical bilinear form, and every finite-dimensional subspace $\mathscr{B} \leq V^*$ is orthogonally closed. The Galois connection gives an order-reversing bijection between $\mathbb{P}(V)$ and the poset of all orthogonally closed subspaces of V^*. So we can describe any subspace $V_1 \leq V$ implicitly by the corresponding orthogonally closed subspace $V_1^\perp$. We denote the poset of all orthogonally closed subspaces of V^* with $\overline{\mathbb{P}}(V^*)$.

The projective geometry $\mathbb{P}(V)$ is a modular lattice, where join and meet are defined as the sum and intersection of subspaces. Modularity means that for all $V_1, V_2, V_3 \in \mathbb{P}(V)$ with $V_1 \leq V_3$ we have

$$V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap V_3. \tag{A.2}$$

Moreover, for spaces $V_1 \leq V_3$ and $V_2 \leq V_4$, we have

$$V = V_1 + V_2 = V_3 \dotplus V_4 \quad \text{implies} \quad V_1 = V_3 \text{ and } V_2 = V_4, \tag{A.3}$$

since $V_3 \cap V_4 = \{0\}$ implies $V_3 = (V_1 \dotplus V_2) \cap V_3 = V_1$ and $V_4 = (V_1 \dotplus V_2) \cap V_4 = V_2$.

Also $\overline{\mathbb{P}}(V^*)$ is a modular lattice, where the meet is the intersection and the join is the orthogonal closure of the sum of subspaces. The following theorem summarizes Section 9.3 of [23].

Proposition 30. *The map $V_1 \mapsto V_1^\perp$ gives an order-reversing lattice isomorphism with inverse $\mathscr{B}_1 \mapsto \mathscr{B}_1^\perp$ between the complemented modular lattices $\mathbb{P}(V)$ and $\overline{\mathbb{P}}(V^*)$. In particular, the intersection of orthogonally closed subspaces in V^* is orthogonally closed and*

$$(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp \quad \text{and} \quad (\mathscr{B}_1 \cap \mathscr{B}_2)^\perp = \mathscr{B}_1^\perp + \mathscr{B}_2^\perp.$$

The sum of two orthogonally closed subspaces in V^* is orthogonally closed and

$$(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp \quad \text{and} \quad (\mathscr{B}_1 + \mathscr{B}_2)^\perp = \mathscr{B}_1^\perp \cap B_2^\perp.$$

Furthermore, orthogonality preserves direct sums, such that

$$V = V_1 \dotplus V_2 \quad \text{implies} \quad V^* = V_1^\perp \dotplus V_2^\perp$$

and

$$V^* = \mathscr{B}_1 \dotplus \mathscr{B}_2 \quad \text{implies} \quad V = \mathscr{B}_1^\perp \dotplus \mathscr{B}_2^\perp.$$

For a linear map $T\colon V \to W$ between vector spaces, the *transpose* $T^*\colon W^* \to V^*$ is defined by $\gamma \mapsto \gamma \circ T$. The transposition map $A \mapsto A^*$ from $L(V, W)$ to $L(W^*, V^*)$ is linear, and it is injective since for all $w \neq 0$ there exists a linear map $h \in W^*$ with $h(w) \neq 0$. Moreover, the transpose of a composition is given by $(T_1T_2)^* = T_2^*T_1^*$.

The image of an orthogonally closed space under the transpose map is orthogonally closed, and we have following identities [15, Prop. A.6].

Proposition 31. *Let V and W be vector spaces and $A\colon V \to W$ be linear. Then we have*

$$A(V_1)^\perp = (A^*)^{-1}(V_1^\perp), \qquad A(\mathscr{B}_1^\perp) = (A^*)^{-1}(\mathscr{B}_1)^\perp,$$
$$A^*(\mathscr{C}_1)^\perp = A^{-1}(\mathscr{C}_1^\perp), \qquad A^*(W_1^\perp) = A^{-1}(W_1)^\perp,$$

for subspaces $V_1 \leq V$, $W_1 \leq W$, $\mathscr{C}_1 \leq W^*$ and orthogonally closed subspaces $\mathscr{B}_1 \leq V^*$. In particular, we have

$$(\operatorname{Im} A)^\perp = \operatorname{Ker} A^*, \qquad \operatorname{Im} A = (\operatorname{Ker} A^*)^\perp,$$
$$(\operatorname{Im} A^*)^\perp = \operatorname{Ker} A, \qquad \operatorname{Im} A^* = (\operatorname{Ker} A)^\perp,$$

for the image and kernel of A and A^*.

The property of being a projector, outer/inner/algebraic generalized inverse carries over to the transpose.

Proposition 32. *A linear map $P\colon V \to V$ is a projector if and only if its transpose P^* is a projector. A linear map $G\colon W \to V$ is an outer/inner/algebraic generalized inverse of $T\colon V \to W$ if and only if G^* is an outer/inner/algebraic generalized inverse of T^*.*

Proof. This follows from the defining equations for these properties. For example, if G is an outer inverse of T, we have

$$G^*T^*G^* = (GTG)^* = G^*,$$

and the reverse implication follows from the injectivity of the transposition map. □

We have seen in Section 2 that algebraic generalized inverses are determined uniquely by their kernel and image. With Proposition 31 we can translate this representation for the transpose

$$G = \mathrm{G}(T, B, E) \iff G^* = \mathrm{G}(T^*, E^\perp, B^\perp).$$

The representation of algebraic generalized inverses via the projectors P and Q reads as

$$G = \mathrm{G}(T, P, Q) \iff G^* = \mathrm{G}(T^*, 1 - Q^*, 1 - P^*).$$

With the results of this section, we obtain the following duality principle for generalized inverses.

Duality Principle 33. Given a valid statement for linear maps on arbitrary vector spaces V involving inclusions, $\{0\}$ and V, sums and intersections, direct sums, kernels and images, projectors, and outer/inner/algebraic generalized inverses, we obtain a valid dual statement by

- reversing the order of the linear maps and the corresponding domains and codomains,
- reversing inclusions and interchanging V and $\{0\}$,
- interchanging sums and intersections,
- interchanging kernels and images.

For example, one easily checks that in Proposition 2, the statements (v) – (vii) are the duals of (ii) – (iv) in this sense, and (iii) and (v) in Proposition 4 are the dual statements of (ii) and (iv).

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