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Local exact boundary controllability of 3D Navier–Stokes equations

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LOCAL EXACT BOUNDARY CONTROLLABILITY OF 3D NAVIER–STOKES EQUATIONS

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ABSTRACT. We consider the Navier–Stokes system in a bounded domain with a smooth boundary. Given a time-dependent solution and an arbitrary open subset of the boundary, we prove the existence of a boundary control, supported in the given subset, that drives the system to the given solution in finite time.

CONTENTS

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$. We consider the controlled Navier–Stokes system in Ω:

$$(1) \qquad \partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p + h = 0, \quad \operatorname{div} u = 0, \quad u|_\Gamma = \gamma + \zeta.$$

Here $u = (u_1, u_2, u_3)$ and p are the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, $\langle u \cdot \nabla \rangle$ stands for the differential operator $u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3$, h and γ are fixed functions, $\operatorname{div} u := \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$ and ζ is a control taking values in the space $\mathcal{E}$ of square-integrable functions in Γ whose support in x is contained in a given open subset $\Gamma_c \subset \Gamma$.

If, instead of system above we consider the case of internal controls under homogeneous Dirichlet boundary conditions, namely

$$(2) \qquad \partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p + h + \eta = 0, \quad \operatorname{div} u = 0, \qquad u|_\Gamma = 0.$$

where now η a control supported in a given open subset $w \subseteq \Omega$, then the problem of exact controllability to trajectories is now rather well understood. Namely, it was

proven that, given a time $T > 0$ and a smooth solution $\hat{u}$ of system (2) with $\eta \equiv 0$, for any initial vector field u_0 sufficiently close to $\hat{u}(0)$ one can find a square integrable control η supported in $(0, T) \times w$ such that the corresponding solution $u(t)$ of system (2), supplemented with the initial condition

$$(3) \qquad\qquad\qquad u(0, x) = u_0(x)$$

is defined on $[0, T]$ and satisfies the relation $u(T) = \hat{u}(T)$. We refer the reader to [4, 6, 9, 11, 13] for the exact statements and the proofs of these results.

From the results on internal controllability we can derive similar results on boundary controllability, at least if we consider the control acts in a connected component of the boundary. The idea is that from [9–11]: (I) the reference solution $\hat{u}$ solving (1) with $\zeta = 0$ is extended to a function $\tilde{u}$ defined in a bigger domain $\mathcal{G}$ containing Ω, and the extension $\tilde{u}$ solves the Navier–Stokes system in $\mathcal{G}$ under homogeneous Dirichlet boundary conditions (with a suitable external force $\tilde{h}$); a given vector field $u_0(x)$ defined in Ω, is also extended to $\bar{u}_0$ in $\mathcal{G}$, $\bar{u}_0$ vanishing at the boundary $\partial \mathcal{G}$. Both extensions being continuous in appropriate spaces. (II) An internal control $\bar{\eta}$ supported in $(0, T) \times \mathcal{G} \setminus (\Omega \cup \Gamma)$ driving the equation in $\mathcal{G}$ from $\bar{u}_0$ to $\tilde{u}(T)$ is taken. (III) Let $\bar{u}$ be the solution associated with the pair $(\bar{u}_0, \bar{\eta})$; then the boundary control $\bar{u}|_{\partial \Omega \setminus \partial \mathcal{G}}$ drives the equation in Ω from $u_0(x)$ to $\hat{u}(T)$. For the details see the referred works.

The fact that the controlled part is a connected component is important in [9–11] to construct the extensions to $\mathcal{G}$ satisfying the homogeneous Dirichlet boundary conditions in $\partial \mathcal{G}$ and then, to be able to apply the internal controllability results known for this type of boundary conditions. A question arises: what can be done if Γ_c, the part of boundary we are able to apply the control in, is not an entire connected component? As an illustration see figure 1. In particular, can we extend a vector field in $\{v \in H^1(\Omega, \mathbb{R}^3) \mid \operatorname{div} v = 0$ in Ω and $v = 0$ on $\partial \Omega \cap \partial \mathcal{G}\}$ to a vector field in $\{v \in H^1(\mathcal{G}, \mathbb{R}^3) \mid \operatorname{div} v = 0$ in $\mathcal{G}$ and $v = 0$ on $\partial \mathcal{G}\}$? This last question has a partial answer in the case $\Omega \subseteq \mathbb{R}^2$: it is shown in [7, Theorem 4.2] that we can extend the vector fields that satisfy an additional Sobolev regularity condition on a neighborhood of the points in $\partial(\partial \Omega \setminus \partial \mathcal{G})$. The extended domain $\mathcal{G}$ being constructed also in a suitable (simple) way. Here we are not going to answer this extension problem. Instead, with a remark on a result from [6], we claim we can overcome the boundary conditions problem, by taking a boundary control supported in $(0, T - \rho) \times \partial \mathcal{G} \setminus \partial \Omega$, together with the internal control $\bar{\eta}$.
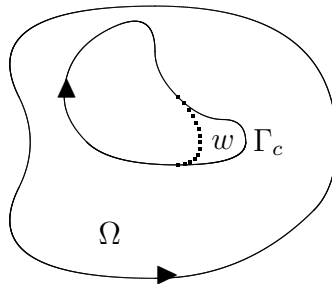

FIGURE 1. Extension of Ω to $\mathcal{G} = \Omega \cup w \cup \Gamma_c$. $\Gamma_c = \partial \Omega \setminus \partial \mathcal{G}$.

The aim of this paper is to establish the exact boundary controllability of the Navier–Stokes system, where the controls are supported in a given open subset of the boundary. Namely, we will prove the following theorem, whose exact formulation is given in section 5.

MAIN THEOREM: *Let $(\hat{u}, \hat{p})$ be a solution for system (1), with $\zeta = 0$ and $t \in (0, T)$, such that $|\hat{u}|_{L^\infty((0,T) \times \Omega, \mathbb{R}^3)} + |\partial_t \hat{u}|_{L^2((0,T), L^\sigma(\Omega, \mathbb{R}^3))} \leq R$, where $R > 0$ and $\sigma > {}^6/_5$ are constants. Then for any open subset $\Gamma_c \subseteq \Gamma$ and any (admissible) initial condition u_0*

sufficiently close to $\hat{u}(0)$, *there exists a boundary control* $\zeta \in L^2((0, T), L^2(\Gamma_c))$, *such that* $u(T) = \hat{u}(T)$.

Note that, although we confine ourselves to the three-dimensional (3D) case, this theorem remains true for the two-dimensional (2D) Navier–Stokes system.

The rest of the paper is organized as follows. In section 2, we introduce the functional spaces arising in the theory of the Navier–Stokes equations and set up our problem. In section 3 we present some remarks concerning the compatibility conditions to be satisfied by the data of the system, in order to have existence of so-called weak solutions, for a linearized auxiliary system. Section 4 is devoted to the null controllability of that linear system. In section 5, we establish the main result of the paper on local exact controllability of the full Navier–Stokes system. The appendix gathers the proof of two auxiliary results used in the main text.

Notation. We write $\mathbb{N}$ and $\mathbb{R}$ for the sets of nonnegative integers and real numbers, respectively, and we define $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}^3$ a bounded domain with a C^2-smooth boundary $\Gamma = \partial\Omega$. The partial time derivative $\partial u/\partial t$ of a function $u(t, x)$ will be denoted by $\partial_t u$.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm, by X' its dual, and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} := \sup\{\langle f, x \rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$.

If X and Y are normed spaces, we consider the intersection $X \cap Y$ endowed with the norm: $|a|_{X \cap Y} := \left(|a|_X^2 + |a|_Y^2\right)^{1/2}$. If X and Y are endowed with a scalar product then, $(a, b)_{X \cap Y} := (a, b)_X + (a, b)_Y$ defines a scalar product in $X \cap Y$.

Given open interval $I \subseteq \mathbb{R}$, then we write

$$W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\},$$

where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I,X,Y)} := \left(|f|_{L^2(I,X)}^2 + |\partial_t f|_{L^2(I,Y)}^2\right)^{1/2}$. Again, if X and Y are endowed with a scalar product then,

$$(f, g)_{W(I, X, Y)} := \int_I (f(\tau), g(\tau))_X \, d\tau + \int_I (\partial_t a(\tau), \partial_t b(\tau))_Y \, d\tau$$

defines a scalar product in $W(I, X, Y)$.

If $\bar{I} \subset \mathbb{R}$ is a closed interval, then $C(\bar{I}, X)$ stands for the space of continuous functions $f : \bar{I} \to X$ with the norm $|f|_{C(\bar{I},X)} = \max_{t \in \bar{I}} |f(t)|_X$.

$\overline{C}_{[a_1,\dots,a_k]}$ denotes a function of nonnegative variables a_j that increases in each of its arguments.

C, C_i, $i = 1, 2, \dots$, stand for unessential positive constants.

2. Preliminaries

2.1. **Functional spaces.** Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain of class C^2 located locally on one side of its boundary $\Gamma = \partial\Omega$. More precisely we suppose that each point $p \in \Gamma$ has a tubular neighborhood $\mathcal{T}_p \subset \mathbb{R}^3$ that is diffeomorphic to a cylinder $\mathbb{C}_p :=$ $\{(w_1, w_2, w_3) \in \mathbb{R}^3 \mid w_1^2 + w_2^2 < 1 \text{ and } |w_3|_{\mathbb{R}} < \varepsilon_p\}$, for a suitable $\varepsilon_p > 0$: There exists a bijective mapping

$$(4) \qquad \begin{aligned} \Phi_p : \mathbb{C}_p &\rightarrow \mathcal{T}_p \\ (w_1, w_2, w_3) &\mapsto \left(w_1, w_2, \Phi_p^0(w_1, w_2)\right) + w_3 \mathbf{n}_{\Phi_p^0(w_1, w_2)} \end{aligned}$$

where (see Figure 2 as an illustration), for $\mathbb{C}_p^0 := \{(w_1, w_2, w_3) \in \mathbb{C}_p \mid w_3 = 0\}$ and $\mathbb{C}_p^- := \{(w_1, w_2, w_3) \in \mathbb{C}_p \mid w_3 < 0\}$ we have

- both Φ_p and its inverse $\Phi_p^{-1} : \mathcal{T}_p \to \mathbb{C}_p$ are of class C^1,
- $\Phi_p(\mathbb{C}_p^0) = \mathcal{T}_p \cap \Gamma$ and $\Phi_p(\mathbb{C}_p^-) = \mathcal{T}_p \cap \Omega$,
- Φ_p^0 is of class C^2 and $\mathbf{n}_{\Phi_p^0(w_1, w_2)}$ is the unit outward normal vector to Γ at the point $\Phi_p^0(w_1, w_2) \in \Gamma$.
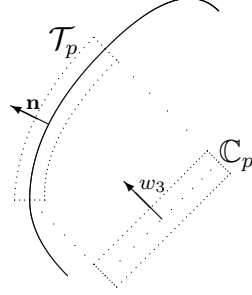

FIGURE 2. Tubular neighborhood.

Let us denote by $L^p(\Omega)$, $q \geq 1$, the usual Lebesgue space of measurable functions f defined in Ω, and such that $|f|^p$ is integrable over Ω. By $W^{s,p}(\Omega)$, for $s \in \mathbb{R}$, denote the usual Sobolev space of order s. Denote also by $L^p(\Omega, \mathbb{R}^3)$, respectively $W^{s,p}(\Omega, \mathbb{R}^3)$, the space of vector fields in Ω whose components belong to $L^p(\Omega)$, respectively $W^{s,p}(\Omega)$. In the case $p = 2$, and as usual we put $H^s(\Omega) := W^{s,2}(\Omega)$ and $H^s(\Omega, \mathbb{R}^3) := W^{s,2}(\Omega, \mathbb{R}^3)$. Recall that $H^0(\Omega) = L^2(\Omega)$ and $H^0(\Omega, \mathbb{R}^3) = L^2(\Omega, \mathbb{R}^3)$.

Due to the incompressibility condition, $\operatorname{div} u = 0$, some important subspaces in the study of the Navier–Stokes system (1) are the Sobolev subspaces

$$H_{\operatorname{div}}^s(\Omega, \mathbb{R}^3) := \{u \in H^s(\Omega, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \Omega\}, \quad s \geq 0.$$

We consider we are able to apply a boundary control through an open subset $\Gamma_c \subseteq \Gamma$. The incompressibility condition allows us to define the trace of $u \cdot \mathbf{n}$ on the boundary Γ, where $\mathbf{n}$ is the unit outward normal vector to the boundary Γ, and then to write

$$H_c := \{u \in L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \Gamma \setminus \Gamma_c\}, \quad H := \{u \in L_{\operatorname{div}}^2(\Omega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

Some spaces of more regular vector fields we find throughout the paper are

$$V := \{u \in H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma\}, \quad V_c := \{u \in H_{\operatorname{div}}^1(\Omega, \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma \setminus \Gamma_c\}.$$

The spaces $H_{\operatorname{div}}^s(\Omega, \mathbb{R}^3)$ are endowed with the scalar product induced by the scalar product in $H^s(\Omega, \mathbb{R}^3)$; the spaces H and H_c with that induced by the scalar product in $L^2(\Omega, \mathbb{R}^3)$; the spaces V and V_c with that induced by the scalar product in $H^1(\Omega, \mathbb{R}^3)$; and $\mathrm{D}(L)$ with that induced by the scalar product in $H^2(\Omega, \mathbb{R}^3)$.

For each $s > 0$, we recall also the dual space $H^{-s}(\Omega) = (H_0^s(\Omega))'$ of

$$H_0^s(\Omega) = \text{ closure of } \{f \in C^\infty(\Omega) \mid \operatorname{supp} f \subset \Omega\} \text{ in } H^s(\Omega).$$

$H^{-s}(\Omega, \mathbb{R}^3)$ is defined analogously.

Finally, fix $\sigma > {}^6\!/\!{}_5$, and introduce the following Banach space of measurable vector fields

$$(5) \quad \mathcal{W} := \Big\{ u \in L^\infty((0, T) \times \Omega, \mathbb{R}^3) \,\Big|\, \operatorname{div} u(t) = 0 \text{ in } \Omega \text{ for a.e. } t \in (0, T),$$

$$\text{and } \partial_t u \in L^2((0, T), L^\sigma(\Omega, \mathbb{R}^3)) \Big\}$$

endowed with the norm $|u|_{\mathcal{W}} := |u|_{L^\infty((0, T) \times \Omega, \mathbb{R}^3)} + |\partial_t u|_{L^2((0, T), L^\sigma(\Omega, \mathbb{R}^3))}$.

2.2. **Setting of the problem.** We start by recalling that the space of the traces $u|_\Gamma$ at the boundary Γ of the elements u in the Hilbert space

$$W((0, T), H^s_{\text{div}}(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$$

is given explicitly in [8, section 2.2], for each $s > 1/2$, $s \notin \{3/2, 5/2\}$. Moreover, denoting that trace space by $G^s((0, T), \Gamma)$, we have that $G^s((0, T), \Gamma)$ in a Hilbert space, the mapping $u \mapsto u|_\Gamma$ is continuous and there exists a continuous extension E_s from $G^s((0, T), \Gamma)$ into $W((0, T), H^s_{\text{div}}(\Omega, \mathbb{R}^3), H^{s-2}(\Omega, \mathbb{R}^3))$ such that $(E_s a)|_\Gamma = a$, for all $a \in G^s((0, T), \Gamma)$.

Together with system (1), (3), consider also the system

$$(6) \qquad \begin{aligned} \partial_t y + \langle y \cdot \nabla \rangle y + \langle z \cdot \nabla \rangle y + \langle y \cdot \nabla \rangle z - \nu \Delta y + \nabla p + f &= 0, & \text{div}\, y &= 0, \\ y|_\Gamma &= 0, & y(0) &= y_0 \end{aligned}$$

where $z \in W((0, T), H^1_{\text{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$.

Definition 2.1. Given $y_0 \in H$ and $f \in L^2((0, T), H^{-1}(\Omega, \mathbb{R}^3))$, we call a function $y \in L^2((0, T), V) \cap L^\infty((0, T), H)$, with $\partial_t y \in L^1((0, T), V')$, a weak solution for system (6) if it is a weak solution in the classical sense of [21] (see also [16]).

Remark 2.1. Recall that V', in [21], is the image $LV = \{Lv \mid v \in V\} \subset H^{-1}(\Omega, \mathbb{R}^3)$ where L is the Stokes operator $L : V \to V'$ with $\langle Lv, u \rangle_{V', V} = ((v, u)) := (\nabla u, \nabla v)_{L^2(\Omega, \mathbb{R}^9)}$. The elements of V' are completely defined by their values in V and, a complement of V' in $H^{-1}(\Omega, \mathbb{R}^3)$ is the set G of distributions vanishing in V. It turns out (see, e.g., [21, Ch. I, Remark 1.9]) that $G = \{\nabla p \mid p \in L^2(\Omega)\}$. Hence we may write

$$(7) \qquad H^{-1}(\Omega, \mathbb{R}^3) = V' \oplus G.$$

To see that the sum is a direct one we take $h \in V' \cap G$, and set $u \in V$ satisfying $Lu = h$; the last identity means that there exists $p \in L^2(\Omega)$ such that $-\nu \Delta u + \nabla p = h$. Then, taking the product with u, we obtain $\nu |\nabla u|^2_{L^2(\Omega, \mathbb{R}^9)} + \langle \nabla p, u \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H^1_0(\Omega, \mathbb{R}^3)} = \langle h, u \rangle_{H^{-1}(\Omega, \mathbb{R}^3), H^1_0(\Omega, \mathbb{R}^3)}$ and, since $h \in G$, we arrive to $|\nabla u|^2_{L^2(\Omega, \mathbb{R}^9)} = 0$. Thus, together with $u|_\Gamma = 0$, we conclude that $u = 0$. Consequently $h = 0$, i.e., $V' \cap G = \{0\}$.

Remark 2.2. The existence of a weak solution for system (6) with $z = 0$, and $f \in L^2((0, T), V')$ is classical (see, e.g., [21, Ch. III]). If $f(t) \in H^{-1}(\Omega, \mathbb{R}^3)$, by (7), we may write $f(t) = f_V(t) + \nabla r(t)$ where $f_V(t) \in V'$ and $r(t) \in L^2(\Omega)$. So, in the first equation in (6), we may replace ∇p by $\nabla(p+r)$ and f by f_V reducing, in this way, the problem to the case the force f is in $L^2((0, T), V')$. In the case $z \neq 0$ we may define weak solution in the same way.

Remark 2.3. Given a weak solution for system (6), the function $p \in L^2((0, T), L^2(\Omega))$ is uniquely defined up to an additive constant.

Definition 2.2. Given $u_0 \in L^2_{\text{div}}(\Omega, \mathbb{R}^3)$, $h \in L^2((0, T), H^{-1}(\Omega, \mathbb{R}^3))$, and $\gamma + \zeta \in G^1((0, T), \Gamma)$, we say that $u \in L^2((0, T), H^1_{\text{div}}(\Omega, \mathbb{R}^3)) \cap L^\infty((0, T), L^2_{\text{div}}(\Omega, \mathbb{R}^3))$, with $\partial_t u \in L^1((0, T), H^{-1}(\Omega, \mathbb{R}^3))$, is a weak solution for system (1), (3) if $y = u - E_1(\gamma + \zeta)$ is a weak solution for system (6) with $z = E_1(\gamma + \zeta)$, $f = h + \partial_t E_1(\gamma + \zeta) + \langle E_1(\gamma + \zeta) \cdot \nabla \rangle E_1(\gamma + \zeta) - \nu \Delta E_1(\gamma + \zeta)$, and $y_0 = u_0 - E_1(\gamma + \zeta)(0) \in H$.

Remark 2.4. The existence of a weak solution for system (1), (3) does not depend on the extension E_1. Indeed if there exists a weak solution in the form $u = y + E_1(\gamma + \zeta)$, and if another extension $F_1(\gamma + \zeta) \in W((0, T), H^1_{\text{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ is given, then we may write $u = y + E_1(\gamma + \zeta) - F_1(\gamma + \zeta) + F_1(\gamma + \zeta)$ and we can check that $k := y + E_1(\gamma + \zeta) - F_1(\gamma + \zeta) = u - F_1(\gamma + \zeta)$ solves system (6) with $z = F_1(\gamma + \zeta)$, $f = h + \partial_t F_1(\gamma+\zeta) + \langle F_1(\gamma+\zeta) \cdot \nabla \rangle F_1(\gamma+\zeta) - \nu \Delta F_1(\gamma+\zeta)$, and $k_0 = u_0 - F_1(\gamma+\zeta)(0) \in H$.

Remark 2.5. The trace $z(0)$ of a function $z \in W((0, T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \subset C([0, T], L^2(\Omega, \mathbb{R}^3))$ at time $t = 0$ is well defined in $L^2(\Omega, \mathbb{R}^3)$ (see, e.g, [17, Ch. 1, Theorem 3.1 and Lemma 12.1]). Clearly we have that that trace must belong to the closure of $H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3)$ in $L^2(\Omega, \mathbb{R}^3)$, which implies $z(0) \in L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$. On the other hand, if $u|_\Gamma = \gamma + \zeta$, then $y := u - E_1(\gamma + \zeta)$ vanishes at the boundary Γ. If y is a weak solution for system (6), then $y(0)$ is well defined in V', since both u_0 and $z(0)$ are in $L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$ we necessarily have that $y(0) = u_0 - E_1(\gamma + \zeta)(0) \in V' \cap L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3) = H$.

Let us fix two functions $h \in L^2((0, T), H^{-1}(\Omega, \mathbb{R}^3))$ and $\gamma \in G^1((0, T), \Gamma)$. Suppose that $\hat{u}$ solves the Navier–Stokes system (1) with $\zeta = 0$ on Γ; that $\hat{u}(0)$ makes sense and that we have $\hat{u}(0) \in L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$.

Given a function $u_0 \in L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$ such that $u_0 - \hat{u}(0) \in H_c$, and an open subset $\Gamma_c \subseteq \Gamma$, our goal is to find a control $\zeta \in G^1((0, T), \Gamma)$ with $\mathrm{supp}\,\zeta(t) \subset \Gamma_c$ for a.e. $t \in (0, T)$, and such that the corresponding solution to system (1), (3), satisfies $u(T) = \hat{u}(T)$.

Let us note that seeking a solution of (1), (3) in the form $u = \hat{u} + v$, we obtain the following equivalent problem for v:

(8)
$$\partial_t v + \mathcal{B}(\hat{u})v + \langle v \cdot \nabla \rangle v - \nu \Delta v + \nabla p = 0, \qquad \mathrm{div}\, v = 0,$$
$$v|_\Gamma = \zeta, \qquad v(0) = u_0 - u(0)$$

where $p = p_u - p_{\hat{u}}$, p_u and $p_{\hat{u}}$ are the pressure functions associated with the solutions u and $\hat{u}$, respectively; and $\mathcal{B}(\hat{u})v$ stands for $\langle \hat{u} \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle \hat{u}$. It is clear that it suffices to consider the problem of null controllability for solutions v of this problem.

We will start by studying the linearization around zero of system (8); for the moment we include an external force:

(9)
$$\partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p + g = 0, \qquad \mathrm{div}\, v = 0,$$
$$v|_\Gamma = \zeta, \qquad v(0) = v_0.$$

Definition 2.3. Given $v_0 \in L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$, and the forces $g \in L^2((0, T), H^{-1}(\Omega, \mathbb{R}^3))$, and ζ in $G^1((0, T), \Gamma)$, we say that $v \in W((0, T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ is a weak solution for system (9), if $v - E_1(\zeta) \in W((0, T), V, V')$ is a weak solution for system

(10)
$$\partial_t y + \mathcal{B}(\hat{u})y - \nu \Delta y + \nabla p + f = 0, \qquad \mathrm{div}\, y = 0,$$
$$y|_\Gamma = 0, \qquad y(0) = y_0$$

with $f = g + \partial_t E_1 \zeta + \mathcal{B}(\hat{u})E_1 \zeta - \nu \Delta E_1 \zeta$, and $y_0 = v_0 - E_1 \zeta(0) \in H$. Again, weak solution for (10) is understood in the classical sense of [21].

Remark 2.6. Note that if v is a weak solution for (9), then necessarily we have $v_0 \cdot \mathbf{n} = E_1 \zeta(0) \cdot \mathbf{n}$ on Γ.

Remark 2.7. Note that in Definition 2.3 we ask more regularity for the solution than in Definition 2.2. It turns out that for the linear system a weak solution in the sense of Definition 2.2 has necessarily the regularity asked in Definition 2.3. Hence the two definitions are equivalent (in the linear case). We "repeat" the definition to stress that in the linear case we (can) look for more regularity. The presentation of two separated "definitions" for the linear and nonlinear problem is classical: compare Problems (3.17)–(3.19) and (1.36)–(1.38) in [21, Chapter III].

Remark 2.8. The existence and uniqueness of a weak solution in $W((0, T), V, V')$ for (10) can be proved by standard arguments as in [21], taking into account that, formally

$$\Big\langle \langle \hat{u} \cdot \nabla \rangle y + \langle y \cdot \nabla \rangle \hat{u}, \, w \Big\rangle_{H^{-1}(\Omega, \mathbb{R}^3), H_0^1(\Omega, \mathbb{R}^3)}$$

$$= \sum_{i,j=1}^3 \int_\Omega \hat{u}_i (\partial_i y_j) w_j \, \mathrm{d}x + \sum_{i,j=1}^3 \int_\Omega y_i (\partial_i \hat{u}_j) w_j \, \mathrm{d}x$$

$$= -\sum_{i,j=1}^3 \int_\Omega \hat{u}_i (\partial_i w_j) y_j \, \mathrm{d}x - \sum_{i,j=1}^3 \int_\Omega y_i (\partial_i w_j) \hat{u}_j \, \mathrm{d}x$$

that leads to the estimate $|\langle \hat{u} \cdot \nabla \rangle y + \langle y \cdot \nabla \rangle \hat{u}|_{H^{-1}(\Omega, \mathbb{R}^3)} \le |\hat{u}|_{L^\infty(\Omega, \mathbb{R}^3)} |y|_{L^2(\Omega, \mathbb{R}^3)}$.

3. Further remarks on the weak solutions

3.1. **On the admissible initial conditions.** From Remark 2.6 we see that the weak solution for system (9) will exist only under some conditions, indeed it gives us one compatibility condition v_0 have to satisfy if we take controls in $G^1((0, T), \Gamma)$. In the weak solutions framework, is this the only condition to be satisfied by v_0? Here we discuss this question for a slightly more general setting, which is useful if we want to change the control space.

Let us fix $\hat{u} \in \mathcal{W}$, and consider the system

$$(11) \qquad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu \Delta v + \nabla p + g &= 0, & \operatorname{div} v &= 0, \\ v\big|_\Gamma &= K_s \zeta, & v(0) &= v_0 \end{aligned}$$

with, $g \in L^2((0, T), H^{-1}(\Omega, \mathbb{R}^3))$, $\mathcal{Z}$ a Hilbert space, $s \in [1, 2] \setminus \{3/2\}$ and $K_s : \mathcal{Z} \to G^s((0, T), \Gamma)$ a continuous linear mapping. The values $s > 1$ are necessary to consider solutions more regular than weak ones. We exclude the case $s = 3/2$ because it presents some singularities, see [8]; it will be not needed in what follows.

Let us start with the following general Lemma:

Lemma 3.1. *Given a linear mapping $\Lambda : Y \to X$ from a Hilbert space Y into a vector space X. Denote by $\ker \Lambda := \{y \in Y \mid \Lambda y = 0\}$ the kernel of Λ and by $\ker \Lambda^\perp$ its orthogonal in Y. Then the range $\Lambda Y := \{z \in X \mid z = \Lambda y$ and $y \in Y\}$ of Λ is a Hilbert space when endowed with the scalar product*

$$(12) \qquad\qquad (a, b)_{\Lambda Y} := (y_a, y_b)_Y$$

where for each $z \in \Lambda Y$, $y_z \in Y$ is defined by

$$(13) \qquad\qquad \Lambda y_z = z \text{ and } y_z \in \ker \Lambda^\perp.$$

Moreover, for the respective induced norms, we have: for each $a \in \Lambda Y$

$$|a|_{\Lambda Y} = \inf\{|y|_Y \mid y \in Y \text{ and } \Lambda y = a\}.$$

Proof. Let $z \in \Lambda Y$ and let $h, g \in \ker \Lambda^\perp$ satisfy $\Lambda h = z$ and $\Lambda g = z$, then $\Lambda(g - h) = 0$ and so, $g - h \in \ker \Lambda \cap \ker \Lambda^\perp = \{0\}$. Hence y_z is unique and so, well defined. Moreover, the mapping $z \mapsto y_z$ is linear, because given $w, z \in \Lambda Y$ and $\alpha, \beta \in \mathbb{R}$, we have $\Lambda(\alpha y_w + \beta y_z) = \alpha w + \beta z = \Lambda y_{\alpha w + \beta z}$; since $\alpha y_w + \beta y_z \in \ker \Lambda^\perp$, necessarily $y_{\alpha w + \beta z} = \alpha y_w + \beta y_z$.

It is straightforward to check that $(\cdot, \cdot)_{\Lambda Y}$ is a scalar product. Now, given a Cauchy sequence (a^m) in ΛY, from the linearity of $z \mapsto y_z$ we have that (y_{a^m}) is a Cauchy sequence in Y. Denoting by z the limit of y_{a^m} in Y, by the closedness of $\ker \Lambda^\perp$ (see [12, section I§12, Theorem 1]) it follows that $z \in \ker \Lambda^\perp$. Then, from $|y_{a^m} - z|_Y := |a^m - \Lambda z|_{\Lambda Y}$, we conclude that Λz is the limit of a^m in ΛY. Hence ΛY is complete.

Finally, given $h \in \{y \in Y \mid \Lambda y = z\}$, writing $h = h^K + h^{K\perp}$ with $h^K \in \ker \Lambda$ and $h^{K\perp} \in \ker \Lambda^\perp$, we have $|h|_Y = |h^K|_Y + |h^{K\perp}|_Y$ and it is clear that $h^{K\perp} = y_z$. Thus $|z|_{\Lambda Y} = |y_z|_Y = |h^{K\perp}|_Y = \inf\{|h|_Y \mid h \in Y \text{ and } \Lambda h = z\}$. □

Remark 3.1. Note that $\Lambda : \ker \Lambda^\perp \to \Lambda Y$ is an isometry.

To simplify the presentation we introduce the following:

Definition 3.1. Given a triple (Λ, Y, X) as in Lemma 3.1 we call the scalar product (12) and associated norm by range scalar product and range norm, respectively.

Let $E_1 : G^1((0, T), \Gamma) \to W((0, T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ be the extension in the beginning of section 2.2. By Lemma 3.1 the space $\mathcal{H}_{K_{1,s}} := E_1 K_s \mathcal{Z}(0) = \{\gamma(0) \mid \gamma \in E_1 K_s \mathcal{Z}\}$ can be endowed with the range scalar product (12), taking $\Lambda \zeta := E_1 K_s \zeta(0)$. Note that $E_1 K_s \mathcal{Z} \subset W((0, T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ and, by classical interpolation results (see, e.g., [17]), we have that $\{\eta(0) \mid \eta \in W((0, T), H^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))\}$ coincides with $L^2(\Omega, \mathbb{R}^3)$. Hence $\mathcal{H}_{K_{1,s}} \subset L^2(\Omega, \mathbb{R}^3)$. From [17, Ch. 1; Theorem 3.2, Remark 3.6 and Lemma 12.1] we also know that the usual $L^2(\Omega, \mathbb{R}^3)$-norm is equivalent to the range norm associated with the mapping from $W((0, T), H^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ into $L^2(\Omega, \mathbb{R}^3)$, sending η to $\eta(0)$. We have $|h|_{\mathcal{H}_{K_{1,s}}} = \inf\{|\zeta|_{\mathcal{Z}} \mid \zeta \in \mathcal{Z} \text{ and } E_1 K_s \zeta(0) = h\}$, for all $h \in \mathcal{H}_{K_{1,s}}$; then from the continuity of the mapping $\zeta \mapsto E_1 K_s \zeta$ we obtain

$$|h|_{\mathcal{H}_{K_{1,s}}} \geq C_1 \inf\left\{ |E_1 K_s \zeta|_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))} \,\Big|\, \zeta \in \mathcal{Z} \text{ and } E_1 K_s \zeta(0) = h \right\}$$

$$\geq C_1 \inf\left\{ \{ |\gamma|_{W((0,T), H^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))} \,\Big| \right.$$

$$\left. \gamma \in W((0, T), H^1(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3)) \text{ and } \gamma(0) = h \right\}$$

$$\geq C_2 |h|_{L^2(\Omega, \mathbb{R}^3)}.$$

Hence, taking the constant $C = C_2^{-1}$, we have that for all $h \in \mathcal{H}_{K_{1,s}}$

$$(14) \qquad\qquad\qquad |h|_{L^2(\Omega, \mathbb{R}^3)} \leq C|h|_{\mathcal{H}_{K_{1,s}}}.$$

However, note that, depending on the space $\mathcal{Z}$ and mapping K_s, we may have that the range norm in $\mathcal{H}_{K_{1,s}}$ is not equivalent to the $L^2(\Omega, \mathbb{R}^3)$-norm.

On the other hand, for example, from classical existence results for weak solutions for the linearized Navier–Stokes system with homogeneous Dirichlet boundary conditions (see, e.g., [21, chapter III, Theorem 1.1]), we can conclude that $H = \{\gamma(0) \mid \gamma \in W((0, T), V, H^{-1}(\Omega, \mathbb{R}^3))\}$. Therefore, for weak solutions and for system (11), the admissible initial conditions are those elements in $\mathcal{A}_{K_{1,s}} = H + \mathcal{H}_{K_{1,s}}$.

Now, considering the addition mapping $\Sigma : H \times \mathcal{H}_{K_{1,s}} \to L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$, defined by $\Sigma(h, v) \mapsto h + v$, where $H \times \mathcal{H}_{K_{1,s}}$ is endowed with the usual Cartesian scalar product:

$$((h_1, v_1), (h_2, v_2))_{H \times \mathcal{H}_{K_{1,s}}} := (h_1, h_2)_H + (v_1, v_1)_{\mathcal{H}_{K_{1,s}}}.$$

We endow its range $H + \mathcal{H}_{K_{1,s}} = \mathcal{A}_{K_{1,s}}$ with the range scalar product, making $\mathcal{A}_{K_{1,s}}$ a Hilbert space. Finally, note that $\mathcal{A}_{K_{1,s}} \subseteq L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$ continuously, i.e.,

$$(15) \qquad\qquad |u|_{L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)} \leq C|u|_{\mathcal{A}_{K_{1,s}}} \quad \text{for all } u \in \mathcal{A}_{K_{1,s}}.$$

Moreover from the definition of the $\mathcal{A}_{K_{1,s}}$-norm, it follows that

$$|u|_{\mathcal{A}_{K_{1,s}}} = \inf\left\{ \left(|h|_H^2 + |v|_{\mathcal{H}_{K_{1,s}}}^2 \right)^{1/2} \,\Big|\, (h, v) \in H \times \mathcal{H}_{K_{1,s}} \text{ and } h + v = u \right\}$$

$$(16) \qquad\qquad \leq |u|_H, \text{ for all } u \in H.$$

and we can conclude that the H-norm and $\mathcal{A}_{K_{1,s}}$-norm are equivalent in H.

3.2. **On the existence and uniqueness of solutions.** Now we can state the following existence result:

Theorem 3.2. *Given $v_0 \in \mathcal{A}_{K_{1,s}}$, $g \in L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))$, and $\zeta \in \mathcal{Z}$, then there exists a unique weak solution $v \in W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ for system (11). Moreover v depends continuously on the given data (v_0, g, ζ):*

$$|v|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}$$
$$\leq \overline{C}_{[|\hat{u}|_{L^\infty((0,T)\times\Omega)}]} \left(|v_0|^2_{L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)} + |g|^2_{L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))} + |\zeta|^2_{\mathcal{Z}} \right).$$

Remark 3.2. Note that (15) and the inequality in the previous theorem implies that $|v|^2_{W((0,T), V_c, H^{-1}(\Omega, \mathbb{R}^3))} \leq C_1 \left(|v_0|^2_{\mathcal{A}_{K_{1,s}}} + |g|^2_{L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))} + |\zeta|^2_{\mathcal{Z}} \right)$, which looks more coherent with the "continuity on the data" statement.

Proof of Theorem 3.2. Let y be the solution of system (10) with $f = g + \partial_t E_1 K_s \zeta + \mathcal{B}(\hat{u}) E_1 K_s \zeta - \nu \Delta E_1 K_s \zeta$, and $y_0 = v_0 - E_1 K_s \zeta(0) \in H$. Then $v = E_1 K_s \zeta + y$ solves (11). Since the mapping $\eta \mapsto \partial_t \eta + \mathcal{B}(\hat{u})\eta - \nu \Delta \eta$ from $W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ into the space $L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))$ is continuous, we can proceed in a standard way to derive the estimate

$$|y|^2_{W((0,T), V, H^{-1}(\Omega, \mathbb{R}^3))}$$
$$\leq C \left(|y_0|^2_H + |g|^2_{L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))} + |E_1 K_s \zeta|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))} \right)$$

for the Oseen-like system (10), with $C = \overline{C}_{[|\hat{u}|_{L^\infty((0,T)\times\Omega)}]}$. From this estimate and the continuity of $E_1 K_s$ we obtain

$$|v|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}$$
$$\leq |y|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))} + |E_1 K_s \zeta|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}$$
$$\leq C|y_0|^2_H + C|g|^2_{L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))} + (C+1)|E_1 K_s \zeta|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}$$
$$\leq C|y_0|^2_H + C|g|^2_{L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))} + C_1|\zeta|^2_{\mathcal{Z}}.$$

From $y_0 = v_0 - E_1 K_s \zeta(0)$, inequality (14), and $|E_1 K_s \zeta(0)|_{\mathcal{H}_{K_{1,s}}} \leq |\zeta|_{\mathcal{Z}}$ we have

$$|y_0|^2_H \leq \left(|v_0|_{L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)} + |E_1 K_s \zeta(0)|_{L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)} \right)^2 \leq 2|v_0|^2_{L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)} + 2C_2|\zeta|^2_{\mathcal{Z}},$$

and we arrive to

$$|v|^2_{W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))}$$
$$\leq \overline{C}_{[|\hat{u}|_{L^\infty((0,T)\times\Omega)}]} \left(|v_0|^2_{L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)} + |g|^2_{L^2((0,T), H^{-1}(\Omega, \mathbb{R}^3))} + |\zeta|^2_{\mathcal{Z}} \right).$$

$\square$

Remark 3.3. From Remark 2.4, we have that the weak solution v for system (11) does not depend on the extension E_1. Hence also the set of admissible initial conditions must be independent of the extension. We can confirm that fact as follows: let $F_1 : K_s \mathcal{Z} \to W((0,T), H^1_{\mathrm{div}}(\Omega, \mathbb{R}^3), H^{-1}(\Omega, \mathbb{R}^3))$ be another extension such that, for all $a \in K_s \mathcal{Z}$, we have $(F_1 a)|_\Gamma = a$. Let $\mathcal{A}^E_{K_{1,s}} = H + \mathcal{H}^E_{K_{1,s}}$ and $\mathcal{A}^F_{K_{1,s}} = H + \mathcal{H}^F_{K_{1,s}}$ stand for the respective admissible sets of initial conditions defined as above. The equality $E_1 K_s \zeta|_\Gamma = F_1 K_s \zeta|_\Gamma$ implies that $E_1 K_s \zeta - F_1 K_s \zeta \in W((0,T), V, H^{-1}(\Omega, \mathbb{R}^3))$, then necessarily $E_1 K_s \zeta(0) - F_1 K_s \zeta(0) \in H$, which means that $\mathcal{H}^E_{K_{1,s}} \subset \mathcal{H}^F_{K_{1,s}} + H = \mathcal{A}^F_{K_{1,s}}$, and we conclude that $\mathcal{A}^E_{K_{1,s}} \subseteq \mathcal{A}^F_{K_{1,s}}$. Analogously we can arrive to $\mathcal{A}^F_{K_{1,s}} \subseteq \mathcal{A}^E_{K_{1,s}}$.

4. EXACT NULL CONTROLLABILITY OF THE LINEAR SYSTEM

Now we consider the case our controls are supported in $(0, T) \times \Gamma_c$. Say, Γ_c is the open subset of the boundary where we are able to act. Let us denote by $G_c^1((0, T), \Gamma) \subset G^1((0, T), \Gamma)$ the space of the traces at Γ of the functions in $W((0, T), V_c, H^{-1}(\Omega, \mathbb{R}^3))$, and consider the system (11) with $g = 0$, $\mathcal{Z} = G_c^1((0, T), \Gamma)$, and $K_1(\zeta) = \zeta$, i.e.,

$$(17) \qquad \begin{aligned} \partial_t v + \mathcal{B}(\hat{u})v - \nu\Delta v + \nabla p &= 0, & \operatorname{div} v &= 0, \\ v|_\Gamma &= \zeta, & v(0) &= v_0 \end{aligned}$$

with $\zeta \in G_c^1((0, T), \Gamma)$ and $v_0 \in \mathcal{A}_{K_{1,1}}$ where $\mathcal{A}_{K_{1,1}} = H + \mathcal{H}_{K_{1,1}}$ is the space of admissible initial conditions for this system with $\mathcal{A}_{K_{1,1}}$ and $\mathcal{H}_{K_{1,1}}$ defined as in section 3.1. Note that since $W((0, T), V_c, H^{-1}(\Omega, \mathbb{R}^3))$ is a closed subspace of $W((0, T), V, H^{-1}(\Omega, \mathbb{R}^3))$, it follows that $G_c^1((0, T), \Gamma)$ is a closed subspace of $G^1((0, T), \Gamma)$ (due to the continuity of extension E_1 and restriction $u \mapsto u|_\Gamma$).

Here we prove the null controllability property for the linear system (17).

Remark 4.1. In this section we find already, in the simpler linear setting, an idea about what we are going to do in the next section for the study of the full system. On the other hand we believe the results on this section, by themselves, may be useful in the study of other controllability problems of the Navier–Stokes system.

Theorem 4.1. *Given* $v_0 \in \mathcal{A}_{K_{1,1}}$, *there exists a control* $\zeta = \zeta(v_0) \in G_c^1((0, T), \Gamma)$ *such that, for the corresponding weak solution* $S(v_0, \zeta)(t)$, *to system* (17), *we have that* $S(v_0, \zeta)(T) = 0$. *Moreover the control may be chosen so that the mapping* $v_0 \mapsto \zeta(v_0)$ *is linear and continuous:* $|\zeta(v_0)|_{G_c^1((0, T), \Gamma)} \leq \overline{C}_{[|\hat{u}|_\mathcal{W}]} |v_0|_{\mathcal{A}_{K_{1,1}}}$.

Let us extend our domain Ω to a bigger domain $\widetilde{\Omega}$ of class C^2 such that

$$\widetilde{\Omega} = \Omega \cup w \cup \Gamma_c, \quad \Omega \cap w = \emptyset;$$

see Figure 1 as an illustration. We choose w such that

$$(18) \qquad w \subseteq \cup\{\mathcal{T}_{p_j} \mid j = 1, 2, \ldots, s, \text{ and } p_j \in \Gamma\}$$

where the $\mathcal{T}_{p_j}$ are the tubular neighborhoods in section 2.

For the proof of Theorem 4.1 we will need the following two auxiliary extension results, whose proofs follow some ideas from [10]. Since we are going to use some tools we do not want to introduce here, the proofs are given in the appendix.

Proposition 4.2. *Recall the set* $\mathcal{W}$ *defined in* (5), *and denote*

$$\widetilde{\mathcal{W}} := \Big\{ u \in L^\infty((0, T) \times \widetilde{\Omega}, \mathbb{R}^3) \,\Big|\, \operatorname{div} u(t) = 0 \text{ in } \widetilde{\Omega} \text{ for a.e. } t \in (0, T),$$

$$\text{and } \partial_t u \in L^2((0, T), L^\sigma(\widetilde{\Omega}, \mathbb{R}^3)) \Big\}$$

Then a given $u \in \mathcal{W}$ *can be continuously extended to* $\tilde{u} \in \widetilde{\mathcal{W}}$: $|\tilde{u}|_{\widetilde{\mathcal{W}}} \leq C|u|_\mathcal{W}$ *for some constant* C *independent of* u. *Moreover the mapping* $u \mapsto \tilde{u}$ *is linear.*

Proposition 4.3. *Let us define* $\widetilde{V}_c := \{u \in H^1(\widetilde{\Omega}, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \widetilde{\Omega} \text{ and } u|_{\partial\widetilde{\Omega}\cap\partial\Omega} = 0\}$. *A given* $\gamma \in W((0, T), V_c, H^{-1}(\Omega, \mathbb{R}^3))$ *can be continuously extended to a* $\breve{\gamma} \in W((0, T), \widetilde{V}_c, H^{-1}(\widetilde{\Omega}, \mathbb{R}^3))$. *Moreover, if* $\gamma \in W((0, T), H^2(\Omega, \mathbb{R}^3)\cap V_c, L^2(\Omega, \mathbb{R}^3))$ *then* $\breve{\gamma} \in W((0, T), H^2(\widetilde{\Omega}, \mathbb{R}^3) \cap \widetilde{V}_c, L^2(\widetilde{\Omega}, \mathbb{R}^3))$, *and*

$$|\breve{\gamma}|_{W((0, T), \widetilde{V}_c, H^{-1}(\widetilde{\Omega}, \mathbb{R}^3))} \leq C|\gamma|_{W((0, T), V_c, H^{-1}(\Omega, \mathbb{R}^3))},$$

$$|\breve{\gamma}|_{W((0, T), H^2(\widetilde{\Omega}, \mathbb{R}^3)\cap\widetilde{V}_c, L^2(\widetilde{\Omega}, \mathbb{R}^3))} \leq C|\gamma|_{W((0, T), H^2(\Omega, \mathbb{R}^3)\cap V_c, L^2(\Omega, \mathbb{R}^3))},$$

for some constant C *independent of* γ. *Moreover the mapping* $\gamma \mapsto \breve{\gamma}$ *is linear.*

We will also need the following:

Lemma 4.4. *Given $u \in H$, its extension to $\widetilde{\Omega}$ by zero outside Ω is in $\widetilde{H} := \{u \in L^2(\widetilde{\Omega}, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \widetilde{\Omega} \text{ and } u \cdot \tilde{\mathbf{n}} = 0 \text{ on } \partial\widetilde{\Omega}\}$. Where $\tilde{\mathbf{n}}$ stands for the unit outward normal vector to the boundary $\partial\widetilde{\Omega}$ of $\widetilde{\Omega}$.*

Proof. Let $\tilde{u}$ be the extension of u by zero outside Ω. We have $(\tilde{u}, \nabla\phi)_{L^2(\widetilde{\Omega}, \mathbb{R}^3)} = (u, \nabla\phi)_{L^2(\Omega, \mathbb{R}^3)} = 0$ for all test function $\phi \in \{\psi \in C^\infty(\widetilde{\Omega}) \mid \operatorname{supp}\psi \subset \widetilde{\Omega}\}$, which means that $\operatorname{div}\tilde{u} = 0$ in $\widetilde{\Omega}$. Now, that $\tilde{u} \cdot \tilde{\mathbf{n}} = 0$ on $\partial\widetilde{\Omega}$ is clear. $\qquad\square$

Proof of Theorem 4.1. Write $v_0 = v_{0H} + v_{0\mathcal{H}}$ where $(v_{0H}, v_{0\mathcal{H}}) \in H \times \mathcal{H}_{K_{1,1}}$ is defined by (13) taking $\Lambda : H \times \mathcal{H}_{K_{1,1}} \to L^2_{\operatorname{div}}(\Omega, \mathbb{R}^3)$ with $\Lambda(h, v) = h + v$. Let also $\gamma_{v_{0\mathcal{H}}} \in W((0, T), V_c, H^{-1}(\Omega))$ be defined by (13) taking $\Lambda : W((0, T), V_c, H^{-1}(\Omega)) \to H_c$ with $\Lambda y = y(0)$.

Extend v_{0H} to the vector $\tilde{v}_{0H}$ defined in $\widetilde{\Omega}$, by zero outside Ω; by Lemma 4.4 $\tilde{v}_{0H} \in \widetilde{H}$. Also, extend both $\hat{u}$ and $\gamma_{v_{0\mathcal{H}}}$ to $\tilde{u}$ and $\breve{\gamma}_{v_{0\mathcal{H}}}$ to the bigger domain $\widetilde{\Omega}$ using Propositions 4.2 and 4.3, respectively. Now, consider the system

$$(19) \qquad \begin{aligned} \partial_t \bar{v} + \mathcal{B}(\tilde{u})\bar{v} - \nu\Delta\bar{v} + \nabla\bar{p} &= \eta, & \operatorname{div}\bar{v} &= 0, \\ \bar{v}|_{\partial\widetilde{\Omega}} &= \zeta, & \bar{v}(0) &= \bar{v}_0 := \tilde{v}_0 + \breve{\gamma}_{v_{0\mathcal{H}}}(0) \end{aligned}$$

where the pair (η, ζ) is our control. Then in the interval $(0, {}^T\!/_2)$ we apply the control $(0, \xi(t)\breve{\gamma}_{v_{0\mathcal{H}}}|_\Gamma(t))$ where ξ is a real smooth function, defined in $[0, {}^T\!/_2]$, taking the value 1 in a neighborhood of $t = 0$, and vanishing in a neighborhood of $t = {}^T\!/_2$. In this way we arrive at time ${}^T\!/_2$ to a point $\bar{v}({}^T\!/_2) \in \widetilde{H}$. Note that $\xi\breve{\gamma}_{v_{0\mathcal{H}}}|_\Gamma \in G^1_c((0, {}^T\!/_2), \partial\widetilde{\Omega})$, where $G^1_c((0, {}^T\!/_2), \partial\widetilde{\Omega})$ is the space of traces $u|_{\partial\widetilde{\Omega}}$, with $u \in W((0, {}^T\!/_2), \widetilde{V}_c, H^{-1}(\widetilde{\Omega}, \mathbb{R}^3))$. By Theorem 3.2, the solution exists in the interval of time $[0, {}^T\!/_2]$.

Now on $({}^T\!/_2, T)$ we apply a control $(\eta_{w_1}, 0)$ where $\eta_{w_1} \in L^2(({}^T\!/_2, T), L^2(\widetilde{\Omega}, \mathbb{R}^3))$ is an internal control with support contained in $w_1 \subset w_1 \cup \partial w_1 \subset w = \widetilde{\Omega} \setminus \Omega$, with w_1 an open set, and driving the solution from $\bar{v}({}^T\!/_2)$ to $\bar{v}(T) = 0$ at time T. The existence of such a control η_{w_1} is proven in [6, section 3, Proposition 1], it follows from an appropriate Carleman estimate, namely [6, section 3, Lemma 1], for the system

$$(20) \qquad \begin{aligned} -\partial_t q + \mathcal{B}^*(\tilde{u})q - \nu\Delta q + \nabla p + f &= 0, & \operatorname{div} q &= 0, \\ q|_\Gamma &= 0, & q(T) &= q_1 \in H \end{aligned}$$

that is, somehow, adjoint to system (17). Where $\mathcal{B}^*(\tilde{u})$ is the formal adjoint to $\mathcal{B}(\tilde{u})$:

$$(21) \qquad (\mathcal{B}^*(\tilde{u})q, v)_{L^2(\widetilde{\Omega}, \mathbb{R}^3)} := \langle q, \mathcal{B}^*(\tilde{u})v \rangle_{H^1_0(\widetilde{\Omega}, \mathbb{R}^3), H^{-1}(\widetilde{\Omega}, \mathbb{R}^3)}, \quad q \in V, v \in V_c.$$

It turns out that

$$(22) \qquad \mathcal{B}^*(\tilde{u})q = \hat{u} \cdot D_s q; \quad D_s q := \left(\nabla q + (\nabla q)^\top\right),$$

where $A^\top$ denotes the transpose matrix of A.

Thus the concatenation of the two controls $(0, \xi\breve{\gamma}_{v_{0\mathcal{H}}}|_{\partial\widetilde{\Omega}})$ and $(\eta_{w_1}, 0)$ drives the system from $\bar{v}(0) = \bar{v}_0$ to $\bar{v}(T) = 0$. We observe that the restriction $v := \bar{v}|_{(0, T)\times\Omega}$ of the corresponding solution $\bar{v}$ to $(0, T) \times \Omega$ solves system (17) with $\zeta = \bar{v}|_{(0, T)\times\Gamma} \in G^1_c((0, T), \Gamma)$, and satisfies $v(T) = 0$. It remains to check that the control driving the system to zero at time T may be chosen depending linearly and continuously on the initial condition.

We note that the mappings $v_0 \mapsto (v_{0H}, v_{0\mathcal{H}})$, $v_{0\mathcal{H}} \mapsto \breve{\gamma}_{v_{0\mathcal{H}}}|_\Gamma$, and $v_{0H} \mapsto \tilde{v}_{0H}$ are linear and continuous. Thus $v_0 \mapsto (0, \xi\breve{\gamma}_{v_{0\mathcal{H}}}|_{\partial\widetilde{\Omega}})$ is linear and continuous, the control $(0, \xi\breve{\gamma}_{v_{0\mathcal{H}}}|_{\partial\widetilde{\Omega}})$ drives the system (19) from $\bar{v}_0$ to a point $v_1 = \bar{v}({}^T\!/_2) \in \widetilde{H}$, and from Theorem 3.2 we can conclude that the mapping $\bar{v}_0 \mapsto v_1$ is linear and continuous. Moreover, from the proof of Proposition 1 in [6] we know that we can find a control ψ_0, driving

the system from v_1 at time $T/2$ to 0 at time T, that is the weak limit, $\psi_{\epsilon_n} \rightharpoonup \psi_0$ in $L^2((T/2, T) \times w_1)$, of a sequence ψ_{ϵ_n} satisfying $|\psi_{\epsilon_n}|_{L^2((T/2,T)\times w_1)} \le \overline{C}_{|\tilde u|_{\widetilde{\mathcal{W}}}} |v_1|_{\widetilde H}$ (see equation (61) in [6]). Thus $|\psi_0|_{L^2((T/2,T)\times w_1)} \le \liminf\limits_{n\to+\infty} |\psi_{\epsilon_n}|_{L^2((T/2,T)\times w_1)} \le \overline{C}_{|\tilde u|_{\widetilde{\mathcal{W}}}} |v_1|_{\widetilde H}$. If v_{ψ_0} is the solution associated to this control we can conclude that, for the pair $(v_{\psi_0}, \psi_0) = (v_{\psi_0}, \psi_0)(v_1)$, we have the bound

$$(23) \qquad |(v_{\psi_0}, \psi_0)(v_1)|_{L^2((T/2,T),\widetilde V) \times L^2(I, L^2(w_1))} \le \overline{C}_{|\tilde u|_{\widetilde{\mathcal{W}}}} |v_1|_{\widetilde H}.$$

Next, we consider the following minimization problem:

Problem 4.1. *Given* $\underline{v}_1 \in \widetilde H$,

$$\begin{aligned} |v|^2_{L^2(I,\widetilde V)} + |\eta|^2_{L^2(I, L^2(w_1))} &\to \min, & A(v, \eta) &= \underline{v}_1, \\ I &= (T/2, T) & (v, \eta) &\in \mathcal{X} \end{aligned}$$

with

$$\mathcal{X} := \left\{ (v, \eta) \in W(I, \widetilde V, H^{-1}(\widetilde\Omega, \mathbb{R}^3)) \times L^2(I, L^2(w_1)) \,\Big|\, u(T) = 0 \text{ and} \right.$$
$$\left. \partial_t v + \mathcal{B}(\tilde u)v - \nu\Delta v + \eta = 0 \right\},$$

and $A : \mathcal{X} \to \widetilde H$ *defined by* $(v, \eta) \mapsto v(T/2)$.

The space $\mathcal{X}$, endowed with the norm inherited from the space $W(I, \widetilde V, H^{-1}(\widetilde\Omega, \mathbb{R}^3)) \times L^2(I, L^2(w_1))$, is a nontrivial Banach space. We can prove that the minimizer exists and is unique. Moreover, if $(v^\mu, \eta^\mu) = (v^\mu(\underline{v}_1), \eta^\mu(\underline{v}_1))$ denotes that minimizer, then the mapping $\underline{v}_1 \mapsto (v^\mu(\underline{v}_1), \eta^\mu(\underline{v}_1))$ is linear (see, for example, [2, section A.2], for more details). The continuity follows from (23) because, necessarily

$$|(v^\mu(\underline{v}_1), \eta^\mu(\underline{v}_1))|_{L^2((T/2,T),\widetilde V) \times L^2(I, L^2(w_1))} \le |(v_{\psi_0}, \psi_0)(\underline{v}_1)|_{L^2((T/2,T),\widetilde V)\times L^2(I,L^2(w_1))}.$$

It easily follows that the mapping

$$v_0 \mapsto (\eta, \zeta)(v_0)(t) := \begin{cases} (0, \xi\breve\gamma_{v_0\mathcal{H}}|_{\partial\widetilde\Omega}) & \text{if } t \in [\tau, T/2] \\ (\eta^\mu(v_1), 0) & \text{if } t \in (T/2, T) \end{cases}$$

is linear and continuous. Hence the weak solution $\bar v = S(\eta, \zeta)(v_0)$ for system (19), corresponding to the control $(\eta, \zeta)(v_0)$ and initial condition $\bar v_0$ depends continuously on v_0, from which we can conclude the linearity and continuity of the mapping $v_0 \mapsto \bar v|_{(0,T)\times\Gamma} \in G^1_c((0, T), \Gamma)$, with $|v|_{(0,T)\times\Gamma}|_{G^1_c((0,T),\Gamma)} \le \overline{C}_{[|\hat u|_{\mathcal{W}}]} |v_0|_{\mathcal{A}_{K_{1,1}}}$. $\qquad\square$

Remark 4.2. In [6] the authors consider, at the very beginning, that the reference solution $\bar y$ solve the Navier–Stokes equation under homogeneous Dirichlet boundary conditions. Here our extension $\tilde u$, of $\hat u$, takes the role of $\bar y$ but, we can see that it does not necessarily vanish on $\partial\widetilde\Omega$, even if $\hat u$ takes its values in V. It turns out that, although in [6] the homogeneous Dirichlet boundary conditions are supposed for the reference solution $\bar y$, this fact is never used to derive the inequality in [6, Lemma 1] that is used to derive the controllability results. Only the fact that $\bar y \in \widetilde{\mathcal{W}}$, $|\bar y|_{\widetilde{\mathcal{W}}} = R < +\infty$, plays a role in the derivation of the several auxiliary estimates to arrive to the inequality.

Also, to go from the referred inequality to the controllability results, "mainly" we need to have that the operator in (22) is the formal adjoint of $\mathcal{B}$ in the sense of (21) but, we can see that that is the case because our extension $\tilde u$ is divergence free in $\widetilde\Omega$.

For a reference solution $\bar y$ satisfying nonhomogeneous Dirichlet boundary conditions we can also use the null controllability result in [15]. Again we must remark that, the

authors suppose that the reference solution satisfy the nonhomogeneous Dirichlet boundary conditions on an open subset $\Gamma_0 \subseteq \Gamma$, and some nonstandard boundary conditions on the interior Γ_1 of $\Gamma \setminus \Gamma_c$ but, now the case $\Gamma_0 = \Gamma$ is allowed.

We have chosen to follow the result in [6], because it asks less regularity for the reference solution. In [15] the reference solution $\bar{y}$ is supposed to be in the space $W^{1,\infty}((0, T), W^{2,\alpha}(\tilde{\Omega}, \mathbb{R}^3))$ with $\alpha > 3$.

5. EXACT NULL CONTROLLABILITY OF THE FULL SYSTEM

In this section we prove the local null controllability of the nonlinear system (8). As a corollary it will follow the local exact controllability to the desired reference solution $\hat{u}$. This case is much more complicated than the linear case because we have to take into account the problem of possible nonexistence and/or nonuniqueness of solutions. Nevertheless, the idea is, again, to reduce the problem in order to apply the results in [6].

It is known that the uniqueness holds if the solutions are regular enough. Having this in mind we will start with more regular data: let us denote by $G_c^2((0, T), \Gamma) \subset G^2((0, T), \Gamma)$ the Hilbert space of the traces at Γ of the functions in $W((0, T), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$.

Let us set $\mathcal{A}_{L_H^4} = L_H^4(\Omega, \mathbb{R}^3) + \mathcal{H}$ as the set of initial conditions, with $L_H^4(\Omega, \mathbb{R}^3) := L^4(\Omega, \mathbb{R}^3) \cap H$, and $\mathcal{H} := \{E_2\zeta(0) \mid \zeta \in G_c^2((0, T), \Gamma)\}$. The space $L_H^4(\Omega, \mathbb{R}^3)$ is supposed to be endowed with the norm inherited from $L^4(\Omega, \mathbb{R}^3)$, which makes it a Banach space. Analogously as in section 3.1 we endow $\mathcal{H}$, with the range scalar product and norm induced by the mapping $\zeta \mapsto E_2\zeta(0)$, with $\zeta \in \mathcal{Z}$ (now taking E_2 instead of E_1, see beginning of section 2.2). While the space $\mathcal{A}_{L_H^4}$ is endowed with the norm defined by

$$(24) \qquad |a|_{\mathcal{A}_{L_H^4}} := \inf\left\{ \left(|a_1|^2_{L_H^4(\Omega, \mathbb{R}^3)} + |a_2|^2_{\mathcal{H}}\right)^{1/2} \, \middle| \, (a_1, a_2) \in L_H^4(\Omega, \mathbb{R}^3) \times \mathcal{H} \right.$$

$$\left. \text{and } a_1 + a_2 = a \right\}$$

making it a Banach space (see, e.g., [24, section 1.2.1]).

Remark 5.1. In [24, section 1.2.1] the author considers $|a_1|_{L_H^4(\Omega, \mathbb{R}^3)} + |a_2|_{\mathcal{H}}$, instead of $\left(|a_1|^2_{L_H^4(\Omega, \mathbb{R}^3)} + |a_2|^2_{\mathcal{H}}\right)^{1/2}$ in (24) but, since these two norms are equivalent in $L_H^4(\Omega, \mathbb{R}^3) \times \mathcal{H}$, the statement on completeness follows. We use the norm in (24) just to be coherent with the "choice" made before for the case of Hilbert spaces, see (16).

Remark 5.2. Note that $\mathcal{A}_{L_H^4} \subset \mathcal{A}_{K_{1,2}} = H + \mathcal{H}$ (see Remark 3.3) is a subset of the set of admissible initial conditions for weak solutions for system (11), with $s = 2$, $\mathcal{Z} = G_c^2((0, T), \Gamma)$ and $K_2 : \zeta \mapsto \zeta$, found in section 3.1. Moreover the inclusion is continuous.

The next Theorem states the local exact null controllability property of system (8), in the case we take initial conditions in $\mathcal{A}_{L_H^4} := L_H^4(\Omega, \mathbb{R}^3) \times \mathcal{H}$.

Theorem 5.1. *There exists $\epsilon > 0$ such that for all $v_0 \in \{v \in \mathcal{A}_{L_H^4} \mid |v|_{\mathcal{A}_{L_H^4}} < \epsilon\}$, we can find a control $\zeta \in G_c^1((0, T), \Gamma)$ such that, the corresponding weak solution $S(v_0, \zeta)(t)$ to system (8) exists and is uniquely defined in $L^4((0, T), L^{12}(\Omega))$, and satisfies $S(v_0, \zeta)(T) = 0$.*

Before the proof, we introduce some tools and make some remarks.

Let us fix a domain $w_1 \subset w_1 \cup \partial w_1 \subset w = \widetilde{\Omega} \setminus (\Omega \cup \partial\Omega)$ and consider the following Banach spaces (see [6, section 3]):

$$\mathbb{E}_0 := \Big\{ (y,\, p,\, \eta) \,\big|\, \psi_1 y \in L^2((0,\, T) \times \widetilde{\Omega},\, \mathbb{R}^3); \quad \psi_2 \eta|_{w_1} \in L^2((0,\, T) \times \widetilde{\Omega},\, \mathbb{R}^3);$$

$$\psi_3 y \in L^2((0,\, T),\, \widetilde{V}) \cap L^\infty((0,\, T),\, \widetilde{H}) \cap L^4((0,\, T),\, L^{12}(\widetilde{\Omega},\, \mathbb{R}^3));$$

$$\text{and } \psi_3 p \in L^2((0,\, T),\, L^2_\mu(\widetilde{\Omega},\, \mathbb{R}^3)) \Big\},$$

$$\mathbb{E} := \Big\{ (y,\, p,\, \eta) \in \mathbb{E}_0 \,\big|\, \psi_3^2(L_{\tilde{u}} y + \nabla p + \eta|_{w_1}) \in L^2((0,\, T),\, W^{-1,6}(\widetilde{\Omega},\, \mathbb{R}^3));$$

$$\text{and } y(0) \in L^4_{\widetilde{H}}(\widetilde{\Omega},\, \mathbb{R}^3) \Big\},$$

$$\mathbb{G} := \Big\{ g \,\big|\, \psi_3^2 g \in L^2((0,\, T),\, W^{-1,6}(\widetilde{\Omega},\, \mathbb{R}^3)) \Big\},$$

where $\widetilde{V} := \{ v \in H^1(\widetilde{\Omega},\, \mathbb{R}^3) \mid \operatorname{div} v = 0 \text{ and } v|_{\partial\widetilde{\Omega}} = 0 \}$; $\widetilde{H} := \{ v \in L^2(\widetilde{\Omega},\, \mathbb{R}^3) \mid \operatorname{div} v = 0 \text{ and } v \cdot \tilde{\mathbf{n}} = 0 \}$; $L^2_\mu(\widetilde{\Omega},\, \mathbb{R}^3) := \{ p \in L^2(\widetilde{\Omega},\, \mathbb{R}^3) \mid \int_{\widetilde{\Omega}} p \, \mathrm{d}x = 0 \}$; $L^4_{\widetilde{H}}(\widetilde{\Omega},\, \mathbb{R}^3) := L^4(\widetilde{\Omega},\, \mathbb{R}^3) \cap \widetilde{H}$; $\tilde{u}$ is the extension of $\hat{u}$ given by Proposition 4.2; $L_{\tilde{u}} y = \partial_t y + \mathcal{B}(\tilde{u})y - \nu\Delta y$; and ψ_1, ψ_2 and ψ_3 are suitable weight functions from $[0,\, T)$ into $\mathbb{R}$. For further details of these functions we refer to [6], here we just remark some properties we will need:

- Given $\rho \in (0,\, T)$, there exists $M_\rho \in \mathbb{R}_0$ such that $|\psi_i|_{C^1([0,\, T-\rho],\, \mathbb{R})} \leq M_\rho$, for all $i \in \{1,\, 2,\, 3\}$;
- $\psi_i^{-1} \in C^1([0,\, T],\, \mathbb{R})$, for all $i \in \{1,\, 2,\, 3\}$;
- $\psi_i(t) = C_i$, for a suitable constant $C_i > 0$, for all t in the interval $[0,\, T/2]$, and for all $i \in \{1,\, 2,\, 3\}$;
- For all $(y,\, p,\, \eta) \in \mathbb{E}$ we have $y(T) = 0$.

The spaces $\mathbb{E}_0$, $\mathbb{E}$ and $\mathbb{G}$ are supposed to be endowed with the "natural" norms

$$|(y,\, p,\, \eta)|_{\mathbb{E}_0} := \Big(|\psi_1 y|^2_{L^2((0,\, T) \times \widetilde{\Omega},\, \mathbb{R}^3)} + |\psi_3 y|^2_{L^2((0,\, T),\, \widetilde{V}) \cap L^\infty((0,\, T),\, \widetilde{H}) \cap L^4((0,\, T),\, L^{12}(\widetilde{\Omega},\, \mathbb{R}^3))}$$

$$+ |\psi_2 \eta|_{w_1}|^2_{L^2((0,\, T) \times \widetilde{\Omega},\, \mathbb{R}^3)} + |\psi_3 p|^2_{L^2((0,\, T),\, L^2(\widetilde{\Omega},\, \mathbb{R}^3))} \Big)^{\frac{1}{2}};$$

$$|(y,\, p,\, \eta)|_{\mathbb{E}} := \Big(|(y,\, p,\, \eta)|^2_{\mathbb{E}_0} + |\psi_3^2(L_{\tilde{u}} y + \nabla p + \eta|_{w_1})|^2_{L^2((0,\, T),\, W^{-1,6}(\widetilde{\Omega},\, \mathbb{R}^3))}$$

$$+ |y(0)|^2_{L^4_{\widetilde{H}}(\widetilde{\Omega},\, \mathbb{R}^3)} \Big)^{\frac{1}{2}};$$

$$|g|_{\mathbb{G}} := |\psi_3^2 g|_{L^2((0,\, T),\, W^{-1,6}(\widetilde{\Omega},\, \mathbb{R}^3))}.$$

Remark 5.3. Notice that in [6], the pressure p is taken depending on $(y,\, \eta)$ and so, the elements of $\mathbb{E}$ are just the pairs $(y,\, \eta)$. Here we consider the elements of $\mathbb{E}$ as the set of corresponding triples $(y,\, p,\, \eta)$ just to simplify the exposition. Note that if we take p depending on $(y,\, \eta)$, by Remark 2 in [6], we have that y is a weak solution and so p is in $L^2((0,\, T),\, L^2(\widetilde{\Omega},\, \mathbb{R}^3))$; then by definition of the spaces we can also derive that $\psi_3 p \in L^2((0,\, T),\, L^2(\widetilde{\Omega},\, \mathbb{R}^3))$. On the other side, in [6], the condition $y(0) \in L^4_{\widetilde{H}}(\widetilde{\Omega},\, \mathbb{R}^3)$ does not appear in the definition of $\mathbb{E}$ but, it is implicitly used in the analysis; its addition to the definition of $\mathbb{E}$, as well as the addition of the term $|y(0)|^2_{L^4_{\widetilde{H}}(\widetilde{\Omega},\, \mathbb{R}^3)}$ in the definition of the norm of $\mathbb{E}$, is appropriate for us to make the exposition here clearer.

It is not difficult to check that $\mathbb{E}_0$ and $\mathbb{G}$ are Banach spaces. That $\mathbb{E}$ is a Banach space follows from the following Lemma 5.2, taking $X = \mathbb{E}_0$, $Y = L^2_{\psi_3^2}((0,\, T),\, W^{-1,6}(\widetilde{\Omega},\, \mathbb{R}^3)) \times$

$L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3)$, $Z = L^2((0, T), H^{-1}(\widetilde{\Omega}, \mathbb{R}^3)) \times \widetilde{H}$, and $A : (y, p, \eta) \mapsto \left(L_{\tilde{u}}y + \nabla p + \eta|_{w_1}, y(0)\right)$, where $L^2_{\psi_3^2}((0, T), W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3)) := \{f \mid \psi_3^2 f \in L^2((0, T), W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3))\}$.

Lemma 5.2. *Let X, Y and Z be normed spaces such that X and Y are complete, and $Y \subseteq Z$ continuously. Given a continuous linear mapping $A : X \to Z$, then the space $A^{-1}Y := \{x \in X \mid Ax \in Y\}$ is complete, if endowed with the norm $|x|_{A^{-1}Y} := \left(|x|_X^2 + |Ax|_Y^2\right)^{1/2}$.*

Proof. It is clear that $|\cdot|_{A^{-1}Y}$ is a norm in the vector space $A^{-1}Y$. To prove the completeness we take an arbitrary Cauchy sequence (x_n) in $A^{-1}Y$, $n \in \mathbb{N}$. Then, from $|x_n - x_m|^2_{A^{-1}Y} := |x_n - x_m|_X^2 + |Ax_n - Ax_m|_Y^2$, we have that necessarily (x_n) is also a Cauchy sequence in X, and (Ax_n) is a Cauchy sequence in Y. Since X and Y are complete, the limit x of (x_n) exists in X, and the limit y of (Ax_n) exists in Y. The continuity of the inclusion $Y \subseteq Z$ implies that y is also the limit of (Ax_n) in Z. Finally, from the continuity of A we have that Ax_n converges to Ax in Z, from which we derive $y = Ax$. Therefore we can conclude that x_n converges to x in $A^{-1}Y$. $\square$

Given $b \in \mathcal{H}$, let $\gamma_b \in W((0, T), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$ be defined by (13), taking $\Lambda : W((0, T), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3)) \to V_c$ with $\Lambda k = k(0)$, and extend γ_b to $\breve{\gamma}_b$, to the bigger domain $\widetilde{\Omega}$, using Proposition 4.3. Let $\xi : [0, T] \to \mathbb{R}$ be a smooth mapping taking the value 1 in a neighborhood of $t = 0$ and the value 0 in a neighborhood of $t = T$.

Now, for $a \in L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3)$, consider the following system in $(0, T) \times \widetilde{\Omega}$:

$$(25) \quad \partial_t y + \langle y \cdot \nabla \rangle y + \mathcal{B}(\tilde{u})y + \mathcal{B}(\xi\breve{\gamma}_b)y - \nu\Delta y + \nabla p + \eta|_{w_1} + g_b = 0, \quad \operatorname{div} y = 0,$$
$$y|_{\partial\widetilde{\Omega}} = 0, \quad y(0) = a,$$

with $g_b = \partial_t\xi\breve{\gamma}_b + \langle \xi\breve{\gamma}_b \cdot \nabla \rangle \xi\breve{\gamma}_b - \nu\Delta\xi\breve{\gamma}_b + \mathcal{B}(\tilde{u})\xi\breve{\gamma}_b$.

Lemma 5.3. *There exists $\delta > 0$ such that for all (a, b) in $\{(x, y) \in L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3) \times \mathcal{H} \mid |(x, y)|_{L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3) \times \mathcal{H}} < \delta\}$ we can find a control $\bar{\eta}$ with $\psi_2\bar{\eta}|_{w_1} \in L^2((0, T) \times \widetilde{\Omega}, \mathbb{R}^3)$, such that there exists a corresponding weak solution $\bar{y}$ to system (25) with $\bar{y}(T) = 0$. Moreover $\bar{y} \in L^4((0, T), L^{12}(\widetilde{\Omega}, \mathbb{R}^3))$ and is unique in this space: $\bar{y} = \bar{y}(a, b, \bar{\eta})$.*

In the proof we will follow the ideas used in the proof of Theorem 2 in [6, section 4]; the following inverse mapping Lemma (see [6, Theorem 3]) will be one of the tools:

Lemma 5.4. *Let E and G be two Banach spaces and let $A : E \to G$ satisfy $A \in C^1(E, G)$. Assume that $e_0 \in E$, $A(e_0) = g_0$, and the derivative $DA|_{e_0} : E \to G$ is surjective. Then there exists a $\delta > 0$ such that, for every g in the ball $\{g \in G \mid |g - g_0|_G < \delta\}$, the equation $A(e) = g$ has a solution $e \in E$.*

The next Lemma shows the setting where we will use Lemma 5.4.

Lemma 5.5. *Setting the spaces*

$$(26) \qquad\qquad E = \mathbb{E} \times \mathcal{H}, \qquad G = \mathbb{G} \times L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3) \times \mathcal{H},$$

and the mapping $A : E \to G$, defined by

$$(27) \qquad A(y, p, \eta, b) = (L_{\tilde{u}}y + \langle y \cdot \nabla \rangle y + \nabla p + \eta|_{w_1} + \mathcal{B}(\xi\breve{\gamma}_b)y + g_b, y(0), b),$$

Then $A \in C^1(E, G)$ and $DA|_{(0,0,0,0)}$ is surjective.

Proof. It is clear that the mapping $(y, p, \eta, b) \mapsto (L_{\tilde{u}}y + \nabla p + \eta|_{w_1}, y(0), b)$ is in $C^1(E, G)$. The continuity of the mapping $\left((y_1, p_1, \eta_1, v_{b1}), (y_2, p_2, \eta_2, v_{b2})\right) \mapsto \langle y_1 \cdot \nabla \rangle y_2$ was shown

in [6, section 4, Proposition 3]. Hence to show that $A \in C^1(E, G)$, it remains to prove that

(28) $$(y, p, \eta, b) \mapsto \mathcal{B}(\xi\breve{\gamma}_b)y + g_b \quad \text{is in} \quad C^1(E, \mathbb{G}).$$

From the boundedness of $b \mapsto \breve{\gamma}_b \in W((0, T), V_c \cap H^2(\Omega, \mathbb{R}^3), L^2(\Omega, \mathbb{R}^3))$, by standard estimates, we easily derive the boundedness of

$$(b_1, b_2) \mapsto \partial_t \xi\breve{\gamma}_{b_1} + \langle \xi\breve{\gamma}_{b_2} \cdot \nabla \rangle \xi\breve{\gamma}_{b_1} - \nu\Delta\xi\breve{\gamma}_{b_1}$$

from $\mathcal{H} \times \mathcal{H}$ into $L^2((0, T), L^2(\widetilde{\Omega}, \mathbb{R}^3))$. Then, since ξ vanishes in neighborhood of $t = T$, it turns out that the mapping

$$b \mapsto \partial_t \xi\breve{\gamma}_b + \langle \xi\breve{\gamma}_b \cdot \nabla \rangle \xi\breve{\gamma}_b - \nu\Delta\xi\breve{\gamma}_b$$

is in $C^1(\mathcal{H}, \mathbb{G})$, because $\psi_3^2 \in C^1([0, T - \rho], \mathbb{R})$, for any $\rho \in (0, T)$.

On the other side, we can also easily derive that $|\mathcal{B}(\tilde{u})\xi\breve{\gamma}_b|_{W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3)}$ is bounded by $C|\tilde{u}|_{L^\infty(\widetilde{\Omega})}|\xi\breve{\gamma}_b|_{L^6(\widetilde{\Omega}, \mathbb{R}^3)}$ and,

$$|\psi_3^2\mathcal{B}(\tilde{u})\xi\breve{\gamma}_b|_{L^2((0, T), W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3))} \leq C|\tilde{u}|_{L^\infty((0, T)\times\widetilde{\Omega})}|\psi_3^2\xi\breve{\gamma}_b|_{L^2((0, T), L^6(\widetilde{\Omega}, \mathbb{R}^3))}.$$

From the continuity of the inclusion $H^1(\widetilde{\Omega}, \mathbb{R}^3) \subset L^6(\widetilde{\Omega}, \mathbb{R}^3)$ (see, e.g., [19, Chapter 2, Theorem 3.6])

$$|\psi_3^2\xi\breve{\gamma}_b|_{L^2((0, T), W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3))} \leq C_1|\tilde{u}|_{L^\infty((0, T)\times\widetilde{\Omega})}|\breve{\gamma}_b|_{L^2((0, T), H^1(\widetilde{\Omega}, \mathbb{R}^3))}$$
$$\leq C_2|b|_{\mathcal{H}}$$

showing that the linear mapping $b \mapsto \mathcal{B}(\tilde{u})\xi\breve{\gamma}_b$ is in $C^1(\mathcal{H}, \mathbb{G})$. Therefore we may conclude that

(29) $$b \mapsto g_b \in C^1(\mathcal{H}, \mathbb{G}).$$

Now, by the Agmon inequality (see, e.g., [22, Chapter II, Section 1.4]) we have

$$|\mathcal{B}(\xi\breve{\gamma}_b)y|_{W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3)} \leq C|\xi\breve{\gamma}_b|_{L^\infty(\widetilde{\Omega}, \mathbb{R}^3)}|y|_{L^6(\widetilde{\Omega}, \mathbb{R}^3)} \leq C_1|\xi\breve{\gamma}_b|_{H^1(\widetilde{\Omega}, \mathbb{R}^3)}^{1/2}|\xi\breve{\gamma}_b|_{H^2(\widetilde{\Omega}, \mathbb{R}^3)}^{1/2}|y|_{L^6(\widetilde{\Omega}, \mathbb{R}^3)}$$

and then

$$|\psi_3^2\mathcal{B}(\xi\breve{\gamma}_b)y|_{L^2((0, T), W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3))}$$
$$\leq C_2|\psi_3^2\xi\breve{\gamma}_b|_{C([0, T], H^1(\widetilde{\Omega}, \mathbb{R}^3))}^{1/2}|\psi_3^2\xi\breve{\gamma}_b|_{L^2((0, T), H^2(\widetilde{\Omega}, \mathbb{R}^3))}^{1/2}|y|_{L^4(0, T), L^6(\widetilde{\Omega}, \mathbb{R}^3))}$$
$$\leq C_3|b|_{\mathcal{H}}|y|_{L^4(0, T), L^6(\widetilde{\Omega}, \mathbb{R}^3))}$$

from which, we can conclude that $(y, p, \eta, b) \mapsto \mathcal{B}(\xi\breve{\gamma}_b)y \in C^1(E, \mathbb{G})$. Therefore, from (29), we have that (28) holds and so, $A \in C^1(E, G)$.

Now, let us set $e_0 = (0, 0, 0, 0)$. The surjectiveness of the linear mapping

$$DA|_{e_0} : (y, p, \eta, b) = (L_{\tilde{u}}y + \nabla p + \eta|_{w_1} + g_b^l, y(0), b),$$

with $g_b^l := \partial_t \xi\breve{\gamma}_b - \nu\Delta\xi\breve{\gamma}_b + \mathcal{B}(\tilde{u})\xi\breve{\gamma}_b$, follows from Proposition 2 in [6, section 3]. $\quad\square$

Proof of Lemma 5.3. From Lemmas 5.4 and 5.5 it follows, in particular, that there exists $\delta > 0$ such that, for every (a, b) in $\{(x, y) \in L_{\widetilde{H}}^4(\widetilde{\Omega}, \mathbb{R}^3) \times \mathcal{H} \mid |(x, y)|_{L_{\widetilde{H}}^4(\widetilde{\Omega}, \mathbb{R}^3)\times\mathcal{H}} < \delta\}$, the equation $A(e) = (0, a, b)$ has a solution $e = (\bar{y}, \bar{p}, \bar{\eta}, b) \in E$ but, this implies that $\bar{y}$

is a weak solution for system (25) because, from the differentiability of the functions ψ_i, $i \in \{1, 2, 3\}$, we have that

$$\mathbb{E} \subseteq \Big\{ (y, p, \eta) \,\big|\, y \in L^2((0, T) \times \widetilde{\Omega}, \mathbb{R}^3); \quad \eta|_{w_1} \in L^2((0, T) \times \widetilde{\Omega}, \mathbb{R}^3);$$

$$y \in L^2((0, T), \widetilde{V}) \cap L^\infty((0, T), \widetilde{H}) \cap L^4((0, T), L^{12}(\widetilde{\Omega}, \mathbb{R}^3)),$$

$$y(0) \in L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3); \quad p \in L^2((0, T), L^2_\mu(\widetilde{\Omega}, \mathbb{R}^3));$$

$$L_{\tilde{u}} y + \nabla p + \eta|_{w_1} \in L^2((0, T), W^{-1,6}(\widetilde{\Omega}, \mathbb{R}^3)) \Big\},$$

from which we can conclude that necessarily we have $y_t \in L^2((0, T), H^{-1}(\widetilde{\Omega}, \mathbb{R}^3))$. Moreover from $(\bar{y}, \bar{p}, \bar{\eta}) \in \mathbb{E}$ we also have $\bar{y}(T) = 0$, and from $\bar{y} \in L^4((0, T), L^{12}(\widetilde{\Omega}))$, it follows that $\bar{y}$ is uniquely defined, within this regularity, by the data (a, b, η) (see [16, Chapter 1, Theorem 6.9]): $\bar{y} = \bar{y}(a, b, \bar{\eta})$. $\qquad\square$

Proof of Theorem 5.1. Given $v_0 \in \mathcal{A}_{L^4_{\widetilde{H}}}$ we write $v_0 = v_{0L^4} + v_{0\mathcal{H}}$ where $(v_{0L^4}, v_{0\mathcal{H}}) \in L^4_H(\Omega, \mathbb{R}^3) \times \mathcal{H}$. It is clear, from (24), that we may choose the pair $(v_{0L^4}, v_{0\mathcal{H}})$ such that $|v_{0L^4}|^2_{L^4_H(\Omega, \mathbb{R}^3)} + |v_{0\mathcal{H}}|^2_{\mathcal{H}} < 4|v_0|^2_{\mathcal{A}_{L^4_{\widetilde{H}}}}$. Then if δ is like in Lemma 5.3 we set $\epsilon = \delta/2$. Hence given $|v_0|_{\mathcal{A}_{L^4_{\widetilde{H}}}} < \epsilon$ we choose $(v_{0L^4}, v_{0\mathcal{H}}) \in L^4_H(\Omega, \mathbb{R}^3) \times \mathcal{H}$, with $|(v_{0L^4}, v_{0\mathcal{H}})|_{L^4_H(\Omega, \mathbb{R}^3) \times \mathcal{H}} < \delta$. Also, if $\tilde{v}_{0L^4} \in L^4_{\widetilde{H}}(\widetilde{\Omega}, \mathbb{R}^3)$ is the extension of v_{0L^4} by zero outside Ω, we still have

$$|(\tilde{v}_{0L^4}, v_{0\mathcal{H}})|_{L^4_H(\Omega, \mathbb{R}^3) \times \mathcal{H}} < \delta.$$

Now, from Lemma 5.3 there exists a control $\eta|_{w_1} \in L^2((0, T) \times \Omega, \mathbb{R}^3)$ such that, the corresponding solution $\bar{y} = \bar{y}(\tilde{v}_{0L^4}, v_{0\mathcal{H}}, \eta)(t)$ to system (25), with the data $(a, b) = (\tilde{v}_{0L^4}, v_{0\mathcal{H}})$ exists, is uniquely defined in $L^4((0, T), L^{12}(\widetilde{\Omega}))$, and satisfies $\bar{y}(T) = 0$. Hence $k := \bar{y} + \xi \breve{\gamma}_b$ solves the system

$$(30) \qquad \begin{aligned} \partial_t k + \langle k \cdot \nabla \rangle k - \nu \Delta k + \nabla p + \eta|_{w_1} &= 0, & \operatorname{div} k &= 0, \\ k|_{\partial \widetilde{\Omega}} &= \xi \breve{\gamma}_b|_{\partial \widetilde{\Omega}}, & k(0) &= \tilde{v}_{L^4} + \breve{\gamma}_b(0), \end{aligned}$$

and the restriction $v = k|_{(0, T) \times \Omega}$ of k to $(0, T) \times \Omega$ solves the system

$$(31) \qquad \begin{aligned} \partial_t v + \langle v \cdot \nabla \rangle v - \nu \Delta v + \nabla p &= 0, & \operatorname{div} v &= 0, \\ v|_\Gamma &= k|_\Gamma, & v(0) &= v_{L^4} + v_{0\mathcal{H}}. \end{aligned}$$

Since it is clear that $\zeta := k|_\Gamma \in G^1_c((0, T), \Gamma)$, to complete the proof it remains to prove that $v(T) = 0$. From $\bar{y}(T) = 0$ it follows that $k(T) := \xi \breve{\gamma}_b(T)$ but, from its definition we know that $\xi(T) = 0$, then $k(T) = 0$, which implies $v(T) = 0$. Moreover v is unique in $L^4((0, T), L^{12}(\Omega))$. $\qquad\square$

Finally we can prove the main Theorem:

Corollary 5.6. *Let $(\hat{u}, \hat{p})$ be a weak solution for system (1), with $\zeta = 0$, such that $|\hat{u}|_{L^\infty((0, T) \times \Omega, \mathbb{R}^3)} + |\partial_t \hat{u}|_{L^2((0, T), L^\sigma(\Omega, \mathbb{R}^3))} < +\infty$, where $\sigma > 6/5$ is a constant. Then for any open subset $\Gamma_c \subseteq \Gamma$ there exists a constant $\epsilon > 0$ such that, for any vector field $u_0 \in L^2_{\operatorname{div}}(\Omega)$ satisfying $u_0 - \hat{u}(0) \in \mathcal{A}_{L^4_{\widetilde{H}}}$ with $|u_0 - \hat{u}(0)|_{\mathcal{A}_{L^4_{\widetilde{H}}}} < \epsilon$, we can find a boundary control $\zeta \in G^1_c((0, T), \Gamma)$, supported in $(0, T) \times \Gamma_c$ such that, the corresponding weak solution to system (1), with $u(0) = u_0$ exists, is uniquely defined by the data (u_0, ζ) in the affine space $\hat{u} + L^4((0, T), L^{12}(\Omega))$, and satisfies $u(T) = \hat{u}(T)$.*

Proof. The difference $v = u - \hat{u}$ solves system (8) with $v_0 = u_0 - \hat{u}(0)$. By Theorem 5.1, there exists $\epsilon > 0$ such that, if $|v_0|_{\mathcal{A}_{L^4_{\widetilde{H}}}} < \epsilon$, we can find a control $\zeta \in G^1_c((0, T), \Gamma)$ such that, the corresponding weak solution $v(t)$, unique in $L^4((0, T), L^{12}(\Omega))$, satisfies

$v(T) = 0$. This means that the same control ζ drives system (1) from $u(0) = u_0$ at time $t = 0$ to $u(T) = \hat{u}(T)$ at time $t = T$. $\square$

APPENDIX: PROOF OF PROPOSITIONS 4.2 AND 4.3

We assume some familiarity with some basic tools from differential geometry. We refer to [3, 5, 14, 23].

Proof of Proposition 4.2 Given $u \in \mathcal{W}$ we have in particular, by definition, that $u \in L^\infty((0, T) \times \Omega)$. Then for a.e. $t \in (0, T)$ we have that $u(t)$ is integrable (see, e.g., [18, section X.2, proof of Theorem 2]), and then $u(t) \in L^\infty(\Omega) \subseteq L^2(\Omega)$.

Following the idea from [10], we start by solving the system

(A.1) $\operatorname{curl} k(t) = u(t)$ and $\operatorname{div} k(t) = 0$ in Ω; $k(t) \cdot \mathbf{n} = 0$ on Γ.

The existence of a solution $k(t)$ is known, for example, from [1, section 12]. The solution is, in general, not unique but, there exists (choosing $\tilde{\kappa} = 0$ in [1, equation 12.20]) a well defined linear choice $u(t) \mapsto k_{u(t)}$ that is continuous in the following sense: $k_{u(t)}$ solves (A.1) and $|k_{u(t)}|_{H^1(\Omega, \mathbb{R}^3)} \leq C|u(t)|_{L^2(\Omega, \mathbb{R}^3)}$ with the constant C independent of u. From now we consider this choice when making reference to *the* solution of (A.1).

Let $p \in \Gamma_c$ and consider the diffeomorphism Φ_p, in (4), mapping the cylinder $\mathbb{C}_p$ onto the tubular neighborhood $\mathcal{T}_p$. We shall start by extending u to $\Omega \cup \mathcal{T}_p$. First we recall that we may see the open subset $\mathcal{T}_p \subset \mathbb{R}^3$ with its induced Euclidean metric as the manifold $(\mathbb{C}_p, g)$ for a suitable Riemannian metric tensor $g = \sum_{i, j=1}^{3} g_{ij} \mathrm{d}w^i \otimes \mathrm{d}w^j$.

Remark A.4. Here we use superscripts to denote the coordinates of the vector fields w because that notation is often used, and convenient to deal with some tools, in differential geometry.

Let $\partial/\partial w^i$ be the vector field induced in $\mathcal{T}_p$ by the new coordinate function w^i, $i = 1, 2, 3$. If (x^1, x^2, x^3) are the Euclidean coordinate functions in $\mathcal{T}_p \subset \mathbb{R}^3$ we find that

(A.2) $\begin{aligned} \partial/\partial w^i \big|_{(w^1, w^2, w^3)} &= \partial/\partial x^i + \partial \Phi_p^0/\partial w^i \big|_{(w^1, w^2, w^3)} \partial/\partial x^3 \text{ for } i = 1, 2 \\ \partial/\partial w^3 \big|_{(w^1, w^2, w^3)} &= \mathbf{n}_{\Phi_p^0(w^1, w^2)}. \end{aligned}$

Since $\Phi_p^0(w^1, w^2) \in \Gamma$ we can see that both $\partial/\partial w^1$ and $\partial/\partial w^2$ are orthogonal to $\partial/\partial w^3$. Moreover the length of $\partial/\partial w^i$, for $i = 1, 2$ is given by $\left(1 + \left(\frac{\partial \Phi_p^0}{\partial w^i}\right)^2\right)^{1/2}$, the Euclidean scalar product $(\partial/\partial w^1, \partial/\partial w^2)$ is equal to $\frac{\partial \Phi_p^0}{\partial w^1}\frac{\partial \Phi_p^0}{\partial w^2}$, and the length of $\partial/\partial w^3$ is 1. Thus the metric tensor becomes

$$g = \left(1 + \left(\frac{\partial \Phi_p^0}{\partial w^1}\right)^2\right) \mathrm{d}w^1 \otimes \mathrm{d}w^1 + \frac{\partial \Phi_p^0}{\partial w^1}\frac{\partial \Phi_p^0}{\partial w^2}(\mathrm{d}w^1 \otimes \mathrm{d}w^2 + \mathrm{d}w^2 \otimes \mathrm{d}w^1)$$

$$+ \left(1 + \left(\frac{\partial \Phi_p^0}{\partial w^2}\right)^2\right) \mathrm{d}w^2 \otimes \mathrm{d}w^2 + \mathrm{d}w^3 \otimes \mathrm{d}w^3.$$

The Euclidean volume element in $\mathcal{T}_p$ may then be written as

$$d\mathbb{C}_p = \sqrt{\bar{g}} \, \mathrm{d}w^1 \wedge \mathrm{d}w^2 \wedge \mathrm{d}w^3, \quad \text{with } \bar{g} := \det[g_{ij}] = 1 + \left(\frac{\partial \Phi_p^0}{\partial w^1}\right)^2 + \left(\frac{\partial \Phi_p^0}{\partial w^2}\right)^2.$$

Remark A.5. We may suppose that the ordered triple $(\partial/\partial w^1, \partial/\partial w^2, \partial/\partial w^3)$ preserves the orientation of $\mathcal{T}_p$ (otherwise we just reorder the coordinate functions as (w^2, w^1, w^3)).

Now we extend the vector field $k(t) = k = \sum_{i=1}^{3} k^i \partial/\partial w^i$ that is defined in $\mathbb{C}_p^-$ to a vector field $\bar{k} = \sum_{i=1}^{3} \bar{k}^i \partial/\partial w^i$ defined in all $\mathbb{C}_p$ as follows

$$(A.3) \quad \bar{k}(w^1, w^2, w^3) = \begin{cases} k(w^1, w^2, w^3), & \text{if } w^3 \leq 0, \\ k^1(w^1, w^2, -w^3)\partial/\partial w^1 + k^2(w^1, w^2, -w^3)\partial/\partial w^2 \\ \qquad\qquad - k^3(w^1, w^2, -w^3)\partial/\partial w^3, & \text{if } w^3 > 0, \end{cases}$$

with the $\partial/\partial w^i$s evaluated at (w^1, w^2, w^3). Note that the extension is well defined for $H^1(\mathbb{C}_p, T\mathbb{C}_p^-)$-vector fields $k = \sum_{i=1}^{3} k^i \partial/\partial w^i$ with $k^3(w^1, w^2, 0) = 0$. It is moreover continuous from $H^1(\mathbb{C}_p^-, T\mathbb{C}_p^-)$ into $H^1(\mathbb{C}_p, T\mathbb{C}_p)$, and from $L^\sigma(\mathbb{C}_p^-, T\mathbb{C}_p^-)$ into $L^\sigma(\mathbb{C}_p, T\mathbb{C}_p)$ for $\sigma \geq 1$.

Now we may compute $\operatorname{curl} \bar{k}$ using the Hodge star $\star$, sharp $\sharp$, flat $\flat$, and exterior derivative d mappings: $\operatorname{curl} \bar{k} = (\star d\bar{k}^\sharp)^\flat$. To simplify the writing we use the *Einstein summatiom convention*, i.e., repeated indices are implicitly summed over, from 1 to 3. For example, with this convention the metric tensor becomes just $g = g_{ij} dw^i \otimes dw^j$.

We find

$$\bar{k}^\sharp = g_{ij}\bar{k}^i dw^j, \quad d\bar{k}^\sharp = \frac{\partial g_{ij}\bar{k}^i}{\partial w^k} dw^k \wedge dw^j,$$

$$\star d\bar{k}^\sharp = \operatorname{sign}(k, i, r)\frac{\partial g_{ij}\bar{k}^i}{\partial w^k}\partial/\partial w^{k\sharp}, \quad \operatorname{curl} \bar{k} = \operatorname{sign}(k, i, r)\frac{\partial g_{ij}\bar{k}^i}{\partial w^k}\partial/\partial w^k,$$

where $\operatorname{sign}(k, i, r)$ stands for the sign of the permutation $(1, 2, 3) \mapsto (k, i, r)$ in the case $\{k, i, r\} = \{1, 2, 3\}$, and $\operatorname{sign}(k, i, r) = 0$ otherwise. We omit the details of this computations; they may be found, together with some remarks on the mappings involved, for example, in [20, Section 5.7].

Hence we find

$$\operatorname{curl} \bar{k} = \left(\frac{\partial(g_{3i}\bar{k}^i)}{\partial w^2} - \frac{\partial(g_{2i}\bar{k}^i)}{\partial w^3}\right)\partial/\partial w^1 + \left(\frac{\partial(g_{1i}\bar{k}^i)}{\partial w^3} - \frac{\partial(g_{3i}\bar{k}^i)}{\partial w^1}\right)\partial/\partial w^2$$

$$+ \left(\frac{\partial(g_{2i}\bar{k}^i)}{\partial w^1} - \frac{\partial(g_{1i}\bar{k}^i)}{\partial w^2}\right)\partial/\partial w^3$$

$$= \left(\frac{\partial \bar{k}^3}{\partial w^2} - \frac{\partial(g_{21}\bar{k}^1 + g_{22}\bar{k}^2)}{\partial w^3}\right)\partial/\partial w^1 + \left(\frac{\partial(g_{11}\bar{k}^1 + g_{12}\bar{k}^2)}{\partial w^3} - \frac{\partial \bar{k}^3}{\partial w^1}\right)\partial/\partial w^2$$

$$+ \left(\frac{\partial(g_{21}\bar{k}^1 + g_{22}\bar{k}^2)}{\partial w^1} - \frac{\partial(g_{11}\bar{k}^1 + g_{12}\bar{k}^2)}{\partial w^2}\right)\partial/\partial w^3;$$

in particular, for $w^3 > 0$, we have the identity

$$\operatorname{curl} \bar{k}|_{(w^1, w^2, w^3)} = \left(-\frac{\partial k^3}{\partial w^2} + \frac{\partial(g_{21}k^1 + g_{22}k^2)}{\partial w^3}\right)\Big|_{(w^1, w^2, -w^3)}\partial/\partial w^1$$

$$+ \left(-\frac{\partial(g_{11}k^1 + g_{12}k^2)}{\partial w^3} + \frac{\partial k^3}{\partial w^1}\right)\Big|_{(w^1, w^2, -w^3)}\partial/\partial w^2$$

$$+ \left(\frac{\partial(g_{21}k^1 + g_{22}k^2)}{\partial w^1} - \frac{\partial(g_{11}k^1 + g_{12}k^2)}{\partial w^2}\right)\Big|_{(w^1, w^2, -w^3)}\partial/\partial w^3.$$

Note that, from (A.2), we see that $\partial/\partial w^i$ is independent of w^3, for $i = 1, 2, 3$. Then, for $w_3 > 0$ we see that $\operatorname{curl} \bar{k}|_{(w^1, w^2, w^3)}$ differs from $\operatorname{curl} k|_{(w^1, w^2, -w^3)}$ only on the signs of the components of $\partial/\partial w^1$ and $\partial/\partial w^2$. Therefore, since $\operatorname{curl} k \in L^\infty(\mathbb{C}_p^-, T\mathbb{C}_p^-)$ we have that $\operatorname{curl} \bar{k} \in L^\infty(\mathbb{C}_p, T\mathbb{C}_p)$. From $\partial_t k \in L^\sigma(\mathbb{C}_p^-, T\mathbb{C}_p^-)$ it follows that also $\partial_t \bar{k} \in L^\sigma(\mathbb{C}_p, T\mathbb{C}_p)$.

We take $\tilde{u} := \operatorname{curl} \bar{k}$ as the extension of u. Then we have that, for $w_3 > 0$,

$$(A.4) \quad \tilde{u}|_{(w^1,\,w^2,\,w^3)} = -u_1|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^1 - u_2|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^2 + u_3|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^3.$$

From the extension in each tubular neighborhood, now we construct an extension to the open set w. From (18) we have that $w \subset \cup\{\mathcal{T}_{p_j} \mid j = 1,\, 2,\, \ldots,\, s,$ and $p_j \in \Gamma\}$. Since $\partial/\partial w^i|_{(w^1,\,w^2,\,-w^3)} = \partial/\partial w^i|_{(w^1,\,w^2,\,w^3)}$, then from (A.4) it follows that the extension $\tilde{u}$ does not depend on the chart $(\mathcal{T}_{p_j}, \Phi_{p_j})$. Indeed from (A.4) we see that the extension $\tilde{u}$ is obtained in the following way: given $x = \Phi_p(w^1,\, w^2,\, w^3) \in w$ we find its "symmetric" $x^- = \Phi_p(w^1,\, w^2,\, -w^3)$ in a given chart, then write

$$u(x^-) = u_1|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^1 + u_2|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^2 + u_3|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^3$$
$$= u_\mathbf{t} + u_3|_{(w^1,\,w^2,\,-w^3)}\, \partial/\partial w^3 = u_\mathbf{t} + u_3|_{(w^1,\,w^2,\,-w^3)}\, \mathbf{n}_{\mu_x}$$

where $\mu_x \in \Gamma$ is the "middle" point $\mu_x := \Phi_p(w^1,\, w^2,\, 0)$, $\mathbf{n}_{\mu_x}$ is the unit outward normal at μ_x, and $u_\mathbf{t} \in T_{\mu_x}\Gamma$ is in the tangent space to Γ at μ_x. Of course the decomposition $u(x^-) = u_\mathbf{t} + u_3\mathbf{n}_{\mu_x}$ is unique and does not depend in any chart. The extension $\tilde{u}$, accordingly to (A.4), is given by $\tilde{u}(x) = -u_\mathbf{t} + u_3\mathbf{n}_{\mu_x}$, and so it is linear and independent of the chart.

It is now clear that $|\tilde{u}|_{\widetilde{\mathcal{W}}} \leq C|u|_\mathcal{W}$ and that the constant C can be taken independent of u. Finally, from $\tilde{u} = \operatorname{curl} \bar{k}$ it follows that $\operatorname{div} \tilde{u} = 0$. The proof of the Lemma is complete. $\qquad\square$

Remark A.6. Note that in general $\tilde{u}$ is not in $H^1(\widetilde{\Omega}, \mathbb{R}^3)$. Indeed, from (A.4), we can see that $\tilde{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$ iff $u \in H^1(\Omega, \mathbb{R}^3)$ and u is normal to Γ.

Proof of Proposition 4.3 We proceed analogously as in the proof of Proposition 4.2. First solve the system (A.1) for $\gamma(t)$:

$$\operatorname{curl} k(t) = \gamma(t) \text{ and } \operatorname{div} k(t) = 0 \text{ in } \Omega; \quad k(t) \cdot \mathbf{n} = 0 \text{ on } \Gamma$$

the only difference is that now we extend $k = k(t)$ as in [10]:

$$(A.5) \qquad \bar{k}_{st}(w^1,\, w^2,\, w^3) = \begin{cases} k(w^1,\, w^2,\, w^3), & \text{if } w^3 \leq 0, \\ 6k(w^1,\, w^2,\, -w^3) - 32k(w^1,\, w^2,\, -w^3/2) \\ \qquad + 27k(w^1,\, w^2,\, -w^3/3), & \text{if } w^3 > 0. \end{cases}$$

Note that (A.5) is a classical linear extension to guarantee the continuity of $k \mapsto \bar{k}_{st}$ in both L^2-norm, H^1-norm, H^2-norm, and H^3-norms. Then we set $\check{\gamma}(t) := \operatorname{curl} \bar{k}_{st}(t)$. The statement of the Lemma follows. $\qquad\square$

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