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Abstract: In this paper a posteriori error estimates for space-time finite element discretizations for optimal control problems governed by the dynamical Lamé system are considered using the dual weighted residual method (DWR). We apply techniques developed in Kröner (2011a), where optimal control problems for second order hyperbolic equations are considered. The provided error estimator separates the influences of different parts of the discretization (time, space, and control discretization). This allows us to set up an adaptive algorithm which improves the accuracy of the computed solutions by construction of locally refined meshes. We present a numerical example showing a speedup in cpu-time as well as a reduction in degrees of freedom in comparison to uniform mesh refinement.

Keywords: optimal control, dynamical Lamé system, a posteriori error estimates, finite elements, dynamic meshes

1. INTRODUCTION

In this paper we present the dual weighted residual method (DWR, cf. Becker and Rannacher (2001)) for finite element methods for optimal control problems governed by the dynamical Lamé system. Thereby we apply techniques from Kröner (2011a). The presented numerical results were developed in Kröner (2011b).

The dynamical Lamé system describes the propagation of elastic waves in an elastic medium. The elasticity of the material provides the restoring force of the wave. The Lamé system can be interpreted as a model for acoustic waves in solid materials or seismic waves. Seismic waves are caused by earthquakes. For the numerical treatment of seismic waves and related inverse problems we refer the reader to the publications Komatitsch et al. (2008) and Komatitsch and Tromp (1999). For an adaptive finite element method for an inverse problem governed by the elastic wave equation, which can be obtained from the Lamé system, we refer the reader to Beilina (2002). In Belishev and Lasiecka (2002) regularity results for controllability of the dynamical Lamé system are derived.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polygonal domain, $T > 0$ and

$$Q = (0,T) \times \Omega, \quad \Sigma = (0,T) \times \partial\Omega.$$

Further, let

$$V = H_0^1(\Omega)^d,$$

$$H = L^2(\Omega)^d,$$

and set

$$X = L^2(V) \cap H^1(H) \cap H^2(V^*)$$

(we use the usual notations for Sobolev and Lebegues spaces leaving out the time interval I, i.e., e.g., we write $L^2(V)$ for $L^2(I,V)$). Then the dynamical Lamé system is given by

$$\begin{cases} y_{tt} - \operatorname{div}\sigma(y) = f & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = 0 & \text{in } \Sigma \end{cases} \tag{1}$$

with the function of displacement $y \in X$, stress tensor

$$\sigma_{ij} = \lambda\delta_{ij}\operatorname{tr}(\varepsilon) + 2\mu\varepsilon_{ij} \tag{2}$$

($\operatorname{tr}\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ denotes the usual trace operator) with Lamé parameters $\lambda, \mu > 0$, strain tensor

$$\varepsilon_{ij}(v) = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j\right),$$

for $i,j \in \{1,2,\ldots,d\}$, $v \in H^1(\Omega)^d$, a given force

$$f \in L^2(H),$$

and initial data $y_0 \in V$, $y_1 \in H$. The relation between the strain and stress tensor can be derived from the general Navier-Lamé system after some linearizations and assuming that the material is homogeneous and isotropic, cf. Braess (2007) and Hughes (2000). The well-possedness of the system is discussed in Section 2.

Let the control space U be a Hilbert space. We consider optimal control problems governed by the dynamical Lamé system of the following type

$$\begin{cases} \text{Minimize} \quad J(u,y), \quad u \in U, \quad y \in X, \\ \qquad \text{s.t.} \quad y_{tt} - \operatorname{div}\sigma(y) = \mathcal{B}(u) \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad\qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad\qquad y = 0 \quad \text{in } \Sigma \end{cases} \tag{P^{DWR}}$$

for given three times Fréchet differentiable operator

$$\mathcal{B}\colon U \to L^2(H), \tag{3}$$

cost functional $J\colon U \times X \to \mathbb{R}^+$ defined by

$$J(u,y) = \int_0^T J_1(y(t))dt + \frac{\alpha}{2}\|u\|_U^2$$

for $\alpha > 0$, $u \in U$, $y \in X$, and a three times Fréchet differentiable functional $J_1 \colon H \to \mathbb{R}^+$. We assume that the control problem admits a locally unique solution; for a discussion of this aspect in more detail see Kröner (2011b).

The optimal control problem (P^{DWR}) is discretized by space-time finite elements; cf. Section 3. Let (u, y) be the solution of the continuous problem and (u_σ, y_σ) of the discretized one. Then the aim is to estimate the error

$$J(u, y) - J(u_\sigma, y_\sigma)$$

in the cost functional by separating the influences of time, space, and control discretization. This allows to set up an efficient adaptive algorithm for local mesh refinement of the space and time discretizations to improve the accuracy of the error in the cost functional. We present a numerical example which shows a speedup in cpu-time of the proposed method as well as a reduction in the number of degrees of freedom to reach a given error level in comparison to uniform refinement. For a discussion of these methods in detail we refer to Kröner (2011b) and Kröner (2011a). For adaptive finite element methods using the DWR approach for optimal control problems governed by parabolic equations we refer the reader to Meidner and Vexler (2007).

2. VARIATIONAL FORMULATION OF THE SYSTEM

In this section we recall the variational formulation of the dynamical Lamé system. We begin with some notations. Let

$$(u, v)_J = \int_J (u(t), v(t))_H \, dt$$

for an open interval $J \subset I$, $u, v \in L^2(H)$, and denote by

$$(\cdot, \cdot) = (\cdot, \cdot)_H$$

the inner product in H. Further, we use the usual notation

$$A : B = \operatorname{tr}(A^T B)$$

for matrices $A, B \in \mathbb{R}^{\nu \times \nu}$, $\nu \in \mathbb{N}$ and write

$$\mathcal{D}v = \varepsilon(v)$$

for $v \in V$.

The variational formulation of the dynamical Lamé system given in the control problem (P^{DWR}) reads as follows: We look for a solution $y \in X$ of

$$\begin{cases} (y_{tt}(t), \xi) + \lambda(\operatorname{div} y(t), \operatorname{div} \xi) + 2\mu(\mathcal{D}y(t) : \mathcal{D}\xi) \\ \qquad = ((\mathcal{B}(u))(t), \xi) \quad \forall \xi \in V, \quad \text{a.e. in } I, \\ y(0) = y_0 \qquad \text{in } \Omega, \\ y_t(0) = y_1 \qquad \text{in } \Omega \end{cases} \tag{4}$$

for given initial data $y_0 \in V$, $y_1 \in H$, and control $u \in U$. Using Korn's second inequality existence and uniqueness of a solution in X can be obtained by standard arguments. For a discussion of the proof in more detail as well as the derivation of the weak formulation we refer to Kröner (2011b).

To shorten notations we introduce the semilinear form

$$a \colon U \times X \times X \to \mathbb{R},$$
$$a(u, y)(\xi) = \lambda(\operatorname{div} y, \operatorname{div} \xi)_I + 2\mu(\mathcal{D}y : \mathcal{D}\xi)_I - (\mathcal{B}(u), \xi)_I.$$

Setting

$$\bar{X} = L^2(H) \cap H^1(V^*),$$
$$Y = X \times \bar{X}$$

we formulate the system (4) equivalently as a first order system in time, i.e. we look for a solution $(y^1, y^2) \in Y$ of

$$\begin{cases} (y_t^2, \xi^1)_I + a(u, y^1)(\xi^1) + (y^2(0) - y_1(u), \xi^1(0))_H \\ \qquad = 0 \quad \forall \xi^1 \in X, \\ (y_t^1, \xi^2)_I - (y^2, \xi^2)_I - (y_0(u) - y^1(0), \xi^2(0))_H \\ \qquad = 0 \quad \forall \xi^2 \in \bar{X}. \end{cases} \tag{5}$$

for given initial data $y_0 \in V$, $y_1 \in H$. Then $y = y^1$ is the solution of system (4).

3. DISCRETIZATION

The Lamé system is discretized by space-time finite elements.

Let a partition of the time interval $\bar{I} = [0, T]$ be given by

$$\bar{I} = \{0\} \cup I_1 \cup \cdots \cup I_M$$

with subintervals $I_m = (t_{m-1}, t_m]$ of size k_m and time points

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T.$$

We define the time discretization parameter k as a piecewise constant function by setting $k|_{I_m} = k_m$ for $m = 1, \ldots, M$.

Further, we define the semi-discrete spaces

$$X_k = \{ v_k \in C(\bar{I}, H) \mid v_k|_{I_m} \in \mathcal{P}_1(I_m, V) \},$$
$$\widetilde{X}_k = \{ v_k \in L^2(I, V) | v_k|_{I_m} \in \mathcal{P}_0(I_m, V) \text{ and } v_k(0) \in H \},$$

where $\mathcal{P}_r(I_m, V)$ denotes the space of all polynomials of degree smaller or equal to $r \in \{0, 1\}$ defined on I_m with values in V.

For given control $u_k \in U$ we call $(y_k^1, y_k^2) \in X_k \times X_k$ a solution of the semi-discrete state equation if

$$\begin{cases} \sum_{m=1}^{M} (\partial_t y_k^2, \xi^1)_{I_m} + a(u, y_k^1)(\xi^1) \\ \qquad + (y_k^2(0) - y_1(u_k), \xi^1(0))_H = 0 \quad \forall \xi^1 \in \widetilde{X}_k, \\ \sum_{m=1}^{M} (\partial_t y_k^1, \xi^2)_{I_m} - (y_k^2, \xi^2)_I - (y_0(u_k) - y_k^1(0), \xi^2(0))_H \\ \qquad = 0 \quad \forall \xi^2 \in \widetilde{X}_k. \end{cases} \tag{6}$$

After these considerations we formulate the semi-discrete optimal control problem

$$\begin{cases} \text{Minimize} \quad J(u_k, y_k^1), \quad (u_k, y_k^1) \in U \times X_k^1, \\ \qquad \text{s.t.} \quad (6). \end{cases} \tag{P_k^{DWR}}$$

The semi-discrete optimal control problem is assumed to admit a (locally) unique solution. For results on existence of solutions of the semi-discrete state system and corresponding control problem see Kröner (2011b).

We continue with the discretization in space. For spatial discretization we will consider two- or three-dimensional regular meshes; see, e.g., Ern and Guermond (2004). A mesh consists of quadrilateral or hexahedral cells K, which constitute a non-overlapping cover of the computational domain Ω. Cells may have hanging nodes, but at most one is allowed for each face in two dimensions (lying on midpoints of faces of neighboring cells) and five in three dimensions. The corresponding mesh is denoted by $\mathcal{T}_h = \{K\}$, where we define the discretization parameter h as a cellwise constant function by setting $h|_K = h_K$ with the diameter h_K of the cell K. We construct on the mesh $\mathcal{T}_h$ conforming finite element spaces $V_h \subset V$ in a standard way by

$$V_h = \left\{ v \in V \mid v|_K \in (\mathcal{Q}(K))^d \text{ for } K \in \mathcal{T}_h \right\}.$$

Here, $\mathcal{Q}(K)$ consists of shape functions obtained by bi- or trilinear transformations of polynomials in $\widehat{\mathcal{Q}}(\hat{K})$ defined on the reference cell $\hat{K} = (0,1)^d$ with

$$\widehat{\mathcal{Q}}(\hat{K}) = \operatorname{span}\left\{ \prod_{j=1}^{d} x_j^{k_j} \;\middle|\; k_j \in \{0,1\} \right\}.$$

In analogy to Schmich and Vexler (2008) we allow dynamic mesh change in time and keep the time steps k_m constant in space. We associate with each time point t_m a mesh $\mathcal{T}_h^m$ and a corresponding (spatial) finite element space V_h^m.

Let $\{\tau_0, \tau_1\}$ be a basis of $\mathcal{P}_1(I_m, \mathbb{R})$ with the following property:

$$\tau_0(t_{m-1}) = 1, \quad \tau_0(t_m) = 0, \quad \tau_1(t_{m-1}) = 0.$$

Then we define

$$\begin{aligned}
X_{k,h}^m &= \operatorname{span}\{ \tau_1 v_1 \mid v_0 \in V_h^{m-1},\ v_1 \in V_h^m \},\\
X_{k,h} &= \{ v_{kh} \in C(\bar{I}, H) \mid v_{kh}|_{I_m} \in X_{k,h}^m \} \subset X_k,\\
\widetilde{X}_{k,h} &= \Big\{ v_{kh} \in L^2(I,V) | v_{kh}|_{I_m} \in \mathcal{P}_0(I_m, V_h^m)\\
&\qquad \text{and } v_{kh}(0) \in V_h^0 \Big\} \subset \tilde{X}_k.
\end{aligned}$$

After this preparation we can formulate the discretized state equation: For given $u_{kh} \in U$ we call $(y_{kh}^1, y_{kh}^2) \in X_{k,h} \times X_{k,h}$ a solution of the discrete state equation if

$$\begin{cases}
\displaystyle\sum_{m=1}^{M} (\partial_t y_{kh}^2, \xi^1)_{I_m} + a(u_{kh}, y_{kh}^1)(\xi^1)\\
\quad + (y_{kh}^2(0) - y_1(u_{kh}), \xi^1(0))_H = 0 \qquad \forall \xi^1 \in \widetilde{X}_{k,h},\\
\displaystyle\sum_{m=1}^{M} (\partial_t y_{kh}^1, \xi^2)_{I_m} - (y_{kh}^2, \xi^2)_I\\
\quad - (y_0(u_{kh}) - y_{kh}^1(0), \xi^2(0))_H = 0 \qquad \forall \xi^2 \in \widetilde{X}_{k,h}.
\end{cases} \tag{7}$$

The discretized equation (7) is assumed to admit a unique solution; cf. the semi-discrete case.

Thus, we can state the optimal control problem discretized in time and space.

$$\begin{cases}
\text{Minimize } \ J(u_{kh}, y_{kh}^1), \quad u_{kh} \in U, \quad y_{kh}^1 \in X_{k,h}\\
\qquad\qquad \text{s.t.} \quad (7).
\end{cases} \tag{P_{kh}^{DWR}}$$

The discretized control problem (P_{kh}^{DWR}) is assumed to admit a (locally) unique solution; cf. the semi-discrete case.

For the control discretization we choose a finite dimensional subspace

$$U_d = U$$

with control discretization parameter d. In case of distributed control we may choose, e.g., $U_d = \widetilde{X}_{k,h}$ with mesh parameters k and h as for the state discretization. If the control is a time dependent parameter with values in $\mathbb{R}^n$, $n \in \mathbb{N}$, we may discretize the control by piecewise constants in time with values in $\mathbb{R}^n$. For a discussion of these aspects in more detail cf. (Meidner, 2008, pp. 37).

The fully-discretized problem reads as

$$\begin{cases}
\text{Minimize } \ J(u_\sigma, y_\sigma^1), \quad u_\sigma \in U_d, \quad y_\sigma^1 \in X_{k,h},\\
\qquad\qquad \text{s.t.} \quad (7).
\end{cases} \tag{$P_{\sigma}^{\mathrm{DWR}}$}$$

Thereby, the discrete solutions are denoted with the index σ collecting the discretization parameters k, h and d. We assume that the corresponding solutions exist; cf. the semi-discrete case.

4. A POSTERIORI ERROR ESTIMATE

To separate the errors in the cost functional with respect to time, space, and control discretization we split the error in the following way

$$\begin{aligned}
J(u, y^1) - J(u_\sigma, y_\sigma^1) &= (J(u, y^1) - J(u_k, y_k^1))\\
&\quad + (J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1))\\
&\qquad + (J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1)),
\end{aligned}$$

where (u, y^1) is the solution of the continuous problem (P^{DWR}), (u_k, y_k^1) of the time discretized problem (P_k^{DWR}), (u_{kh}, y_{kh}^1) the solution of the time and space discretized problem (P_{kh}^{DWR}) and (u_σ, y_σ^1) is the solution of the fully discretized problem (P_σ^{DWR}). To derive an a posteriori error estimate we introduce the Lagrangian

$$\begin{aligned}
\mathcal{L} \colon U \times &\Big(\operatorname{span}(X \cup X_k^r) \times \operatorname{span}(\bar{X} \cup X_k^r) \Big)\\
&\times \Big(\operatorname{span}(X \cup \widetilde{X}_k^{r-1}) \times \operatorname{span}(\bar{X} \cup \widetilde{X}_k^{r-1}) \Big) \longrightarrow \mathbb{R},
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{L}(u, y, p) &= J(u, y^1) - \sum_{m=1}^{M} (\partial_t y^2, p^1)_{I_m}\\
&\quad - a(y^1, p^1) - \sum_{m=1}^{M} (\partial_t y^1, p^2)_{I_m} + (y^2, p^2)_I\\
&\quad - (y^2(0) - y_1(u), p^1(0))_H + (y_0(u) - y^1(0), p^2(0))_H
\end{aligned} \tag{8}$$

for $(u, y, p) \in U \times \big(\operatorname{span}(X \cup X_k) \times \operatorname{span}(\bar{X} \cup X_k)\big) \times \big(\operatorname{span}(X \cup \widetilde{X}_k) \times \operatorname{span}(\bar{X} \cup \widetilde{X}_k)\big)$.

Then, according to Kröner (2011b) there holds

$$\begin{aligned}
J(u, y^1) - J(u_k, y_k^1) &\approx \frac{1}{2}\Big(\mathcal{L}'_{\underline{y}}(u_k, \underline{y}_k, \underline{p}_k)(\underline{y} - \hat{\underline{y}}_k)\\
&\qquad + \mathcal{L}'_{\underline{p}}(u_k, \underline{y}_k, \underline{p}_k)(\underline{p} - \hat{\underline{p}}_k) \Big),\\
J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) &\approx \frac{1}{2}\big(\mathcal{L}_{\underline{y}'}(u_{kh}, \underline{y}_{kh}, \underline{p}_{kh})(\underline{y}_k - \hat{\underline{y}}_{kh})\\
&\qquad + \mathcal{L}'_{\underline{p}}(u_\sigma, \underline{y}_{kh}, \underline{p}_{kh})(\underline{p}_k - \hat{\underline{p}}_{kh}) \big),\\
J(u_{kh}, y_{kh}^1) - J(u_\sigma, \underline{y}_\sigma^1) &\approx \frac{1}{2} \mathcal{L}'_u(u_\sigma, \underline{y}_\sigma, \underline{p}_\sigma)(u_{kh} - \hat{u}_\sigma)
\end{aligned} \tag{9}$$

with arbitrary functions $(\hat{u}_k, \hat{\underline{y}}_k, \hat{\underline{p}}_k) \in U \times (X_k)^2 \times (\widetilde{X}_k)^2$, $(\hat{u}_{kh}, \hat{\underline{y}}_{kh}, \hat{\underline{p}}_{kh}) \in U \times (X_{k,h})^2 \times (\widetilde{X}_{k,h})^2$, $(\hat{u}_\sigma, \hat{\underline{y}}_\sigma, \hat{\underline{p}}_\sigma) \in$

$U_d \times (X_{k,h})^2 \times (\widetilde{X}_{k,h})^2$. Here, we used the notation $\underline{y} = (y^1, y^2)$ and accordingly for the adjoint state and the corresponding discrete functions.

A posteriori error estimates can be derived from (9) by standard techniques and can be used in an adaptive algorithm for local mesh refinement, cf. Kröner (2011b).

5. NUMERICAL EXAMPLE

In this section we present a numerical example. Therefore, we specify the cost functional in (P^{DWR}). For $U = L^2(\mathbb{R}^l)$ we consider

$$J(u,y) = \frac{1}{2}\|y - y_{\text{d}}\|^2_{L^2(H)} + \frac{\alpha}{2}\|u\|^2_{L^2(\mathbb{R}^l)}, \tag{10}$$

with $u \in U$, $y \in X$, and given $y_{\text{d}} \in L^2(H)$. Further, we choose the operator in (3) by

$$\mathcal{B}\colon U \to L^2(H), \quad \mathcal{B}u = \sum_{i=1}^{l} u_i(t) g_i(x)$$

for given functions $g_i \in H$, $i = 1, \ldots, l$, $l \in \mathbb{N}$. Thus, in this example the control is time-dependent with values in $\mathbb{R}^l$.

For the discretization we proceed as in Section 3. The discrete control space is chosen as

$$U_d = \{\, u \in L^2(\mathbb{R}^l) \mid u|_{I_m} \in \mathcal{P}_0(I_m, \mathbb{R}^l), \quad u(0) \in \mathbb{R}^l \,\},$$

where the time intervals I_m are the same as used for the discretization of the state.

We verify that in this case the estimator $\mathcal{L}'_u(u_\sigma, \underline{y}_\sigma, \underline{p}_\sigma)(\cdot)$ (cf. (9)) vanishes. We introduce the adjoint operator $\mathcal{B}^*$ given by

$$\mathcal{B}^*\colon L^2(H) \to U,$$
$$(\mathcal{B}^* q)(t)_i = (g_i, q)_H \quad (i = 1, \ldots, l),$$

since

$$\begin{aligned}(u, \mathcal{B}^* q)_U &= \int_0^T \sum_{i=1}^{l} u_i (\mathcal{B}^* q)(t)_i dt \\ &= \int_0^T \sum_{i=1}^{l} u_i(t)(g_i, q(t))_H dt \\ &= (\mathcal{B}u, q)_I\end{aligned}$$

for $q \in L^2(H)$.

Lemma 1. Under the assumptions from above, there holds

$$\mathcal{L}'_u(u_\sigma, \underline{y}_\sigma, \underline{p}_\sigma)(\cdot) = 0. \tag{11}$$

Proof. By standard arguments there holds the optimality condition given by

$$(\alpha u_\sigma + \mathcal{B}^* \underline{p}_\sigma, \delta u)_U = 0 \quad \forall \delta u \in U_d. \tag{12}$$

Further, there holds $\mathcal{B}^* \underline{p}_\sigma \in U_d$ for all $\underline{p}_\sigma \in X^{r,s}_{k,h}$, since

$$\underline{p}_\sigma(t) = \sum_{k=0}^{r_d} p_{\sigma,k} t^k, \quad p_{\sigma,k} \in V_h^s, \quad t \in I_m$$

and so

$$(\mathcal{B}^* \underline{p}_\sigma)_i|_{I_m} = \sum_{k=0}^{r_d} \left(\int_\Omega g_i p_{\sigma,k} dx \right) t^k \in \mathcal{P}_{r_d}(I_m, \mathbb{R}^l).$$

Thus, we can choose

$$\delta u = \alpha u_\sigma + \mathcal{B}^* \underline{p}_\sigma$$

in (12) and obtain (11).

Remark 1. The previous lemma can be generalized to the case of different temporal meshes for the control and state discretization if the set of time points of the state discretization is a subset of the time points of the control discretization.

For the computations we choose the data as follows

$$y_0(x) = \begin{cases} (\sin(8\pi(x_1 - 0.125)) \\ \qquad \cdot \sin(8\pi(x_2 - 0.125)), 0)^T, \\ \qquad\qquad \text{for } 0.125 < x_1, x_2 < 0.25, \\ (0,0)^T, \qquad \text{else}, \end{cases}$$

$$y_1(x) = (0,0)^T,$$

$$y_{\text{d}}(t,x) = 0,$$

$$g_1(x) = \begin{cases} (1,1)^T, \text{ for } x_1 < 0, \\ (0,0)^T, \text{ else} \end{cases}$$

$$g_2(x) = \begin{cases} (1,1)^T, \text{ for } x_1 > 0, \\ (0,0)^T, \text{ else} \end{cases}$$

$$\alpha = 0.001, \quad d = 2, \quad l = 2, \quad \lambda = 1, \quad \mu = 1 \tag{13}$$

for $(t,x) = (t, x_1, x_2) \in [0,T] \times \Omega = [0, 0.5] \times [-1,1]^2$.

In Figure 1 we present a comparison of the error in the cost functional for adaptive and uniform refinement. It illustrates that in case of adaptive refinement we need less degrees of freedom than in case of uniform refinement to reach a given error tolerance. Further, in Table 1 we compare the cpu-time and the degrees of freedom to reach an error less than $6.5 \cdot 10^{-8}$ normalizing the values for uniform refinement to 100%. We have an essential gain in cpu-time and number of unknowns in case of adaptive refinement. In Figure 2-6 the spatial meshes at different time points are given.

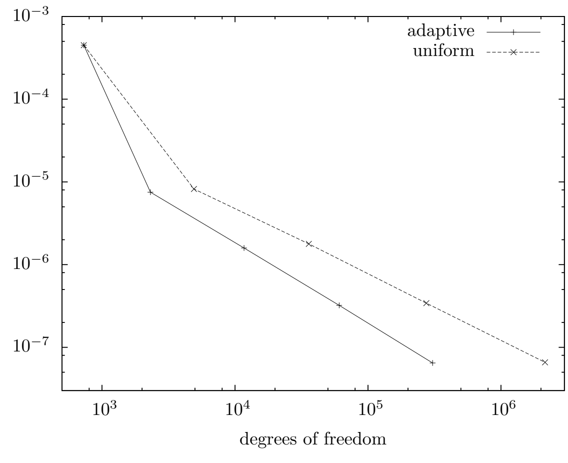

Fig. 1. Error for adaptive and uniform refinement

refinement	cpu-time	dof	error
uniform	100%	100%	$6.6 \cdot 10^{-8}$
adaptive	34%	15 %	$6.5 \cdot 10^{-8}$

Table 1. Comparison of the cpu-time for uniform and adaptive refinement

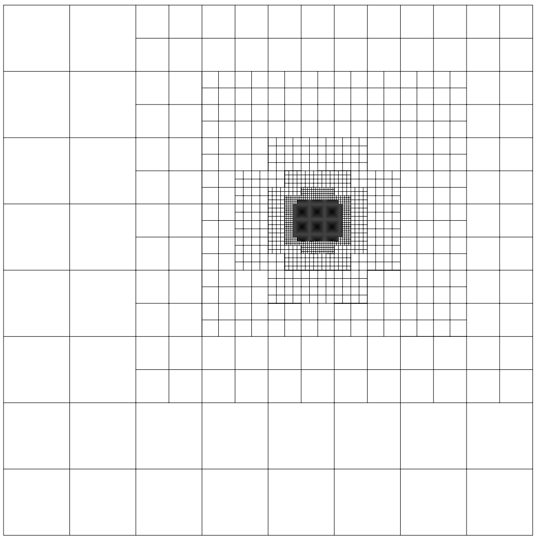

Fig. 2. Spatial mesh at time $t = 0$

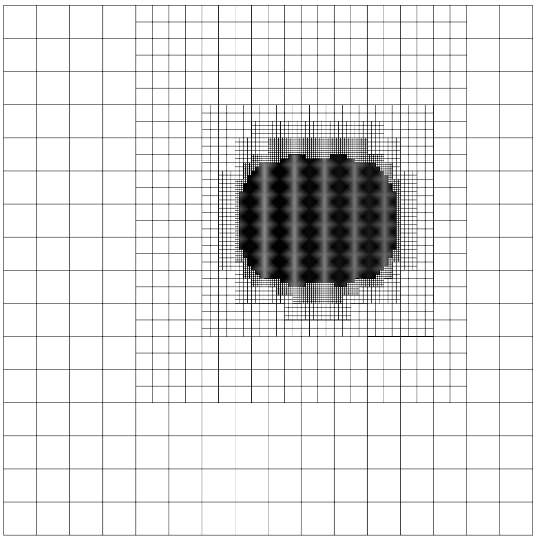

Fig. 3. Spatial mesh at time $t = 0.25$

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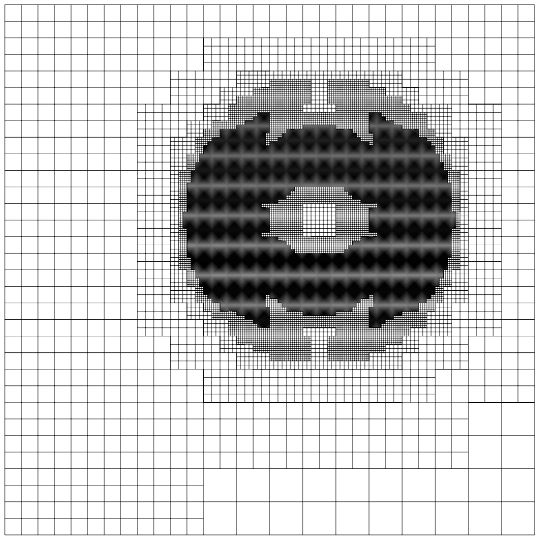

Fig. 4. Spatial mesh at time $t = 0.5$

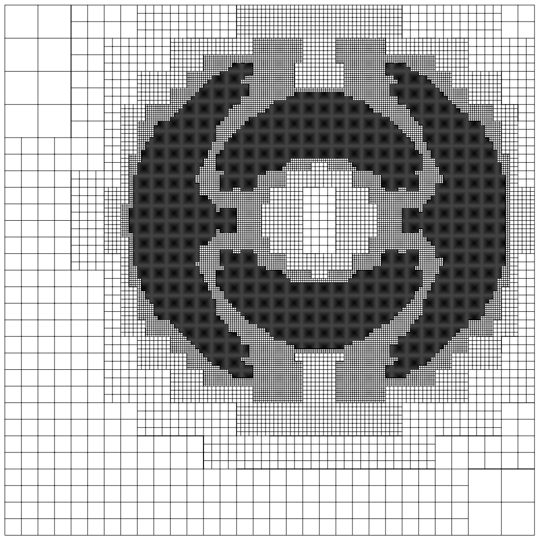

Fig. 5. Spatial mesh at time $t = 0.75$

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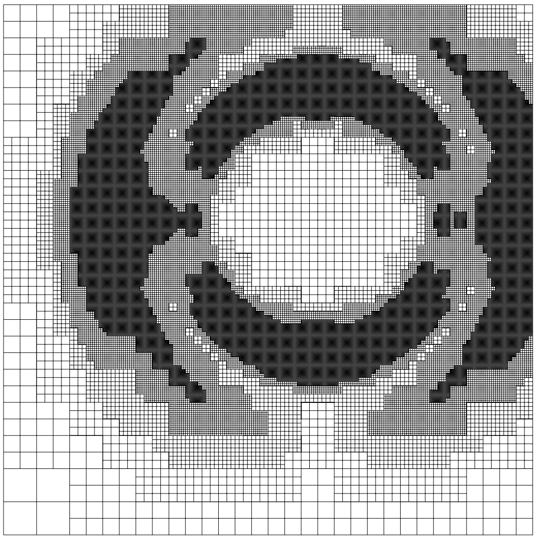

Fig. 6. Spatial mesh at time $t = 1$

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