Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

Multilevel Preconditioning of Graph-Laplacians: Polynomial Approximation of the Pivot Blocks Inverses

P. Boyanova, I. Georgiev, S. Margenov, L. Zikatanov

RICAM-Report 2011-30

Multilevel Preconditioning of Graph-Laplacians: Polynomial Approximation of the Pivot Blocks Inverses

P. Boyanova[1,2], I. Georgiev[3,4], S. Margenov[2], L. Zikatanov[5]

[1]Uppsala University, Box 337, 751 05 Uppsala, Sweden
[2]IICT – BAS, Acad. G. Bonchev Str., bl. 25A, 1113 Sofia, Bulgaria
[3]IMI – BAS, Acad. G. Bonchev Str., bl. 8 1113 Sofia, Bulgaria
[4]RICAM – AAS, Altenbergerstraße 69, A-4040 Linz, Austria
[5]Department of Mathematics, Penn State,
University Park, PA 16802, USA

Abstract

We consider the discrete system resulting from mixed finite element approximation of a second-order elliptic boundary value problem with Crouzeix-Raviart nonconforming elements for the vector valued unknown function and piece-wise constants for the scalar valued unknown function. Since the mass matrix corresponding to the vector valued variables is diagonal, these unknowns can be eliminated exactly. Thus, the problem of designing an efficient algorithm for the solution of the resulting algebraic system is reduced to one of constructing an efficient algorithm for a system whose matrix is a graph-Laplacian (or weighted graph-Laplacian).

We propose a preconditioner based on an algebraic multilevel iterations (AMLI) algorithm. The hierarchical two-level transformations and the corresponding 2×2 block splittings of the graph-Laplacian needed in an AMLI algorithm are introduced locally on macroelements. Each macroelement is associated with an edge of a coarser triangulation. To define the action of the preconditioner we employ polynomial approximations of the inverses of the pivot blocks in the 2×2 splittings. Such approximations are obtained via the best polynomial approximation of x^{-1} in L_∞ norm on a finite interval. Our construction provides sufficient accuracy and moreover, guarantees that each pivot block is approximated by a positive definite matrix polynomial.

One possible application of the constructed efficient preconditioner is in the numerical solution of unsteady Navier-Stokes equations by a projection method. It can also be used to design efficient solvers for problems corresponding to other mixed finite element discretizations.

Keywords: preconditioning · AMLI · polynomial matrix approximation

1 Introduction

Let us consider the elliptic problem in mixed form:

$$\begin{aligned} \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$, $\boldsymbol{f}$ is a given smooth vector valued function, $\boldsymbol{n}$ is the outward unit normal vector to the boundary. We refer to the vector valued unknown function $\boldsymbol{u}$ as a velocity and to the scalar valued unknown p as a pressure. The preconditioner that we construct is for the finite element (FE) discretization with Crouzeix-Raviart elements for $\boldsymbol{u}$ and piece-wise constants for p. It is known that such choice of spaces is stable (see [4]), and such spaces are used as a discretization in a projection method for Navier-Stokes equations (see e.g. [2]). The Crouzeix-Raviart mass matrix is diagonal and the velocity unknowns can be eliminated exactly (see e.g. [8]). The matrix A of the reduced system for the pressure is a weighted graph-Laplacian. We consider 2-D uniform mesh of right-angled triangles. Then A corresponds to the T-shaped four point stencil shown in Fig. 1.

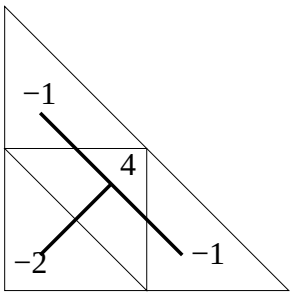

Figure 1: Schur complement four point stencil for the pressure

We propose a preconditioner for the weighted graph-Laplacian system based on algebraic multilevel iterations (AMLI) method. The framework for this method was originally introduced in [1] (see also [10], [6]). AMLI is a recursive generalization of two-level preconditioners that has optimal computational complexity due to a proper Schur complement stabilization using Chebyshev polynomials. We consider a sequence of nested triangulations $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_l$ of the domain Ω, constructed by recursive uniform refinement of a given initial mesh. We denote by $A^{(0)}, A^{(1)} \cdots A^{(l)}$ the corresponding system matrices. Regarding a hierarchical 2x2 block partitioning of the system matrix $A^{(k)}$ at a refinement level k,

$$\widehat{A}^{(k)} = J^{(k)} A^{(k)} J^{(k)^T} = \begin{bmatrix} \widehat{A}_{11}^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & \widehat{A}_{22}^{(k)} \end{bmatrix} \begin{matrix} \} \text{degrees of freedom added by refinement} \\ \} \text{coarse mesh degrees of freedom} \end{matrix}$$

the AMLI preconditioner is defined as follows: $C^{(0)} = A^{(0)}$ at the coarsest mesh level with index 0,

$$C^{(k)} = J^{(k)^{-1}} \begin{bmatrix} \widehat{C}_{11}^{(k)} & 0 \\ \widehat{A}_{21}^{(k)} & \tilde{A}^{(k-1)} \end{bmatrix} \begin{bmatrix} I & \widehat{C}_{11}^{(k)^{-1}} \widehat{A}_{12}^{(k)} \\ 0 & I \end{bmatrix} J^{(k)^{-T}}$$

at successively refined levels k, where $\widehat{C}_{11}^{(k)}$ are symmetric positive definite approximations of $\widehat{A}_{11}^{(k)}$ that satisfy

$$\boldsymbol{v}^T \widehat{A}_{11}^{(k)} \boldsymbol{v} \leq \boldsymbol{v}^T \widehat{C}_{11}^{(k)} \boldsymbol{v} \leq (1+b) \boldsymbol{v}^T \widehat{A}_{11}^{(k)} \boldsymbol{v}, \text{ for all } \boldsymbol{v}, \tag{2}$$

and $\tilde{A}^{(k-1)^{-1}} = Q_{\beta-1}\left(C^{(k-1)^{-1}} A^{(k-1)}\right) C^{(k-1)^{-1}}$, where $Q_{\beta-1}$ is a properly chosen polynomial of degree $\beta - 1$.

It is known that under conditions, detailed in [1], the relative condition number satisfies $\kappa(C^{(l)^{-1}} A^{(l)}) \approx (1+b)/(1-\gamma^2)$, where γ is the Cauchy-Bunyakowski-Schwarz (CBS) constant, that characterizes the two-level hierarchical partitionings.

For the case $\beta = 2$ the coefficients q_0 and q_1 of the optimal stabilization polynomial $Q_1(y) = q_0 + q_1 y$, which has to be evaluated in the AMLI W-cycle, are given by (see [1, 5]):

$$q_0 = \frac{2}{\xi}, \quad q_1 = \frac{-1}{1-\gamma^2+b(1-2\xi)}, \quad \text{with} \quad \xi = \sqrt{1+b+b^2-\gamma^2} - b \tag{3}$$

In this article we propose and justify the use of polynomial approximation of the pivot blocks inverses $(\widehat{A}_{11}^{(k)})^{-1}$ in the AMLI setting for weighted graph-Laplacians, that yields good optimal order preconditioned conjugate gradient (PCG) convergence rates.

The rest of the paper is organized as follows. In Section 2 we present the general idea of constructing approximation of a matrix inverse, using the best polynomial approximation of x^{-1} in L_∞ norm on a finite interval. In Section 3 we show how this idea is applied for the pivot block inverses in AMLI method for weighted graph-Laplacians. Section 4 is devoted to analysis of the numerical results of applying PCG method with the constructed preconditioner, as well as a comparison with the case when conjugate gradient (CG) iterations are used for solving the pivot block systems in the AMLI algorithm. Concluding remarks and brief summary of the results are found in Section 5.

2 Polynomial approximation of a matrix inverse

Let H be $n \times n$ symmetric positive definite (SPD) matrix with a set of eigenvalue-eigenvector pairs $\{\lambda_i, \bar{\boldsymbol{v}}_i\}_{i=1}^n$, $0 < \lambda_1 \leq \cdots \leq \lambda_n$. We construct a polynomial approximation (preconditioner) C^{-1} of H^{-1} (i.e. preconditioner C of H) such that $C^{-1} = P_\nu(H)$, $P_\nu \in \mathcal{P}_\nu$, and $\kappa(C^{-1}H)$ is close to 1.

Theorem 2.1. *Let $P_\nu(x) \in \mathcal{P}_\nu$, and $0 < m \leq P_\nu(x)x \leq M$ for all x in an interval $S, [\lambda_1, \lambda_n] \subset S$. Then $m\boldsymbol{v}^T\boldsymbol{v} \leq \boldsymbol{v}^T P_\nu(H) H \boldsymbol{v} \leq M\boldsymbol{v}^T\boldsymbol{v}, \forall \boldsymbol{v} \in \mathbb{R}^n$.*

Proof. For all $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \bar{\boldsymbol{v}}_i$, we have:

$$\begin{aligned}(P_\nu(H)H\,\boldsymbol{v}, \boldsymbol{v}) &= (P_\nu(H)H \sum_{i=1}^{n} \alpha_i \bar{\boldsymbol{v}}_i, \sum_{i=1}^{n} \alpha_i \bar{\boldsymbol{v}}_i) = \sum_{i=1}^{n} P_\nu(\lambda_i)\lambda_i \alpha_i^2 \\ &\leq M \sum_{i=1}^{n} \alpha_i^2 = M(\boldsymbol{v}, \boldsymbol{v}).\end{aligned}$$

The left inequality follows in the same way. □

We would like to have $P_\nu(H)H \sim I$, i.e. $P_\nu(x)x \sim 1, \forall x \in S$.
Let us assume that $|P_\nu(x) - 1/x| < \epsilon, \forall x \in S \subset (0, +\infty)$. Then

$$1 - \epsilon x < P_\nu(x)x < 1 + \epsilon x, \quad \forall x \in S.$$

If we know some estimate of the spectrum of H, i.e., $\lambda_{\min} \leq \lambda_i \leq \lambda_{\max}, i = 1, \ldots, n$, we can take $S = [\lambda_{\min}, \lambda_{\max}]$ and get $1 - \epsilon\lambda_{\max} < P_\nu(x)x < 1 + \epsilon\lambda_{\max}$. In general, if ϵ is small enough, or equivalently, the degree ν of the polynomial is large enough, we have $P_\nu(x)x > 0$ for all $x \in S$, $P_\nu(H)H$ is SPD, and

$$\kappa(P_\nu(H)H) < \frac{1 + \epsilon\lambda_{\max}}{1 - \epsilon\lambda_{\max}}.$$

We define $P_\nu(x)$ to be the best polynomial approximation of x^{-1} in L_∞ norm on the finite interval $[\lambda_{min}, \lambda_{max}]$, i.e.

$$\left\| \frac{1}{x} - P_\nu \right\|_{\infty, [\lambda_{\max}, \lambda_{\min}]} = \min_{P \in \mathcal{P}_\nu} \left\| \frac{1}{x} - P \right\|_{\infty, [\lambda_{\max}, \lambda_{\min}]} = E(\nu).$$

It is known (see e.g. [9, 7]), that for $\nu \geq 1$

$$P_\nu(x) = \frac{1}{x}\left(1 + \frac{2(-\theta)^{-\nu}}{(\theta - \theta^{-1})^2} R_{\nu+1}(2\sigma x - a)\right),$$

where

$$\sigma = \frac{1}{\lambda_{\max} - \lambda_{\min}}, \quad a = \frac{\lambda_{\max} + \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}, \quad \theta = a + \sqrt{a^2 - 1},$$

$$R_{\nu+1}(x) = \theta T_{\nu+1}(x) + 2T_\nu(x) + \theta^{-1} T_{\nu-1}(x),$$

and $T_\nu(x) \in \mathcal{P}_\nu$ is the Chebyshev polynomial of degree ν. The error of best polynomial approximation $E(\nu)$ is

$$E(\nu) = \frac{8\sigma\theta^{-\nu}}{(\theta - \theta^{-1})^2}.$$

A three term recurrence for $P_\nu(x)$ then exists (see [7]), that let us use the following algorithm for computing $P_\nu(H)$:

$$\begin{aligned} P_0(H) &= \frac{\eta(1+\delta)}{(1-\delta)^2} I, \quad P_1(H) = -\left(\frac{\eta}{1-\delta}\right)^2 A + \frac{2\eta}{(1-\delta)^2} I, \\ P_{k+1}(H) &= [(1+\delta)I - \eta A] P_k(H) - \delta P_{k-1}(H) + \eta I, \end{aligned}$$

where

$$\eta = \frac{4}{(\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}})^2}, \quad \delta = \left(\frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}\right)^2.$$

3 AMLI with polynomial approximation of the pivot blocks inverses for weighted graph-Laplacians

We examine the case of two space dimensions and uniform mesh of right-angled triangles for the discretization of (1). Each refined mesh is obtained by dividing the current triangles in four congruent ones by connecting the midpoints of the sides. We define the hierarchical two-level transformation and corresponding 2×2 splitting of the related graph-Laplacian for the reduced pressure system locally for macroelements associated with the edges of the coarse triangulation. We define $A_e^{(k)}$ using a weight parameter $t \in (0,1)$ so that the contribution of the links between the interior nodes among the (macroelement) edge matrices of the current coarse triangle are distributed correctly. Following the numbering from Fig. 2, we introduce the local (macroelement) matrix $A_e^{(k)}$, corresponding to a hypotenuse, in the form:

$$A_e^{(k)} = A_{e;H}^{(k)} = \left[\begin{array}{cccc|cccc} t+1 & -2t & \frac{t-1}{2} & \frac{t-1}{2} & & & & \\ -2t & 2t & & & & & & \\ \frac{t-1}{2} & & \frac{5-t}{2} & & & & -2 & \\ \frac{t-1}{2} & & & \frac{5-t}{2} & & & & -2 \\ \hline & & & & t+1 & -2t & \frac{t-1}{2} & \frac{t-1}{2} \\ & & & & -2t & 2t & & \\ & & -2 & & \frac{t-1}{2} & & \frac{5-t}{2} & \\ & & & -2 & \frac{t-1}{2} & & & \frac{5-t}{2} \end{array}\right].$$

The local (macroelement) matrix $A_e^{(k)} = A_{e;C}^{(k)}$, corresponding to a cathetus, is introduced

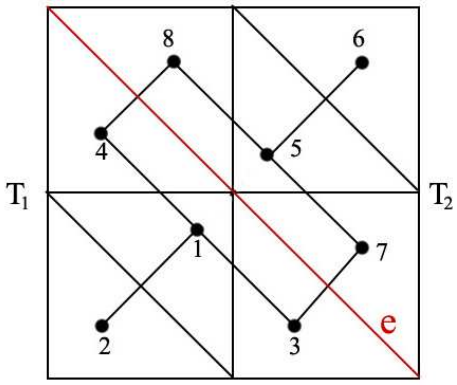

Figure 2: Macroelement of two adjacent coarse triangles with a common hypotenuse

in a similar way. Then we define the (macroelement) local transformation matrix $J_e^{(k)}$ as:

$$J_e^{(k)} = \begin{bmatrix} 1 & c & d & d & & & & \\ 1 & d & c & d & & & & \\ 1 & d & d & c & & & & \\ & & & & 1 & c & d & d \\ & & & & 1 & d & c & d \\ & & & & 1 & d & d & c \\ r & r & r & r & & & & \\ & & & & r & r & r & r \end{bmatrix},$$

where c, d are parameters. As shown in [3] this two-level hierarchical partitioning complies with the conditions of the main theorem in AMLI theory (see [1]) when $r = \sqrt{2}/2$ and the AMLI stabilization polynomial degree β equals 2 or 3. An estimate for the CBS constant $\gamma^2 \leq 0.58$ for $c = 1$, $d = -0.1$ is derived there too.

We now examine $P_\nu(\widehat{A}_{11}^{(k)})$, where $P_\nu(x)$ is the best polynomial approximation of x^{-1} on the interval $S = [\lambda_{\min}, \lambda_{\max}]$ that contains all eigenvalues of $\widehat{A}_{11}^{(k)}$. The hierarchical construction insures that the minimal and maximal eigenvalues of the pivot block do not depend on the problem size (see e.g. [3]). Therefore it is enough to calculate them for a rather coarse mesh and use the obtained values for all k. All results presented in this paper are for $S = [1.3, 10.55]$. In Fig. 3 the behavior of the error $P_\nu(x) - 1/x$ for polynomial degrees $\nu \in \{2, 3, 4\}$ can be seen. In all cases, there are $\nu + 2$ alternations that comply with approximation theory.

Using Theorem 2.1, one can verify that

$$\boldsymbol{v}^T \widehat{A}_{11}^{(k)} \boldsymbol{v} \leq (1 + E(\nu)\lambda_{\max}) \boldsymbol{v}^T (P_\nu(\widehat{A}_{11}^{(k)}))^{-1} \boldsymbol{v} \leq \frac{(1 + E(\nu)\lambda_{\max})}{(1 - E(\nu)\lambda_{\max})} \boldsymbol{v}^T \widehat{A}_{11}^{(k)} \boldsymbol{v}.$$

We define $\widehat{C}_{11}^{(k)} = (1 + E(\nu)\lambda_{\max})(P_\nu(\widehat{A}_{11}^{(k)}))^{-1}$, and then

$$b = (1 + E(\nu)\lambda_{\max})/(1 - E(\nu)\lambda_{\max}) - 1. \tag{4}$$

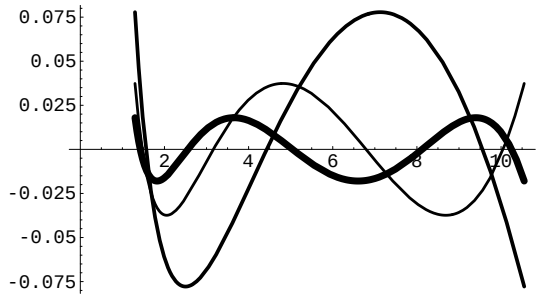

Figure 3: The error $P_\nu(x) - 1/x$ for polynomial degrees $\nu \in \{2, 3, 4\}$

4 Numerical results

The presented numerical tests show the convergence behavior of the PCG method to solve the weighted graph-Laplacian system for the case of Dirichlet boundary conditions in the unit square $\Omega = (0, 1) \times (0, 1)$, on a uniform grid with characteristic mesh size h. The tests concern the AMLI W-cycle with polynomial approximation for the pivot blocks inverses, as well as a comparison of such preconditioner with an AMLI W-cycle, which uses some CG iterations for the pivot block systems instead.

In the presentation of the numerical results we have used the following notation: ϵ is the PCG stopping criteria; n_ℓ, $\ell = 1, \ldots, 5$, denotes the total number of degrees of freedom on the finest grid $n_\ell = 4^{\ell-1} \times 2048$. Thus, the largest number of fine grid degrees of freedom that we have used in the numerical experiments is $n_5 = 524288$. For all the numerical tests the coarsest grid size is $h = 1/16$, which corresponds to $n_0 = 512$ degrees of freedom. In Tables 1–4 the number of PCG iterations needed to reduce the energy norm of the error by a factor ϵ are given.

The proposed construction for the system under consideration guarantees that the matrices $\widehat{C}_{11}^{(k)}$ are SPD for degree $\nu \geq 2$. For the experiments summarized in Table 1 we use the value of b as in (4) to calculate the coefficients (3). The convergence behavior for $\nu = 3$ and $\nu = 4$ is much better than for $\nu = 2$ and in these cases the method stabilizes earlier. This is an expected result, since the estimate and the related value of b is considerably bigger for $\nu = 2$. It is important to note that b is a measurement of how well $\widehat{C}_{11}^{(k)}$ approximates $\widehat{A}_{11}^{(k)}$ and it also affects to a large extend the stabilization properties of the polynomial $Q_{\beta-1}$.

The proposed AMLI preconditioner uses the same polynomials for the pivot block systems at each PCG iteration, thus ensuring the linearity of the process and preserving the orthogonality of the search directions. We compare it with the case of an AMLI PCG method that uses some CG iterations for approximate solution of the pivot block systems. There are several specific aspects of such experiments. CG automatically adapts to find the best approximation for a given right-hand side but this means that the matrix polynomial action that corresponds to the specified number of inner CG iterations varies at each PCG iteration. This destroys the linearity of the AMLI method process - an effect that could be taken care of by using generalized conjugate gradient (GCG) method instead of PCG

			Number of PCG iterations				
ν	b	ϵ	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
2	9.166	10^{-3}	8	12	13	13	13
		10^{-6}	14	26	28	28	28
		10^{-9}	21	40	43	43	44
3	1.303	10^{-3}	5	6	6	6	6
		10^{-6}	10	11	11	11	11
		10^{-9}	15	17	17	17	17
4	0.467	10^{-3}	4	5	6	5	6
		10^{-6}	8	11	11	11	11
		10^{-9}	12	16	16	16	16

Table 1: W-cycle preconditioner with polynomial approximation of degree ν for $[\widehat{A}_{11}^{(k)}]^{-1}$.

			Number of PCG iterations				
ν	b	ϵ	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
3	1.303	10^{-3}	6	9	10	10	10
		10^{-6}	14	$*$	$*$	$*$	29
		10^{-9}	23	$*$	$*$	48	$*$
4	0.467	10^{-3}	5	6	7	7	7
		10^{-6}	10	21	25	25	15
		10^{-9}	16	44	$*$	45	24

Table 2: W-cycle preconditioner. The approximate action of $[\widehat{A}_{11}^{(k)}]^{-1}$ is computed by applying ν CG iterations. The symbol $*$ means that more than 300 PCG iterations were needed to achieve the prescribed error reduction ϵ.

at the price of more memory and CPU usage. For the results, presented in Table 2, we assume that we can use for the case of ν CG pivot block iterations the same estimate for b as in the case of polynomial of degree ν pivot blocks inverses. The symbol $*$ used in the tables means that the PCG method did not converge within a limit of 300 iterations. As can be seen, the settings presented in Table 2 do not obtain good results, no stabilization effect is visible and in some cases the process does not converge at all.

The experiments presented in Tables 3 and 4 are motivated by observations that we have made from many different numerical tests (see also [5, 6]). As it turns out, a value of $b = 0$ for the proposed AMLI method does not slow down the convergence, but on the contrary, even leads to better stabilization numbers when polynomial pivot approximations are used. The results in Table 4 show a better behavior of the preconditioner than the results in Table 2. Nevertheless, the polynomial approximation of $[\widehat{A}_{11}^{(k)}]^{-1}$ results in much better (more stable) algorithm.

		Number of PCG iterations				
ν	ϵ	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
2	10^{-3}	8	8	8	8	8
	10^{-6}	14	15	15	15	15
	10^{-9}	21	22	22	22	22
3	10^{-3}	5	5	5	6	6
	10^{-6}	10	10	11	11	11
	10^{-9}	15	16	16	16	16
4	10^{-3}	4	5	5	5	5
	10^{-6}	8	9	9	9	9
	10^{-9}	12	13	13	13	13

Table 3: W-cycle preconditioner ($b = 0$) with polynomial approximation of degree ν for $[\widehat{A}_{11}^{(k)}]^{-1}$.

		Number of PCG iterations				
ν	ϵ	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
3	10^{-3}	6	6	11	8	$*$
	10^{-6}	14	34	36	22	19
	10^{-9}	23	71	79	78	36
4	10^{-3}	5	6	6	6	7
	10^{-6}	10	18	12	12	10
	10^{-9}	16	31	32	19	18

Table 4: W-cycle preconditioner ($b = 0$). The approximate action of $[\widehat{A}_{11}^{(k)}]^{-1}$ is computed by applying ν CG iterations. The symbol $*$ means that more than 300 PCG iterations were needed to achieve the prescribed error reduction ϵ.

5 Concluding remarks

The proposed construction for a polynomial matrix approximation leads to a considerable improvement of the conditioning of the system, while providing sufficient accuracy and positive definiteness. Applying such approximation to the pivot blocks inverses in the AMLI preconditioner for weighted graph-Laplacians yields a theoretically justified linear PCG algorithm of optimal computational complexity with experimentally confirmed good convergence rates. The studied approach is applicable in a rather general setting of hierarchical multilevel methods where the spectrum of pivot blocks is uniformly bounded with respect to the number of the refinement levels.

6 Acknowledgments

Research has been partially supported by the Bulgarian NSF Grants DO 02-338/08 and DO 02-147/08, and by the Austrian Science Fund, Grant P22989-N18. The work of the fourth author has been supported in part by the US National Science Foundation, Grants DMS-0810982, and OCI-0749202.

References

[1] Axelsson, O., Vassilevski, P. S., 1990. Algebraic multilevel preconditioning methods. II. SIAM J. Numer. Anal. 27 (6), 1569–1590.

[2] Bejanov, B., Guermond, J.-L., Minev, P. D., 2005. A locally DIV-free projection scheme for incompressible flows based on non-conforming finite elements. Internat. J. Numer. Methods Fluids 49 (5), 549–568.

[3] Boyanova, P., Margenov, S., 2009. Multilevel splitting of weighted graph-laplacian arising in non-conforming mixed FEM elliptic problems. In: Margenov, S., Vulkov, L., Wasniewski, J. (Eds.), Numerical Analysis and Its Applications. Vol. 5434 of Lecture Notes in Computer Science. Springer, pp. 216–223.

[4] Brezzi, F., Fortin, M., 1991. Mixed and hybrid finite element methods. Vol. 15 of Springer Series in Computational Mathematics. Springer-Verlag, New York.

[5] Georgiev, I., Kraus, J., Margenov, S., 2008. Multilevel preconditioning of rotated bilinear non-conforming FEM problems. Comput. Math. Appl. 55 (10), 2280–2294.

[6] Kraus J., Margenov S., 2009. Robust Algebraic Multilevel Methods and Algorithms. Radon Series Comp. Appl. Math., vol. 5.

[7] Kraus J., Pillwein V., Zikatanov L., 2009. Algebraic multilevel iteration methods and the best approximation to 1/x in the uniform norm. RICAM Technical report no. 2009-17.

[8] Margenov, S., Minev, P., 2007. On a MIC(0) preconditioning of non-conforming mixed FEM elliptic problems. Math. Comput. Simulation 76 (1-3), 149–154.

[9] Rivlin, T. J., 1981. An introduction to the approximation of functions. Dover Publications Inc., New York, corrected reprint of the 1969 original, Dover Books on Advanced Mathematics.

[10] Vassilevski, P. S., 2008. Multilevel block factorization preconditioners. Springer, New York, matrix-based analysis and algorithms for solving finite element equations.