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Additive Schur complement approximation and application to multilevel preconditioning

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ADDITIVE SCHUR COMPLEMENT APPROXIMATION AND APPLICATION TO MULTILEVEL PRECONDITIONING

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Abstract. In the present paper we introduce an algorithm for Additive Schur Complement Approximation (ASCA). This approximation technique can be applied in various iterative methods for solving systems of linear algebraic equations arising from finite element (FE) discretization of Partial Differential Equations (PDE).

Here we will show how the ASCA can be used to construct a nonlinear Algebraic Multi-Level Iteration (AMLI) method. The construction involves a linear (multiplicative) two-level preconditioner at each level, which is computed in the course of a simultaneous exact two-by-two block factorization of local (stiffness) matrices associated with a covering of the entire domain by overlapping subdomains. Unlike in Schwarz type domain decomposition methods this method does not require a global coarse problem but instead uses local coarse problems to provide global communication. We prove a robust condition number bound and present numerical tests that demonstrate that the ASCA when combined with a proper AMLI-cycle results in a multilevel method of optimal order of computational complexity.

1. Introduction

The numerical solution of partial differential equations (PDE) in many cases involves algorithms in which sparse Schur complement approximations play a key role. Constructing a good sparse preconditioner to the exact Schur complement is crucial in the development of optimal or nearly optimal order iterative solution methods, see, e.g., [18] and the references therein, as typically the exact Schur complements arising in classical multilevel block factorization preconditioners are dense matrices. The purpose of the present work is to develop an Additive Schur Complement Approximation (ASCA) that can be recursively applied in the construction of various multilevel preconditioners.

In this presentation we will assume that the linear system of algebraic equations, whose Schur complement we are interested to approximate, stems from a finite element (FE) discretization of a given PDE. Further, we will suggest that we have access to the element stiffness matrices. It is important to note that these requirements do not restrict us in general but just simplify the description of the considered method and also the algorithm for its practical application.

The ASCA can be considered as a generalization of the method first described in [9] and later studied in [2, 11, 12] for various classes of problems/matrices. What distinguishes the approximation technique presented in this paper from similar approaches considered earlier is that the ASCA it is based on coverings of the domain by subdomains that in general are permitted to overlap each other. This important feature of the algorithm, as proved and demonstrated in Sections 4–6, not only enhances the quality of the approximation substantially but also effects the desired robustness when solving problems arising in the modeling of highly heterogeneous media.

There is a similarity between the construction of the ASCA and the procedure of building interpolation in element-based algebraic multigrid (AMGe), cf. [4], especially, in the case when the AMGe algorithm is based on element agglomeration, see [7]. As distinguished from AMGe, however, the ASCA that we propose in this paper does not uniquely define interpolation weights but results in local minimum-energy extensions that create ambiguity on the overlapping of the subdomains.

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Let us note that this does not cause any problems because here we are not interested in determining an interpolation operator that is used then to form a coarse-grid operator via the Galerkin relation (triple-matrix product) but instead we want to compute the coarse-grid operator (Schur complement approximation) directly from local contributions (from exact local Schur complements).

Let us mention that preliminary numerical tests have indicated that the proposed ASCA method can be applied successfully in a very general setting, i.e., for various types of PDE. The present paper, however, serves the purpose to introduce the basic algorithm and to give a flavor of its potential. For that reason we will consider a model problem of a scalar elliptic PDE with highly oscillatory (piecewise constant) coefficient. Similar problems have been addressed recently by different authors, see, e.g., [5, 16, 17].

The paper is organized as follows. We continue with stating a model problem. In Section 3 we formulate the general algorithm of Additive Schur Complement Approximation (ASCA) and give three specific examples differing in the construction of the coverings of the domain Ω. These examples will serve as a basis for comparing the qualitative behavior of the related additive approximations later. In Section 4 we prove a condition number estimate for one of these examples. Section 5 summarizes the theoretical framework of nonlinear Algebraic Multi-Level Iteration (AMLI) methods (in their multiplicative variant), which form one potential field of application of the proposed ASCA. Finally, we present some numerical results in Section 6 to demonstrate the high quality and the robustness of the ASCA, as well as the efficiency of the related AMLI preconditioner for problems with highly oscillatory coefficients.

2. Problem formulation

Let us consider the following second-order elliptic boundary value problem

$$-\nabla\cdot(\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}) \quad in\ \Omega, \tag{2.1a}$$

$$u = 0 \quad on\ \Gamma_D, \tag{2.1b}$$

$$(\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x}))\cdot\mathbf{n} = 0 \quad on\ \Gamma_N, \tag{2.1c}$$

where Ω denotes a bounded polygonal (polyhedral) domain in $\mathbb{R}^d$, $d=2,3$. The source term $f(\mathbf{x})$ is a given function in $L_2(\Omega)$ and $\mathbf{n}$ is the outward unit vector normal to the boundary $\Gamma=\partial\Omega$ where $\Gamma=\Gamma_D\cup\Gamma_N$ and Γ_D and Γ_N are the parts of the boundary where Dirichlet and Neumann boundary conditions are imposed. The coefficient matrix $\mathbf{a}(\mathbf{x})=(a_{ij}(\mathbf{x}))_{i,j=1}^d$ is symmetric positive definite (SPD) and uniformly bounded in Ω, i.e., $\mathbf{a}(\mathbf{x})=(\mathbf{a}(\mathbf{x}))^T$ and $c_1|\mathbf{x}|^2\le(\mathbf{a}(\mathbf{x})\,\mathbf{x})\cdot\mathbf{x}\le c_2|\mathbf{x}|^2$ for all $\mathbf{x}\in\mathbb{R}^d$, and $0<c_1\le c_2<\infty$. In the following we will consider two-dimensional problems only as this simplifies the presentation, that is, we will stick to the case $d=2$. However, the presented techniques naturally transfer and can be applied to d-dimensional problems for $d\ge 3$ as well.

We assume that the partition of the domain Ω has been accomplished in such a way that over each element $e\in\mathcal{T}_h$ the functions $a_{ij}(\mathbf{x})$ are smooth and hence $\mathbf{a}(\mathbf{x})$ can be well approximated by a piecewise constant SPD diffusion tensor $\mathbf{a}(e)=\mathbf{a}_e$, i.e.,

$$\mathbf{a}(\mathbf{x})\approx\mathbf{a}_e=\begin{bmatrix} a_{e:11} & a_{e:12}\\ a_{e:21} & a_{e:22}\end{bmatrix},\qquad \forall\mathbf{x}\in e\ \forall e\in\mathcal{T}_h. \tag{2.2}$$

In particular we will consider Problem (2.1) with diffusion tensor

$$\mathbf{a}(\mathbf{x})=\alpha(\mathbf{x})I=\alpha_e I\quad\forall e\in\mathcal{T}_h \tag{2.3}$$

where $\alpha_e>0$ is a scalar quantity that may vary over several orders of magnitude across element interfaces and I denotes the identity matrix. Without loss of generality, (after rescaling) we may assume that $\alpha_e\in(0,1]$ for all $e\in\mathcal{T}_h$.

In this case the weak formulation of (2.1) reads: Find $u \in H^1_D(\Omega) := \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_D\}$, such that

$$\int_\Omega \alpha(\mathbf{x}) \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H^1_D(\Omega). \tag{2.4}$$

To find a numerical solution to (2.4) we consider a conforming FE method in the subspace of continuous piecewise bilinear functions. As a result we obtain a system of linear algebraic equations with a sparse SPD stiffness matrix $A_h \in \mathbb{R}^{n\times n}$, i.e.,

$$A_h \boldsymbol{u}_h = \boldsymbol{f}_h. \tag{2.5}$$

3. Additive Schur complement approximation

3.1. **Preliminaries and notation.** Before we explain the basic idea of additive Schur complement approximation let us agree on the notation. Let $\mathcal{T} = \mathcal{T}_h = \cup_e \{e\}$ be a non-overlapping partition of $\Omega = \Omega_h$ into elements e, which we will refer to as the *(fine) mesh*.

Definition 3.1. *A structure $F = \cup_{k\in\mathcal{I}_F} e_k$ is a union of elements $e = e_k \in \mathcal{T}$ where k is taken from an index set $\mathcal{I}_F = \{k_1, k_2, \ldots, k_{n_F}\}$; By $\mathcal{F} = \mathcal{F}_h = \{F_i : \ i = 1, 2, \ldots, n_\mathcal{F}\}$ we denote a set of structures that covers $\mathcal{T}$, i.e., for all $e \in \mathcal{T}$ there exists a structure $F = F_i \in \mathcal{F}$ such that $e \subset F$.*

Definition 3.2. *A macro structure $G = \cup_{k\in\mathcal{I}_G} F_k$ is a union of structures $F = F_k \in \mathcal{F}$ where k is taken from an index set $\mathcal{I}_G = \{k_1, k_2, \ldots, k_{n_G}\}$; By $\mathcal{G} = \mathcal{G}_h = \{G_i : \ i = 1, 2, \ldots, n_\mathcal{G}\}$ we denote a set of macro structures covering $\mathcal{F}$, i.e., for all $F \in \mathcal{F}$ there exists a macro structure $G = G_i \in \mathcal{G}$ such that $F \subset G$.*

Definition 3.3. *If any two distinct structures (or macro structures) have an empty intersection, i.e., $F_i \cap F_j = \emptyset$ (or $G_i \cap G_j = \emptyset$) for all $i \neq j$, the set $\mathcal{F}$ (or $\mathcal{G}$) is called a non-overlapping covering. Otherwise it is called an overlapping covering.*

Remark 3.4. *Structures and macro structures can be interpreted as (sub) graphs. The graph of any macro structure then is obtained as the union of the graphs of the structures it is composed of. If elements are viewed as the smallest structures, and interpreted as graphs, consisting of vertices and edges, all bigger structure and macro structure graphs are obtained as unions of the elemental graphs. The interpretation of (macro) structures as graphs will be useful when deriving condition number estimates in Section 4.*

As mentioned in the introduction of this article, we will assume that we have access to the individual element (stiffness) matrices, which we will denote by A_e. In general, for $X \in \{e, F, G\}$ by A_X, or B_X, or S_X, we will denote a small-sized ("local") matrix that is associated with either an element e or a structure F or a macro structure G. The corresponding restriction operator $R_X : \mathbb{R}^n \mapsto \mathbb{R}^{n_X}$ restricts a global vector $\boldsymbol{v} \in \mathbb{R}^n$ to a local vector $\boldsymbol{v}_X \in \mathbb{R}^{n_X}$, which is defined only on X, Similarly, we will denote by $R_{Y\mapsto X}$ the restriction from $Y \in \{F, G\}$ to $X \in \{e, F, G\}$. Note that the transpose of the restriction operator, e.g., R_X^T defines the natural inclusion, which maps any vector $\boldsymbol{v}_X$ that is defined on X to a global vector $\boldsymbol{v} \in \mathbb{R}^n$ by extending $\boldsymbol{v}_X$ with zeros outside of X. Hence, the assembly of the stiffness matrix $A = A_h$ can be written in the form

$$A = \sum_{e\in\mathcal{T}_h} R_e^T A_e R_e. \tag{3.1}$$

As is readily seen, A can also be represented in terms of local matrices A_F where $F \in \mathcal{F}$, i.e.,

$$A = \sum_{F\in\mathcal{F}} R_F^T A_F R_F \ , \tag{3.2}$$

or, A can be assembled from local matrices A_G where G runs over all elements of the covering $\mathcal{G}$ by macro structures, i.e.,

$$A = \sum_{G\in\mathcal{G}} R_G^T A_G R_G \ . \tag{3.3}$$

The local matrices

$$A_F = \sum_{e\subset F} \sigma_{e,F} R_{F\mapsto e}^T A_e R_{F\mapsto e} \ , \tag{3.4a}$$

$$A_G = \sum_{F\subset G} \sigma_{F,G} R_{G\mapsto F}^T A_F R_{G\mapsto F} \ , \tag{3.4b}$$

we will refer to as *structure* and *macro structure matrices*, respectively. It is important to note that we will choose the non-negative scaling factors $\sigma_{e,F}$ and $\sigma_{F,G}$ in (3.4) in such a way that the assembling properties (3.2) and (3.3) are satisfied. This implies

$$\sum_{F\supset e} \sigma_{e,F} = 1 \quad \forall e\in\mathcal{T} \ , \tag{3.5a}$$

$$\sum_{G\supset F} \sigma_{F,G} = 1 \quad \forall F\in\mathcal{F} \ . \tag{3.5b}$$

Alternatively, any *macro structure matrix* A_G can also be assembled directly from element matrices, i.e.,

$$A_G = \sum_{e\subset G} \sigma_{e,G} R_{G\mapsto e}^T A_e R_{G\mapsto e}, \tag{3.6}$$

which requires the weights for the element matrices to satisfy the condition

$$\sum_{G\supset e} \sigma_{e,G} = 1 \quad \forall e\in\mathcal{T} \ . \tag{3.7}$$

The simplest choice of the scaling factors that ensures the conditions (3.5) and (3.7) is

$$\sigma_{e,F} = \frac{1}{\sum_{F'\supset e} 1} \ , \tag{3.8a}$$

$$\sigma_{F,G} = \frac{1}{\sum_{G'\supset F} 1} \ , \tag{3.8b}$$

$$\sigma_{e,G} = \frac{1}{\sum_{G'\supset e} 1} \ . \tag{3.8c}$$

Then the assembling property is transferred from $\mathcal{A}_\mathcal{F} := \{A_F : \ F\in\mathcal{F}\}$ to $\mathcal{A}_\mathcal{G} := \{A_G : \ G\in\mathcal{G}\}$, that is, $\mathcal{A}_\mathcal{G}$ satisfies (3.3).

Remark 3.5. *Note that in general it is not necessary that for instance $\sigma_{e,F_1} = \sigma_{e,F_2}$ if $e\subset F_1$ and $e\subset F_2$. However, we prefer to use the* balanced *distribution of the weights according to (3.8), which we experienced to result in the best approximation properties.*

Figure 1 illustrates the composition of one macro structure $G = G_{2\times 2}$ from 2×2 non-overlapping structures $F = F_{2\times 2}$ of size 2×2 elements (left picture), and an overlapping covering $\mathcal{G} = \mathcal{G}_{1/2}$ of a set $\mathcal{F}$ of $4\times 4 = 16$ structures F by $3\times 3 = 9$ macro structures G (right picture).

3.2. **Algorithm.** The algorithmic representation of the method of additive Schur complement approximation (ASCA) requires

- a set $\mathcal{D}$ of degrees of freedom (DOF) which is the union of a set $\mathcal{D}_\mathrm{c}$ of coarse degrees of freedom (CDOF) and its complement $\mathcal{D}_\mathrm{f} := \mathcal{D}\setminus\mathcal{D}_\mathrm{c}$ in $\mathcal{D}$;
- a non-overlapping or an overlapping covering $\mathcal{F}$ of $\mathcal{T}$;
- a set of structure matrices $\mathcal{A}_\mathcal{F} := \{A_F : \ F\in\mathcal{F}\}$ satisfying the assembling property (3.2).

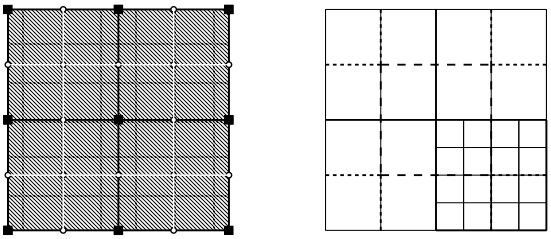


FIGURE 1. Macro structure $G_{2\times 2}$ composed of 2×2 non-overlapping structures $F_{2\times 2}$ (left); Covering $\mathcal{G}_{1/2}$ with an overlap of $1/2$ of the width of one macro structure (right picture)

Subsequently, a permutation of the rows and columns of A is performed according to a two-level partitioning of $\mathcal{D}$, i.e. the rows and columns corresponding to $\mathcal{D}_{\mathrm{f}} := \mathcal{D} \setminus \mathcal{D}_{\mathrm{c}}$ are numbered first and those related to $\mathcal{D}_{\mathrm{c}}$ are numbered last, as represented in (3.9)

$$A = A_h = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := \begin{bmatrix} A_{\mathrm{ff}} & A_{\mathrm{fc}} \\ A_{\mathrm{cf}} & A_{\mathrm{cc}} \end{bmatrix} \begin{matrix} \} \ \mathcal{D}_{\mathrm{f}} \\ \\ \} \ \mathcal{D}_{\mathrm{c}} \end{matrix} \tag{3.9}$$

Then if we denote the corresponding Schur complement by

$$S = A_{\mathrm{cc}} - A_{\mathrm{cf}} A_{\mathrm{ff}}^{-1} A_{\mathrm{fc}}, \tag{3.10}$$

the following algorithm can be used for the computation of an additive Schur complement approximation Q for the exact Schur complement S.

Algorithm 3.6 (Additive Schur complement approximation)**.** *ASCA($\mathcal{F}$, $\mathcal{A}_{\mathcal{F}}$, $\mathcal{D}_{\mathrm{f}}$, $\mathcal{D}_{\mathrm{c}}$)*

1. *Define a global two-level numbering for the set $\mathcal{D} = \mathcal{D}_{\mathrm{f}} \cup \mathcal{D}_{\mathrm{c}}$ of all DOF with consecutive numbering of the CDOF in the second block and apply the corresponding permutation to the matrix A, which results in (3.9).*
2. *Determine a (non-overlapping or overlapping) covering $\mathcal{G}$ of $\mathcal{F}$ and a set of scaling factors $\{\sigma_{F,G} : F \in \mathcal{F}\}$ satisfying (3.5b).*
3. *For all $G \in \mathcal{G}$ accomplish the following steps:*
   (a) *Determine a "local" two-level numbering of the DOF of G and assemble the corresponding macro structure matrix A_G according to (3.4b). Then*

$$A_G = \begin{bmatrix} A_{G:\mathrm{ff}} & A_{G:\mathrm{fc}} \\ A_{G:\mathrm{cf}} & A_{G:\mathrm{cc}} \end{bmatrix} \begin{matrix} \} \ \mathcal{D}_{G:\mathrm{f}} \\ \\ \} \ \mathcal{D}_{G:\mathrm{c}} \end{matrix} \tag{3.11}$$

   where $\mathcal{D}_{G:\mathrm{f}}$ and $\mathcal{D}_{G:\mathrm{c}}$ stand for the "local" sets of FDOF and CDOF.
   (b) *Compute the "local" Schur complement*

$$S_G = A_{G:\mathrm{cc}} - A_{G:\mathrm{cf}} A_{G:\mathrm{ff}}^{-1} A_{G:\mathrm{fc}}. \tag{3.12}$$

   (c) *Determine the "local-to-global" mapping for the CDOF in $\mathcal{D}_{G:\mathrm{c}}$, i.e., determine the global numbers of the "local" CDOF and define $R_{G:\mathrm{c}}$.*
4. *Assemble a global Schur complement approximation Q from the "local" contributions S_G:*

$$Q = \sum_{G\in\mathcal{G}} R_{G:\mathrm{c}}^T S_G R_{G:\mathrm{c}} \tag{3.13}$$

The following will describe three different examples for constructing an ASCA and furthermore illustrate the generality of this concept.

3.3. **Examples.** As already noted any ASCA is specified by its corresponding covering $\mathcal{G}$ and by the scaling factors $\sigma_{F,G}$ used in the assembly (3.4) of macro structure matrices from structure matrices.

For all three examples given below we consider a conforming FE discretization of Problem (2.1) based on bilinear elements where $\Omega = \Omega_h = (0,1)^2$, and $\mathcal{T}_h$ is a uniform mesh consisting of square elements with mesh size $h = h_x = h_y = 1/(N+1)$, i.e., $n = (N+1)^2$, where N is of the form $N = 2k$ and $k \in \mathbb{N}$. The sets $\mathcal{D}_\mathrm{f}$ and $\mathcal{D}_\mathrm{c}$ of FDOF and CDOF can be associated with the vertices of the elements (nodes) of $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, where $\mathcal{T}_h$ is derived from a uniform coarse mesh $\mathcal{T}_H$ by subdividing into four similar elements of size $h = H/2$ each coarse element of size H.

Example 1. Let $\mathcal{F} = \mathcal{F}_0 = \{F_{2\times 2}\}$ be the set of 2×2 structures giving a non-overlapping covering of Ω. Next, consider a non-overlapping covering $\mathcal{G} = \mathcal{G}_0$ of $\mathcal{F} = \mathcal{F}_0$ by macro structures $G_{2\times 2}$. The index 0 for $\mathcal{F}_0$ and $\mathcal{G}_0$ indicates that neither structures nor macro structures overlap in this example.

Example 2. Consider the overlapping covering $\mathcal{G} = \mathcal{G}_{1/2}$ for $\mathcal{F} = \mathcal{F}_0 = \{F_{2\times 2}\}$ by macro structures $G_{2\times 2}$, as illustrated in Figure 1. The set of structures provides a non-overlapping partition as in Example 1. The index 1/2 of $\mathcal{G}_{1/2}$ indicates that macro structures overlap with 1/2 of their width, see right picture of Figure 1. The reciprocal scaling factors $1/\sigma_{e,G}$ according to (3.8) are shown in the left picture of Figure 3 for a mesh of size 8×8. Note that due to the specific strategy of overlapping macro structures, these weights are constant on blocks of size 2×2 elements.

The purpose of the second example is to reveal that introducing an overlap in the covering $\mathcal{G}$ of $\mathcal{F}$ is the key to the robustness of the ASCA with respect to inter-element jumps of the PDE coefficient.

Example 3. In this example a covering $\mathcal{F} = \mathcal{F}_{1/2} = \{F_{3\times 3}\}$ of Ω is made by overlapping structures (with an "overlap of width 1/2"). Then consider the overlapping covering $\mathcal{G} = \mathcal{G}_{1/2}$ of $\mathcal{F} = \mathcal{F}_{1/2}$ by macro structures $G_{3\times 3}$ of size 3×3 (structures), as shown in Figure 2, i.e., neighboring structures $F \in \mathcal{F}_{1/2}$ overlap each other with half their width, and neighboring macro structures $G \in \mathcal{G}_{1/2}$ overlap each other with half their width as well. The reciprocal scaling factors $1/\sigma_{e,G}$ according to (3.8) are shown in the right picture of Figure 3 for a mesh of size 16×16.

The third example should demonstrate two things. First, in general, both, structures and macro structures may overlap, and second, by increasing the size of the structures (and hence that of the macro structures and their overlap) the quality of the ASCA improves but with the cost of a gradual loss of sparsity.

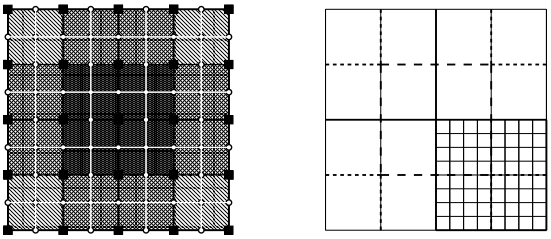


FIGURE 2. Macro structure $G_{3\times 3}$ composed of 3×3 overlapping structures $F_{3\times 3}$ (left); Covering $\mathcal{G}_{1/2}$ with an overlap of 1/2 of the width of one macro structure (right picture)

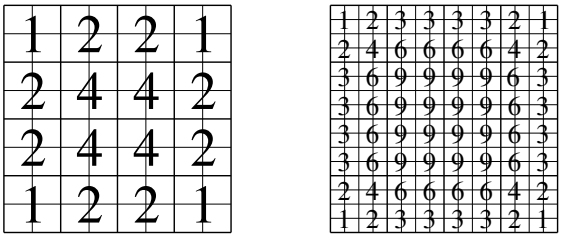


FIGURE 3. Reciprocal scaling factors $\sigma_{e,G}^{-1}$: Example 2 (left) and Example 3 (right picture)

4. Condition number estimates

For convenience we will use the notation $A \geq B$ for two symmetric positive semidefinite (SPSD) matrices A and B equivalentely to the statement of $A - B$ being SPSD in the remainder of this paper.

4.1. **Local estimates based on the CBS constant.** As it has been shown in [2] (see also [10]), for the case of non-overlapping macro structures (like in Example 1) the additive Schur complement approximation Q satisfies

$$(1 - \hat{\gamma}^2)S \leq Q \leq S \tag{4.1}$$

where $\hat{\gamma}$ is the constant in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality

$$|\boldsymbol{v}_1^T \hat{A}_{12} \boldsymbol{v}_2| \leq \hat{\gamma} \left(\boldsymbol{v}_1^T A_{11} \boldsymbol{v}_1 \, \boldsymbol{v}_2^T \hat{A}_{22} \boldsymbol{v}_2 \right)^{1/2}$$

for the two-level hierarchical basis (HB) matrix

$$\hat{A} = \begin{bmatrix} A_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = J^T A J \tag{4.2}$$

where $A = A_h$ is given by (3.9). This estimate in principal also applies to the case of overlapping macro structures, see Remark 4.8 below. The (classical) HB transformation results in the relation $\hat{A}_{22} = A_H$ where A_H is the coarse-level stiffness matrix arising from FE discretization of the original problem, e.g., (2.4), on a coarse mesh with mesh size H, where often one chooses $H = 2h$. The HB transformation matrix

$$J = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix} \tag{4.3}$$

in (4.2) corresponds to a particular choice of the off-diagonal block W in (4.3), which we will denote by $W = \hat{W}$. Note that typically $\hat{W}$ is a sparse matrix.

As is easily seen the exact Schur complement S is invariant to a transformation with a matrix J of the form (4.3), i.e.,

$$S = \hat{S} = \hat{A}_{22} - \hat{A}_{21} A_{11}^{-1} \hat{A}_{12}.$$

In contrast, the lower right block

$$\hat{A}_{22} = A_{22} + W^T A_{11} W + W^T A_{12} + A_{21} W$$

of $\hat{A}$ and thus also

$$1 - \hat{\gamma}^2 = \inf_{\boldsymbol{v}_2 \in V_2 \setminus \ker(\hat{A}_{22})} \frac{\boldsymbol{v}_2^T S \boldsymbol{v}_2}{\boldsymbol{v}_2^T \hat{A}_{22} \boldsymbol{v}_2}$$

clearly depend on the particular choice of W. It is therefore natural to raise the questions whether the estimate (4.1) is sharp, and whether there exists (a similar) transformation invariant estimate. We will answer both of these questions in the remainder of this subsection.

The following well-known lemma will be useful for our considerations [6] (see also [1, 10]).

Lemma 4.1. *Let A be an SPSD matrix of 2×2 block form for which the upper left block A_{11} is SPD, and let γ be the CBS constant associated with A. Then*

$$(1-\gamma^2)A_{22} \leq S \leq A_{22}.$$

Now let us consider the class of generalized hierarchical basis (GHB) matrices $\bar{\mathcal{A}} = \bar{\mathcal{A}}(A)$ that consists of all matrices $\bar{A}$ that can be obtained from A by a *compatible* basis transformation

$$\bar{A} = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \bar{J}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \bar{J} \tag{4.4}$$

where $\bar{J}$ is obtained from (4.3) by choosing $W{:=}\bar{W}$.

Definition 4.2. *A basis transformation of the form (4.4) is called compatible if*

$$\bar{A}_{22} = \bar{A}_{22}(\bar{W}) = [\bar{W}^T, I] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{W} \\ I \end{bmatrix} = \sum_{G \in \mathcal{G}} R_{G:\mathrm{c}}^T \bar{A}_{G:22} R_{G:\mathrm{c}}$$

and

$$\bar{A}_{G:22} = \bar{A}_{G:22}(\bar{W}_G) = [\bar{W}_G^T, I_G] \begin{bmatrix} A_{G:11} & A_{G:12} \\ A_{G:21} & A_{G:22} \end{bmatrix} \begin{bmatrix} \bar{W}_G \\ I_G \end{bmatrix}$$

where $\bar{W}_G = \bar{W}|_G := R_{G:\mathrm{f}} \bar{W} R_{G:\mathrm{c}}^T$ is the restriction of $\bar{W}$ to G for all $G \in \mathcal{G}$. Here $R_{G:\mathrm{f}}$ and $R_{G:\mathrm{c}}$ are obtained from R_G by deleting all its rows and columns corresponding to CDOF (for $R_{G:\mathrm{f}}$) and FDOF (for $R_{G:\mathrm{c}}$).

Then we have the following theorem.

Theorem 4.3. *Let A be an SPSD matrix in two-level block form and A_{11} be SPD. Further, let $\bar{A} \in \bar{\mathcal{A}}(A)$ be a GHB matrix arising from A via a compatible basis transformation, and let $\bar{\gamma}$ denote the related CBS constant. Then*

$$(1-\bar{\gamma}^2)S \leq Q \leq S = A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{4.5}$$

where Q is the ASCA corresponding to $\mathcal{G}$ and $\bar{\gamma} = \max_{G \in \mathcal{G}} \bar{\gamma}_G$.

Proof. The upper bound in (4.5) follows directly from the well-known minimization property of Schur complements, i.e.,

$$\begin{aligned}
\boldsymbol{v}_2^T Q \boldsymbol{v}_2 &= \boldsymbol{v}_2^T \left(\sum_{G \in \mathcal{G}} R_{G:\mathrm{c}}^T S_G R_{G:\mathrm{c}} \right) \boldsymbol{v}_2 = \sum_{G \in \mathcal{G}} \boldsymbol{v}_{G:\mathrm{c}}^T S_G \boldsymbol{v}_{G:\mathrm{c}} \\
&= \sum_{G \in \mathcal{G}} \min_{\boldsymbol{v}_{G:\mathrm{f}}} \begin{bmatrix} \boldsymbol{v}_{G:\mathrm{f}} \\ \boldsymbol{v}_{G:\mathrm{c}} \end{bmatrix}^T A_G \begin{bmatrix} \boldsymbol{v}_{G:\mathrm{f}} \\ \boldsymbol{v}_{G:\mathrm{c}} \end{bmatrix} = \sum_{G \in \mathcal{G}} \min_{\boldsymbol{v}_1} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T R_G^T A_G R_G \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} \\
&\leq \min_{\boldsymbol{v}_1} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T \left(\sum_{G \in \mathcal{G}} R_G^T A_G R_G \right) \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \min_{\boldsymbol{v}_1} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T A \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \boldsymbol{v}_2^T S \boldsymbol{v}_2.
\end{aligned}$$

In order to prove the lower bound, let $\bar{J}$ define an arbitrary compatible basis transformation of the form (4.4), i.e., let $W = \bar{W}$ satisfy Definition 4.2. Then in view of Lemma 4.1 we have

$$(1-\bar{\gamma}_G^2)\bar{A}_{G:22}(\bar{W}_G) \leq S_G \leq \bar{A}_{G:22}(\bar{W}_G) \quad \forall W_G \; \forall G \in \mathcal{G}$$

from which, by summation over all $G \in \mathcal{G}$, and using the compatibility assumption, it follows that

$$(1-\bar{\gamma}^2)\bar{A}_{22}(\bar{W}) \leq Q \leq \bar{A}_{22}(\bar{W})$$

where $\bar{\gamma} = \max_{G\in\mathcal{G}} \bar{\gamma}_G$. Finally, since $S \leq \bar{A}_{22}$ (by Lemma 4.1) it follows that $(1-\bar{\gamma}^2)S \leq Q$. □

The following corollary is an immediate consequence of Theorem 4.3.

Corollary 4.4. *Let W^* denote the matrix that provides the minimum energy coarse basis subject to the compatibility constraint (of local support), i.e., $A^* \in \bar{\mathcal{A}}(A)$ and $A^*_{G:22}(W^*_G) \leq \bar{A}_{G:22}(\bar{W}_G)$ for all $\bar{W}_G$ and for all $G \in \mathcal{G}$. Then*

$$(1-\gamma^{*2})S \leq Q \leq S \tag{4.6}$$

where $\gamma^* := \inf_{\bar{A}\in\bar{\mathcal{A}}(A)} \bar{\gamma}(\bar{A})$.

Remark 4.5. *As we see from (4.5) or (4.6), the ASCA approximates the exact global Schur complement from below, i.e., $Q \leq S$ whereas a coarse-grid operator of the form $A_H = P^T AP$ computed via the Galerkin relation, as well as A_H in variational multigrid typically approximate S from above, i.e., $S \leq A_H$.*

Remark 4.6. *Note that due to the definition of γ^* the bound (4.6) does no longer depend on the particular choice of W in (4.3) and thus is transformation invariant.*

Remark 4.7. *In many cases, (4.6) improves the corresponding bound (4.1). Typically this occurs when each macro structure contains more than 2^d elements, e.g., when an m-fold regular mesh refinement procedure is applied, which results in $H = mh$ for $m > 2$. Another situation in which the bound based on the CBS constant associated with the HB transformation in general is not sharp is when the discretization is performed using elements of order $p \geq 2$. If we consider a uniform mesh of mesh size $h_x = h_y = h$ and a coarse mesh with mesh size $H = 2h$, a discretization of the Laplace operator (equation (2.1a) with $\mathbf{a}(\mathbf{x}) = I$) using biquadratic elements results in $1-\hat{\gamma}_G^2 = 9(97-\sqrt{2369})/1408 \approx 0.308912$ whereas $1-{\gamma^*_G}^2 = 1485/3682 \approx 0.403313$. Using the FE space consisting of piecewise bilinear functions, however, one finds $1-\hat{\gamma}_G^2 = 1-{\gamma^*_G}^2 = 5/8$ and it can easily be seen that (4.1) is sharp in this case. The situation is similar for piecewise quadratic and piecewise linear conforming FE spaces, where in the first case (4.1) is improved by (4.6) whereas in the second case (4.1) is already sharp.*

Remark 4.8. *Both of the condition number estimates based on the CBS constant ((4.1) and (4.6)) also apply to ASCAs based on overlapping macro structures. Clearly, the definition of the classical HB transformation matrix (4.3) remains unchanged and for any reasonable covering $\mathcal{G}$ by overlapping macro structures G, the specific choice $W = \hat{W}$ will result in a compatible basis transformation. In general, the overlap of macro structures will result in a richer set of compatible transformations and hence one can expect an improvement of (4.1) by (4.6) in many cases.*

We will not go into details of constructing compatible basis transformations for overlapping coverings here but present an alternative technique of deriving condition number estimates instead.

4.2. **A robust uniform bound.** In this subsection we will prove a robust uniform condition number estimate for Example 2 from Section 3.3 that does not explicitely depend on the CBS constant. The presented local analysis purely relies on properties of Schur complements.

Only very few basic tools are needed, which we summarize below. For the sake of self-containedness, we also include the proofs of Lemma 4.9 and Lemma 4.11.

Lemma 4.9. *Let A and B be two SPSD matrices in two-level block form (relating to the same two-level partitioning $\mathcal{D} = \mathcal{D}_{\mathrm{f}} \oplus \mathcal{D}_{\mathrm{c}}$) satisfying $\underline{\alpha}A \leq B \leq \bar{\alpha}A$. Further, let A_{11} and B_{11}, the blocks corresponding to $\mathcal{D}_{\mathrm{f}}$, be SPD. Then the Schur complements $S_A = A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $S_B = B_{22} - B_{21}B_{11}^{-1}B_{12}$ satisfy $\underline{\beta}S_A \leq S_B \leq \bar{\beta}S_A$ where $[\underline{\beta}, \bar{\beta}] \subset [\underline{\alpha}, \bar{\alpha}]$.*

Proof. Under the assumptions of the lemma, the matrix A permits the block factorization

$$A = \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & S_A \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ & I \end{bmatrix},$$

and so does B. Hence, from $B \leq \bar{\alpha}A$ it follows that

$$\begin{aligned} \begin{bmatrix} B_{11} & \\ & S_B \end{bmatrix} &\leq \bar{\alpha} \begin{bmatrix} I & \\ -B_{21}B_{11}^{-1} & I \end{bmatrix} A \begin{bmatrix} I & -B_{11}^{-1}B_{12}+ \\ & I \end{bmatrix} \\ &= \bar{\alpha} \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} - B_{21}B_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & S_A \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} - B_{11}^{-1}B_{12} \\ & I \end{bmatrix}. \end{aligned}$$

Let $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$ be an arbitrary vector and $\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix} := \begin{bmatrix} \boldsymbol{v}_1 + (A_{11}^{-1}A_{12} - B_{11}^{-1}B_{12})\boldsymbol{v}_2 \\ \boldsymbol{v}_2 \end{bmatrix}$.
Then we have

$$\boldsymbol{v}_1^T B_{11}\boldsymbol{v}_1 + \boldsymbol{v}_2^T S_B \boldsymbol{v}_2 \leq \bar{\alpha}(\boldsymbol{w}_1^T A_{11}\boldsymbol{w}_1 + \boldsymbol{w}_2^T S_A \boldsymbol{w}_2).$$

Thus, for $\boldsymbol{v}_1 = (B_{11}^{-1}B_{12} - A_{11}^{-1}A_{12})\boldsymbol{v}_2$ we obtain

$$[(B_{11}^{-1}B_{12} - A_{11}^{-1}A_{12})\boldsymbol{v}_2]^T B_{11}(B_{11}^{-1}B_{12} - A_{11}^{-1}A_{12})\boldsymbol{v}_2 + \boldsymbol{v}_2^T S_B \boldsymbol{v}_2 \leq \bar{\alpha}\boldsymbol{v}_2^T S_A \boldsymbol{v}_2. \tag{4.7}$$

Since $B_{11} \geq 0$ the relation (4.7) shows that $S_B \leq \bar{\beta}S_A$ for some constant $\bar{\beta}$ satisfying $0 \leq \bar{\beta} \leq \bar{\alpha}$.

In a similar way one shows that $\underline{\beta}S_A \leq S_B$ for some constant $\underline{\beta}$ satisfying $\underline{\beta} \geq \underline{\alpha}$, which completes the proof of the lemma. □

The following corollary is an immediate consequence of Lemma 4.9.

Corollary 4.10. *Let A and B be as in Lemma 4.9. If $A \leq B$ then $S_A \leq S_B$.*

Moreover, we have the following inequality, which follows directly from the minimization property of Schur complements.

Lemma 4.11. *Let A and B be two SPSD matrices in two-level block form relating to the same two-level partitioning $\mathcal{D} = \mathcal{D}_{\mathrm{f}} \oplus \mathcal{D}_{\mathrm{c}}$. Let A_{11} and B_{11} be SPD and $S_A = A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $S_B = B_{22} - B_{21}B_{11}^{-1}B_{12}$ denote the two Schur complements. Then*

$$S_A + S_B \leq S_{A+B}.$$

Proof.

$$\begin{aligned} \boldsymbol{v}_2^T(S_A + S_B)\boldsymbol{v}_2 &= \boldsymbol{v}_2^T S_A \boldsymbol{v}_2 + \boldsymbol{v}_2^T S_B \boldsymbol{v}_2 \\ &= \inf_{\boldsymbol{v}_1} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T A \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} + \inf_{\boldsymbol{v}_1} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T B \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} \\ &\leq \inf_{\boldsymbol{v}_1} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T (A + B) \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = S_{A+B} \end{aligned}$$

□

As an example of local estimation of the relative condition number of the ASCA defined via equation (3.13) we will analyze Example 2 from Section 3.3 now.

In this analysis it will be convenient to interprete structures and macro structures as graphs and to associate FDOF and CDOF with the *fine* nodes (f-nodes) and *coarse* nodes (c-nodes) of these graphs (where *node* is a synonym for *vertex* of a graph). The edges of the graphs F or G correspond to nonzero off-diagonal entries in the related matrices A_F or A_G, respectively.

Coming back to Example 2 (from Section 3.3), first we note that each structure F has exactly one interior f-node. In the global mesh these f-nodes form an independent set and hence we can eliminate the corresponding unknowns independently. Performing this reduction step on the matrices A_F, for all $F \in \mathcal{F}$, the resulting local structure Schur complements, let us denote them by A'_F, can be used to assemble the related global Schur complement A', i.e.,

$$A' = \sum_{F \in \mathcal{F}} R_F^T A'_F R_F.$$

Now we start with this new set of SPSD structure matrices $\mathcal{A}'_{\mathcal{F}} := \{A'_F : F \in \mathcal{F}\}$ and we will show below (in the proof of Theorem 4.12) that there exists a set of SPSD auxiliary structure matrices $\mathcal{B}'_{\mathcal{F}} := \{B'_F : F \in \mathcal{F}\}$ and macro structure matrices $\mathcal{B}'_{\mathcal{G}} := \{B'_G : G \in \mathcal{G}\}$ with the following properties. First,

$$\frac{1}{c_G} A'_G \leq B'_G \leq A'_G \quad \forall G \in \mathcal{G}, \tag{4.8}$$

where $A'_G := \sum_{F \subset G} R_{G \mapsto F}^T A'_F R_{G \mapsto F}$. Second, our aim is to construct $\mathcal{B}'_{\mathcal{G}}$ in such a way that the related Schur complements have the assembling property

$$S_{B'} = \sum_{G \in \mathcal{G}} R_{G:c}^T S_{B'_G} R_{G:c}, \tag{4.9}$$

where $S_{B'}$ denotes the exact global Schur complement of the matrix

$$B' = \sum_{G \in \mathcal{G}} R_G^T B'_G R_G.$$

In the following we will show that the constants c_G in (4.8) satisfy $1 \leq c_G < \infty$ for all $G \in \mathcal{G}$. Then

$$c := \max_{G \in \mathcal{G}} c_G < \infty, \tag{4.10}$$

and the following theorem holds true.

Theorem 4.12. *Consider the discretization of Problem (2.4) based on conforming bilinear elements on a uniform mesh with mesh size h. If the overlapping covering $\mathcal{G} = \mathcal{G}_{1/2}$ has been chosen according to Example 2 then the additive Schur complement approximation Q on the mesh with mesh size $H = 2h$ is spectrally equivalent to the exact Schur complement S, i.e.,*

$$\frac{1}{c} S \leq Q \leq S. \tag{4.11}$$

The constant c in (4.11) does neither depend on the jumps of a piecewise constant coefficient $\alpha = \alpha_e$ nor on the mesh size h. In particular, in Example 2 the estimate (4.11) holds for $c = 4$.

Proof. The upper bound in (4.11) has already been deduced from the energy minimization property of the Schur complement in the proof of Theorem 4.3. It remains to prove the lower bound $S \leq cQ$. Let us denote by $S_{A'}$ the Schur complement of $A' = \sum_{G \in \mathcal{G}} R_G^T A'_G R_G$ where A'_G (and A') are obtained from A_G (and A) by static condensation of the unknowns corresponding to the interior

nodes of structures $F \subset G$ (and $F \in \mathcal{F}$). Assuming that there exists a set $\mathcal{B}'_{\mathcal{G}}$ of matrices B'_G with the properties (4.8)–(4.10) the left inequality in (4.11) can readily be seen from

$$\begin{aligned} S &= S_A = S_{A'} \leq cS_{B'} \\ &= c\sum_{G\in\mathcal{G}} R_{G:c}^T S_{B'_G} R_{G:c} \leq c\sum_{G\in\mathcal{G}} R_{G:c}^T S_{A'_G} R_{G:c} \\ &= \sum_{G\in\mathcal{G}} R_{G:c}^T S_{A_G} R_{G:c} = cQ \end{aligned} \tag{4.12}$$

where for the first inequality in (4.12) we have additionally used Corollary 4.10. Hence the proof of the theorem will be complete if we succeed to construct a set $\mathcal{B}'_{\mathcal{G}} = \{B'_G : G \in \mathcal{G}\}$ satisfying (4.8)–(4.10). Note that in Example 2 we have $A_F = \sum_{e\subset F} \alpha_e R_{F\mapsto e}^T A_e R_{F\mapsto e}$ where $\{e : e \subset F\} = \{e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}\}$. Without loss of generality, we may assume that $\alpha_{e_{i_4}} = 1$ and $\alpha_{e_{i_k}} = \epsilon_k \in (0, 1]$ for $1 \leq k \leq 3$. Then A_F depends on the three parameters ϵ_k, $k = 1, 2, 3$, only.

In order to accomplish our task, in a first step we construct one auxiliary structure matrix $B_{F'}$ for every matrix $A_{F'}$ satisfying the relation

$$\frac{1}{c_F} A'_F \leq B'_F \leq A'_F \quad \forall F \in \mathcal{F}. \tag{4.13}$$

This is done by vanishing in $A_{F'}$ all off-diagonal entries a'_{ij} for edges between two f-nodes i and j (with respect to the local numbering) and adding them to the corresponding diagonal entry a'_{ii}, that is, $a'_{ii} \leftarrow a'_{ii} + a'_{ij}$, and $a'_{ij} \leftarrow 0$ iff $\{i, j\} \subset \mathcal{D}_{F:\mathrm{f}}$. The graphs of the structure matrices $A_{F'}$ and $B_{F'}$ are illustrated in Figure 4(a) and Figure 4(b). For the matrices B'_F resulting from this procedure, it turns out that $c_F = 4$ is the smallest positive number for which

$$M := B'_F - \frac{1}{c_F} A'_F \geq 0 \tag{4.14}$$

independently of $\epsilon_k \in (0, 1]$ for $k = 1, 2, 3$ and thus (4.13) holds for $c_F = 4$. One way of proving inequality (4.14) for $c_F \geq 4$ is by representing M in the form

$$M = M_0 + \epsilon_1(M_{10} + \epsilon_1 M_{11} + \epsilon_2 M_{12} + \epsilon_3 M_{13}) + \epsilon_2(M_{20} + \epsilon_2 M_{22} + \epsilon_3 M_{23}) + \epsilon_3(M_{30} + \epsilon_3 M_{33})$$

and verifying that the involved parameter-independent matrices M_0, M_{i0}, and M_{ij}, for $i = 1, 2, 3$ and $j \geq i$ are all SPSD.

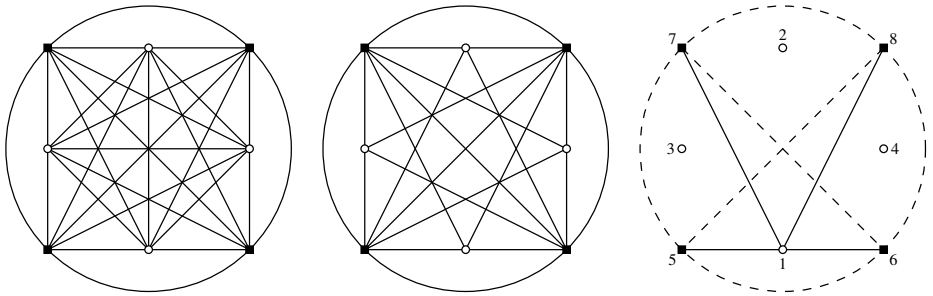
Finally, B'_F can be split into the sum of four SPSD matrices $B_F^{(1)}, \ldots, B_F^{(4)}$, i.e., $B'_F = \sum_{p=1}^4 B_F^{(p)}$, which can be used as building blocks in the construction of B'_G. We choose $B_F^{(p)}$, $1 \leq p \leq 4$, as the matrix arising from B'_F by scaling its lower right 4×4 diagonal block, which corresponds to the coarse nodes, with 1/4–the corresponding subgraph of $B_F^{(1)}$ is indicated by dashed lines in Figure 4(c)–and vanishing in the off-diagonal blocks of $B_F^{(p)}$ (by diagonal compensation) all entries that represent edges between any c-node and any f-node $k \neq p$. That is, the only f-c (or c-f) connections that remain involve the f-node p, see Figure 4(c) for the example of $B_F^{(1)}$. It is important to note that this part of the construction guarantees that the matrices $B_F^{(p)}$, $1 \leq p \leq 4$ can be used to assemble (auxiliary) macro structure matrices B'_G that satisfy the assembling property (4.9) for their Schur complements. If the structure G has the form

F_{i_3}	F_{i_4}
F_{i_1}	F_{i_2}

we define the auxiliary matrices via

$$B'_{F_{i_3}} := B^{(1)}_{F_{i_3}} + B^{(4)}_{F_3}, \qquad B'_{F_{i_4}} := B^{(1)}_{F_{i_4}} + B^{(3)}_{F_4},$$
$$B'_{F_{i_1}} := B^{(2)}_{F_{i_1}} + B^{(4)}_{F_1}, \qquad B'_{F_{i_2}} := B^{(2)}_{F_{i_2}} + B^{(3)}_{F_2},$$

according to the local numbering of nodes (DOF) in each structure F_{i_k} for $1 \le k \le 4$, see Figure 4(c). The resulting matrices $B'_G = \sum_{k=1}^{4} \sigma_{F_{i_k},G} R^T_{G \mapsto F_{i_k}} B'_{F_{i_k}} R_{G \mapsto F_{i_k}}$ satisfy (4.8) and (4.9) for $c_G = 4$ for all $G \in \mathcal{G}$, which completes the proof. □



(a) Graph of A'_F (b) Graph of B'_F (c) Graph of $B^{(1)}_F$

FIGURE 4. Graphs of structure matrices used in the proof of Theorem 4.12

5. Multilevel preconditioning

In this section we give an example of a multilevel block-factorization method in which the proposed ASCA can be incorporated. For the sake of a self-contained presentation we recall the definition of the nonlinear (variable-step) algebraic multilevel method first, and then summarize some convergence properties based on a spectral equivalence relation of a (linear) multiplicative two-level preconditioner.

5.1. **Nonlinear AMLI-cycle method.**

Let us consider a sequence of two-by-two block matrices

$$A^{(k)} = \begin{bmatrix} A^{(k)}_{11} & 0 \\ A^{(k)}_{21} & S^{(k-1)} \end{bmatrix} \begin{bmatrix} I & (A^{(k)}_{11})^{-1} A^{(k)}_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A^{(k)}_{11} & A^{(k)}_{12} \\ A^{(k)}_{21} & S^{(k-1)} + A^{(k)}_{21} (A^{(k)}_{11})^{-1} A^{(k)}_{12} \end{bmatrix} \tag{5.1}$$

associated with a (nested) sequence of meshes $\mathcal{T}_k$ where $k = \ell, \ell-1, \ldots, 1$, and ℓ denotes the level of the finest discretization (fine-mesh level). Here $S^{(k-1)}$ denotes the Schur complement in the exact block factorization (5.1) of $A^{(k)}$. Moreover, we define the following abstract (linear) multiplicative two-level preconditioner

$$\bar{B}^{(k)} = \begin{bmatrix} B^{(k)}_{11} & 0 \\ A^{(k)}_{21} & Q^{(k-1)} \end{bmatrix} \begin{bmatrix} I & (B^{(k)}_{11})^{-1} A^{(k)}_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} B^{(k)}_{11} & A^{(k)}_{12} \\ A^{(k)}_{21} & Q^{(k-1)} + A^{(k)}_{21} (B^{(k)}_{11})^{-1} A^{(k)}_{12} \end{bmatrix} \tag{5.2}$$

to $A^{(k)}$ at level $k = \ell, \ell-1, \ldots, 1$. Here $B^{(k)}_{11}$ is a preconditioner to $A^{(k)}_{11}$ and $Q^{(k-1)}$ is an approximation to $S^{(k-1)}$. In order to relate the two sequences $(A^{(k)})_{k=\ell,\ldots,1}$ and $(B^{(k)})_{k=\ell,\ldots,1}$ to each other we define

$$A^{(\ell)} := A_h = A, \tag{5.3}$$

where A_h is the stiffness matrix in (2.5), and set

$$A^{(k-1)} := Q^{(k-1)}, \quad k = \ell, \ldots, 1, \tag{5.4}$$

where $Q^{(k-1)}$ is the additive Schur complement approximation to the exact Schur complement $S^{(k-1)}$ in (5.1).

Next we define the nonlinear AMLI-cycle preconditioner $B^{(k)}[\cdot] : \mathbb{R}^{N_k} \mapsto \mathbb{R}^{N_k}$ for $1 \leq k \leq \ell$ by

$${B^{(k)}}^{-1}[\boldsymbol{y}] := U^{(k)} D^{(k)} [L^{(k)} \boldsymbol{y}], \tag{5.5}$$

where

$$L^{(k)} := \begin{bmatrix} I & 0 \\ -A_{21}^{(k)} {B_{11}^{(k)}}^{-1} & I \end{bmatrix}, \tag{5.6}$$

$U^{(k)} = {L^{(k)}}^T$, and

$$D^{(k)}[\boldsymbol{z}] = \begin{bmatrix} {B_{11}^{(k)}}^{-1} \boldsymbol{z}_1 \\ {Z^{(k-1)}}^{-1}[\boldsymbol{z}_2] \end{bmatrix}. \tag{5.7}$$

The (nonlinear) mapping ${Z^{(k-1)}}^{-1}[\cdot]$ is defined by

$$\begin{aligned} {Z^{(0)}}^{-1}[\cdot] &= {A^{(0)}}^{-1}, \\ {Z^{(k)}}^{-1}[\cdot] &:= {B^{(k)}}^{-1}[\cdot] \quad \text{if } \nu = 1 \text{ and } k > 0, \\ {Z^{(k)}}^{-1}[\cdot] &:= {B_\nu^{(k)}}^{-1}[\cdot] \quad \text{if } \nu > 1 \text{ and } k > 0, \end{aligned} \tag{5.8}$$

with

$${B_\nu^{(k)}}^{-1}[\boldsymbol{d}] := \boldsymbol{x}_{(\nu)}$$

where $\boldsymbol{x}_{(\nu)}$ is the ν-th iterate obtained when applying the generalized conjugate gradient (GCG) algorithm, see [3, 14], to the linear system $A^{(k)}\boldsymbol{x} = \boldsymbol{d}$ thereby using $B^{(k)}[\cdot]$ as a preconditioner and starting with the initial guess $\boldsymbol{x}_{(0)} = \boldsymbol{0}$. The vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_\ell)^T$ specifies how many inner GCG iterations are performed at each of the levels $k = 1, 2, \ldots, \ell - 1$, and $\nu_\ell > 0$ denotes the maximum number of orthogonal search directions at level ℓ (the fine-grid level). The additional GCG-type variable-step iterations on certain levels (those levels k for which $\nu_k > 1$) involve the use of the same type of variable-step preconditioner. We restrict our considerations to the case in which a fixed number ν of inner GCG-type iterations is performed at every intermediate level, that is, employing the vector $\boldsymbol{\nu} = \boldsymbol{\nu}_W = (\nu, \nu, \ldots, \nu, m_{\max})^T$ where the algorithm is restarted at level ℓ after every $m_{\max}$ iterations. This method is referred to as (nonlinear) ν-fold W-cycle AMLI method.

5.2. **Convergence properties.** Let us summarize the main results of the convergence analysis of the nonlinear AMLI method as presented in [8].

We are interested in the error reduction factor in A-norm, that is, if $\boldsymbol{x}_{(i)}$ denotes the i-th iterate generated by the nonlinear AMLI, we will show how to derive a bound of the form

$$\frac{\|\boldsymbol{x} - \boldsymbol{x}_{(i+1)}\|_A}{\|\boldsymbol{x} - \boldsymbol{x}_{(i)}\|_A} \leq \delta < 1. \tag{5.9}$$

The analysis in [8], which we recall here, assumes the spectral equivalence of the two-level preconditioners $\bar{B}^{(k)}$ and the matrices $A^{(k)}$ defined in (5.2) and (5.1), respectively, i.e., an approximation property of the form

$$\bar{B}^{(k)} \leq A^{(k)} \leq \vartheta_k \bar{B}^{(k)} \leq \vartheta \bar{B}^{(k)} \quad k = 1, 2, \ldots, \ell. \tag{5.10}$$

A slightly different approach to analyze the nonlinear AMLI-cycle method is based on the assumption that all fixed-length V-cycle multilevel methods from any coarse level $k-k_0$ to level k with exact solution at level $k-k_0$ are uniformly convergent in k with an error reduction factor $\delta_{k_0} \in [0,1)$, see [15, 18]. Both approaches, however, are based on the idea to estimate the deviation of the nonlinear preconditioner $B^{(k)}[\cdot]$ from an SPD matrix $\bar{B}^{(k)}$.

The analysis of the nonlinear AMLI cycle method, as included in the remainder of this section, makes use of the following GCG convergence rate estimate that has been proven in [14].

Theorem 5.1. *Let A, $\bar{B}$ be SPD matrices of size $N\times N$ and $B^{-1}[\cdot]$ a mapping from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let $\boldsymbol{d}$, $\boldsymbol{x}_{(0)}$ be vectors of $\mathbb{R}^N$ and let $\boldsymbol{r}_{(n)}$, $\boldsymbol{x}_{(n)}$ be the sequences of iterates and residuals generated by applying the GCG (FCG) algorithm with preconditioner $B[\cdot]$ to the linear system $A\boldsymbol{x}=\boldsymbol{d}$. If for any n,*

$$\frac{\|B^{-1}[\boldsymbol{r}_{(n)}]-\bar{B}^{-1}\boldsymbol{r}_{(n)}\|_{\bar{B}}}{\|\bar{B}^{-1}\boldsymbol{r}_{(n)}\|_{\bar{B}}} \le \epsilon_n < 1$$

then

$$\frac{\|\boldsymbol{x}-\boldsymbol{x}_{(n+1)}\|_A}{\|\boldsymbol{x}-\boldsymbol{x}_{(n)}\|_A} \le \sqrt{1-\frac{4\kappa(1-\epsilon_n)^2}{(\kappa+\epsilon_n^2(\kappa-1)+(1-\epsilon_n)^2)^2}}$$

where $\kappa=\kappa(\bar{B}^{-1}A)$.

For a proof of the following useful corollary we refer the reader to Reference [8].

Corollary 5.2. *Consider a matrix $A^{(k)}$ and its approximations $\bar{B}^{(k)}$ defined by (5.2) both of which are assumed to be SPD. If ν is a positive integer, $B^{(k)}[\cdot]$ denotes the preconditioner defined by (5.5)–(5.8), and*

$$\frac{\|\bar{B}^{(k)}{B^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}}{\|\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}} \le \epsilon_k < 1 \quad \forall \boldsymbol{v}\neq\boldsymbol{0} \tag{5.11}$$

then

$$\frac{\|A^{(k)}{B_\nu^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v}\|_{(A^{(k)})^{-1}}}{\|\boldsymbol{v}\|_{(A^{(k)})^{-1}}} \le \delta_k(\nu) \quad \forall \boldsymbol{v}\neq\boldsymbol{0} \tag{5.12}$$

where

$$\delta_k(\nu)=\left(1-\frac{4\kappa(1-\epsilon_k)^2}{(1+\kappa-2\epsilon_k+\kappa\epsilon_k^2)^2}\right)^{\nu/2} \tag{5.13}$$

and $\kappa=\kappa((\bar{B}^{(k)})^{-1}A^{(k)})$.

The next lemma relates the accuracy of the approximation of $A^{(k-1)}$ by the preconditioner $B_\nu^{(k-1)}[\cdot]$ at level $k-1$ to the accuracy of the approximation of $\bar{B}^{(k)}$ by $B^{(k)}[\cdot]$.

Lemma 5.3. *Consider the same operators as in Corollary 5.2. If*

$$\frac{\|A^{(k-1)}{B_\nu^{(k-1)}}^{-1}[\boldsymbol{u}]-\boldsymbol{u}\|_{(A^{(k-1)})^{-1}}}{\|\boldsymbol{u}\|_{(A^{(k-1)})^{-1}}} \le \delta_{k-1} \quad \forall \boldsymbol{u}\neq\boldsymbol{0}$$

then

$$\frac{\|\bar{B}^{(k)}{B^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}}{\|\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}} \le \delta_{k-1} \quad \forall \boldsymbol{v}\neq\boldsymbol{0}.$$

Proof. Let $\boldsymbol{v}$ be an arbitrary (but fixed) nonzero vector. First we observe that

$$\frac{\|\bar{B}^{(k)}{B^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}}{\|\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}}=\frac{(\bar{B}^{(k)}{B^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v})^T({B^{(k)}}^{-1}[\boldsymbol{v}]-(\bar{B}^{(k)})^{-1}\boldsymbol{v})}{\boldsymbol{v}^T(\bar{B}^{(k)})^{-1}\boldsymbol{v}}.$$

Let $\boldsymbol{y}=(\boldsymbol{y}_1^T,\boldsymbol{y}_2^T)^T=L^{(k)}\boldsymbol{v}$, where the partitioning of $\boldsymbol{y}$ is according to the splitting at level k. Then for ${Z^{(k-1)}}^{-1}[\cdot]={B_\nu^{(k-1)}}^{-1}[\cdot]$ we find

$$\bar{B}^{(k)}{B^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v}=L^{(k)^{-1}}\begin{bmatrix}\mathbf{0}\\ A^{(k-1)}{B_\nu^{(k-1)}}^{-1}[\boldsymbol{y_2}]-\boldsymbol{y_2}\end{bmatrix},$$

$$ {B^{(k)}}^{-1}[\boldsymbol{v}]-(\bar{B}^{(k)})^{-1}\boldsymbol{v}=L^{(k)^T}\begin{bmatrix}\mathbf{0}\\ {B_\nu^{(k-1)}}^{-1}[\boldsymbol{y_2}]-(A^{(k-1)})^{-1}\boldsymbol{y_2}\end{bmatrix},$$

and

$$(\bar{B}^{(k)})^{-1}\boldsymbol{v}=L^{(k)^T}\begin{bmatrix}(B_{11}^{(k)})^{-1}\boldsymbol{y_1}\\ (A^{(k-1)})^{-1}\boldsymbol{y_2}\end{bmatrix}.$$

Hence,

$$\begin{aligned}&\frac{\|\bar{B}^{(k)}{B^{(k)}}^{-1}[\boldsymbol{v}]-\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}}{\|\boldsymbol{v}\|_{(\bar{B}^{(k)})^{-1}}}\\ =\;&\frac{(A^{(k-1)}{B_\nu^{(k-1)}}^{-1}[\boldsymbol{y_2}]-\boldsymbol{y_2})^T({B_\nu^{(k-1)}}^{-1}[\boldsymbol{y_2}]-(A^{(k-1)})^{-1}\boldsymbol{y_2})}{\boldsymbol{y_1}^T(B_{11}^{(k)})^{-1}\boldsymbol{y_1}+\boldsymbol{y_2}^T(A^{(k-1)})^{-1}\boldsymbol{y_2}}\\ \le\;&\frac{\|A^{(k-1)}{B_\nu^{(k-1)}}^{-1}[\boldsymbol{y_2}]-\boldsymbol{y_2}\|_{(A^{(k-1)})^{-1}}}{\|\boldsymbol{y_2}\|_{(A^{(k-1)})^{-1}}},\end{aligned}$$

which completes the proof. □

The main convergence result is stated in the following theorem (cf. Theorem 5.5 in [8]).

Theorem 5.4. *Consider the linear system $A^{(\ell)}\boldsymbol{x}=\boldsymbol{d}^{(\ell)}$ where $A^{(\ell)}$ is an SPD stiffness matrix, and, let $\boldsymbol{x}_{(i)}$ be the sequence of iterates generated by the nonlinear AMLI algorithm. Further, assume that the approximation property (5.10) holds. If ν, the number of inner GCG iterations at every coarse level (except the coarsest where $A^{(0)}=Q^{(0)}$ is inverted exactly), is chosen such that*

$$\delta(\nu):=\left(1-\frac{4\vartheta(1-\epsilon)^2}{(1+\vartheta-2\epsilon+\vartheta\epsilon^2)^2}\right)^{\nu/2}\le\epsilon \tag{5.14}$$

for some positive $\epsilon<1$ then

$$\frac{\|\boldsymbol{x}-\boldsymbol{x}_{(i+1)}\|_{A^{(\ell)}}}{\|\boldsymbol{x}-\boldsymbol{x}_{(i)}\|_{A^{(\ell)}}}\le\sqrt{1-\frac{4\vartheta(1-\epsilon)^2}{(1+\vartheta-2\epsilon+\vartheta\epsilon^2)^2}}=\delta(1)=:\delta<1. \tag{5.15}$$

Proof. From the definition of ${B^{(k)}}^{-1}[\cdot]$, see (5.5)–(5.8), it follows that in the first step of recursion we have

$${B^{(1)}}^{-1}[\cdot]=(\bar{B}^{(1)})^{-1}$$

and thus (5.11) holds for $k=1$ and $\epsilon_1=0$. Then Corollary 5.2 shows that the inequality (5.12) is true for $\delta_1(\nu)$ given by (5.13) using $\epsilon_1=0$ and $\kappa=\vartheta$ where ϑ is the constant from the approximation property (5.10). Next, Lemma 5.3 shows that (5.11) holds for $k=2$ and $\epsilon_2\le\delta_1(\nu)$ where $\delta_1(\nu)$

is chosen according to (5.13). Thus (5.12) is true for $\delta_1(\nu)$ by Corollary 5.2. Repeating these arguments we conclude that (5.11) and (5.12) hold true for $\delta_k(\nu)$ given by (5.13) where

$$\epsilon_k \le \delta_{k-1}(\nu) \le \left(1 - \frac{4\vartheta(1-\epsilon_{k-1})^2}{(1+\vartheta-2\epsilon_{k-1}+\vartheta\epsilon_{k-1}^2)^2}\right)^{\nu/2} \tag{5.16}$$

for any $k > 1$. Moreover, since the right-hand side of (5.16) approaches zero when ν increases, the sequence $(\epsilon_k)_{k=1,2,\dots}$ is uniformly bounded by some $\epsilon < 1$ if ν is sufficiently large. Assuming that $\epsilon_{k-1} \le \epsilon$ we find from (5.16) and (5.14) that $\epsilon_k \le \delta(\nu) \le \epsilon$ for all $k \ge 1$. But then, since (5.11) also holds for $k = \ell$ and $\epsilon_\ell = \epsilon$, we have

$$\frac{\|{B^{(\ell)}}^{-1}[\boldsymbol{r}_{(n)}] - (\bar{B}^{(\ell)})^{-1}\boldsymbol{r}_{(n)}\|_{\bar{B}^{(\ell)}}}{\|(\bar{B}^{(\ell)})^{-1}\boldsymbol{r}_{(n)}\|_{\bar{B}^{(\ell)}}} = \frac{\|\bar{B}^{(\ell)}{B^{(\ell)}}^{-1}[\boldsymbol{r}_{(n)}] - \boldsymbol{r}_{(n)}\|_{(\bar{B}^{(\ell)})^{-1}}}{\|\boldsymbol{r}_{(n)}\|_{(\bar{B}^{(\ell)})^{-1}}} \le \epsilon < 1$$

and thus Theorem 5.1 yields the desired result. □

In particular, for $\nu \ge 2$ condition (5.14) is satisfied if

$$1 - \epsilon \le \frac{4\vartheta(1-\epsilon)^2}{(1+\vartheta-2\epsilon+\vartheta\epsilon^2)^2},$$

that is, if

$$\vartheta \le f(\epsilon) = \frac{1 - \epsilon^2 + 2\epsilon^3 + 2\sqrt{\epsilon - 2\epsilon^2 + 3\epsilon^3 - 2\epsilon^4}}{(1+\epsilon^2)^2}. \tag{5.17}$$

Remark 5.5. *Note that the relative condition number $\kappa({Q^{(k-1)}}^{-1}S^{(k-1)})$ affects the approximation property (5.10). In the simplest case in which the multiplicative two-level preconditioner (5.2) is considered under the assumption $B_{11}^{(k)} = A_{11}^{(k)}$ this results in $\vartheta = \kappa({Q^{(k-1)}}^{-1}S^{(k-1)})$.*

6. Numerical results

In the first set of numerical experiments we consider the model problem (2.1) with a scalar coefficient $\alpha = \alpha_e = 10^{-p}$ where $p \in \{0, 1, \dots, q\}$ is uniformly distributed, that is, the coefficient is log-uniformly distributed. The jumps that appear across element interfaces on the fine mesh are up to q orders of magnitude.

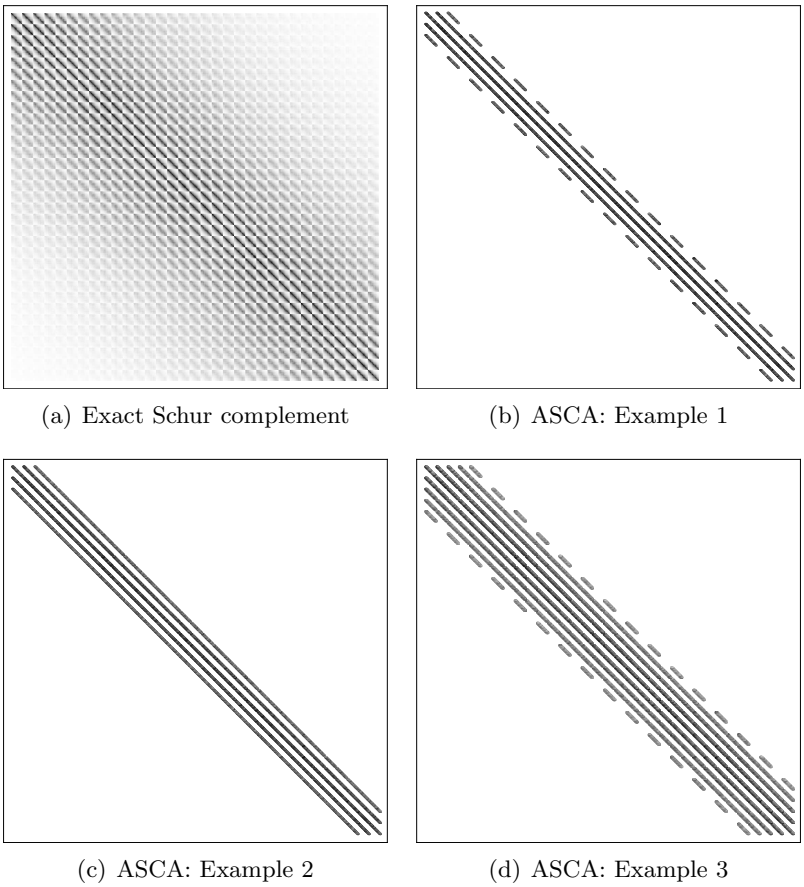
First we compare the (effective) relative condition numbers for the three different additive Schur complement approximations described in Examples 1 to 3. The results are summarized in Table 1. We observe that while in case of Example 1 the relative condition number deteriorates for increasing jumps, the overlapping coverings by macro structures, as used in Examples 2 and 3, are the key to keep the ASCA spectrally equivalent to the exact Schur complement independently of the jumps. As illustrated in Figure 5 this robustness comes at the price of a moderate increase of the number of nonzero entries in the approximation Q. For a mesh consisting of 64×64 elements, the matrix Q_1 depicted in Figure 5(b) has 16641 nonzero entries whereas the approximation Q_2, whose nonzero pattern is shown in Figure 5(c), has 25281 nonzeros. In general, the number of nonzero entries of the ASCA according to Example 2 is bounded from above by $25 \times \dim(S)$, which amounts to $25 \times 1089 = 27225$ in this example.

The next numerical experiment confirms that the two-level preconditioner in which we replace only the Schur complement S by its approximation Q_2 according to Example 2 is uniform. As we see from Table 2 the number of preconditioned conjugate gradient iterations to achieve a prescribed residual reduction is (nearly) constant when varying the problem size (mesh parameter h) and the contrast of the coefficient (jump parameter q).

Since in practice it is not feasible to invert the pivot block A_{11} of the two-level preconditioner exactly, we study the effect of using different preconditioners B_{11} for A_{11} in combination with the

κ	Example 1	Example 2	Example 3
$q=0$	1.3	1.3	1.1
$q=1$	2.1	1.6	1.2
$q=2$	6.9	1.9	1.4
$q=3$	31.4	2.1	1.5
$q=4$	101.6	2.4	1.6
$q=5$	167.9	2.6	1.6
$q=6$	403.2	2.7	1.7
$q=7$	1530.3	2.6	1.6
$q=8$	23439.7	2.6	1.7

TABLE 1. $\kappa(Q_i, S)$ for a uniform mesh of 64×64 elements: Log-uniformly distributed coefficient



(a) Exact Schur complement (b) ASCA: Example 1

(c) ASCA: Example 2 (d) ASCA: Example 3

FIGURE 5. Nonzero pattern of additive Schur complement approximations

Schur complement approximations Q_2 and Q_3 (from Examples 2 and 3). We consider three different preconditioners $B_{11} \in \{B_{11}^{\mathrm{I}}, B_{11}^{\mathrm{II}}, B_{11}^{\mathrm{III}}\}$ that preserve the row sums of A_{11}, i.e.,

$$A_{11}\mathbf{1} = B_{11}^{i}\mathbf{1} \quad \text{for } i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \tag{6.1}$$

	Two-level convergence: Example 2			
k_0	$h = 1/32$	$h = 1/64$	$h = 1/128$	$h = 1/256$
$q = 0$	10	10	10	10
$q = 1$	10	10	10	10
$q = 2$	10	10	10	11
$q = 4$	10	11	12	12
$q = 6$	10	11	12	12
$q = 8$	10	11	12	12

TABLE 2. Number of iterations k_0 for residual reduction by 10^8: Log-uniformly distributed coefficient

which is a desirable property in this setting, see [9, 13]. The preconditioner B_{11}^{I} is defined as the tridiagonal part of A_{11} with modification of the diagonal entries (diagonal compensation) such that (6.1) is fulfilled. B_{11}^{II} is obtained from modified incomplete LU factorization of A_{11} where fill-in terms in the triangular factors L and U are allowed only in those positions where A_{11} has nonzero entries. Finally, B_{11}^{III} is a preconditioner based on local exact factorization with a posteriori modification of the diagonal (of U) in order to satisfy (6.1); This method has been introduced in [9] and is a natural choice in the present context since the local LU factorizations can be retrieved in the course of computing the local Schur complements. Here the macro structure matrices $A_G \in \mathcal{A}_G$ are factorized exactly in the first step of the procedure. The convergence results for the related two-level preconditioners are summarized in Table 3. The mesh consists of 256×256 elements in this experiment.

	Example 2			Example 3		
B_{11}	B_{11}^{I}	B_{11}^{II}	B_{11}^{III}	B_{11}^{I}	B_{11}^{II}	B_{11}^{III}
$q = 0$	21	10	10	22	7	6
$q = 1$	21	11	11	22	8	7
$q = 2$	22	11	11	23	9	7
$q = 4$	22	13	13	23	10	8
$q = 6$	22	14	13	23	10	8
$q = 8$	22	13	13	23	10	8

TABLE 3. Number of iterations k_0 for residual reduction by 10^8: Log-uniformly distributed coefficient

Next we consider a nonlinear W-cycle algebraic multilevel iteration method based on the two-level preconditioner with $B_{11} = B_{11}^{\mathrm{III}}$ and ASCA according to Example 3. In the first multilevel experiment we vary the mesh size h and the jump parameter q. The finest mesh consists of $N \times N$ elements and the number of levels ℓ is chosen in such a way that the coarsest mesh has 8×8 elements in each case, i.e., $N = 16, 32, \ldots, 512$, which corresponds to $\ell = 2, 3, \ldots, 7$. As it can be seen from

Table 4 we observe uniform convergence. Note that the number of arithmetic operations for the set-up and for the application of the AMLI preconditioner grows linearly with the size of the problem, which finally results in an optimal order solution process.

W-cycle	AMLI convergence: Example 3					
$B_{11} = B_{11}^{\mathrm{III}}$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$q = 0$	5	6	6	6	6	6
$q = 1$	6	6	6	7	7	7
$q = 2$	6	7	7	7	7	7
$q = 3$	6	7	7	8	8	8
$q = 4$	6	7	8	8	8	8
$q = 5$	7	7	8	8	9	9
$q = 6$	7	8	8	9	9	9
$q = 7$	7	8	8	9	9	9
$q = 8$	7	8	8	9	9	9

TABLE 4. Number of outer iterations for residual reduction by 10^8: Log-uniformly distributed coefficient

In the second multilevel experiment we fix the mesh size and consider a discretization using 256×256 elements. The number of levels is varied from $\ell = 2$ to $\ell = 6$. The coefficient distribution that we use in this numerical test has been taken from Reference [5]. The main difference to the

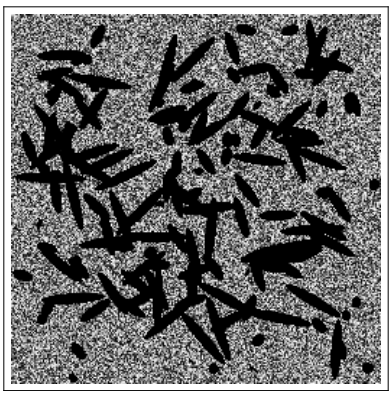


FIGURE 6. Coefficient distribution with random background and randomly distributed inclusions (see Figure 6.2 and Table 6.6 in [5])

previous setting of a log-uniformly distributed coefficient is that in this experiment we have islands with a high diffusion coefficient $\alpha_e = \alpha_{\max}$ on a macro scale and inbetween a highly oscillatory coefficient $\alpha_e = 10^{p\,\mathrm{rand}}$ with $p \in \{1, 2, \ldots, q\}$, i.e., $\alpha_{\max}/\alpha_{\min} \approx 10^q$, see Figure 6. Again we observe uniform convergence at optimal order of computational complexity, see Table 5.

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W-cycle	AMLI convergence: Example 3				
$B_{11} = B_{11}^{\mathrm{III}}$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
$q = 1$	6	6	6	6	6
$q = 2$	7	7	7	7	7
$q = 3$	7	7	7	7	7
$q = 4$	8	7	8	8	8
$q = 5$	8	8	8	8	8
$q = 6$	8	8	8	8	8

TABLE 5. Number of iterations k_0 for residual reduction by 10^8: Random background and randomly distributed inclusions as illustrated in Figure 6

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