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On BPS type domain decomposition preconditioner for finite element discretizations of 3-d elliptic equations

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On BPS type domain decomposition preconditioner for finite element discretizations of 3-d elliptic equations*†

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Abstract

BPS is a well known an efficient and rather general domain decomposition Dirichlet-Dirichlet type preconditioner, suggested in the famous series of papers Bramble, Pasciak and Schatz (1986-1989). Since then, it has been serving as the origin for the whole family of domain decomposition Dirichlet-Dirichlet type preconditioners-solvers as for h so hp discretizations of elliptic problems. For its original version, designed for h discretizations, the named authors proved the bound $\mathcal{O}(1+\log^2 H/h)$ for the relative condition number under some restricting conditions on the domain decomposition and finite element discretization. Here H/h is the maximal relation of the characteristic size H of a decomposition subdomain to the mesh parameter h of its discretization. It was assumed that subdomains are images of the reference unite cube by trilinear mappings. Later similar bounds related to h discretizations were proved for more general domain decompositions, defined by means of coarse tetrahedral meshes. These results, accompanied by the development of some special tools of analysis aimed at such type of decompositions, were summarized in the book of Toselli and Widlund (2005). This paper is also confined to h discretizations. We further expand the range of admissible domain decompositions for constructing BPS preconditioners, in which decomposition subdomains can be convex polyhedrons, satisfying some conditions of shape regularity. We prove the bound for the relative condition number with the same dependence on H/h as in the bound given above. In the part, related to the analysis of the key problem of interface preconditioning, our technical tools are an expansion of those used

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by Bramble, Pasciak and Schatz to more general geometries. The proof of the so called abstract bound for the relative condition number of the domain decomposition preconditioner is produced in a traditional, but more standardized way.

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1 Introduction

In the famous series of papers on the method of domain decomposition, Bramble, Pasciak & Schatz [3]-[6] presented an efficient and rather general DD (domain decomposition) preconditioner for h finite element discretizations of 3-d elliptic partial differential equations. Since then, it is commonly referred as BPS preconditioner with the abbreviation coming from first letters in surnames of the authors. BPS preconditioner has been the origin of the whole family of DD (domain decomposition) Dirichlet-Dirichlet type efficient preconditioners-solvers as for h so hp discretizations of elliptic problems. At first, it was expanded to more general h discretizations and domain decompositions with the state of art in this area well reflected in recent books by [28] Smith, Bjorstad & Gropp [33] and Toselli & Widlund [34], see also the review paper by Korneev & Langer [18]. Vast bibliography of related papers can be also found in these publications. For the developments of fast DD preconditioners-solvers of BPS type for hp discretizations of 3-d elliptic problems, apart from the cited books we refer to Pavarino & Widlund [29]-[31], Casarin [10], Korneev, Langer & Xanthis [19]-[20], Korneev & Rytov [22]-[24].

As we noted, generalizations of BPS preconditioners were related not only to the types of finite element discretizations, but also to domain decompositions. Bramble *et al.* assumed that subdomains are images of the reference unite cube by trilinear mappings, see A.1-A.4 in [6, p. 4]. Respectively, the discretization mesh of each subdomain was assumed to be a rectangular mesh on the reference cube. Toselli & Widlund [34] considered decompositions, which are obtained by means of coarser tetrahedral meshes, see Assumption 4.3, p. 90, and more general finite element discretization meshes. Decompozitions, studied by the latter authors, required development of new technical tools for analysis of some components of DD algorithms. In this paper, we widen the family of domain decompositions to which BPS type preconditioners can be efficiently applied. It is assumed that there is a finite number of the *reference convex polyhedrons* $\tau^\circ_\varkappa$, $\varkappa = 1, 2, \dots, \varkappa_\circ$, which have plain faces, are shape regular in definite sense, made specific in Section 2.1, and have diameters equal to unity. We consider decompositions

$$\overline{\Omega} = \bigcup\nolimits_{j=1}^{J} \overline{\Omega}_j \tag{1.1}$$

into subdomains Ω_j, which are images relative to corresponding reference polyhedrons $\tau^\circ_{\varkappa(j)}$ by the mappings $x = \Upsilon_j(y) : \overline{\tau}^\circ_{\varkappa(j)} \to \overline{\Omega}_j$, $\Upsilon_j(y) \in [L^1_\infty(\overline{\tau}^\circ_{\varkappa(j)})]^3$. These mappings can be characterized by scaling parameters, controlling the changes of lengths and angles and satisfy compatibility conditions for the finite element discretization and the domain decomposition. Since the mappings are continuous, in each boundary $\partial\Omega_j$ one can distinguish faces, edges and vertices, which are uniquely defined as the images of faces, edges and vertices of the polyhedron $\tau^\circ_{\varkappa(j)}$. For simplicity, for these polyhedrons we use also the notation $\tau_{0,j} = \tau^{\circ}_{\varkappa(j)}$. Compatibility of the decomposition assumes the property:

$\mathcal{A}$.1. *For each pair $i \neq j$ the intersection $\overline{\Omega}_i \cap \overline{\Omega}_j$ is or empty or common for the pair of subdomains face, or edge or vertex.*

Under these general conditions, finite elements of the FE (finite element) assemblage can be curvilinear. Nevertheless, for the FE mesh, we use the term *triangulation* and assume that the following natural requirement is fulfilled:

$\mathcal{A}$.2. *Let $\mathcal{T}_j$ and $\mathcal{V}(\Omega_j) \subset H^1(\Omega_j)$ are triangulations $\mathcal{T}_j$ of subdomains Ω_j and corresponding finite element spaces. Then they define the triangulation $\mathcal{T}$ of the domain Ω and the finite element space $\mathcal{V}(\Omega) \subset H^1(\Omega)$ in such a way that $\mathcal{T}_j$ and $\mathcal{V}(\Omega_j)$ are the restrictions to Ω_j of $\mathcal{T}$ and $\mathcal{V}(\Omega)$, respectively.*

In Subsection 2.1 we will add specific details pertaining the requirements to the regularity of decompositions, used in this paper.

Suppose, we need to solve the problem

$$-\nabla \cdot \varrho \nabla u = f\,, \qquad \varrho = \varrho_j = \text{const} > 0 \quad \text{for} \quad x \in \Omega_j\,, \qquad u\big|_{\partial\Omega} = 0\,, \tag{1.2}$$

which weak formulation is: find $u \in \overset{\circ}{H}{}^1(\Omega)$ satisfying the integral identity

$$a_\Omega(u, v) = (f, v)_\Omega\,, \qquad u, \forall\, v \in \overset{\circ}{H}{}^1(\Omega)\,, \tag{1.3}$$

where

$$a_\Omega(w, v) \equiv \int_\Omega \varrho \nabla w \cdot \nabla v dx\,, \qquad (w, v)_\Omega \equiv \int_\Omega wv dx \qquad \forall\, w, v \in H^1(\Omega)\,,$$

and the definitions of the spaces $H^1(\Omega)$ and $\overset{\circ}{H}{}^1(\Omega)$ are reminded at the end of the Introduction.

The system of FE (finite element) algebraic equations

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{1.4}$$

is equivalent to the integral identity for the FE solution u_{fem}

$$a_\Omega(u_{\text{fem}}, v) = (f, v)_\Omega\,, \qquad u_{\text{fem}}, \forall\, v \in \mathcal{V}(\Omega)\,, \tag{1.5}$$

on the subspace $\mathcal{V}(\Omega) \subset \overset{\circ}{H}{}^1(\Omega)$ of finite element functions. For obtaining an efficient iterative solver for (1.4), it is necessary to create an efficient preconditioner-solver for the FE stiffness matrix $\mathbf{K}$. We will construct DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ with the relative condition number satisfying the bound

$$\text{cond}[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}] \leq c(1 + \log^2 1/\hbar)\,, \tag{1.6}$$

where $\hbar$ is the parameter, characterizing the finite element mesh, and c is the constant independent of $\hbar$ and ϱ. This bound is of the same form as in Bramble *et al.* [6] and Toselli & Widlund [34], and it testifies that $\boldsymbol{\mathcal{K}}$ is the preconditioner almost optimal in the respect of the relative condition number.

Suppose, the system of algebraic equations $\boldsymbol{\mathcal{K}}\mathbf{v} = \boldsymbol{\phi}$ with any vector $\boldsymbol{\phi}$ can be solved for $\mathcal{O}(\hbar^{-3}(1+\log 1/\hbar)^\gamma)$ arithmetic operations with γ of the order one. Then, according to (1.6), the system (1.4) can be solved, *e.g.,* by the preconditioned conjugate gradient method, for $\mathcal{O}(\hbar^{-3}(1+$

$\log 1/\hbar)^{\gamma+1}$) arithmetic operations, *i.e.,* we have the DD solver of almost optimal/linear numerical complexity. As we will see from the later description of the BPS type preconditioner $\boldsymbol{\mathcal{K}}$, it contains five components or modules, which completely define its arithmetical cost. Among them practically only one, commonly referred as the face component, requires additional optimization in order the above bounds of the arithmetical work were not violated. For the rest four there are a few suggestions in the literature the use of which do not compromise these bounds. However, in this paper, we do not elaborate on complete specifications of all components and concentrate on the analysis of the relative condition number.

The proof of (1.6) is separated in the paper in two parts. Formally, the preconditioner, similarly looking to the DD preconditioner, can be defined for the general s.p.d. (symmetric positive definite) matrix. The bound for the relative condition number of such preconditioner is conventionally termed the *abstract bound*, since it does not reflect properties of the boundary value problem and its discretization. The derivation of such a bound, convenient for the further use, makes first part of the proof. In second part, we establish the dependence of the values, entering the abstract bound on the properties of the boundary value problem, its discretization and decomposition. In general setting, abstract estimates for the boundaries of the relative spectrum of DD preconditioners are traditionally obtained by means of the space decomposition approach, *i.e.*, as a direct consequence of the space decomposition stability, see for this notion, *e.g.*, Zhang [39, 40] and the survey paper by Korneev & Langer [18]. It has been developing in a numerous papers with the results and bibliography well reflected in the cited above books and papers. We refer here additionally only to the pioneering papers of Matsokin & Nepomnyaschikh [26] and Lions [25]. In this paper, we employ a purely algebraic approach, based on Theorem 3.1 and Corollaries 3.1 and 3.2. It seems to be the fastest way to find out main factors influencing the quality of the DD preconditioner. In second part of the proof, the most involved is the analysis of consequences of splitting subdomain faces in the inter-subdomain boundary Schur complement precondtitioner – which is a module of the DD preconditioner being designed – and of some other features of this preconditioner. At that, the difference of our assumptions on the domain decomposition from ones used earlier has most significant implications in the proof. Among some seemingly possible ways of analysis in the second part of the proof, we choose to develop further the approach of Bramble *et al.* [6], adapting it to more general domain decompositions.

The DD preconditioner-solver, we deal with in this section, belongs to the Dirichlet-Dirichlet type and possesses several advantages. At each primal DD iteration, it reduces solution of the finite element system of algebraic equations to solution of local problems on suddomains of decomposition, local problems on faces, also local prolongation and restriction operations, and the wire-basket problem, which is a single global problem. Only computational properties of the latter problem are influenced by jumps of the coefficient ϱ on the inter-subdomain boundary, but as a rule it has a much smaller dimension, than the initial finite element problem. DD algorithm, can be treated as having three or four hierarchical stages, depending on the arrangement of the wire basket solver. At the corresponding stages of it, solution of local problems and local operations of prolongations/restrictions can be performed in parallel.

The paper is arranged as follows. Section 2 is primarily devoted to the description of the structure and main components of DD preconditioner, although in this section some components are analyzed alongside with their specification. Our main assumptions on the type of the domain decompositions and finite element discretizations under consideration are formulated in

Subsection 2.1. Subsection 2.2 outlines the general structure of the DD preconditioner, whereas Subsections 2.3–2.5 present respectively

preconditioners for solvers of local Dirichlet problems on subdomains and prolongation operators from inter-subdomain boundary,

arrangement of the preconditioner for the subproblem arising on faces of the inter-subdomain boundary, and

the preconditioner-solver for the subproblem arising on the wire basket of the decomposition, which is the union of edges and vertices of subdomains of decomposition.

The abstract bound for the relative condition number is derived in Subsection 3.1, while Theorem 3.3, representing the main result of the paper is proved in Subsection 3.2. Subsection 3.3 contains some subsidiary results, used at the proof of the main bounds.

Let us underline that principle features of components of the BPS preconditioner have not changed since time of its appearance. Their descriptions are found in many papers and several books, see, *e.g.,* Toselli & Widlund [34]. We repeat expositions of the components for completeness, since the purpose of the paper is to justify them under different conditions.

Let us list some notations used in the paper.

Additionally to the introduced above abbreviations, we use SLAE for *system of linear algebraic equations*, a.o. for *arithmetic operations*, d.o.f. for *degree of freedom.*

Capital letters of the styles $\mathbf{A}, \mathbb{A}, \boldsymbol{\mathcal{A}}$ are used for matrices, small boldface letters – for vectors, whereas $\mathbf{I}$ denotes identity matrices. A s.p.d. matrix $\mathbf{A}$ induces the norm $\|\cdot\|_{\mathbf{A}} = (\cdot, \mathbf{A}\cdot)^{1/2}$. The notation $\|\cdot\|_{\mathbf{A}}$ is used also in the case of a symmetric nonnegative matrix $\mathbf{A}$, when $\|\cdot\|_{\mathbf{A}}$ is indeed a seminorm. Whenever we write $\mathbf{A}^{-1}\mathbf{y}$, we imply some procedure of finding the vector $\mathbf{x}$, which solves SLAE $\mathbf{A}\mathbf{x} = \mathbf{y}$.

$(\cdot,\cdot)_\Omega$, and $\|\cdot\|_{0,\Omega}$ are the scalar product and the norm in $L^2(\Omega)$.

Throughout the paper we consider domains which are Lipshits continuous and as a rule subjected additional conditions of reqularity.

$|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ are the semi-norm and the norm in the Sobolev space $H^k(\Omega)$, *i.e.*,

$$|v|_{k,\Omega}^2 = \sum_{|q|=k} \int\limits_\Omega (D_x^q v)^2 dx, \qquad \|v\|_{k,\Omega}^2 = \|v\|_{0,\Omega}^2 + \sum_{l=1}^{k} |v|_{l,\Omega}^2 \,,$$

where

$$D_x^q v := \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2} \dots x_d^{q_d}, \quad q = (q_1, q_2, \dots, q_d), \quad q_1, q_2, \dots, q_d \geq 0, \quad |q| = \sum_{k=1}^{d} q_k \,.$$

$\mathring{H}^1(\Omega)$ is the subspace of $H^1(\Omega)$ containing functions having zero traces on $\partial\Omega$.

For the norm in the space $H^{1/2}(\partial\Omega)$ of functions on the boundary $\partial\Omega$, we use

$$\|v\|_{1/2,\partial\tau_0} = \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{(v(x)-v(y))^2}{|x-y|^3} \, ds(x) ds(y) + \|v\|_{0,\partial\Omega}^2 \right)^{1/2} . \tag{1.7}$$

If $\mathcal{F} \subset \partial\Omega$ is a proper subset of $\partial\Omega$, the space $H_{00}^{1/2}(\mathcal{F})$ is defined as containing functions, which continuations by zero on $\partial\Omega$ belong to $H^{1/2}(\partial\Omega)$. The notation ${}_{00}\|\cdot\|_{1/2,\mathcal{F}}$ stands for the norm in $H_{00}^{1/2}(\mathcal{F})$, and since this space coincides with the interpolation space $[\mathring{H}^1(\mathcal{F}), L_2(\mathcal{F})]_{1/2}$, the norm in it can be also defined by the interpolation as

$$ {}_{00}\|\cdot\|_{1/2,\mathcal{F}} = \|\cdot\|_{[\mathring{H}^1(\mathcal{F}),L_2(\mathcal{F})]_{1/2}}\,, $$

see, *e.g.,* Nečas & Hlavaćek [27] and Toselli & Widlund [34, Lemma A.8]. In the context of this paper the role of Ω is often played by a shape regular polyhedron, whereas $\mathcal{F}$ is its plain face. In this case, there are no difficulties in defining norms $\|\cdot\|_{1,\mathcal{F}}$, ${}_{00}\|\cdot\|_{1/2,\mathcal{F}}$. The other type of domains, we deal with through the paper, are obtained by means of nondegenerate differentiable mappings of shape regular polyhedrons.

If $\mathcal{V}(\Omega)$ is a finite element space, $\mathbf{v} \leftrightarrow v \in \mathcal{V}(\Omega)$ assumes that entries of the vector $\mathbf{v}$ are nodal values of FE function v, τ_r stands for the domain of a finite element and $\mathbb{R} = \{r = 1, 2, \dots, \mathcal{R}\}$ is the set of their numbers.

2 Outline of domain decomposition algorithm and its main components

2.1 Decomposition and finite element meshes

We will use (1.2), where ϱ_j can be arbitrary positive numbers, as a model problem for constructing an efficient DD preconditioner-solver, in which only the component of the smallest dimension depends on the jumps of the coefficient ϱ. We will make no distinction between the domain of the problem and the computational domain, occupied by the assemblage of finite elements. Assumptions $\mathcal{A}.1$, $\mathcal{A}.2$ mean that the inter-subdomain boundary, by which we mean the set

$$ \Gamma_B = \left(\bigcup\nolimits_{j=1}^{J} \partial\Omega_j\right) \setminus \partial\Omega $$

fits the inter-element boundary, and that jumps of the coefficient ϱ are allowed to occur only on inter-subdomain boundary. Next assumptions, we are going to formulate, will be on the *quasiuniformity* of the FE (finite element) discretization and *shape quasiuniformity* of the domain decomposition.

In this paper, quasiuniformity of the FE (finite element) discretization is understood not in a common, but in a different sense and, probably, would be better termed as relative quasiuniformity. Let $\mathcal{T}_{0,j}$, δ^r be the inverse images of the triangulation $\mathcal{T}_j$ of a subdomain Ω_j and of a FE domain τ_r, $r \in \mathbb{R}_j$, where $\mathbb{R}_j = \{r : \tau_r \subset \Omega_j\}$ is the subset of numbers of finite elements, belonging to subdomain Ω_j. We assume that each triangulation $\mathcal{T}_{0,j}$ is quasiuniform with the same for all $j = 1, 2, \dots, J$ mesh parameter $\hbar$ and that FE domains δ^r have plain faces. Basically this means that in general domains τ_r are associated with the unite tetrahedra or cube τ_0 by linear or trilinear mappings, respectively, and that usual geometric quasiuniformity conditions are fulfilled. Let $\overline{y^{(i)}, y^{(j)}}$ be the edge joining the vertices $y^{(i)}, y^{(j)} \in \overline{\delta}^r$, $\hbar_{i,j}^{(r)}$ be its length and $\hat{\theta}_{i,j}^{(r)}$ be the angle between the edge $\overline{y^{(i)}, y^{(j)}}$ and the plane, containing the rest edges converging at the vertex $y^{(i)}$. Then for some constants $\hat{\alpha}^{(1)}, \hat{\theta} > 0$, the quasiuniformity conditions for the triangulation $\mathcal{T}_{0,j}$ can be written in the form

$$\hat{\alpha}^{(1)}\hbar \le \hbar_{i,j}^{(r)} \le \hbar\,, \qquad \hat{\theta} \le \hat{\theta}_{i,j}^{(r)} \le \pi - \hat{\theta}\,, \tag{2.1}$$

or in the equivalent form

$$0 < c_{\triangle} \le \underline{\rho}_r/\overline{\rho}_r\,, \qquad \hat{\alpha}^{(1)}\hbar \le \overline{\rho}_r \le \hbar\,, \tag{2.2}$$

where $\underline{\rho}_r$ and $\overline{\rho}_r$ are the radii of the largest inscribed and the smallest circumscribed spheres for the tetrahedron τ_r. The mesh parameter $\hbar$ is termed also mesh size.

For each mapping $x = \Upsilon_j(y)$, let $\mathbf{i}_{j,k}(y)$ be the unit vector in the space of variables x tangent to the line $y_l = \text{const}$, $l \ne k$, and directed towards the growth of y_k. By $\theta_{j,k}(y)$ we denote the angle between $\mathbf{i}_{j,k}(y)$ and the plane containing $\mathbf{i}_{j,l}(y)$, $l \ne k$, and let $\mathcal{H}_{k,j}$ be the Lame's coefficients

$$\mathcal{H}_{k,j} = \left[\sum_{l=1}^{3}\left(\frac{\partial \Upsilon_{l,j}}{\partial y_k}\right)^2\right]^{1/2}\,. \tag{2.3}$$

Shape quasiuniformity of the decomposition implies existence of such positive constants $\underline{\alpha}_D, \underline{\theta}_D$ and $H_{D,j},\ j = 1, 2, \ldots, J,$ that

$$\underline{\alpha}_D\, H_{D,j} \le \mathcal{H}_{j,k} \le H_{D,j}\ , \qquad \underline{\theta}_D \le \theta_{j,k} \le \pi - \underline{\theta}_D\ , \qquad 0 < \underline{\alpha}_D, \underline{\theta}_D = \text{const}\,, \tag{2.4}$$

and that reference polyhedrons $\tau_{0,j}$ satisfy some conditions of shape and size regularity. Let $\underline{\wp}_j$, $\underline{\wp}_j^{(k)}$ and $\overline{\wp}_j$, $\overline{\wp}_j^{(k)}$ are the radii of the largest inscribed and the smallest circumscribed spheres for the reference polyhedron $\tau_{0,j}$ and its faces $\hat{F}_j^{(k)}$, respectively. Let $\hat{l}_j^{(k)}$ be the lengths of edges of the reference $\tau_{0,j}$. Then a part of these reference subdomain regularity conditions are expressed by the inequalities

$$0 < c_{\circ} \le \underline{\wp}_j/\overline{\wp}_j\,, \qquad \overline{\wp}_j = 1\,, \tag{2.5}$$

$$0 < c_{\circ} \le \underline{\wp}_j^{(k)}/\overline{\wp}_j^{(k)}\,, \qquad 0 < c_{\circ} \le \overline{\wp}_j^{(k)}, \hat{l}_j^{(k)} \le 1\,. \tag{2.6}$$

Indeed, if reference subdomains are tetrahedrons, then (2.5) are sufficient to provide all needed properties of the decompositions and, in particular, quasiuniformity of faces, *i.e.,* (2.6), and angles. In a more general case additional conditions should be imposed on the angles, namely angles between adjacent faces and between adjacent edges of the faces of polyhedrons $\tau_{0,j}$. These angles should be separated from zero and from 2π by some constant $\hat{\theta}_D > 0$.

For obtaining some bounds of Subsection 3.1, we use one property, which apparently is a consequence of the shape regularity of the domain decomposition and quasiuniformity of the reference triangulations. According to it, each triangulation $\mathcal{T}_{0,j}$ can be expanded on c-vicinity of $\tau_{0,j}$, $c = \text{const}$, in such a way that the formulated above conditions of quasiuniformity remain fulfilled, possibly with new constants $\hat{\alpha}^{(1)}$, $\hat{\theta}$, $c_{\triangle}$, for which, however, we retain the same notations. We do not prove this property under conditions, imposed above, and take it for an additional assumption.

Additional requirements, introduced above for the decomposition and the finite element discretizations, we summarize in the form of the assumptions below.

$\mathcal{A}.3$. *The domain decomposition is shape quasiuniform so that conditions (2.4), (2.5), (2.6) and the condition on the angles are fulfilled.*

$\mathcal{A}$.4. *Triangulations* $\mathcal{T}_{0,j}$ *of reference subdomains have the same mesh size* $\hbar$ *and satisfy quasiuniformity conditions (2.1),(2.2).*

$\mathcal{A}$.5. *Each triangulation* $\mathcal{T}_{0,j}$ *can be expanded on c-vicinity of* $\tau_{0,j}$, $c = \text{const} \geq 1$ *in such a way that quasiuniformity of the triangulation will be retained.*

The description of the BPS preconditioner, presented in this Section, practically does not depend on the reference element, used in the FE discretization. The proof of the estimate for the relative condition number in the frame of h-version is also almost not affected by the type of the reference element. However for simplicity and definiteness, we assume that finite elements are tetrahedrons with linear coordinate functions.

2.2 Structure of DD preconditioner

Orderings of d.o.f.'s (degrees of freedom), adapted to the substructuring algorithms and to the corresponding DD Dirichlet-Dirichlet type algorithms, are reflected in representations of FE stiffness matrices in the block forms

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{I,B} \\ \mathbf{K}_{B,I} & \mathbf{K}_B \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{I,F} & \mathbf{K}_{I,W} \\ \mathbf{K}_{F,I} & \mathbf{K}_F & \mathbf{K}_{F,W} \\ \mathbf{K}_{W,I} & \mathbf{K}_{W,F} & \mathbf{K}_W \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{I,F} & \mathbf{K}_{I,E} & \mathbf{K}_{I,V} \\ \mathbf{K}_{F,I} & \mathbf{K}_F & \mathbf{K}_{F,E} & \mathbf{K}_{F,V} \\ \mathbf{K}_{E,I} & \mathbf{K}_{E,F} & \mathbf{K}_E & \mathbf{K}_{E,V} \\ \mathbf{K}_{V,I} & \mathbf{K}_{V,F} & \mathbf{K}_{V,E} & \mathbf{K}_V \end{pmatrix}, \quad (2.7)$$

where lower indices mark the sets of d.o.f.'s, living on

- I – interiors of subdomains Ω_j,
- B – the inter-subdomain boundary,
- F – interiors of the faces of subdomains,
- W – the wire basket of DD mesh, *i.e.,* union of edges and vertices of subdomains of decomposition,
- E – interiors of the edges, and
- V – vertices of subdomains of decomposition.

In the sets of vertices, edges and faces are included only those, which are not on the part of the boundary, where the Dirichlet boundary condition is imposed. If on a part of $\partial\Omega$ natural boundary conditions are prescribed, it is usually included in the interface boundary. At the same time it is necessary to note that in analysis of DD algorithms sometimes it is convenient to include all vertices, edges and faces on $\partial\Omega$ into consideration. In what follows it is done without special remarks. As usual, additional indexation will be used reflecting different ways of ordering of elements of the introduced sets. For instance, whereas F_j and F_j^k will denote the union of faces of subdomain Ω_j and its separate faces, respectively, at the same time separate faces ordered by some global ordering will be denoted F^l. Similar indices will be used for the sets related to edges.

According to the given above block forms of the matrix $\mathbf{K}$, the vector space of FE d.o.f.'s, which is denoted by V, can be decomposed into the direct sums of corresponding subspaces

$$V = V_I \oplus V_B\,, \qquad V = V_I \oplus V_F \oplus V_W\,,$$

$$V = V_I \oplus V_F \oplus V_E \oplus V_V\,,$$

where $V_B = V_F \oplus V_E \oplus V_V$ and $V_W = V_E \oplus V_V$ are the subspaces of the inter-subdomain boundary and wire basket d.o.f.'s. In the notations of the corresponding subspaces of FE functions, we replace V by $\mathcal{V}$ so that, *e.g.*, $V_E \leftrightarrow \mathcal{V}_E(\Omega)$, whereas $\mathcal{V}_E(E)$ is the subspace of traces on E. Corresponding subspaces for a particular subdomain Ω_j are supplied with additional indices j. In agreement with this $\mathcal{V}_F(\Omega_j)$ is the subspace of face functions from $\mathcal{V}(\Omega_j)$ and $\mathcal{V}_F(F_j)$ is the space of their traces on F_j. The FE spaces on the pre-image domain $\tau_{0,j}$, on the union $\hat{F}_j$ of its faces, and on the wire basket $\hat{W}_j$, similar notations are used with $\mathbb{V}$ standing instead of $\mathcal{V}$. Hence, $\mathbb{V}_E(\tau_{0,j})$ and $\mathbb{V}_E(E_j)$ are the subspace of edge functions from $\mathbb{V}(\tau_{0,j})$ and the space of their traces on $\hat{E}_j$.

Let us turn to the 3×3 block form (2.7) of the matrix $\mathbf{K}$ and vectors $\mathbf{v} \in V = V_I \oplus V_F \oplus V_W$. The inverse to the DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ for the FE stiffness matrix $\mathbf{K}$ can be expressed by the general formulas

$$\begin{aligned} \boldsymbol{\mathcal{K}}^{-1} &= \boldsymbol{\mathcal{K}}_I^{+} + \mathbf{P}_{V_B \to V} \boldsymbol{\mathcal{S}}_B^{-1} \mathbf{P}_{V_B \to V}^{\top} , && (2.8) \\ \boldsymbol{\mathcal{S}}_B^{-1} &= \boldsymbol{\mathcal{S}}_F^{+} + \mathbf{P}_{V_W \to V_B} (\boldsymbol{\mathcal{S}}_W^{F})^{-1} \mathbf{P}_{V_W \to V_B}^{\top} . && (2.9) \end{aligned}$$

S.p.d. matrices $\boldsymbol{\mathcal{K}}_I, \boldsymbol{\mathcal{S}}_F$ and $\boldsymbol{\mathcal{S}}_W^F$ are defined on the subspaces V_I, V_F and V_W of d.o.f.'s living on interior parts of subdomains of decomposition, their faces and on the wire basket, respectively. Indeed, these matrices can be defined implicitly, and, in what follows, we need only to define procedures realizing multiplications of vectors by $\boldsymbol{\mathcal{K}}_I^{-1}$, by the pseudo inverse $\boldsymbol{\mathcal{S}}_F^{+}$ of $\boldsymbol{\mathcal{S}}_F$, continued by zero entries up to the quadratic form on $V_B \times V_B$, and by $(\boldsymbol{\mathcal{S}}_W^F)^{-1}$. Rectangular matrices

$$\mathbf{P}_{V_B \to V} : V_B \to V , \qquad \mathbf{P}_{V_W \to V_B} : V_W \to V_B$$

realize prolongation operations.

The structure of the preconditioning operation, which is the multiplication by $\boldsymbol{\mathcal{K}}^{-1}$, mimics the multiplication by $\mathbf{K}^{-1}$ at the implementation of the three stage block Gauss elimination procedure. In it, d.o.f.'s internal for subdomains of decomposition are eliminated at the first stage, d.o.f.'s living on faces – at the second stage and d.o.f.'s of the wire basket – at the third stage. In order to become certain of this, let us look at the factorization

$$\mathbf{K} = \mathbf{C}_I \begin{pmatrix} \mathbf{K}_I & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_B \end{pmatrix} \mathbf{C}_I^{\top} , \qquad \mathbf{C}_I = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{K}_{BI}\mathbf{K}_I^{-1} & \mathbf{I} \end{pmatrix} , \tag{2.10}$$

where $\mathbf{S}_B$ is also factored into the product of three matrices

$$\mathbf{S}_B = \begin{pmatrix} \mathbf{S}_F & \mathbf{S}_{FW} \\ \mathbf{S}_{WF} & \mathbf{S}_W \end{pmatrix} = \mathbf{C}_B \begin{pmatrix} \mathbf{S}_F & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_W^F \end{pmatrix} \mathbf{C}_B^{\top} , \qquad \mathbf{C}_B = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_{WF}\mathbf{S}_F^{-1} & \mathbf{I} \end{pmatrix} . \tag{2.11}$$

and where, clearly, $\mathbf{S}_B$ and $\mathbf{S}_W^F$ are Schur complements

$$\mathbf{S}_B = \mathbf{K}_B - \mathbf{K}_{BI}\mathbf{K}_I^{-1}\mathbf{K}_{IB} , \qquad \mathbf{S}_W^F = \mathbf{S}_W - \mathbf{S}_{WF}\mathbf{S}_F^{-1}\mathbf{S}_{FW} . \tag{2.12}$$

Resulting from (2.10)-(2.12) factorization of $\mathbf{K}^{-1}$ is

$$\mathbf{K}^{-1} = (\mathbf{C}_I^{\top})^{-1} \begin{pmatrix} \mathbf{K}_I^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_B^{-1} \end{pmatrix} \mathbf{C}_I^{-1} , \qquad \mathbf{C}_I^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_{BI}\mathbf{K}_I^{-1} & \mathbf{I} \end{pmatrix} , \tag{2.13}$$

$$\mathbf{S}_B^{-1} = \begin{pmatrix} \mathbf{S}_F^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{S}_W^F)^{-1} \end{pmatrix} \mathbf{C}_B^{-1}\,, \qquad \mathbf{C}_B^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{S}_{WF}\mathbf{S}_F^{-1} & \mathbf{I} \end{pmatrix}\,. \tag{2.14}$$

If to introduce the prolongation matrices

$$\mathbf{P}_{B,\text{exact}} = \begin{pmatrix} -\mathbf{K}_I^{-1}\mathbf{K}_{IB} \\ \mathbf{I} \end{pmatrix}, \qquad \mathbf{P}_{W,\text{exact}} = \begin{pmatrix} -\mathbf{S}_F^{-1}\mathbf{S}_{FW} \\ \mathbf{I} \end{pmatrix}, \tag{2.15}$$

based on exact solution procedures, then $\mathbf{K}^{-1}$ can be defined by the equivalent to (2.13),(2.14) expressions

$$\mathbf{K}^{-1} = \mathbf{K}_I^{+} + \mathbf{P}_{B,\text{exact}}\mathbf{S}_B^{-1}\mathbf{P}_{B,\text{exact}}^{\top}\,, \tag{2.16}$$

$$\mathbf{S}_B^{-1} = \mathbf{S}_F^{+} + \mathbf{P}_{W,\text{exact}}(\mathbf{S}_W^F)^{-1}\mathbf{P}_{W,\text{exact}}^{\top}\,. \tag{2.17}$$

Therefore, formally (2.8), (2.9) is the result of the replacement in (2.16), (2.17) of the s.p.d. matrices $\mathbf{K}_I, \mathbf{S}_F, \mathbf{S}_W^F$ by their preconditioners and replacement of the rectangular prolongation matrices $\mathbf{P}_{B,\text{exact}}, \mathbf{P}_{W,\text{exact}}$ by their appropriate approximations $\mathbf{P}_{V_B\to V}$ and $\mathbf{P}_{V_W\to V_B}$.

The DD preconditioner-solver will be designed in such a way that the procedure of the multiplication by $\boldsymbol{\mathcal{K}}^{-1}$ of any vector

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_I \\ \mathbf{v}_F \\ \mathbf{v}_W \end{pmatrix}$$

will assume the sequence of operations

$$\boldsymbol{\mathcal{K}}^{-1}\mathbf{v} = \{\boldsymbol{\mathcal{K}}_I^{+} + \mathbf{P}_{V_B\to V}\,[\,\boldsymbol{\mathcal{S}}_B^{-1} = \boldsymbol{\mathcal{S}}_F^{+} + \mathbf{P}_{V_W\to V_B}\boldsymbol{\mathcal{S}}_W^{-1}\mathbf{P}_{V_W\to V_B}^{\top}\,]\,\mathbf{P}_{V_B\to V}^{\top}\}\,\mathbf{v}\,, \tag{2.18}$$

performed consequently. Let *f.-wise* means *face-wise*, *vector assembl.* means *vector assembling*, which assumes that, after completing local operations, the obtained vectors are assembled in the vector of a larger dimension. Then (2.18) consists of the following operations:

1) $\mathbf{v}_I^{(1)} = \boldsymbol{\mathcal{K}}_I^{-1}\mathbf{v}_I \;\Leftarrow$ subdomain-wise ; f.-wise + vector assembl. $\Rightarrow\; \mathbf{v}_B^{(1)} = \mathbf{P}_{V_B\to V}^{\top}\mathbf{v}\,;$

2) f.-wise $\Rightarrow\; \mathbf{v}_F^{(2)} = \boldsymbol{\mathcal{S}}_F^{-1}\mathbf{v}_F^{(1)}$; f.-wise + vector assembl. $\Rightarrow\; \mathbf{v}_W^{(2)} = \mathbf{P}_{V_W\to V_B}^{\top}\mathbf{v}_B^{(1)}\,;$

3) $\mathbf{v}_W^{(3)} = \boldsymbol{\mathcal{S}}_W^{-1}\mathbf{v}_W^{(2)}\,;$

4) $\mathbf{v}_B^{(4)} = \mathbf{P}_{V_W\to V_B}\mathbf{v}_W^{(3)} \;\Leftarrow$ f.-wise ;

5) $\mathbf{v}_B^{(5)} = \mathbf{v}_F^{(2)} + \mathbf{v}_B^{(4)} \;\Leftarrow$ f.-wise ;

6) $\mathbf{v}^{(6)} = \mathbf{P}_{V_B\to V}\mathbf{v}_B^{(5)} \;\Leftarrow$ subdomain-wise ;

7) $\Delta\mathbf{v} = \mathbf{v}_I^{(1)} + \mathbf{v}^{(6)} \;\Leftarrow$ subdomain-wise .

In 2), $\mathbf{v}_F^{(1)}$ is subvector of $\mathbf{v}_B^{(1)}$, sums of vectors in 5) and 7) are understood as topological sums, *i.e.,* $\mathbf{v}_F^{(2)}$ and $\mathbf{v}_I^{(1)}$ are considered as continued by zero entries on all nodes, participating in the respective sum.

2.3 Local Dirichlet problems for subdomains and prolongation from inter-subdomain boundary

2.3.1 Local Dirichlet problems

In view of the assumption $\mathcal{A}.1$ and the accepted ordering of d.o.f.'s, the block $\mathbf{K}_I = \mathrm{diag}\,[\mathbf{K}_{I_j}]_{j=1}^J$ is the block diagonal matrix with each block $\mathbf{K}_{I_j}$ related to the internal unknowns of the corresponding subdomain Ω_j. Naturally, the preconditioner-solver for the local Dirichlet problems has the same typical block diagonal form, so that

$$\boldsymbol{\mathcal{K}}_I^+ := \begin{pmatrix} \boldsymbol{\mathcal{K}}_I^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad \text{with} \qquad \boldsymbol{\mathcal{K}}_I = \mathrm{diag}\,[\,\boldsymbol{\mathcal{K}}_{I_1}, \boldsymbol{\mathcal{K}}_{I_2}, \dots, \boldsymbol{\mathcal{K}}_{I_J}\,]\,, \tag{2.19}$$

and it is assumed that

$$\underline{\gamma}_{I_j} \boldsymbol{\mathcal{K}}_{I_j} \prec \mathbf{K}_{I_j} \prec \overline{\gamma}_{I_j} \boldsymbol{\mathcal{K}}_{I_j}\,, \qquad 0 < \underline{\gamma}_I \le \underline{\gamma}_{I_j}\,, \qquad \overline{\gamma}_{I_j} \le \overline{\gamma}_I\,, \tag{2.20}$$

for some $\underline{\gamma}_I, \overline{\gamma}_I$ independent of j.

The preconditioners involved in DD algorithm should satisfy two contradictory requirements, which for $\boldsymbol{\mathcal{K}}_{I_j}$ have the following form.

α) The values of $\underline{\gamma}_I, \overline{\gamma}_I$ should be as close to unity as possible, uniformly in j.

β) There exist sufficiently fast solvers for systems of algebraic equations with matrices $\boldsymbol{\mathcal{K}}_{I_j}$. In general, when we have no fast exact solver for the system with the matrix $\mathbf{K}_{I_j}$ and it is not reasonable to take $\boldsymbol{\mathcal{K}}_{I_j} = \mathbf{K}_{I_j}$, there indeed exists a number of options for defining subdomain preconditioners $\boldsymbol{\mathcal{K}}_{I_j}$ efficient in a sense of the requirements $\alpha), \beta)$. The situation is one of the simplest, when reference subdomains are the unite cube, *i.e.*, $\tau_{0,j} \equiv \tau_0$, triangulated by the uniform rectangular mesh of sizes $\hbar_1 \times \hbar_2 \times \hbar_3$. The preconditioner $\boldsymbol{\mathcal{K}}_{I_j}$ can be defined as the matrix of the FD approximation of the operator $-\varrho_j \Delta v,\ v\Big|_{\partial\tau_0} = 0$, multiplied by $\hbar_1\hbar_2\hbar_3$. In this case $\underline{\gamma}_I, \overline{\gamma}_I$ are constants depending only on the shape regularity conditions for the mappings Υ_j. Different multilevel preconditioners for FE stiffness matrices have been developed, which can be efficient at the use as $\boldsymbol{\mathcal{K}}_{I_j}$ for rather general decompositions and triangulations. We refer here only to early papers on BPX preconditioners of Bramble Pasciak and Xu [7], hierarchical preconditioners of Yserentant [35], and the book on multigrid methods of Bramble & Zhang [9]. The preconditioner $\boldsymbol{\mathcal{K}}_{I_j}$ can be defined also implicitly by means of some inexact iterative solver for the SLAE with the matrix $\mathbf{K}_{I_j}$ or some subsidiary preconditioner $\widetilde{\boldsymbol{\mathcal{K}}}_{I_j}$. For example one multigrid iteration can produce very efficient local preconditioner, see Jung & Langer [15], Jung *et al.* [16]. Additional references as well as brief descriptions of some of fast solvers for the Dirichlet problems on domains of decomposition can be found in the survey paper of Korneev & Langer [18].

2.3.2 Prolongation from inter-subdomain boundary

Obviously, the matrices $\mathbf{P}_{B,\mathrm{exact}} : V_B \to V$ and $\mathbf{P}_{B_j,\mathrm{exact}} : V_W \to V_B$ of the discrete harmonic global and local prolongations satisfy equalities

$$\|\mathbf{P}_{B,\mathrm{exact}}\mathbf{v}_B\|_{\mathbf{K}} = \|\mathbf{v}_B\|_{\mathbf{S}_B}\,, \qquad \|\mathbf{P}_{B_j,\mathrm{exact}}\mathbf{v}_{B_j}\|_{\mathbf{K_j}} = \|\mathbf{v}_{B_j}\|_{\mathbf{S}_{B_j}}\,,$$

for $\forall\, \mathbf{v}_B \in \mathbf{V}_B$ and $\forall\, \mathbf{v}_{B_j} \in \mathbf{V}_{B_j}$, respectively. As a rule, applicable to DD preconditioner under consideration, for obtaining good DD algorithms it is sufficient to implement prolongations, satisfying

$$\|\mathbf{P}_{V_B\to V}\mathbf{v}_B\|_{\mathbf{K}} \le c_{P_B}\|\mathbf{v}_B\|_{\mathbf{S}_B}\,, \qquad \forall\, \mathbf{v}_B \in \mathbf{V}_B\,, \tag{2.21}$$

for a good, *e.g.,* constant c_{P_B}. The global prolongation operator is defined on the basis of local prolongation operators $\mathbf{P}_{V_{B_j}\to V_j}$ in such a way that the restriction of $\mathbf{P}_{V_B\to V}$ to any subdomain $\overline{\Omega}_j$ is $\mathbf{P}_{V_{B_j}\to V_j}$. Therefore, for providing (2.21) it is sufficient that

$$\|\mathbf{P}_{V_{B_j}\to V_j}\mathbf{v}_{B_j}\|_{\mathbf{K_j}} \le c_{P_B}\|\mathbf{v}_{B_j}\|_{\mathbf{S}_{B_j}}\,, \qquad \forall\, \mathbf{v}_{B_j} \in \mathbf{V}_{B_j}\,, \tag{2.22}$$

uniformly for all subdomains.

Let $\mathcal{P}_{B_j} : \mathcal{V}_B(\partial\Omega_j) \to \mathcal{V}(\Omega_j)$ be the prolongation operator, whose matrix is $\mathbf{P}_{V_{B_j}\to V_j}$. Under stated conditions on the function ϱ, reference domains, their triangulations and FE assemblage, inequality (2.22) is equivalent to

$$|\mathcal{P}_{B_j} v_{B_j}|_{1,\Omega_j} \le c_{\mathcal{P}_B} |v_{B_j}|_{1/2,\partial\Omega_j}\,, \qquad \forall\, v_{B_j} \in \mathcal{V}_B(\partial\Omega_j)\,. \tag{2.23}$$

It immediately follows from (2.22), definition of the local FE matrices $\mathbf{K_j}$ and the right inequality of lemma below.

Lemma 2.1. *Let assumptions $\mathcal{A}_k$, $k = 1, 2, 3, 4$, are fulfilled. Then for positive constants $\underline{\gamma}_S, \overline{\gamma}_S$ depending only on $\alpha_D^{(1)}, \theta_D$ and on the reference subdomains $\tau^\circ_\varkappa$, $\varkappa = 1, 2, \ldots, \varkappa_\circ$, we have*

$$\underline{\gamma}_S \varrho_j\, |v_{B_j}|^2_{1/2,\partial\Omega_j} \le \|\mathbf{v}_{B_j}\|^2_{\mathbf{S}_{B_j}} \le \varrho_j \overline{\gamma}_S\, |v_{B_j}|^2_{1/2,\partial\Omega_j}\,, \qquad \forall\, \mathbf{v}_{B_j} \leftrightarrow v_{B_j} \in \mathcal{V}_B(\partial\Omega_j)\,. \tag{2.24}$$

Proof. According to our assumptions, reference subdomains are such that for any $v_{B_j} \in H^{1/2}(\partial\tau_{0,j})$ the function $u \in H^1(\tau_{0,j})$, satisfying

$$|\, u\,|^2_{1,\tau_{0,j}} = \inf_{v\big|_{\partial\tau_{0,j}} = v_{B_j}} |\, v\,|^2_{1,\tau_{0,j}}\,,$$

exists and the inequalities

$$\underline{\beta}\,|\, v_{B_j}\,|^2_{1/2,\partial\tau_{0,j}} \le |\, u\,|^2_{1,\tau_{0,j}} \le \overline{\beta}\,|\, v_{B_j}\,|^2_{1/2,\partial\tau_{0,j}}\,, \qquad v \in H^1(\tau_{0,j})\,, \tag{2.25}$$

hold for all $j = 1, 2, \ldots, J$ with positive $\underline{\beta}, \overline{\beta}$. From (2.25), we come to its discrete counterpart: for $\forall\, v_{B_j} \in \mathcal{V}_{B_j}(\partial\tau_{0,j})$ and v, which minimizes $|\, v\,|_{1,\tau_{0,j}}$ among all $v \in \mathcal{V}(\tau_{0,j})$ coinciding with v_{B_j} on $\partial\tau_{0,j}$, we have

$$\underline{\beta}\,|\, v_{B_j}\,|^2_{1/2,\partial\tau_{0,j}} \le \inf_{v\big|_{\partial\tau_{0,j}} = v_{B_j}} |\, v\,|^2_{1,\tau_{0,j}} \le \hat{\beta}\,|\, v_{B_j}\,|^2_{1/2,\partial\tau_{0,j}}\,, \qquad v \in \mathcal{V}(\tau_{0,j})\,. \tag{2.26}$$

Indeed, the left inequality (2.26) is a particular case of the left inequality (2.25). In order to get the right inequality (2.26), it is sufficient to use a suitable quasi-interpolation operator. We pick up the quasi-interpolation operator $\mathcal{I}_h$, suggested by Scott & Zhang [32], the properties of which are summarized in Lemma 2.2. Now, for arbitrary $v_{B_j} \in \mathcal{V}_B(\partial\tau_{0,j})$, we consider u satisfying (2.25), its interpolation $\mathcal{I}_h u$ and v which is equal to v_{B_j} on the boundary $\partial\tau_{0,j}$ and minimize $|\, v\,|_{1,\tau_{0,j}}$. The right inequality (2.26) with $\hat{\beta} = \overline{\beta} c_{\mathrm{int}}$ follows by taking into account properties b),d), formulated in in Lemma 2.2, and (2.25):

$$|\, v\,|^2_{1,\tau_{0,j}} \le |\, \mathcal{I}_h u\,|^2_{1,\tau_{0,j}} \le c_{\mathrm{int}} |\, u\,|^2_{1,\tau_{0,j}} \le \overline{\beta} c_{\mathrm{int}} \,|\, u\,|^2_{1/2,\partial\tau_{0,j}} = \overline{\beta} c_{\mathrm{int}} \,|\, v_{B_j}\,|^2_{1/2,\partial\tau_{0,j}} .$$

In order to complete the proof, it is left to transform (2.26) in (2.24) by means of the mapping $x = \Upsilon_j(y) : \overline{\tau}_{0,j} \to \overline{\Omega}_j$ and produce necessary estimates with the use of (2.4). □

Let $\Omega \subset R^n$ be the n-dimensional domain of the arbitrary quasiuniform triangulation $\mathcal{S}_h$ with nodal points $x^{(i)}$, $i = 1, 2, \ldots, I$, and maximal edge size h. To each node $x^{(i)}$, we relate the $(n-1)$-dimensional simplex τ_i, which is the face of one of the n-dimesional simplices of the triangulation $\mathcal{S}_h$ having the vertex $x^{(i)}$. For n vertices of the simplex τ_i we use also notations the $z^{(i)}_l$, $l = 1, 2, \ldots, n$, assuming that $z^{(i)}_1 = x^{(i)}$. The choice of τ_i is not unique, but for $x^{(i)} \in \partial\Omega$ we always take $\tau_i \subset \partial\Omega$. By $\mathcal{V}_\triangle(\Omega)$ and $\mathcal{V}_{\mathrm{tr}}(\partial\Omega)$ are denoted the space of functions, which are continuous on $\overline{\Omega}$ and linear on each simplex of the triangulation, and the space of their traces on $\partial\Omega$, respectively. Let $\theta_i \in \mathcal{P}(\tau_i)$ be the function, satisfying

$$\int_{\tau_i} \theta_i \lambda^{(i)}_l \, dx = \delta_{1,l} , \qquad l = 1, 2, .., n ,$$

where $\lambda^{(i)}_l$ are the barycentric coordinates in τ_i related to its vertices $z^{(i)}_l$ and $\delta_{i,l}$ is Kronecker's symbol. If $\phi_i \in \mathcal{V}_\triangle(\Omega)$ are the Galerkin FE basis functions such that $\phi_i(x_j) = \delta_{i,j}$, $i, j = 1, 2, .., I$, then for each $v \in H^1(\Omega)$ the quasi-interpolation $\mathcal{I}_h v \in \mathcal{V}_\triangle(\Omega)$ is defined as

$$\mathcal{I}_h v = \sum_{i=1}^{I} \left(\int_{\tau_i} \theta_i v \, dx \right) \phi_i(x) .$$

Lemma 2.2. *The quasi-interpolation operator $\mathcal{I}_h$ satisfies*

a) $\mathcal{I}_h v : H^1(\Omega) \mapsto \mathcal{V}_\triangle(\Omega)$, *and, if* $v \in \mathcal{V}_\triangle(\Omega)$, *then* $\mathcal{I}_h v = v$,

b) $(v - \mathcal{I}_h v) \in \overset{\circ}{H}{}^1(\Omega)$, *if* $v|_{\partial\Omega} \in \mathcal{V}_{\mathrm{tr}}(\partial\Omega)$,

c) $\|v - \mathcal{I}_h v\|_{t,\Omega} \le c_{\mathrm{int}} h^{s-t} \|v\|_{s,\Omega}$ *for* $t = 0, 1,$ *and* $s = 1, 2,$

d) $|\mathcal{I}_h v|_{1,\Omega} \le c_{\mathrm{int}} |\, v\,|_{1,\Omega}$ *and* $\|\mathcal{I}_h v\|_{1,\Omega} \le c_{\mathrm{int}} \|\, v\,\|_{1,\Omega}$ *for all* $v \in H^1(\Omega)$, *where by* c_{int} *are denoted constants, depending only on* $\hat{\alpha}^{(1)}, \hat{\theta}$ *from the quasiuniformity conditions (2.1).*

Proof. The proof was given by Scott & Zhang [32], in the above form Lemma is found in Xu & Zou [38]. □

There are suggestions in the literature on fast prolongation operators in 3-d, which provide inequalities (2.22),(2.23) with good constant c_{P_B} and have linear or almost linear computational complexity, but mostly they are related to the subdomains of a canonincal geometrical form or their images. Below, we formulate result on the efficiency of the use for prolongations from the inter-subdomain boundary upon the whole domain by means of inexact two-layer solvers for local Dirichlet problems on subdomains Ω_j. They have at least two advantages against others, since the same solvers can be used in two main components of DD algorithm and since they are applicable for rather general subdomains of decomposition and FE meshes.

Suppose, $\boldsymbol{\mathcal{K}}_I = \mathrm{diag}\,[\,\boldsymbol{\mathcal{K}}_{I_1}, \boldsymbol{\mathcal{K}}_{I_2}, \ldots, \boldsymbol{\mathcal{K}}_{I_J}\,]$ is the preconditioner-solver, for which (2.20) hold with good $\underline{\gamma}_I, \overline{\gamma}_I$, and solving SLAE with matrices $\boldsymbol{\mathcal{K}}_{I_j}$ is relatively cheap. In this case, a natural way of defining the prolongation operator $\mathbf{P}_{V_B \to V} : V_B \to V$ is by means of the inexact solver with the preconditioner $\boldsymbol{\mathcal{K}}_{I_j}$ for each local discrete Dirichlet problem governed by the matrix $\mathbf{K}_{I_j}$. In order to define such a prolongation, we introduce some new notations.

Definition 2.1. *Let* $\mathbf{A}$ *be a s.p.d. matrix,* $\boldsymbol{\mathcal{B}}$ *be its s.p.d. preconditioner and let the system* $\mathbf{Ax} = \mathbf{y}$ *be solved inexactly by the Richardson iterative process*

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \sigma_{k+1}\boldsymbol{\mathcal{B}}^{-1}(\mathbf{A}\mathbf{x}^k - \mathbf{y})\,, \qquad \mathbf{x}^0 = \mathbf{0}\,, \tag{2.27}$$

with Chebyshev iteration parameters σ_k*. The notation* $\boldsymbol{\mathcal{I}}\,[\mathbf{A}, \boldsymbol{\mathcal{B}}, \nu]$ *stands for the matrix implicitly defined for* ν *iterations (2.27),* i.e., *such that* $\boldsymbol{\mathcal{I}}\,[\mathbf{A}, \boldsymbol{\mathcal{B}}, \nu]\,\mathbf{v}^\nu = \mathbf{y}$ *and*

$$(\boldsymbol{\mathcal{I}}\,[\mathbf{A}, \boldsymbol{\mathcal{B}}, \nu])^{-1} = [\mathbf{I} - \prod_{k=1}^{\nu}(\mathbf{I} - \sigma_k\boldsymbol{\mathcal{B}}^{-1}\mathbf{A})]\,\mathbf{A}^{-1}\,.$$

Besides, we set $\boldsymbol{\mathcal{I}}_\circ[\mathbf{A}, \boldsymbol{\mathcal{B}}] := \boldsymbol{\mathcal{I}}\,[\mathbf{A}, \boldsymbol{\mathcal{B}}, \nu_{1/2}]$*, where* ν_ϵ *is the least number of iterations providing the bound*

$$\|\mathbf{x} - \mathbf{x}^{\nu_\epsilon}\|_{\mathbf{A}} \le \epsilon\|\mathbf{x}\|_{\mathbf{A}}\,.$$

We introduce vectors $\mathbf{1}_{B_j} \in V_{B_j}$ and $\mathbf{1}_j \in V_j$, with all entries equal to the unity and the mass matrix $\mathbf{M}_{B_j}$ induced by $\|v_{B_j}\|^2_{0,\partial\tau_{0,j}}$ and the space of FE traces $\mathbb{V}_{\hat{B}_i}(\partial\tau_{0,j})$. In addition, we introduce matrices

$$\boldsymbol{\Theta}_{B_j} = \frac{1}{\mathrm{mes}_1(\partial\tau_{0,j})}\mathbf{1}_{B_j}\mathbf{1}_{B_j}^\top\mathbf{M}_{B_j}\,, \qquad \boldsymbol{\Theta}_j = \frac{1}{\mathrm{mes}_1(\partial\tau_j)}\mathbf{1}_j\mathbf{1}_{B_j}^\top\mathbf{M}_{B_j}\,,$$

$$\mathbf{K}_{I_j,\mathrm{it}} = \boldsymbol{\mathcal{I}}_\circ\,[\mathbf{K}_{I_j}, \boldsymbol{\mathcal{K}}_{I_j}]\,,$$

such that for a given vector $\mathbf{v}_{B_j}$ the products $\boldsymbol{\Theta}_{B_j}\mathbf{v}_{B_j}$ and $\boldsymbol{\Theta}_j\mathbf{v}_{B_j}$ yield vectors in the spaces V_{B_j} and V_j, respectively, with all entries equal to the mean value of v_{B_j} on $\partial\tau_j$. Thus, the matrix $\mathbf{K}_{I_j,\mathrm{it}}$ is produced by the inexact iterative process which is formally defined by Definition 2.1. Now, for the prolongation matrix one can take

$$\mathbf{P}_{V_{B_j} \to V_j} = \boldsymbol{\Theta}_j + \begin{pmatrix} -\mathbf{K}^{-1}_{I_j,\mathrm{it}}\mathbf{K}_{I_j,B_j}(\mathbf{I}_{B_j} - \boldsymbol{\Theta}_{B_j}) \\ \mathbf{I}_{B_j} \end{pmatrix}\,, \tag{2.28}$$

here $\mathbf{I}_{B_j}$ is the unity matrix.

Lemma 2.3. *Let assumptions* $\mathcal{A}_k$*,* $k = 1, 2, 3, 4$*, for the finite element discretization of the elliptic problem (1.2) and the domain decomposition be fulfilled and inequalities (2.20) hold. Let also the prolongation matrix is (2.28). Then*

$$\nu_{j,1/2} = c(1 + \log(\hat{\alpha}^{(1)}\hbar)^{-1})/(\log\rho_j^{-1})$$

and for any k_j iterations, $k_j \geq \nu_{j,1/2}$, one has

$$\|\mathbf{P}_{V_{B_j}\to V_j}\mathbf{v}_{B_j}\|_{\mathbf{K}_j} \leq 2\|\mathbf{v}_{B_j}\|_{\mathbf{S}_{B_j}}\,, \tag{2.29}$$

where $\rho_j = (1-\vartheta_j)/(1+\vartheta_j)$, $\vartheta_j = \sqrt{\underline{\gamma}_{I_j}/\overline{\gamma}_{I_j}}$, and c is the constant independent of $\hbar$.

Proof. We omit the proof, since it is quite similar to the proofs of the bounds for analogous prolongation operators derived in Korneev *et al.* [20] and Korneev & Rytov [24]. □

Remark 2.1. *Suppose the arithmetical cost of solving the system $\boldsymbol{\mathcal{K}}_I\mathbf{w}_I = \mathbf{w}_B$ is less than $cJ\hbar^{-3}(1+\log\hbar^{-1})^{\gamma}$. Since the matrix-vector multiplication $\mathbf{K}_I\mathbf{w}_I$, $\forall\mathbf{w}_I$, requires $cJ\hbar^{-3}$ a.o., the arithmetical cost of the prolongation $\mathbf{P}_{V_B\to V}\mathbf{v}_B$ is $\mathcal{O}(\hbar^{-3}(1+\log\hbar^{-1})^{1+\gamma})$.*

In some situations, met, *e.g.*, in the *hp*-version, the cost of the matrix-vector multiplications by $\mathbf{K}_{I_j}$ can be not cheap, but there exists a preconditioner cheap for matrix-vector multiplications. In this case, it can be advantageous to use two preconditioners for the internal stiffness matrices $\mathbf{K}_{I_j}$. If $\boldsymbol{\mathcal{K}}_{I_j}$ and $\boldsymbol{\mathcal{B}}_{I_j}$ are preconditioners for $\mathbf{K}_{I_j}$, the prolongation matrix can be defined by expressions

$$\mathbf{P}_{V_{B_j}\to V_j} = \boldsymbol{\Theta}_j + \begin{pmatrix} -\boldsymbol{\mathcal{K}}_{I_j,\mathrm{it}}^{-1}\boldsymbol{\mathcal{K}}_{I_j,B_j}(\mathbf{I}_{B_j}-\boldsymbol{\Theta}_{B_j}) \\ \mathbf{I}_{B_j} \end{pmatrix}, \qquad \boldsymbol{\mathcal{K}}_{\mathrm{it}} = \boldsymbol{\mathcal{I}}_{\circ}\left[\boldsymbol{\mathcal{K}}_j, \boldsymbol{\mathcal{B}}_j\right], \tag{2.30}$$

assumed that $\boldsymbol{\mathcal{K}}_{I_j}$ is cheap for matrix-vector multiplications, but solving systems of SLAE with the matrix $\boldsymbol{\mathcal{B}}_{I_j}$ is cheaper than solving systems with the matrix $\boldsymbol{\mathcal{K}}_{I_j}$. If the triangulation of the subdomain is topologically equivalent to a regular triangulation, a good preconditioner $\boldsymbol{\mathcal{K}}_{I_j}$ can be a finite-difference preconditioner. Still systems with the matrix $\boldsymbol{\mathcal{K}}_{I_j}$ can be not easy to solve. In this case, the preconditioner $\boldsymbol{\mathcal{B}}_{I_j}$ can be the matrix resulting from, *e.g.*, incomplete Holesky decomposition for $\boldsymbol{\mathcal{K}}_{I_j}$, the matrix corresponding a few multigrid or two grid iterations applied to $\boldsymbol{\mathcal{K}}_{I_j}$, see Jung & Langer [15], Jung *et al.* [16], some kind of multilevel preconditioners like the one of Bramble *et al.* [7] or others.

2.4 Face component

Among difficult problems at designing DD algorithms for 3-d elliptic problems are those related to the preconditioning the interface and wire basket subproblems: it is necessary to derive preconditioners-solvers $\boldsymbol{\mathcal{S}}_F$, $\boldsymbol{\mathcal{S}}_W^F$ for Schur complements $\mathbf{S}_F$, $\mathbf{S}_W^F$ and the prolongation matrix $\mathbf{P}_{V_W\to V_B}$, which possess two basic properties. In particular, Schur complement preconditioners should provide good values of gammas in the inequalities

$$\underline{\gamma}_B\boldsymbol{\mathcal{S}}_B \leq \mathbf{S}_B \leq \overline{\gamma}_B\boldsymbol{\mathcal{S}}_B\,, \tag{2.31}$$

$$\underline{\gamma}_W\boldsymbol{\mathcal{S}}_W^F \leq \mathbf{S}_W^F \leq \overline{\gamma}_W\boldsymbol{\mathcal{S}}_W^F \tag{2.32}$$

and at the same time allow fast solution of SLAE with these peconditioners for the matrices..

Let us note that one can define preconditioner $\boldsymbol{\mathcal{S}}_B$ (or $\boldsymbol{\mathcal{S}}_F$) by an inexact iterative solver for $\mathbf{S}_B$ (or $\mathbf{S}_F$) or even to reduce solving (1.4) to solving system with the Schur complement

$\mathbf{S}_B$. At an iterative solution of the SLAE with the matrix $\mathbf{S}_B$, this matrix is used only for the matrix-vector multiplications $\mathbf{S}_B\mathbf{v}_B$. If there is a fast solver for the SLAE with the matrix $\mathbf{K}_I$, then it is reasonable not to store $\mathbf{S}_B$, but at each iteration for the former SLAE to perform the sequence of operations $(\mathbf{K}_B - \mathbf{K}_{BI}\mathbf{K}_I^{-1}\mathbf{K}_{IB})\mathbf{v}_B$ with the use of the fast solver for realization of $\mathbf{K}_I^{-1}$. Clearly, in this case the computational cost of the operation, implicitly realizing $\mathbf{S}_B\mathbf{v}_B$, is not less, than the cost of the operation $\boldsymbol{\mathcal{K}}_I^{-1}\mathbf{v}_I$, but nevertheless it can be significantly cheaper, than the multiplication by the explicitly given $\mathbf{S}_B$. The cost is usually reduced by means of defining these preconditioners-solvers in a more elaborate way.

Alike (2.19), $\boldsymbol{\mathcal{S}}_F$ may be defined as the block diagonal matrix

$$\boldsymbol{\mathcal{S}}_F = \operatorname{diag}\left[\boldsymbol{\mathcal{S}}_{F^1}, \boldsymbol{\mathcal{S}}_{F^2}, \ldots, \boldsymbol{\mathcal{S}}_{F^Q}\right], \tag{2.33}$$

where Q is the number of the FE faces inside the computational domain and each block $\boldsymbol{\mathcal{S}}_{F^k}$ corresponds to d.o.f.'s of one face F^k. The Schur complement $\mathbf{S}_F$ is not block diagonal like $\boldsymbol{\mathcal{S}}_F$ in (2.33), and, in general, all faces of the subdomain Schur complement $\mathbf{S}_{F_j}$ are coupled. Decoupling faces in the preconditioner $\boldsymbol{\mathcal{S}}_F$ results in losses in the realtive condition number, including the case when one takes blocks $\mathbf{S}_{F^k}$ on the diagonal of $\mathbf{S}_F$ for the blocks $\boldsymbol{\mathcal{S}}_{F^k}$ in the preconditioner (2.33). However, these losses are not significant, and under the stated conditions we will show that they are estimated by the multiplier $\mathcal{O}(\log^2 \hbar^{-1})$.

All preconditioners $\boldsymbol{\mathcal{K}}_I$, $\boldsymbol{\mathcal{S}}_B$, $\boldsymbol{\mathcal{S}}_F$, $\boldsymbol{\mathcal{S}}_W^F$ and prolongation matrices $\mathbf{P}_{V_B\to V}$, $\mathbf{P}_{V_W\to V_B}$ can be assembled from preconditioners and prolongation matrices $\boldsymbol{\mathcal{K}}_{I_j}$, $\boldsymbol{\mathcal{S}}_{B_j}$, $\boldsymbol{\mathcal{S}}_{F_j}$, $\boldsymbol{\mathcal{S}}_{W_j}^F$, $\mathbf{P}_{V_{B_j}\to V_j}$, $\mathbf{P}_{V_{W_j}\to V_{B_j}}$, similarly defined for each single subdomain Ω_j of decomposition. In accordance with (2.33), the preconditioner for the face subproblem is the block diagonal matrix

$$\boldsymbol{\mathcal{S}}_{F_j} = \operatorname{diag}\left[\boldsymbol{\mathcal{S}}_{F_j^1}, \boldsymbol{\mathcal{S}}_{F_j^2}, \ldots, \boldsymbol{\mathcal{S}}_{F_j^{Q_j}}\right], \tag{2.34}$$

where the notation F_j^l stands for a face of the subdomain Ω_j, Q_j is the number of free faces (implying $F_j^l \subset \Omega$) of Ω_j and l is the local number of these faces. Let i and j are the numbers of two subdomains such that $F^k = \overline{\Omega}_i \bigcap \overline{\Omega}_l$ and s, l are numbers of the face F^k in local orderings of faces for subdomains Ω_i, Ω_j, respectively. Therefore, $F^k = F_i^s = F_j^l$ and $\boldsymbol{\mathcal{S}}_{F^k}$ is the sum of the corresponding blocks of Schur complement preconditioners $\boldsymbol{\mathcal{S}}_{F_i}$ and $\boldsymbol{\mathcal{S}}_{F_j}$:

$$\boldsymbol{\mathcal{S}}_{F^k} = \boldsymbol{\mathcal{S}}_{F_i^s} + \boldsymbol{\mathcal{S}}_{F_j^l}\,, \tag{2.35}$$

where it is assumed d.o.f.'s, living on face F^k, have the same local numbers on the faces F_i^s and F_j^l.

Let $\mathbf{S}_{00,F_j^l}$ be the matrix of the quadratic form such that

$$\mathbf{v}^\top \mathbf{S}_{00,F_j^l}\,\mathbf{v} \equiv {}_{00}|v|^2_{1/2,F_j^l}\,, \qquad \mathbf{v} \leftrightarrow v \in \mathring{\mathcal{V}}(F_j^l)\,, \tag{2.36}$$

and $\mathring{\mathcal{V}}(F_j^l)$ is the subspace of the traces on F_j^l of FE functions vanishing on the wire basket W_j. Then one can set

$$\boldsymbol{\mathcal{S}}_{F^k} = \varrho_i \mathbf{S}_{00,F_i^s} + \varrho_l \mathbf{S}_{00,F_l^j}\,. \tag{2.37}$$

Matrices $\mathbf{S}_{00,F_j^p}$ can be replaced by some preconditioners close to them in the spectrum. Suppose that each pair of the FE spaces $\mathcal{V}(\Omega_j)$ and $\mathbb{V}(\tau_{0,j})$ are associated by mapping $\Upsilon_j(y) \in [\mathbb{V}_j(\tau_{0,j})]^3$ satisfying the shape quasiuniformity conditions. Then matrices $\mathbf{S}_{00,F_j^p}$ can be replaced by matrices $\mathbb{S}_{00,\hat{F}_j^p}$ of simpler quadratic forms

$$\mathbf{v}^\top \mathbb{S}_{00,\hat{F}_j^p} \mathbf{v} \equiv \, {}_{00}|v|^2_{1/2,\hat{F}_j^p} \,, \qquad \mathbf{v} \leftrightarrow v \in \mathring{\mathcal{V}}(\hat{F}_j^p) \,, \tag{2.38}$$

multiplied by $H_{D,j}$, where $\mathring{\mathcal{V}}(\hat{F}_j^p)$ is the space of traces of FE functions on the face $\hat{F}_j^p$ of the reference subdomain $\tau_{0,j}$, vanishing on its wire basket. More over, there are hold the inequalities

$$\underline{\gamma}_\circ H_{D,j}\, \mathbb{S}_{00,\hat{F}_j^p} \,\le\, \mathbf{S}_{00,F_j^p} \,\le\, \overline{\gamma}_\circ \, H_{D,j}\, \mathbb{S}_{00,\hat{F}_j^p} \,, \qquad \underline{\gamma}_\circ > 0 \,, \tag{2.39}$$

with positive constants $\underline{\gamma}_\circ, \overline{\gamma}_\circ$ depending only on shape quasiuniformity conditions, in other words, on $\alpha_D^{(1)}, \theta_D$ from (2.4), $c_\circ$ from (2.5) and $\theta_\circ$. In spite of all these simplifications, in general, the one face preconditioner

$$\boldsymbol{\mathcal{S}}_{F^k} = \varrho_i \, H_{D,i}\, \mathbb{S}_{00,\hat{F}_i^s} \,+\, \varrho_j \, H_{D,j}\, \mathbb{S}_{00,\hat{F}_l^j} \tag{2.40}$$

still does not provide good procedures for solving SLAE, governed by it, as well as for its evaluation. In view of this, further simplifications are usually done which take into account particular forms of reference subdomains and their faces and reference discretizations. As their result, some preconditioners $\boldsymbol{\mathcal{S}}_{00,\hat{F}_j^p}$ satisfying

$$\underline{\gamma}_{00} \boldsymbol{\mathcal{S}}_{00,\hat{F}_j^p} \,\le\, \mathbb{S}_{00,\hat{F}_j^p} \,\le\, \overline{\gamma}_{00}\, \boldsymbol{\mathcal{S}}_{00,\hat{F}_j^p} \,, \qquad \underline{\gamma}_{00} > 0 \,, \tag{2.41}$$

are found, and the matrices

$$\boldsymbol{\mathcal{S}}_{F^k} = \varrho_i \, H_{D,i}\, \boldsymbol{\mathcal{S}}_{00,\hat{F}_i^s} \,+\, \varrho_j \, H_{D,j}\, \boldsymbol{\mathcal{S}}_{00,\hat{F}_j^l} \tag{2.42}$$

are taken for the face preconditioners.

Analysis, completed in Section 3, allows us to conclude that under assumptions $\mathcal{A}.1 - \mathcal{A}.5$ and $\underline{\gamma}_{00}, \overline{\gamma}_{00} = \mathrm{const}$ the presented way of preconditioning of face problems guarantees the bounds

$$\underline{\gamma}_F \boldsymbol{\mathcal{S}}_F \le \mathbf{S}_F \le \overline{\gamma}_F \boldsymbol{\mathcal{S}}_F \,, \tag{2.43}$$

for

$$\underline{\gamma}_F \ge \frac{c}{1 + \log^2 \hbar^{-1}} \,, \qquad c, \overline{\gamma}_F = \mathrm{const} \,. \tag{2.44}$$

Let us note that (2.44) is a collateral result, and for the control of the relative condition number of the DD preconditioner $\boldsymbol{\mathcal{K}}$ and inter-subdomain Schur complement preconditioner $\boldsymbol{\mathcal{S}}_B$ we do not use directly and prove the left inequality (2.44), see, *e.g.,* Lemma 3.2.

2.5 Wire basket component

The wire basket preconditioner $\boldsymbol{\mathcal{S}}_W^F$ is a single global matrix governing subsystems of algebraic equations to be solved at each iteration of the DD algorithm. Obviously, it can be assembled from subdomain preconditioners $\boldsymbol{\mathcal{S}}_{W_j}^F,\ j = 1, 2, .., J,$ and, in view of assumptions $\mathcal{A}.1 - \mathcal{A}.3$, these subdomain preconditioners can be defined by means of the reference subdomains. Until now, no invertible trace theorem has been proved for the traces on $\partial\mathcal{D}$ of functions from the space $H^{1/2}(\mathcal{D})$ on domains $\mathcal{D} \in R^2$ with piece-wise smooth boundaries. This posed one of the obstacles in designing efficient wire basket preconditioners. Some useful results in DD analysis has been established with the use of the "seminorm" $|\cdot|_{0,\partial\mathcal{D}}$ in the space $L_2(\partial\mathcal{D})$ as the norm in the space of traces of FE functions from $L_2^{1/2}(\mathcal{D})$. This seminorm is introduced by the equality

$$|\, v\,|_{0,\partial\mathcal{D}}^2 \overset{\text{def}}{=} \inf_{c\in R} \|\, v - c\,\|_{0,\partial\mathcal{D}}^2\,, \qquad \forall\, v \in L_2(\partial\mathcal{D})\,. \tag{2.45}$$

In the DD algorithm the role of $\mathcal{D}$ and $\partial\mathcal{D}$ are played by faces and their boundaries and, respectively, by the interface boundary and the wire basket, whereas for the reference subdomain $\tau_{0,j}$ – by its boundary $\partial\tau_{0,j}$ and the wire basket $\hat{W}_j$. Later, in Lemma 3.3, we give the bounds of seminorms $|\, v\,|_{0,\hat{W}_j}$ for the traces on $\hat{W}_j$ of FE functions by their seminorms $|\, v\,|_{1/2,\partial\tau_{0,j}}$ and the bounds in the opposite direction for the appropriately chosen prolongations. Motivated by these bounds the wire basket preconditioner is usually obtained by means of preconditioning of the quadratic forms

$$\mathbf{v}_{W_j}^\top \mathbf{W}_j\, \mathbf{v}_{W_j} = |\, v\,|_{0,\hat{W}_j}^2 \overset{\text{def}}{=} \inf_{c_j\in R} \|\, v - c_j\,\|_{0,\hat{W}_j}^2\,, \qquad \forall\, \mathbf{v}_{W_j} \leftrightarrow v \in \mathbb{V}_W(\hat{W}_j)\,. \tag{2.46}$$

In particular, it may be adopted

$$\boldsymbol{\mathcal{S}}_{W_j}^F = \varrho_j\, H_{D,j}\, (1 + \log \hbar^{-1})\, \boldsymbol{\mathcal{W}}_j\,, \tag{2.47}$$

where $\boldsymbol{\mathcal{W}}_j$ is some good preconditioner for $\mathbf{W}_j$. The space of traces $\mathbb{V}_W(\hat{W}_j)$ contains continuous piece wise linear functions. For this reason, if $\mathbf{M}_{\hat{W}_j}$ is the mass matrix defined by the identity

$$\mathbf{v}_{W_j}^\top \mathbf{M}_{\hat{W}_j} \mathbf{v}_{W_j} = ||v||_{0,\hat{W}_j}^2\,, \qquad \forall\ \mathbf{v}_{W_j} \leftrightarrow v \in \mathbb{V}_W(\hat{W}_j)\,,$$

and vector $\mathbf{z}_j$ contains unity for all entries, then simple computations give

$$|v|_{L_2(W)} = \int_{\hat{W}_j} v^2\, ds - \frac{(\int_{\hat{W}_j} v\, ds)^2}{\int_{\hat{W}_j}\, ds} = \mathbf{v}_{W_j}^\top \left(\mathbf{M}_{\hat{W}_j} - \frac{(\mathbf{M}_{\hat{W}_j}\mathbf{z}_j)(\mathbf{M}_{\hat{W}_j}\mathbf{z}_j)^\top}{\mathbf{z}_j^\top \mathbf{M}_{\hat{W}_j}\mathbf{z}_j} \right) \mathbf{v}_{W_j}\,.$$

Therefore,

$$\mathbf{W}_j = \mathbf{M}_{\hat{W}_j} - \frac{1}{\mathrm{mes}_1(\hat{W}_j)} (\mathbf{M}_{\hat{W}_j}\mathbf{z}_j)(\mathbf{M}_{\hat{W}_j}\mathbf{z}_j)^\top\,, \tag{2.48}$$

where $\mathrm{mes}_1(\hat{W}_j)$ is the length of the wire basket $\hat{W}_j$ of the reference subdomain.

Solution procedure for the S.L.A.E. with the matrix $\boldsymbol{\mathcal{S}}_W^F$ simplifies, if we replace the mass matrix $\mathbf{M}_{\hat{W}_j}$ by its diagonal preconditioner, for which we use notation $\mathbf{D}_j$. We assemble it from matrices $\mathbf{D}_{\hat{E}_j^k}$ defined for each edge $\hat{E}_j^k$ of the reference domain. It is sufficient to describe the matrix $\mathbf{D}_{\hat{E}_j^k}$ for one edge $\hat{E}_j^k$, which without loss of generality can be described as the set $\hat{E}_j^k = \{x : 0 < x_1 < \ell,\ x_2, x_3 \equiv 0\}$. Suppose, points $x_1 = x_1^{(i)},\ i = 0, 1, .., n,\ \hbar_i = x_1^{(i)} - x_1^{(i-1)} > 0,\ x^{(0)} = 0,\ x^{(n)} = \ell,$ are the nodes of discretization and

$$\eta_0 = \hbar_1/2, \quad \eta_i = (\hbar_i + \hbar_{i+1})/2 \quad \text{for} \quad i = 1, 2.., n-1, \quad \eta_n = \hbar_n/2\,.$$

Then we set

$$\mathbf{D}_{\hat{E}_j^k} = \text{diag}\,[\eta_i]_{i=0}^n\,. \tag{2.49}$$

Note that at assembling the matrix $\mathbf{D}_j$, assembling is indeed needed only for vertices of the polygon $\tau_{0,j}$ and not for other wire basket nodes. If vertex $x^{(i)} \in \hat{W}_j$ is common for $\kappa \geq 3$ edges, then the entry on the diagonal of $\mathbf{D}_j$, corresponding to this vertex, is the sum of κ those single entries in κ edge matrices $\mathbf{D}_{\hat{E}_j^k}$, which correspond to the same vertex.

Defining the matrix $\boldsymbol{\mathcal{W}}_j$ by the relation

$$\mathbf{v}_{W_j}^\top \boldsymbol{\mathcal{W}}_j\, \mathbf{v}_{W_j} \equiv \inf_{c_j \in R} \sum_{y^{(i)} \in \hat{W}_i} \eta_i (v_i - c_j)^2 \tag{2.50}$$

with summation over the nodes $y^{(i)}$ on the wire basket $\hat{W}_i$, we come to the equality

$$\boldsymbol{\mathcal{W}}_j = \mathbf{D}_j - \frac{1}{\text{mes}_1(\hat{W}_j)} (\mathbf{D}_j \mathbf{z}_j)(\mathbf{D}_j \mathbf{z}_j)^\top \tag{2.51}$$

in the same way as above. Then we define $\boldsymbol{\mathcal{S}}_{W_j}^F$ by (2.47). As a consequence of the spectral equivalence of matrices $\mathbf{M}_{\hat{W}_j}$ and $\mathbf{D}_j$, the two matrices $\mathbf{W}_j$ and $\boldsymbol{\mathcal{W}}_j$, defined in (2.48) and in (2.51), are also spectrally equivalent. The wire basket Schur complement preconditioner is defined now by assembling subdomain preconditioners $\boldsymbol{\mathcal{S}}_{W_j}^F$:

$$\begin{aligned} \boldsymbol{\mathcal{S}}_W^F &= \biguplus_{j=1}^J \boldsymbol{\mathcal{S}}_{W_j}^F = \biguplus_{j=1}^J\ \varrho_j\, H_{D,j}\, (1 + \log \hbar^{-1})\, \boldsymbol{\mathcal{W}}_j = \\ &= \biguplus_{j=1}^J\ \varrho_j\, H_{D,j}\, (1 + \log \hbar^{-1}) \left(\mathbf{D}_j - \frac{1}{\text{mes}_1(\hat{W}_j)} (\mathbf{D}_j \mathbf{z}_j)(\mathbf{D}_j \mathbf{z}_j)^\top \right)\,, \end{aligned} \tag{2.52}$$

where the sign $\biguplus_{j=1}^J$ stands for assemblig procedure over all subdomains.

The solution procedure for the system of algebraic equations

$$\mathbf{S}_W^F \mathbf{v} = \mathbf{f}_W \tag{2.53}$$

still require consideration due to the presence of the second term in the right part of (2.53). It is equivalent to the minimization of the quadratic functional, *i.e.*, finding

$$\min_{\mathbf{v}} I(\mathbf{v}) = \min_{\mathbf{v}} \left(\frac{1}{2} \sum_{j=1}^J \min_{c_j} [(\mathbf{v}^{(j)} - \mathbf{z}_j c_j)^\top \mathbf{D}_W^{(j)} (\mathbf{v}^{(j)} - \mathbf{z}_j c_j)] - \mathbf{v}^\top \mathbf{f} \right)\,, \tag{2.54}$$

where $\mathbf{D}_W^{(j)} = \varrho_j\, H_{D,j}\,(1+\log h^{-1})\mathbf{D}_j$ and $\mathbf{v}^{(j)}$ denotes the restriction to the wire basket of the reference domain. When it does not cause confusion, we omit index "W" in the notations of vectors, living on the wire basket. We describe first the procedure for finding the minimizer of $I(\mathbf{v})$ in the case of the Neumann boundary conditions on $\partial\Omega$ and all nodes of the discretization are free nodes.

Let $\mathbf{D}_W$ be the matrix assembled from matrices $\mathbf{D}_W^{(j)}$. Writing down the conditions of minimum of quadratic functional (2.54) with respect to components of the vector $\mathbf{v}$ and constants c_j leads to the system of algebraic equations

$$
\begin{aligned}
&\mathbf{D}_W\mathbf{v} - \biguplus_j(\mathbf{D}_W^{(j)}\mathbf{z}_j c_j) = \mathbf{f}\,, \\
&\mathbf{z}_j^\top \mathbf{D}_W^{(j)}\mathbf{v}^{(j)} = c_j a_j\,, \qquad a_j = \mathbf{z}_j^\top \mathbf{D}_W^{(j)}\mathbf{z}_j = \varrho_j\, H_{D,j}\,(1+\log h^{-1})\,\mathrm{mes}_1(\hat{W}_j)\,.
\end{aligned}
\tag{2.55}
$$

Since $\mathbf{D}_W$ is a diagonal matrix, one can easily express $\mathbf{v}$ through unknown constants c_j

$$
\mathbf{v} = \mathbf{D}_W^{-1}\left(\biguplus_j(\mathbf{D}_W^{(j)}\mathbf{z}_j c_j) + \mathbf{f}\right) \tag{2.56}
$$

and substitute in the second group of equations (2.55). This produces the system of algebraic equations

$$
c_j a_j - \mathbf{z}_j^\top \mathbf{D}_W^{(j)}\left[\mathbf{D}_W^{-1}\biguplus_l(\mathbf{D}_W^{(l)}\mathbf{z}_l c_l)\right]_j = \mathbf{z}_j^\top \mathbf{D}_W^{(j)}\left[\mathbf{D}_W^{-1}\mathbf{f}\right]_j\,, \tag{2.57}
$$

where $[\mathbf{s}]_j$ stands for the subvector $[\mathbf{s}]_j \in V_{W_j}$ of the vector $\mathbf{s} \in V_W$. The product $\mathbf{d}_j := \mathbf{D}_W^{(j)}\mathbf{z}_j$ is the vector, containing diagonal elements of $\mathbf{D}_W^{(j)}$ for the entries. On this account, systems (2.56),(2.57) can be rewritten as

$$
\mathbf{v} = \mathbf{D}_W^{-1}\left(\biguplus_j(\mathbf{d}_j c_j) + \mathbf{f}\right)\,, \tag{2.58}
$$

$$
c_j a_j - \mathbf{d}_j^\top\left[\mathbf{D}_W^{-1}\biguplus_l(\mathbf{d}_l c_l)\right]_j = \mathbf{d}_j^\top\left[\mathbf{D}_W^{-1}\mathbf{f}\right]_j\,. \tag{2.59}
$$

Having solved system (2.59), we substitute solution in (2.58) and calculate vector $\mathbf{v}$.

It is necessary to make several remarks. The system (2.59) resembles FE/FD systems of algebraic equations. The matrix of this system is nonnegative, the sum of its coefficients in each row is zero. In order to establish that, it is necessary to substitute all $c_l = 1$ and note that entries of the vector inside square brackets are first of all positive and are equal to 1. From that it follows that the vector $\mathbf{c} = \{c_j\}_{j=1}^{j}$ is found up to an arbitrary constant vector $c\mathbf{1}$, *i.e.*, if $\mathbf{c}$ is the solution, then $\mathbf{c} + c\mathbf{1}$ is also solution. As a consequence, the vector, which solves (2.58), is also defined up to the same constant vector $c\mathbf{1}$. At the Dirichlet boundary condition the values of a_j, see (2.55), are not changed. This means that in expression (2.55) for a_j matrix $\mathbf{D}_W^{(j)}$ is written as no boundary conditions are imposed on $\partial\hat{\Omega}_j$ and has the dimension $N_{W_j}^{+} \times N_{W_j}^{+}$, where $N_{W_j}^{+}$

is the number of all nodes on W_j. However, in the rest relations (2.55)-(2.59) matrices $\mathbf{D}_W$ and $\mathbf{D}_W^{(i)}$ take into account the Dirichlet boundary conditions and, *e.g.*, the latter have dimensions $N_{W_i} \times N_{W_i}$, where N_{W_i} is the number of free nodes on the wire basket W_j. Respectively, vectors $\mathbf{d}_i$ have dimension N_{W_i}. As a consequence, in all rows, corresponding to c_j with subdomain boundary $\partial\Omega_j$ having nodes on $\partial\Omega$, there is some diagonal predominance, the matrix of the system (2.59) is s.p.d., and this system has a unique solution.

Remark 2.2. *If all coefficients before c_j in (2.59) are calculated, then solution of (2.59) by the general Gauss elimination procedure will require cJ^3=const a.o. More efficient solution procedures can be used, and, in particular, sparsity of the matrix of the system (2.59) can be taken into account. If the vector in brackets in (2.58) is known, then one has to spend $cJ\hbar^{-1} = \mathcal{O}(\hbar^{-1})$ a.o. in order to find $\mathbf{v}$. It easy to see, that the arithmetical cost of the coefficients before c_j in (2.59) and the vector in brackets in (2.58) is $cJ\hbar^{-1} = \mathcal{O}(\hbar^{-1})$ a.o. Therefore, at J fixed the arithmetical cost of solving the system (2.53) is $\mathcal{O}(\hbar^{-1})$*

The prolongation matrix $\mathbf{P}_{V_W \to V_B}$ can be defined face by face, *i.e.,* by means of the matrices

$$\mathbf{P}_{V_{W^q} \to V_{B^q}}\,, \qquad q = 1, 2, .., Q\,,$$

where $\mathbf{P}_{V_{W^q} \to V_{B^q}}$ is the prolongation matrix from the boundary $W^q = \partial F^q$ on the closure $\overline{F}^q$ of a face F^q. If $\mathbf{v}_{W^q}$ is a vector living on W^q, then by $\widetilde{\mathbf{P}}^{(q)}$ is denoted such prolongation matrix that $\widetilde{\mathbf{P}}^{(q)}\mathbf{v}_{W^q}$ is the prolongation by zero entries to all internal nodes of the face. We set

$$\mathbf{P}_{V_{W^q} \to V_{\overline{F}^q}} \mathbf{v}_{W^q} = \mathrm{m}_{\mathrm{ean}}(\mathbf{v}_{W^q})\mathbf{1}_{\overline{F}^q} + \widetilde{\mathbf{P}}^{(q)}(\mathbf{v}_{W^q} - \mathrm{m}_{\mathrm{ean}}(\mathbf{v}_{W^q})\mathbf{1}_{W^q})\,, \tag{2.60}$$

where $\mathbf{1}_{\overline{F}^q}$ and $\mathbf{1}_{W^q}$ are vectors living on $\overline{F}^q$ and W^q, respectively, and having unities for components, and $\mathrm{m}_{\mathrm{ean}}(\mathbf{v})$ is the arithmetic mean value of components of the vector $\mathbf{v}$. Obviously, the so defined prolongation $\mathbf{P}_{V_W \to V_B}\mathbf{v}$ is cheap and requires $J\hbar^{-2}$ a.o.

3 Relative condition number of DD preconditioner and numerical complexity

We start from abstract estimates of the boundaries of the relative spectrum of DD preconditioners. They are expressed through the relative spectrum bounds of the component preconditioners and bounds for the norms of prolongation operators. Then in Subsection 3.2 we carry out a more accurate analysis of the multipliers, entering these abstract bounds, and illuminate the dependence of the relative condition numbers of DD preconditioners on the parameters characterizing the elliptic problem, FE discretizations and decompositions of the domain into subdomains. Several subsidiary results are collected in Subsection 3.3.

3.1 Abstract bound for relative condition number

According the above outline, derivation of the abstract relative spectrum bounds below is completed in two steps. At first step, we basically apply twice one result, which is Corollary 3.2, in compliance with the representation (2.8), (2.9) of the DD preconditioner. This allows us to estimate in Theorem 3.2 the relative condition number of the DD preconditioner under general assumptions, pertaining all five its components: three component preconditioners and two prolongation operators. The second step, which is more involved, is the proof of Theorem 3.3. It takes into account a more detailed representation of the interface Schur complement preconditioner $\boldsymbol{\mathcal{S}}_B$, which is more adjusted to fast and parallelized computations. According to the previously given description, it is composed of the face preconditioner with split d.o.f.'s living on different faces, specific wire basket preconditioner, and the prolongation operator from the wire basket upon interface boundary.

3.1.1 Purely algebraic considerations

Let $\mathbf{S} = \mathbf{A}_2 - \mathbf{A}_{21}\mathbf{A}_1^{-1}\mathbf{A}_{12}$ be the Schur complement for the $n \times n$ nonnegative symmetric matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_2 \end{pmatrix}$$

with the $n_k \times n_k$ blocks $\mathbf{A}_k$ on diagonal, $n_1 + n_2 = n$, and $\mathbf{A}_1$ be a positive matrix. We remind that if $\ker[\mathbf{A}] \neq \mathbf{0}$, then $\ker[\mathbf{A}] = \ker[\mathbf{S}]$. In the case of $\mathbf{A}$ with a nontrivial kernel, we term $(\cdot,\cdot)_{\mathbf{A}} := (\cdot, \mathbf{A}\cdot)$ and $\|\cdot\|_{\mathbf{A}} := \sqrt{(\cdot,\cdot)_{\mathbf{A}}}$ by the scalar product and the norm, respectively, with understanding that they are such in the factor space $V/(\ker[\mathbf{A}])$, and $\|\cdot\|_{\mathbf{A}}$ is only a seminorm in V. Let also $\boldsymbol{\mathcal{P}}_2 : R^{n_2} \to R^n$ be the "prolongation" matrix

$$\boldsymbol{\mathcal{P}}_2 = \begin{pmatrix} \boldsymbol{\mathcal{P}}_{21} \\ \boldsymbol{\mathcal{P}}_{22} \end{pmatrix}, \qquad \boldsymbol{\mathcal{P}}_{22} = \mathbf{I},$$

such that for some $c_{_S} > 0$

$$\|\boldsymbol{\mathcal{P}}_2\mathbf{v}_2\|_{\mathbf{A}}^2 \le c_{_S}\, \|\mathbf{v}_2\|_{\mathbf{S}}^2, \qquad \forall\, \mathbf{v}_2 \in R^{n_2}. \tag{3.1}$$

Note that, if $\mathbf{v}_2 \in \ker[\mathbf{S}]$, this inequality assumes the inclusion $\boldsymbol{\mathcal{P}}_2\mathbf{v}_2 \in \ker[\mathbf{A}]$ and otherwise it can't be fulfilled. By means of the transformation matrix $\mathbf{C}$, we introduce the matrix $\mathbf{B} = \mathbf{C}^T\mathbf{A}\mathbf{C}$, where

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & \boldsymbol{\mathcal{P}}_{21} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_2 \end{pmatrix},$$

and the block diagonal matrix $\mathbf{B}_{\rm diag} = {\rm diag}\,[\mathbf{B}_1, \mathbf{B}_2]$. Obviously $\mathbf{B}_1 = \mathbf{A}_1$. We denote by V and V_k the spaces R^n and R^{n_k}, respectively, and introduce the space $\widetilde{V}_2 := \boldsymbol{\mathcal{P}}_2 V_2$. According to the theorem below, the splitting of the space V into the direct sum of the subspaces $\widetilde{V}_1 = V_1$ and $\widetilde{V}_2$ is stable, under condition that $c_{_S}$ is a constant.

Theorem 3.1. *Let* $\mathbf{A}$ *be a positive symmetric matrix, the prolongation matrix* $\boldsymbol{\mathcal{P}}_2$ *satisfy (3.1), and* $\boldsymbol{\mathcal{A}}$ *be the matrix defined through its inverse*

$$\boldsymbol{\mathcal{A}}^{-1} := \mathbf{A}_1^{+} + \boldsymbol{\mathcal{P}}_2\mathbf{S}^{-1}\boldsymbol{\mathcal{P}}_2^T. \tag{3.2}$$

Then

$$\underline{c}_B\mathbf{B}_{\rm diag} \le \mathbf{B} \le \overline{c}_B\mathbf{B}_{\rm diag}, \qquad \underline{c}_B\boldsymbol{\mathcal{A}} \le \mathbf{A} \le c_{_S}\overline{c}_B\boldsymbol{\mathcal{A}}, \tag{3.3}$$

where $\underline{c}_B = 1 - \sqrt{1 - c_{_S}^{-1}} \ge (2c_{_S})^{-1}$ *and* $\overline{c}_B = 1 + \sqrt{1 - c_{_S}^{-1}} \le 2 - c_{_S}^{-1} \le 2.$

Proof. The basic for the proof of the theorem are the inequalities

$$\gamma_B := \cos\measuredangle(V_1, \widetilde{V}_2) = \sup_{\mathbf{v}_k\in V_k\setminus\{\mathbf{0}\}} \frac{\mathbf{v}_1^T\mathbf{B}\mathbf{v}_2}{\|\mathbf{v}_1\|_{\mathbf{B}}\|\mathbf{v}_2\|_{\mathbf{B}}} = \sup_{\mathbf{v}_k\in V_k\setminus\{\mathbf{0}\}} \frac{\mathbf{v}_1^T\mathbf{B}_{12}\mathbf{v}_2}{\|\mathbf{v}_1\|_{\mathbf{B}_1}\|\mathbf{v}_2\|_{\mathbf{B}_2}} \le \sqrt{1-c_{_S}^{-1}}\,, \tag{3.4}$$

originally obtained by Haase *et al.* [13] and one of characterizations of γ_B:

$$\gamma_B^2 = \sup_{\mathbf{v}_2\in V_2\setminus\{\mathbf{0}\}} \frac{\mathbf{v}_2^\top\mathbf{B}_{21}\mathbf{B}_1^{-1}\mathbf{B}_{12}\mathbf{v}_2}{||\mathbf{v}_2||_B^2}\,, \tag{3.5}$$

independent proofs of which were given by Axelsson & Vassilevskii [1], Bjorstad & Mandel [2], and Haase *et al.* [14]. For deriving (3.4), we note, that there hold the equalities

$$\mathbf{S} = \mathbf{B}_2 - \mathbf{B}_{21}\mathbf{B}_1^{-1}\mathbf{B}_{12} = \mathbf{A}_2 - \mathbf{A}_{21}\mathbf{A}_1^{-1}\mathbf{A}_{12}\,, \qquad ||\mathbf{v}_2||_B^2 = \|\boldsymbol{\mathcal{P}}_2\mathbf{v}_2\|_{\mathbf{A}}^2\,,$$

which follow from definitions of the involved matrices. By means of these inequalities, we transform the representation (3.5) and come to the squared inequality (3.4) as follows:

$$\gamma_B^2 = \sup_{\mathbf{v}_2\in V_2\setminus\{\mathbf{0}\}} \frac{\mathbf{v}_2^\top\mathbf{B}_{21}\mathbf{B}_1^{-1}\mathbf{B}_{12}\mathbf{v}_2}{||\mathbf{v}_2||_B^2} = \sup_{\mathbf{v}_2\in V_2\setminus\{\mathbf{0}\}} \{1 - \frac{\mathbf{v}_2^\top[\mathbf{B}_2 - \mathbf{B}_{21}\mathbf{B}_1^{-1}\mathbf{B}_{12}]\mathbf{v}_2}{||\mathbf{v}_2||_B^2}\} =$$

$$= \sup_{\mathbf{v}_2\in V_2\setminus\{\mathbf{0}\}} \{1 - \frac{\mathbf{v}_2^\top\mathbf{S}\mathbf{v}_2}{\|\boldsymbol{\mathcal{P}}_2\mathbf{v}_2\|_{\mathbf{A}}^2}\} \le 1 - \inf_{\mathbf{v}_2\in V_2\setminus\{\mathbf{0}\}} \frac{\mathbf{v}_2^\top\mathbf{S}\mathbf{v}_2}{\|\boldsymbol{\mathcal{P}}_2\mathbf{v}_2\|_{\mathbf{A}}^2} \le 1 - c_{_s}^{-1}\,.$$

It is seen from the above relations that, if the constant c_s in (3.1) is sharp, then the estimate (3.4) is also sharp. In this case the sign $\le$ in the last inequality above is replaced by the sign of equality.

The inequalities of the theorem with the matrix $\mathbf{B}$ are the consequence of (3.4) and the equality $\|\boldsymbol{\mathcal{P}}_2\mathbf{v}_2\|_{\mathbf{A}}^2 = \mathbf{v}_2^T\mathbf{B}_2\mathbf{v}_2$. Now we introduce the block diagonal matrix $\mathbf{A}_{_S} = \mathrm{diag}\,[\mathbf{A}_1, \mathbf{S}]$ and note that an equivalent definition of $\boldsymbol{\mathcal{A}}$ is

$$\boldsymbol{\mathcal{A}} = \mathbf{C}^{-T}\mathbf{A}_{_S}\mathbf{C}^{-1}\,.$$

For this reason and in view of (3.1) and the pair of inequalities (3.3) for the matrix $\mathbf{B}$, we can write

$$\mathbf{A} = \mathbf{C}^{-T}\mathbf{B}\,\mathbf{C}^{-1} \le \overline{c}_B\mathbf{C}^{-T}\mathbf{B}_{\mathrm{diag}}\mathbf{C}^{-1} \le \overline{c}_B c_{_S}\mathbf{C}^{-T}\mathbf{A}_{_S}\mathbf{C}^{-1} = \overline{c}_B c_{_S}\boldsymbol{\mathcal{A}}\,,$$

and

$$\mathbf{A} \ge \underline{c}_B\mathbf{C}^{-T}\mathbf{B}_{\mathrm{diag}}\mathbf{C}^{-1} \ge \underline{c}_B\mathbf{C}^{-T}\mathbf{A}_{_S}\mathbf{C}^{-1} = \underline{c}_B\boldsymbol{\mathcal{A}}.$$

This proves the theorem. □

Corollary 3.1. *Suppose matrices $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{S}}$ satisfy the inequalities*

$$\underline{c}_A\boldsymbol{\mathcal{A}}_1 \le \mathbf{A}_1 \le \overline{c}_A\boldsymbol{\mathcal{A}}_1\,, \qquad \underline{c}_S\boldsymbol{\mathcal{S}} \le \mathbf{S} \le \overline{c}_S\boldsymbol{\mathcal{S}}\,, \tag{3.6}$$

and $\boldsymbol{\mathcal{A}}^{-1} = \boldsymbol{\mathcal{A}}_1^{+} + \boldsymbol{\mathcal{P}}_2\boldsymbol{\mathcal{S}}^{-1}\boldsymbol{\mathcal{P}}_2^T$. Then

$$\mathrm{cond}[\,\boldsymbol{\mathcal{A}}^{-1}\mathbf{A}] \le c_{_S}\frac{1+\sqrt{1-c_{_S}^{-1}}}{1-\sqrt{1-c_{_S}^{-1}}}\frac{\max(\overline{c}_A, \overline{c}_S)}{\min(\underline{c}_A, \underline{c}_S)} \le 4c_{_S}^2\frac{\max(\overline{c}_A, \overline{c}_S)}{\min(\underline{c}_A, \underline{c}_S)}\,.$$

If $\boldsymbol{\mathcal{S}}$ is constructed in a way of approximation of the matrix $\mathbf{B}_2$, and instead of inequalities (3.6) for $\mathbf{S}$ there hold the inequalities $\underline{c}_S\boldsymbol{\mathcal{S}} \le \mathbf{B}_2 \le \overline{c}_S\boldsymbol{\mathcal{S}}$, then

$$\mathrm{cond}[\,\boldsymbol{\mathcal{A}}^{-1}\mathbf{A}] \le \overline{c}_B \max(\overline{c}_A, \overline{c}_S)/\underline{c}_B \min(\underline{c}_A, \underline{c}_S) \le 4c_s \max(\overline{c}_A, \overline{c}_S)/\min(\underline{c}_A, \underline{c}_S)\,.$$

Proof. The corollary directly follows from (3.1), (3.3) and (3.6). □

A direct approximation of $\mathbf{B}_2$ at defining the interface preconditioner $\boldsymbol{\mathcal{S}}$ may give better condition estimates, especially if the prolongation operator is not close to a descrete harmonic. For a suitable set of conditions, one can adopt the bounds

$$\underline{c}_A \mathbf{v}_1^\top \boldsymbol{\mathcal{A}}_1 \mathbf{v}_1 \le \mathbf{v}^\top \mathbf{B} \mathbf{v}\,, \qquad \mathbf{A}_1 \le \overline{c}_A \boldsymbol{\mathcal{A}}_1\,, \qquad \underline{c}_S \boldsymbol{\mathcal{S}} \le \mathbf{S}\,, \qquad \mathbf{B}_2 \le \overline{c}_S \boldsymbol{\mathcal{S}}\,. \tag{3.7}$$

Corollary 3.2. *Under conditions (3.7), we have the bound*

$$\mathrm{cond}\,[\,\boldsymbol{\mathcal{A}}^{-1}\mathbf{A}] \le 2(\underline{c}_A^{-1} + \underline{c}_S^{-1}) \max(\overline{c}_A, \overline{c}_S)\,. \tag{3.8}$$

Proof. For the preconditioner $\boldsymbol{\mathcal{B}} = \mathrm{diag}\,[\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{S}}]$ of $\mathbf{B}$, the same energy equivalence estimates hold as for the preconditioner $\boldsymbol{\mathcal{A}}$ of $\mathbf{A}$, from where we immediately get (3.8). □

Remark 3.1. *Assume $\mathbf{A}$ be a nonnegative symmetric matrix and $\mathbf{A}_1$ be positive. Then Theorem 3.1 retains with $\boldsymbol{\mathcal{A}}^+ := \mathbf{A}_1^+ + \boldsymbol{\mathcal{P}}_2 \mathbf{S}^+ \boldsymbol{\mathcal{P}}_2^T$.*

3.1.2 Influence of main components on the bounds of relative spectrum

Now we are ready to estimate the influence of main components of the DD Dirichlet-Dirichlet algorithm on the relative condition number of the DD preconditioner $\boldsymbol{\mathcal{K}}$.

Lemma 3.1. *Let inequalities (2.20) for the preconditioner $\boldsymbol{\mathcal{K}}_I$, inequalities (2.22) for the prolongation operators $\mathbf{P}_{V_{B_j}\to V_j}$ and inequalities (2.31) for the Schur complement preconditioner $\boldsymbol{\mathcal{S}}_B$ hold. Then*

$$\underline{\gamma}\boldsymbol{\mathcal{K}} \le \mathbf{K} \le \overline{\gamma}\boldsymbol{\mathcal{K}} \tag{3.9}$$

with

$$\underline{\gamma} \ge \frac{\underline{\gamma}_I}{2(1 + c_{P_B}^2 + \underline{\gamma}_B^{-1}\underline{\gamma}_I)}\,, \qquad \overline{\gamma} \le 2\max(\overline{\gamma}_I, c_{P_B}^2 \overline{\gamma}_B)\,. \tag{3.10}$$

Proof. Let us consider $\mathbf{v} \in V$ represented in two forms $\mathbf{v} = \mathbf{v}_I + \mathbf{v}_B = \overline{\mathbf{v}}_I + \overline{\mathbf{v}}_B$, where $\overline{\mathbf{v}}_B := \mathbf{P}_{V_B\to V}\mathbf{v}_B$ and $\overline{\mathbf{v}}_I := \mathbf{v} - \overline{\mathbf{v}}_B$. Then by the left inequality (2.20), estimate of the square of a sum by the sum of the squares and by (2.22), we have

$$\overline{\mathbf{v}}_I^\top \boldsymbol{\mathcal{K}}_I \overline{\mathbf{v}}_I \le 2\underline{\gamma}_I^{-1}(\mathbf{v}^\top \mathbf{K}\mathbf{v} + \overline{\mathbf{v}}_B^\top \mathbf{K} \overline{\mathbf{v}}_B) \le 2\underline{\gamma}_I^{-1}(\mathbf{v}^\top \mathbf{K}\mathbf{v} + c_{P_B}^2 \|\mathbf{v}_B\|_{\mathbf{S}_B}^2) \le 2\underline{\gamma}_I^{-1}(1 + c_{P_B}^2)\mathbf{v}^\top \mathbf{K}\mathbf{v}\,.$$

Another needed bound is

$$\|\mathbf{P}_{V_{B_j}\to V_j}\mathbf{v}_{B_j}\|_{\mathbf{K_j}}^2 \le c_{P_B}^2 \overline{\gamma}_B \|\mathbf{v}_{B_j}\|_{\boldsymbol{\mathcal{S}}_{B_j}}\,, \qquad \forall\, \mathbf{v}_{B_j} \in \mathbf{V}_{B_j}^2\,, \tag{3.11}$$

which is a consequence of (2.22) and the right inequality (2.31). Now we see that we can apply Corollary 3.2 by setting

$$\underline{c}_A = \frac{\underline{\gamma}_I}{2(1 + c_{P_B}^2)}\,, \qquad \overline{c}_A = \overline{\gamma}_I\,, \qquad \underline{c}_S = \underline{\gamma}_B\,, \qquad \overline{c}_S = c_{P_B}^2 \overline{\gamma}_B\,,$$

which completes the proof. □

Lemma 3.2. *Suppose, there hold inequalities (2.32) for the wire basket Schur complement preconditioner, the inequalities*

$$\underline{\widetilde{\gamma}}_F \boldsymbol{\mathcal{S}}_F \leq \mathbf{S}_B\,, \qquad \mathbf{S}_F \leq \overline{\gamma}_F \boldsymbol{\mathcal{S}}_F \tag{3.12}$$

for the face subproblem preconditioner, and the inequality

$$\|\mathbf{P}_{V_W \to V_B} \mathbf{v}_W\|^2_{\mathbf{S}_B} \leq c_{P_W} \mathbf{v}_W^\top \boldsymbol{\mathcal{S}}_W^F \mathbf{v}_W \tag{3.13}$$

for the prolongation matrix from the wire basket on the interface boundary.

Then

$$\underline{\gamma}_B \boldsymbol{\mathcal{S}}_B \leq \mathbf{S}_B \leq \overline{\gamma}_B \boldsymbol{\mathcal{S}}_B\,, \tag{3.14}$$

where

$$\underline{\gamma}_B \geq \frac{1}{\underline{\widetilde{\gamma}}_F^{-1} + \underline{\gamma}_W^{-1}}\,, \qquad \overline{\gamma}_B \leq 2\max(\overline{\gamma}_F, c_{P_W})\,, \tag{3.15}$$

Proof. With

$$\underline{c}_A = \underline{\widetilde{\gamma}}_F\,, \qquad \overline{c}_A = \overline{\gamma}_F\,, \qquad \underline{c}_S = \underline{\gamma}_W\,, \qquad \overline{c}_S = c_{P_W}\,,$$

the proof directly follows from Corollary 3.2. □

Now, we will formulate an abstract estimates of the relative spectral bounds for DD preconditioner.

Theorem 3.2. *Under the assumptions of Lemmas 3.1 and 3.2, the inequalities (3.9) hold with*

$$\underline{\gamma} \geq \frac{\underline{\gamma}_I}{2(1 + c_{P_B}^2 + \underline{\gamma}_I(\underline{\widetilde{\gamma}}_F^{-1} + \underline{\gamma}_W^{-1}))}\,, \qquad \overline{\gamma} \leq 4\max(\frac{\overline{\gamma}_I}{2},\, c_{P_B}^2 \overline{\gamma}_F,\, c_{P_B}^2 c_{P_W})\,. \tag{3.16}$$

Proof. For the proof, it is sufficient to plug (3.15) in (3.10). □

3.2 Dependence of condition number on discretization and decomposition parameters

In order to find out the dependence of $\underline{\gamma}, \overline{\gamma}$ on parameters of the FE discretization and domain decomposition, we have to find out the dependence on these parameters of the values entering the right parts of the bounds (3.16). For this purpose, we start from summarizing all our specific assumptions on the component preconditioners and prolongation operators for DD preconditioner (2.8), (2.9). First of all, we fix the set of assumptions allowing to elucidate the dependence on $\hbar$. This dependence, which though can be characterized as weak, is irreducible within the structure of DD preconditioner under consideration.

i) Assumptions $\mathcal{A}.1 - \mathcal{A}.5$ are fulfilled.

ii) Interior subdomain preconditioners $\boldsymbol{\mathcal{K}}_{I_j}$ satisfy inequalities (2.20).

iii) The Schur complement preconditioner $\boldsymbol{\mathcal{S}}_F$ is defined according to (2.33), (2.42) with inequalities (2.41) hold.

iv) The second level Schur complement preconditioner $\boldsymbol{\mathcal{S}}_W^F$ is defined according to (2.51), (2.52).

v) The prolongation matrix $\mathbf{P}_{V_B \to V}$ from the inter-subdomain boundary upon the whole computational domain satisfies (2.22).

vi) The prolongation matrix $\mathbf{P}_{V_{W_j} \to V_{B_j}}$ from the wire basket upon the inter-subdomain boundary is defined by (2.60).

Most multipliers before matrices and before norms, entering the inequalities participating in conditions i) - vi), are considered as constants independent of $\hbar$. Specifically, these are constants $\underline{\alpha}_D, \underline{\theta}_D, c_\circ, \wp_\circ, \theta_\circ$ from the shape regularity conditions for mappings $x = \Upsilon_j(y) : \overline{\tau}_{0,j} \to \overline{\Omega}_j$ and size and shape quasiuniformity conditions for the reference subdomains, constants $\hat{\alpha}^{(1)}, \hat{\theta}$ from the quasiuniformity conditions for FE discretizations of reference subdomains, constants $\underline{\gamma}_I, \overline{\gamma}_I, \underline{\gamma}_{00}, \overline{\gamma}_{00}$ from the energy inequalities (2.20), (2.41), and the constant c_{P_B} from (2.22), which bounds the energy norm of the prolongation operator from the inter-subdomain boundary.

Theorem 3.3. *Under the stated conditions* i) - vi)*, the inequalities*

$$\frac{\underline{c}}{(1 + \log \hbar^{-1})^2} \boldsymbol{\mathcal{K}} \le \mathbf{K} \le \overline{c}\, \boldsymbol{\mathcal{K}} \tag{3.17}$$

hold with positive constants $\underline{c}, \overline{c}$ independent of $\hbar$.

Proof. The bound for the factor c_{P_B} in the inequalities (2.21) and (2.22) by a constant follow from Lemma 2.3 and the definition of the prolongation matrix $\mathbf{P}_{V_B \to V}$. Therefore, for the proof of the theorem, it is necessary to find out the dependence of values $\underline{\gamma}_W, c_{P_W}, c_F, \overline{\gamma}_F$ in (3.16) on $\hbar$. It will be done with the use of subsidiary results, collected in Subsection 3.3.

1. Bound for $\underline{\gamma}_W$. Since the preconditioner $\boldsymbol{\mathcal{S}}_W^F$ defined in (2.47), (2.51), (2.52), and the preconditioner defined in (2.47), (2.48), (2.52) are spectrally equivalent, it is sufficient to consider only the latter. According to (2.47), (2.48) and inequality (3.36) of Lemma 3.3 in Subsection 3.3, we have

$$\mathbf{v}_{W_j}^\top \boldsymbol{\mathcal{S}}_{W_j}^F \mathbf{v}_{W_j} = \varrho_j\, H_{D,j}\, (1 + \log \hbar^{-1}) |v_{\hat{W}_j}|^2_{0,\hat{W}_j} \le c\, \varrho_j\, H_{D,j}\, (1 + \log^2 \hbar^{-1}) |v_{\hat{B}_j}|^2_{1/2,\partial\tau_{0,j}} , \tag{3.18}$$

where $v_{\hat{W}_j}$ is the FE function on $\hat{W}_j$, corresponding to vector $\mathbf{v}_{W_j}$, and $v_{\hat{B}_j} \in \mathbb{V}_{B_j}$ is any FE function coinciding with the entries of $\mathbf{v}_{W_j}$ at the nodes on $\hat{W}_j$. By the conditions of shape regularity (2.4) for mappings $x = \Upsilon_j(y) : \tau_{0,j} \to \Omega_j$, one has

$$c_1 H_{D,j} |v_{\hat{B}_j}|^2_{1/2,\partial\tau_{0,j}} \le |u_{B_j}|^2_{1/2,\partial\Omega_j} \le c_2 H_{D,j} |v_{\hat{B}_j}|^2_{1/2,\partial\tau_{0,j}} \tag{3.19}$$

with the constants $c_k = c_k(\underline{\alpha}_D, \underline{\theta}_D)$ and $u_{B_j}(\Upsilon_j(y)) = v_{\hat{B}_j}(y)$. Therefore, for any $\mathbf{v}_{B_j}$ with the subvector $\mathbf{v}_{W_j}$, it follows by (3.18), (3.19) and (2.24) that

$$\mathbf{v}_{W_j}^\top \boldsymbol{\mathcal{S}}_{W_j}^F \mathbf{v}_{W_j} \le c \varrho_j\, (1 + \log \hbar^{-1})^2 |u_{B_j}|^2_{1/2,\partial\Omega_j} \le c (1 + \log^2 \hbar^{-1}) \mathbf{v}_{B_j}^\top \mathbf{S}_{B_j} \mathbf{v}_{B_j} . \tag{3.20}$$

The inequality (3.20) holds for any vector $\mathbf{v}_{B_j}$ and, consequently, for the vector $\mathbf{v}'_{B_j}$ minimizing the norm $\|\cdot\|_{\mathbf{S}_{B_j}}$ among all vectors $\mathbf{v}_{B_j}$ with the fixed subvector $\mathbf{v}_{W_j}$. This approves the left inequality (2.32) for subdomain Ω_j with the specific value of $\underline{\gamma}_W$:

$$\underline{\gamma}_W \boldsymbol{\mathcal{S}}^F_{W_j} \leq \mathbf{S}^F_{W_j}\,, \qquad \underline{\gamma}_W = \frac{c}{1+\log^2 \hbar^{-1}} > 0\,, \tag{3.21}$$

and the inequality (2.32) follows from (3.21) by assembling.

2. Bound for c_{P_W}. Again it is sufficient to give the proof for one of the preconditioners of Section 2.5, namely for the one defined in (2.47), (2.48), (2.52). Let $\mathbf{v}_W = \mathbf{P}_{V_W \to V_B}\mathbf{v}_W$ and $v_W \in \mathbb{V}_W$, $v_B \in \mathbb{V}_B$ be the corresponding FE functions. Application of the right inequality (2.24), right inequality (3.19), the inequality (3.35) of Lemma 3.3 and the definition of $\boldsymbol{\mathcal{S}}^F_{W_j}$ by (2.47), (2.48) yield the bound:

$$\begin{aligned} \|\mathbf{P}_{V_{W_j}\to V_{B_j}}\mathbf{v}_{W_j}\|^2_{\mathbf{S}_{B_j}} &\leq c_2 \overline{\gamma}_S \varrho_j H_{D,j} |v_{\hat{B}_j}|^2_{1/2,\partial\tau_{0,j}} \leq \\ &\leq c\varrho_j H_{D,j}\, |v_{\hat{W}}|^2_{0,\hat{W}_j} \leq \tfrac{c}{1+\log \hbar^{-1}}\, \mathbf{v}^\top_{W_j} \boldsymbol{\mathcal{S}}^F_{W_j} \mathbf{v}_{W_j}\,. \end{aligned} \tag{3.22}$$

This means that

$$c_{P_W} \leq c/(1+\log \hbar^{-1}) \tag{3.23}$$

and c_{P_W} is small for small $\hbar$.

3. Bound for $\widetilde{\underline{\gamma}}_F$. Alongside with (3.49), for the same FE functions, we get by the factor space argument:

$${}_{00}|v_F|^2_{1/2,\mathcal{F}} \leq c_{\mathcal{F}}(1+\log^2 \hbar^{-1})|v|^2_{1/2,\partial\tau_0}\,. \tag{3.24}$$

In the face preconditioner $\boldsymbol{\mathcal{S}}_{F_j}$ for subdomain Ω_j, we consider one block $\boldsymbol{\mathcal{S}}_{F^p_j}$ on its diagonal corresponding to one of the faces F^p_j. From the definition of this block, see (2.38), (2.41), (2.42), it follows

$$\mathbf{v}^\top_{F^p_j} \boldsymbol{\mathcal{S}}_{F^p_j} \mathbf{v}_{F^p_j} = \rho_j\, H_{D,j} \mathbf{v}^\top_{F^p_j} \boldsymbol{\mathcal{S}}_{00,\hat{F}^p_j} \mathbf{v}_{F^p_j} \leq \underline{\gamma}^{-1}_{00} \rho_j\, H_{D,j} |v_{\hat{F}^p_j}|^2_{1/2,\hat{F}^p_j}\,. \tag{3.25}$$

To arbitrary vector $\mathbf{v}_{F^p_j} \in V_{F^p_j}$ correspond a unique FE functions $v_{\hat{F}^p_j} \in \mathbb{V}_F(\hat{F}^p_j)$ and $u_{F^p_j} \in \mathcal{V}_F(F^p_j)$ such that $u_{F^p_j}(\Upsilon_j(y)) = v_{\hat{F}_j}(y)$. By application of (3.24), conditions (2.4) of shape regularity for mappings $x = \Upsilon_j(y) : \tau_{0,j} \to \Omega_j$, and (2.24)

$$\begin{aligned} \underline{\gamma}^{-1}_{00} \rho_j\, H_{D,j} |v_{\hat{F}^p_j}|^2_{1/2,\hat{F}^p_j} &\leq c_{\mathcal{F}} \underline{\gamma}^{-1}_{00} \rho_j\, H_{D,j} (1+\log^2 \hbar^{-1})|v|^2_{1/2,\partial\tau_{0,j}} \leq \\ \leq cc_{\mathcal{F}} \underline{\gamma}^{-1}_{00} \rho_j\, (1+\log^2 \hbar^{-1})|u|^2_{1/2,\partial\Omega_j} &\leq cc_{\mathcal{F}} \underline{\gamma}^{-1}_{00} \underline{\gamma}^{-1}_S (1+\log^2 \hbar^{-1}) \mathbf{v}^\top_j \mathbf{S}_j \mathbf{v}_j\,. \end{aligned} \tag{3.26}$$

From (3.25) and (3.26), the definition of the face preconditioner $\boldsymbol{\mathcal{S}}_{F_j}$, and the shape regularity of polygons Ω_j, it follows that

$$\widetilde{\underline{\gamma}}^{-1}_F \leq cc_{\mathcal{F}} \underline{\gamma}^{-1}_{00} \underline{\gamma}^{-1}_S (1+\log^2 \hbar^{-1})\,. \tag{3.27}$$

4. Bound for $\overline{\gamma}_F$. Let $\mathcal{F} = F^p$ be one of the faces of the reference subdomain $\tau_{0,j}$, which satisfy the shape regularity conditions. We remind that for $v_{\mathcal{F}} \in H^{1/2}(\tau_{0,j})$, $\operatorname{supp}(v) = \mathcal{F}$, one has

$$\|v_{\mathcal{F}}\|_{1/2,\tau_{0,j}} \approx {}_{00}|v_{\mathcal{F}}|_{1/2,\mathcal{F}} \tag{3.28}$$

and that the number of faces of each subdomain is bounded independently of j. We consider an arbitrary vector $\mathbf{v}_{F_j}$ and two FE functions corresponding to it, namely $v_{\hat{F}_j} \in \mathbb{V}_F(\hat{F}_j)$ and $u_{F_j} \in \mathcal{V}_F(F_j)$ such that $u_{F_j}(\Upsilon_j(y)) = v_{\hat{F}_j}(y)$. Taking into account (2.24), the Cauchy inequality, the shape regularity conditions for mappings $x = \Upsilon_j(y) : \tau_{0,j} \to \Omega_j$, and (3.28), we can write

$$\begin{gathered}\|\mathbf{v}_{F_j}\|_{\mathbf{S}_{F_j}} \leq \overline{\gamma}_S \rho_j |u_{F_j}|_{1/2,\partial\Omega_j} \leq c\overline{\gamma}_S \rho_j H_{D,j} |v_{F_j}|_{1/2,\tau_{0,j}} \leq \\ \leq c\overline{\gamma}_S \rho_j H_{D,j} \sum_p |v_{F_j^p}|_{1/2,\tau_{0,j}} \approx c\overline{\gamma}_S \rho_j H_{D,j} \sum_p {}_{00}|v_{F_j^p}|_{1/2,F_j^p}\,.\end{gathered} \tag{3.29}$$

Taking into account the definition of the face preconditioner, and (2.41), (2.42), we conclude that

$$\|\mathbf{v}_{F_j}\|_{\mathbf{S}_{F_j}} \leq c\,\overline{\gamma}_S\,\overline{\gamma}_{00} \sum_p \|\mathbf{v}_{\hat{F}_j^p}\|_{\boldsymbol{\mathcal{S}}_{00,\hat{F}_j}}\,. \tag{3.30}$$

Together with two other relationships (2.34),(2.40) used at the definition of the face preconditioner, this bound implies

$$\overline{\gamma}_F \leq c\,\overline{\gamma}_S\,\overline{\gamma}_{00}\,. \tag{3.31}$$

Now, we can return directly to the proof of Theorem 3.3. It is completed by substitution in (3.16) the estimates, obtained above for $\underline{\gamma}_W, c_{P_W}, c_F, \overline{\gamma}_F$, and the estimate $c_{P_B} \leq 2$ for the constant in the inequalities (2.21) and (2.22), which follow from (2.29) and the definition of the prolongation matrix $\mathbf{P}_{V_B \to V}$. □

Suppose the preconditioners $\boldsymbol{\mathcal{K}}_I$ and $\boldsymbol{\mathcal{S}}_F$, satisfying assumptions of Theorem 3.3, are such that solving SLAE with each for the matrix requires not more than $\mathcal{O}(\hbar^{-3})$ a.o. Then, with the use of Theorem 3.3 and presented definitions and analysis of other components of BPS preconditioner, solving the SLAE with the matrix $\boldsymbol{\mathcal{K}}$ requires not more than $\mathcal{O}(\hbar^{-3}(1+\log^2\hbar^{-1})^2)$ a.o., see Remarks 2.1, 2.2 and conclusion on the cost of the prolongation (2.60).

3.3 Subsidiary results

The proof of Theorem 3.3, given in the preceding subsection, uses subsidiary results which are summarized below in one theorem and three lemmas. Among them Theorem 3.4 belongs to Xu (1989) [37, Theorem 2.1] and is presented without proof. It is related to the discrete Sobolev type inequalities for the limiting case, when the imbedding theorem for the Sobolev's spaces fails. Another example of inequalities of such type is inequality (3.36). For subdomains of simpler forms it can be found in several sources, see [34] for references, we prove it for convex hexahedrons under the set of the shape regularity conditions of Subsection 2.1. Below we prove also the bound for the prolongation operator from the wire basket and the bound for the FE face functions, *i.e.*, having nonzero nodal values only at the nodes living on faces of the subdomains of the decomposition.

Theorem 3.4. *Let $\Omega \in R^d$ be Lipshitz continuous domain and $p > 1$. Then for any $\epsilon, \lambda \in (0,1]$ and $v \in W_p^{d/p}(\Omega) \cap C^{0,\lambda}(\overline{\Omega})$, there exists such constant $c = c(d,p,\Omega)$, that*

$$\|v\|_{C(\overline{\Omega})} \leq c\,\{|\log\epsilon|^{1-\frac{1}{d}}\|v\|_{W_p^{d/p}(\Omega)} + \epsilon^\lambda \|v\|_{C^{0,\lambda}(\overline{\Omega})}\}\,. \tag{3.32}$$

Proof. We refer for the proof to Xu [37, Theorem 2.1] and to the earlier proof by the same author in [36] for the case $p = d = 2,\ \lambda = 1$. □

For the case $p = d = 2,\ \lambda = 1$ the inequality (3.32) can be written in the form

$$\|v\|_{L_\infty(\Omega)} \leq\ c\,\{|\log \epsilon|^{1/2}\|v\|_{H^1(\Omega)} + \epsilon\|v\|_{W^1_\infty(\Omega)}\}\,. \tag{3.33}$$

Before formulating next lemma, let us remind that $\mathbb{V}_j(\tau_{0,j})$ is the FE space on $\tau_{0,j}$, induced by the FE assemblage satisfying the qusiuniformity conditions, $\mathbb{V}_B(\partial\tau_{0,j})$ is the FE space of traces on $\partial\tau_{0,j}$, $\hat{W}_j$ is the union of edges and vertices of a polygon $\tau_{0,j}$ and $\hat{F}_j = \partial\tau_{0,j} \setminus \hat{W}_j$. By $|v|_{0,\hat{W}_j}$ is denoted the seminorm

$$|v|^2_{0,\hat{W}_j} = \inf_{c\in R} \|v - c\|^2_{0,\hat{W}_j}\,. \tag{3.34}$$

Lemma 3.3. *For any FE function* $v_W \in \mathbb{V}_W(\tau_{0,j})$, i.e., *vanishing at all nodes which do not belong* $\hat{W}_j$, *and any* $u_B \in \mathbb{V}_B(\partial\tau_0)$, *there hold inequalities*

$$|v_W|^2_{1/2,\partial\tau_{0,j}} \leq c|v_W|^2_{0,\hat{W}_j}\,, \tag{3.35}$$

$$|u_B|^2_{0,\hat{W}_j} \leq c(1 + \log \hbar^{-1})|u_B|^2_{1/2,\partial\tau_{0,j}} \tag{3.36}$$

with the constant $c =$ const *independent of* $\hbar$*.*

Clearly, (3.35) bounds the prolongation operator $\mathbf{P}_{V_{W_j}\to V_{B_j}}$ defined for one face in (2.60). Indeed, let us denote by $\hat{\mathcal{P}}_{\mathbb{V}_{W_j}\to\mathbb{V}_{B_j}}$ the prolongation operator in the FE subspace $\mathbb{V}_B(\partial\tau_{0,j})$ on the boundary of the reference subdomain and imply by ϕ_W the FE function specified on W_j and equal there to v_W. Then $v_W = \hat{\mathcal{P}}_{\mathbb{V}_{W_j}\to\mathbb{V}_{B_j}}\phi_W$, and a consequence of (3.35) is

$$\mathbf{v}_W^\top \mathbf{S}_W^F \mathbf{v}_W \leq |\hat{\mathcal{P}}_{\mathbb{V}_{W_j}\to\mathbb{V}_{B_j}} v_W|^2_{1/2,\partial\tau_{0,j}} \leq c|v_W|^2_{0,\hat{W}_j}\,. \tag{3.37}$$

At the same time, (3.36) characterizes traces on $\hat{W}_j$ of FE functions.

Proof. In view of the quasiunifomity conditions for FE discretizaton (2.1) or (2.2), the function v_W is distinct from zero only in $\mathcal{O}(\hbar)$-vicinity of the wire basket $\hat{W}_j$. The proof of the bound

$$\|v_W\|^2_{1,\tau_{0,j}} \leq c\|v_W\|^2_{0,\hat{W}_j}\,. \tag{3.38}$$

is completed in a similar way with the cases of subdomains of a simpler forms, see *e.g.,* Bramble *et al.* [3]-[6] and Toselli & Widlund [34]. From (3.38), by the trace theorem it follows

$$\|v_W\|^2_{1/2,\partial\tau_{0,j}} \leq c\|v_W\|^2_{0,\hat{W}_j}\,. \tag{3.39}$$

For obtaining (3.35), it is left to apply the factor space argument.

The inequality (3.36) for subdomains of simpler types can be found in many sources, see Bramble *et al.* [6], Bramble and Xu [8]. An easy proof of it for the case $\tau_{0,j} = \tau_0 := \{x :\ 0 <$

$x_1, x_2, x_3 < 1\}$ is based on the inequality (3.33). We consider $u \in H^1(\tau_0)$ and write (3.33) for the point $x = (0,0,x_3)$, $0 < x_3 < 1$, of the square $\Pi_{x_3} = \{x : 0 < x_1, x_2 < 1,\ x_3 \equiv \text{const}\}$:

$$|u(0,0,x_3)|^2 \le\ c\,\{|\log\epsilon|\,\|u\|^2_{H^1(\Pi_{x_3})} + \epsilon^2\|u\|^2_{W^1_\infty(\Pi_{x_3})}\}\,. \tag{3.40}$$

The inequality

$$\|u\|^2_{0,\mathcal{E}} \le c\,\{|\log\epsilon|\,\|u\|^2_{H^1(\tau_{0,j})} + \epsilon^2\|u\|^2_{W^1_\infty(\tau_{0,j})}\}$$

for the edge $\mathcal{E} = \{x = (0,0,x_3),\ 0 < x_3 < 1\,\}$ follows by integrating (3.40) with respect to x_3. In the case of the FE h-version, we can use some trivial inequalities. Namely, if $v \in \mathbb{V}_B(\tau_{0,j})$ and τ is the domain of any finite element of a quasiuniform reference FE assemblage on $\tau_{0,j}$, then

$$\hbar^{3/2}|v|_{L_\infty(\tau)} \le c\|v\|_{0,\tau}\,, \qquad \hbar^{3/2}|v|_{W^1_\infty(\tau)} \le c\,|v|_{H^1(\tau)}\,, \qquad \forall\ v \in \mathbb{V}_B(\tau_{0,j})\,, \tag{3.41}$$

see, *e.g.,* Ciarlet [11] and Korneev [17]. Now, by taking $\epsilon = \hbar^2$ and applying (3.41), one comes to

$$\|u\|^2_{0,\mathcal{E}} \le c\,(1 + |\log\hbar|)\,\|u\|^2_{1,\tau_{0,j}}\,, \tag{3.42}$$

and summation overall edges, belonging to $\hat{W}_j$, allows to conclude that

$$\|u\|^2_{0,\hat{W}_j} \le c\,(1 + |\log\hbar|)\,\|u\|^2_{1,\tau_{0,j}}\,, \tag{3.43}$$

Inequality (3.43) also holds for discrete harmonic functions, whence and from (2.24) it follows the bound

$$\|u\|^2_{0,\hat{W}_j} \le c\,(1 + |\log\hbar|)\,\|u\|^2_{1/2,\partial\tau_{0,j}}\,. \tag{3.44}$$

In order to obtain (3.36), it is sufficient to use the definition of the seminorm $|\cdot|_{0,\mathcal{E}}$ and the characterizations of the seminorm in the space $H^{1/2}(\partial\tau_{0,j})$, which is $|u|^2_{1/2,\partial\tau_{0,j}} \approx \inf_{c\in R^1}\|u + c\|^2_{1/2,\partial\tau_{0,j}}$.

Now we will prove the inequality (3.36) for the case of polyhedrons $\tau_{0,j}$, satisfying our assumptions of size and shape quasiuniformity. Again, we start from the consideration of one edge, and, without restriction of generality, we can assume that such an edge is $\mathcal{E} = \{x = (0,0,x_3),\ 0 < x_3 < 1\,\}$. For simplicity the length of the edge is adopted to be unity, since according to the shape regularity condition for the reference subdomains, the length of each edge of the reference polyhedron $\tau_{0,j}$ is $\mathcal{O}(1)$. Let $\pi_{j,\mathcal{E}}$ be the unite cube, which has $\mathcal{E}$ for the edge and which intersection with $\tau_{0,j}$ is the largest. If $\pi_{j,\mathcal{E}} \cap \tau_{0,j} \ne \pi_{j,\mathcal{E}}$, then we consider the least c-vicinity $S_{j,c}$ of $\tau_{0,j}$, which covers $\pi_{j,\mathcal{E}}$ and denote by $\mathcal{T}_{j,\text{ext}}$ an extension of the reference triangulation $\mathcal{T}_{0,j}$ on $S_{j,c}$, satisfying the quasiuniformity conditions (2.1), (2.2). More exactly, by $\mathcal{T}_{j,\text{ext}}$ is understood the triangulation, each nest of which has nonempty intersection with $S_{j,c}$, so that $S_{j,c} \subset \tau_{j,\text{ext}}$, where $\tau_{j,\text{ext}}$ is the domain of the extended triangulation. We consider the FE space $\mathbb{V}_j(\tau_{j,\text{ext}})$, induced by the triangulation $\mathcal{T}_{j,\text{ext}}$, and its restriction $\mathbb{V}_j(\pi_{j,\mathcal{E}})$ to the unite cube $\pi_{j,\mathcal{E}}$ and note that for any function $v \in \mathbb{V}_j(\tau_{0,j})$ there exists an expansion $v_{\text{ext}} \in \mathbb{V}_j(\tau_{j,\text{ext}})$ for which

$$\|v_{\text{ext}}\|_{1,\tau_{j,\text{ext}}} \le\ c\,\|v\|_{1,\tau_{0,j}}\,. \tag{3.45}$$

Note, that under our assumptions the constant does not depend on j.

For small $\epsilon = \mathcal{O}(\hbar^k),\ k > 1,$ which we use, the inequality (3.33) can be rewritten as

$$\|v\|_{L_\infty(\Omega)} \leq\ c\,\{|\log\epsilon|^{1/2}\|v\|_{H^1(\Omega)} + \epsilon|v|_{W^1_\infty(\Omega)}\}\,, \tag{3.46}$$

with the term $\epsilon\|v\|_{L_\infty(\Omega)}$ shifted from the right to the left part of (3.33), that changes only the constant. Let $\sqcap_{x_3}$ be the intersection of the plane $x_3 = \text{const}$ with $\pi_{j,\mathcal{E}}$. Then by integrating (3.46), written for $\sqcap_{x_3}$ and squared, over $x_3 \in (0, 1)$, we obtain

$$\begin{aligned}\|v\|^2_{2,\mathcal{E}} \leq\ & c\,\{|\log\epsilon| \textstyle\int_0^1 \|v_{\text{ext}}\|^2_{1,\sqcap_{x_3}}\, dx_3 + \epsilon^2 \int_0^1 |v_{\text{ext}}|^2_{W^1_\infty(\sqcap_{x_3})}\, dx_3 \leq \\ & \leq\ c\,\{|\log\epsilon|\|v_{\text{ext}}\|^2_{1,\pi_{j,\mathcal{E}}} + \epsilon^2|v_{\text{ext}}|^2_{W^1_\infty(\pi_{j,\mathcal{E}})}\}\,.\end{aligned} \tag{3.47}$$

Inequalities (3.41) hold for any $w \in \mathbb{V}_j(\tau_{j,\text{exp}})$ and $\tau \in \mathcal{T}_{j,\text{ext}}$, so that the choice of $\epsilon = \hbar^2$ and application of (3.41), (3.45) yields (3.43):

$$\|v\|^2_{0,\mathcal{E}} \leq c\,(1 + |\log\hbar|)\,\|v_{\text{ext}}\|^2_{1,\tau_{j,\text{ext}}} \leq c\,(1 + |\log\hbar|)\,\|v\|^2_{1,\tau_{0,j}}\,. \tag{3.48}$$

The transition to (3.36) is the same, as in the case of $\tau_{0,j} = \tau_0$. □

For simplification of notations, in the next Lemma and mostly in its proof, we omit index j, so that, *e.g.*, τ_0 stands for $\tau_{0,j}$.

Lemma 3.4. *Let τ_0 be the size and shape regular convex polyhedron, its triangulation is quasiuniform, $\mathcal{F}$ be any face of it. Let $v \in \mathbb{V}_B(\tau_0)$ be represented by the sum $v = v_F + v_W$ of functions $v_F, v_W \in \mathbb{V}_B(\partial\tau_0)$, which nodal values are not zeroes only on $\mathcal{F}$ and $\mathcal{E}$, respectively. Then*

$$_{00}\|v_F\|^2_{1/2,\mathcal{F}} \leq c(1 + \log^2\hbar^{-1})\|v\|^2_{1/2,\partial\tau_0}\,. \tag{3.49}$$

Proof. The proof can be produced with use of the representation of the norm in $H^{1/2}(\partial\tau_0)$ by the expression (1.7) for $\partial\Omega = \partial\tau_0$

$$\|v\|_{1/2,\partial\tau_0} = \left(\int_{\partial\tau_0}\int_{\partial\tau_0} \frac{(v(x) - v(y))^2}{|x-y|^3}\, ds(x)ds(y) + \|v\|^2_{0,\partial\tau_0}\right)^{1/2}\,, \tag{3.50}$$

in which double integral for the function $v_F \in \mathbb{V}_B(\partial\tau_0)$, vanishing on $\partial\tau_0 \setminus \mathcal{F}$, is represented in the form

$$|v_F|^2_{1/2,\partial\tau_0} = \int_{\mathcal{F}}\int_{\mathcal{F}} \frac{(v_F(x) - v_F(y))^2}{|x-y|^3}\, ds(x)ds(y) + 2\int_{\mathcal{F}}\int_{\partial\tau_0\setminus\mathcal{F}} \frac{v_F^2(y)}{|x-y|^3}\, ds(x)\, ds(y)\,. \tag{3.51}$$

Since by Lemma 3.3

$$\begin{aligned}&\int_{\mathcal{F}}\int_{\mathcal{F}} \frac{(v_F(x) - v_F(y))^2}{|x-y|^3}\, ds(x)ds(y) \leq \\ &\leq |v_F|^2_{1/2,\partial\tau_0} = |v - v_E|^2_{1/2,\partial\tau_0} \leq 2(|v|^2_{1/2,\partial\tau_0} + |v_E|^2_{1/2,\partial\tau_0}) \leq c(1 + \log h^{-1})|v|^2_{1/2,\partial\tau_0}\,,\end{aligned} \tag{3.52}$$

it is left to estimate only second term in the right part of (3.51).

The set $\partial\tau_0 \setminus \mathcal{F}$ can be separated in two parts

$$\mathcal{B}_k\,, \qquad k=1,2\,, \qquad \mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing\,,$$

such that $\overline{\mathcal{B}}_1$ includes the closures of the faces, having with the face $\mathcal{F}$ common vertices or edges and vertices, and $\overline{\mathcal{B}}_2 = \partial\tau_0 \setminus (\overline{\mathcal{B}}_1 \cup \mathcal{F})$. For instance, we have $\mathcal{B}_2 = \varnothing$, if $\tau_{0,j}$ is a tetrahedra, and $\mathcal{B}_2$ contains only the edge opposite to $\mathcal{F}$, if $\tau_{0,j}$ is a cube. Additionally, we subdivide $\mathcal{B}_1$ in two nonoverlapping subsets $\mathcal{B}_1 = \mathcal{B}_{11} \cup \mathcal{B}_{12}$, where $\mathcal{B}_{12}$ contains faces which have common edges with $\mathcal{F}$. Note that $\mathcal{B}_{11} = \varnothing$ for a tetrahedra or a cube. Due to the size and shape regularity of the reference subdomains, we conclude that $\operatorname{dist}[\mathcal{F}, \mathcal{B}_2] \geq c_\vartriangle$ with $c_\vartriangle = \text{const}$. Therefore,

$$\int_{\mathcal{F}} \int_{\mathcal{B}_2} \frac{u^2(y)}{|x-y|^3}\, ds(x)\, ds(y) \leq (c/c_\vartriangle^3)\, \|u\|_{0,\mathcal{F}}^2\,. \tag{3.53}$$

Turning to faces in $\mathcal{B}_1$, we can assume without loss of generality that the face $\mathcal{F}$ is in the plane $x_3 \equiv 0$. Let $\mathcal{F}' = \mathcal{F}'_k$, $k = 1, 2, .., \Bbbk$, be the faces in $\mathcal{B}_1$ and $\pounds = \pounds_k$ be the straight line which is the intersection of the plane, containing the face $\mathcal{F}'$, with the plane $x_3 \equiv 0$. Let for definiteness, $k = 1, 2, .., \Bbbk_\circ$ correspond to $\mathcal{F}' = \mathcal{F}'_k \subset \mathcal{B}_{12}$. Taking again into account that $\tau_{0,j}$ have diameters equal to 1 and are shape regular, in a sense explained in Subsection 2.1, one can establish the inequality

$$\int_{\mathcal{B}_{12}} \frac{1}{|x-y|^3}\, ds(x) \leq \overline{c} \sum_{k=1}^{\Bbbk_2} \frac{1}{\operatorname{dist}[y, \pounds_k]}\,. \tag{3.54}$$

In order to prove (3.54), we consider one face $\mathcal{F}' = \mathcal{F}'_k$, which have a common edge $\mathcal{E}$ with $\mathcal{F}$, and without loss of generality assume that the line $\pounds$ is the axis x_2. It is easy to see, that it is sufficient to consider only the case, when the angle between faces $\mathcal{F}$ and $\mathcal{F}'$ is not greater, than $\pi/2$. In the opposite case, a stronger bound holds.

Let $z = \mathcal{Z}(x) : \tau_{0,j} \to \tau'_{0,j}$ be a nondegenerate mapping, which satisfies the conditions:

A) $\mathcal{Z}(\mathcal{F}) = \mathcal{F}$, whereas the plane of the face $\mathcal{F}'$ is rotated around the axis x_2 at the smallest angle sufficient to make it orthogonal to the plane of the face $\mathcal{F}$,

B) the Jacobian $|\mathbb{J}(\mathcal{Z})|$ of the Jacobi matrix $\mathbb{J}(\mathcal{Z}) = (\partial z_k/\partial x_l)_{k,l=1}^3$ satisfies the inequalities

$$c \leq |\mathbb{J}(\mathcal{Z})| \leq C\,, \qquad \forall\, x \in \overline{\tau}_{0,j}\,, \tag{3.55}$$

where c and C are positive constants, depending only on $\tau_{0,j}$.

For our purpose, it is sufficient to use affine maps. Assume for simplicity that faces $\mathcal{F}, \mathcal{F}'$ are in positive parts of the half-spaces with respect to the planes $x_1 \equiv 0$ and $x_3 \equiv 0$, respectively, and γ is the angle between these faces. Then we can set $z_1 = x_1 + x_3 \tan(\frac{\pi}{2} - \gamma)$, $z_k = x_k$, $k = 2, 3$.

Let $\mathcal{F}'' = \mathcal{Z}(\mathcal{F}')$ and Π be the smallest rectangle containing $\mathcal{F}''$. Without loss of generality one can assume that the origin of the coordinates x, z is at the vertex of Π, so that $\Pi = \{(x_2, x_3) \in (0, a_2) \times (0, a_3)\}$, where a_2, a_3 are the notations for the sizes of the edges of Π. Then, under the introduced assumptions

$$\int_{\mathcal{F}'} \frac{1}{|x-y|^3}\, ds(x) \leq c \int_{\mathcal{F}''} \frac{1}{|z-y|^3}\, ds(z) \leq \int_0^{a_2} \int_0^{a_3} \frac{1}{[y_1^2 + (z_2 - y_2)^2 + z_3^2]^{3/2}}\, dz_3\, dz_2\,.$$

$$\int_0^{a_3} \frac{1}{(y_1^2+(z_2-y_2)^2+z_3^2)^{3/2}}\,dz_3 = \frac{z_3}{(y_1^2+(z_2-y_2)^2)[y_1^2+(z_2-y_2)^2+z_3^2]^{1/2}}\Big|_0^{a_3} =$$

$$= \frac{a_3}{(y_1^2+(z_2-y_2)^2)[y_1^2+(z_2-y_2)^2+a_3^2]^{1/2}} \le \frac{1}{(y_1^2+(z_2-y_2)^2)}\,,$$

$$\int_0^{a_2} \frac{1}{(y_1^2+(z_2-y_2)^2)}\,dz_2 = \frac{1}{y_1}\arctan\frac{z_2-y_2}{y_1}\Big|_0^{a_2} \le \frac{\pi}{y_1}\,.$$

Thus, we have shown that

$$\int_{\mathcal{F}}\int_{\mathcal{F}'} \frac{v_F^2(y)}{|x-y|^3}\,ds(x)\,ds(y) \le c\int_{\mathcal{F}} \frac{v_F^2(y)}{\operatorname{dist}[y,\pounds]}\,dy_1\,dy_2 = c\int_{\mathcal{F}} \frac{v_F^2(y)}{y_1}\,dy_1\,dy_2\,. \tag{3.56}$$

It is easy to note that in the case of convex shape and size regular polyhedrons $\tau_{0,j}$ the proof practically does not depend on whether $\mathcal{F}' \subset \mathcal{B}_{12}$ or $\mathcal{F}' \subset \mathcal{B}_{11}$, and therefore (3.56) holds also in the latter case.

Let $\overline{\mathcal{F}}^\hbar$ be the union of the closures of domains of one layer of finite elements with vertices or edges on the boundary $\partial\mathcal{F}$. Quasiuniformity conditions for the reference triangulation allow us to conclude that there exists $c_\dagger = \text{const} > 0$ for which the intersection of the straight line $x_1 \equiv c_\dagger\hbar,\ x_3 \equiv 0$, with $\overline{\mathcal{F}}^\hbar$ and with $\overline{\mathcal{F}}$ coincide. We denote this intersection by $\mathcal{L}$. Two cases should be distinguished: $\mathcal{L} \subset \overline{\mathcal{F}}^\hbar_{\mathcal{E}}$ and $\mathcal{L}$ not in $\overline{\mathcal{F}}^\hbar_{\mathcal{E}}$, where $\overline{\mathcal{F}}^\hbar_{\mathcal{E}}$ is the union of the closures of domains of one layer finite elements with vertices or edges on $\overline{\mathcal{E}}$. We start from consideration of first case and we subdivide $\mathcal{F}$ in two parts $\mathcal{F} = \mathcal{F}_\alpha \cup \mathcal{F}_\beta,\ \mathcal{F}_\alpha = \{x \in \mathcal{F} : x_1 < c_\dagger\hbar\},\ \mathcal{F}_\beta = \mathcal{F}\backslash\overline{\mathcal{F}}_\alpha$. Accordingly we represent the integral in the right part of (3.56) by the sum

$$\mathcal{I} := \int_{\mathcal{F}} \frac{v_F^2}{x_1}\,dx_1dx_2 = \mathcal{I}_\alpha + \mathcal{I}_\beta\,, \tag{3.57}$$

$$\mathcal{I}_\alpha = \int_{\mathcal{F}_\alpha} \frac{v_F^2(x_1,x_2,0)}{x_1}\,dx_1dx_2\,, \qquad \mathcal{I}_\beta = \int_{\mathcal{F}_\beta} \frac{v_F^2(x_1,x_2,0)}{x_1}\,dx_1dx_2\,.$$

According to Lemma 3.5, proved at the end of this section, first of these integrals is bounded by the integral along the line $\mathcal{L}$:

$$\mathcal{I}_\alpha \le c\int_{\mathcal{L}} v_F^2(x)\,dx_2\,. \tag{3.58}$$

This integral can be estimated in the same way, as the integral in the left part of (3.48), and, therefore,

$$\mathcal{I}_\alpha \le c(1+|\log\hbar|)\|v_F\|^2_{1,\tau_0}\,. \tag{3.59}$$

The needed estimate is obtained by application of (3.48):

$$\begin{aligned}\mathcal{I}_\alpha &\le c(1+|\log\hbar|)\|v-v_W\|^2_{1,\tau_0} \le \\ &\le 2c(1+|\log\hbar|)(\|v\|^2_{1,\tau_0}+\|v_W\|^2_{1,\tau_0}) \le (1+\log^2\hbar)\|v\|^2_{1,\tau_0}\,.\end{aligned} \tag{3.60}$$

For bounding second integral in (3.57), it is convenient again to use prolongations of FE functions on some vicinity of $\tau_{0,j}$ and then cut out an appropriate cube with the edges of the length $\mathcal{O}(1)$. We use the chosen above system of coordinates $x = (x_1, x_2, x_3)$ such that $\mathcal{F}$ is in the plane (x_1, x_2), $\mathcal{F}''$ is in the plane (x_2, x_3), and the line $\pounds$ is on the axis x_2. Let $\pi_j := (0, \ell) \times (a, b) \times (0, \ell)$, $b = a + \ell$, be the least cube, one face of which is in the plane $x_3 \equiv 0$ and covers $\mathcal{F}$. We consider the c-vicinity $S_{c,j}$ of $\tau_{0,j}$ and the extended triangulation $\mathcal{T}_{j,\mathrm{ext}}$, occupying the domain $\tau_{j,\mathrm{ext}}$, where c is such that $\pi_j \in \tau_{j,\mathrm{ext}}$. Besides, we introduce the space $\mathbb{V}_j(\pi_j)$ as the restriction of $\mathbb{V}_j(\tau_{j,\mathrm{ext}})$ to the cube π_j. The extensions v_{ext} of functions $v \in \mathbb{V}(\tau_{0,j})$ in the space $\mathbb{V}_j(\tau_{j,\mathrm{ext}})$ are assumed to be produced in such a way that the inequality (3.45) holds.

Let for $x_2 \in (a, b)$ each line $x_2 \equiv \text{const}$, $x_3 \equiv 0$, has nonempty intersection with the face $\mathcal{F}$ with x_1 in the interval denoted $\Lambda = \Lambda(x_2)$. Then

$$\begin{gathered}
\mathcal{I}_\beta := \int_a^b \int_{\Lambda \setminus (0, c_\dagger \hbar)} \frac{v_F^2(x_1, x_2, 0)}{x_1} \, dx_1 dx_2 \le \\
\le \ln((c_\dagger \hbar)^{-1}) \int_a^b \|v_F(x_1, x_2, 0)\|^2_{L_\infty(\Lambda)} \, dx_2 \le \\
\le 2 \ln((c_\dagger \hbar)^{-1}) \left\{ \int_a^b \|v(x_1, x_2, 0)\|^2_{L_\infty(\Lambda)} \, dx_2 + \int_a^b \|v_W(x_1, x_2, 0)\|^2_{L_\infty(\Lambda)} \, dx_2 \right\} ,
\end{gathered} \tag{3.61}$$

where it is implied that $\| \cdot \|_{L_\infty(\Lambda)} = 0$, if $\Lambda = \Lambda(x_2) = \varnothing$. For the reason that v_W is distinct from zero only at the nodes of the wire basket, it can be shown that

$$\int_a^b \|v_W(x_1, x_2, 0)\|^2_{L_\infty(\Lambda)} \, dx_2 \le c \, \|v\|^2_{2, \hat{W}_j} ,$$

and combining with (3.48) yields

$$\int_0^{a_2} \|v_W(x_1, x_2, 0)\|^2_{L_\infty(\Lambda)} \, dx_2 \le c \, (1 + |\log \hbar|) \, \|v\|^2_{1, \tau_{0,j}} . \tag{3.62}$$

First integral in the right part of (3.61) is estimated similarly to (3.47). Let $\pi_j(x_2)$ be the cross section of π_j by the plane $x_2 \equiv \text{const}$. We write (3.46) squared for $\pi_j(x_2)$ and integrate over $x_2 \in (a, b)$:

$$\begin{gathered}
\int_a^b \|v(x_1, x_2, 0)\|^2_{L_\infty(\Lambda(x_2))} \, dx_2 \le \int_a^b \|v_{\exp}(x_1, x_2, 0)\|^2_{L_\infty(\Lambda(x_2))} \, dx_2 \le \\
\le \; c \, \{ |\log \epsilon| \int_a^b \|v_{\exp}\|^2_{1, \pi_j(x_2)} \, dx_2 + \epsilon^2 \int_a^b |v_{\exp}|^2_{W^1_\infty(\pi_j(x_2))} \, dx_2 \le \\
\le \; c \, \{ |\log \epsilon| \|v_{\exp}\|^2_{1, \pi_j} + \epsilon^2 |v_{\exp}|^2_{W^1_\infty(\pi_j)} \} \le \\
\le \; c \, \{ |\log \epsilon| \|v_{\exp}\|^2_{1, \tau_{j,\exp}} + \epsilon^2 |v_{\exp}|^2_{W^1_\infty(\tau_{j,\exp})} \} .
\end{gathered} \tag{3.63}$$

Inequalities (3.41) are hold for v belonging to the extended FE space, *i.e.,*

$$\hbar^{3/2} |v|_{L_\infty(\tau)} \le c \|v\|_{0,\tau} , \qquad \hbar^{3/2} |v|_{W^1_\infty(\tau)} \le c \, |v|_{H^1(\tau)} , \qquad \forall \; v \in \mathbb{V}(\tau_{j,\exp}) . \tag{3.64}$$

We now set $\epsilon = \hbar^2$, apply (3.64) to the second term in the right part, and then apply (3.45), repeating last steps in deriving (3.48). This is reflected in the inequalities

$$\int_a^b \|v(x_1,x_2,0)\|^2_{L_\infty(\Lambda(x_2))}\, dx_2 \le c\,(1+|\log\hbar|)\,\|v_{\exp}\|^2_{1,\tau_{j,\exp}} \le c\,(1+|\log\hbar|)\,\|v\|^2_{1,\tau_{0,j}}\,. \qquad (3.65)$$

Bounds (3.57), (3.60)–(3.62), (3.65) allow us to write

$$\mathcal{I} \le c\,(1+\log^2\hbar)\,\|v\|^2_{1,\tau_{0,j}}\,. \qquad (3.66)$$

and combining (3.54), (3.56), (3.57), (3.66), we conclude that

$$\int_{\mathcal{F}}\int_{\mathcal{F}'} \frac{v_F^2(y)}{|x-y|^3}\, ds(x)\, ds(y) \le c\,(1+\log^2\hbar)\|v\|^2_{1,\tau_{0,j}}\,. \qquad (3.67)$$

Heretofore our considerations were restricted by the assumption $\mathcal{L}\subset\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}}$. Let the angles between edge $\mathcal{E}$ and the adjacent to $\mathcal{E}$ edges are separated from π by some constant $\theta_\circ,\ 0<\theta_\circ\le\pi/4$. Then at sufficiently small $\hbar$, one can always choose such $c_\dagger = c_\dagger(\theta_\circ) = \text{const}$ that $\mathcal{L}\subset\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}}$. The mentioned assumption will be violated, if the angles between the adjacent to $\mathcal{E}$ edges and the edge $\mathcal{E}$, or one of these angles, approach π. In general the line $\mathcal{L}$ can intersect with several domains $\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}_k}$ neighboring the domain $\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}}$. However, from further considerations it will became clear that sufficiently representative case is when the line $\mathcal{L}$ intersects with two domains. In order to retain some notations we adopt that $\mathcal{E}=\mathcal{E}_1$ and that the line $\mathcal{L}$ intersects with $\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}_1}$ and $\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}_2}$. For definiteness it is implied that edges of the face $\mathcal{F}$ are counted counter-clockwise.

We consider triangles with the common vertex at the point $\overline{\mathcal{E}}_k\cap\overline{\mathcal{E}}_{k+1}$ and denote by $\mathcal{S}_k$ the one of their sides in $\mathcal{F}^{\hbar}$, which is closest to the bisector of the angle between $\mathcal{E}_{k-1}$ and $\mathcal{E}_k$. It is convenient to change the definition of the line $\mathcal{L}$. Let $\overline{\mathcal{F}}^{\hbar}_{\dagger}=\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}_1}\cup\overline{\mathcal{F}}^{\hbar}_{\mathcal{E}_2}$ and $\mathcal{F}^{\hbar}_k,\ k=1,2$, be the nonoverlapping subdomains of $\mathcal{F}^{\hbar}_{\dagger}$ separated by the line $\mathcal{S}_1$.

We define $\mathcal{L}_\dagger\subset\overline{\mathcal{F}}^{\hbar}_{\dagger}$ as the broken line, which on each domain $\overline{\mathcal{F}}^{\hbar}_k,\ k=1,2$, is a straight line parallel to $\mathcal{E}_k$ and such that $\text{dist}[y,\partial\mathcal{F}]\ge c_\dagger\hbar,\ \forall\, y\in\mathcal{L}_\dagger$, for some constant $c_\dagger>0$. Now, the line $\mathcal{L}$ is defined as the intersection of the straight line $x_1\equiv c_\dagger\hbar$ with $\overline{\mathcal{F}}^{\hbar}$. If $\mathcal{L}$ crosses the line $\mathcal{E}_2$, we proceed as follows. The line $\mathcal{L}$ defines the least mesh simply connected domain $\mathcal{F}^{\hbar,2}\subset\mathcal{F}^{\hbar}_2$, the left border of which is $\mathcal{S}_1$ and the right one is defined by the condition that each line $x_2=\text{const}$, crossing $\mathcal{L}$, crosses as well $\mathcal{L}_\dagger$ on $\overline{\mathcal{F}}^{\hbar,2}$. We introduce also the sets $\overline{\mathcal{F}}^{\hbar}(\mathcal{E}) := \overline{\mathcal{F}}^{\hbar}_1\cup\overline{\mathcal{F}}^{\hbar,2}$ and $\text{Ł}_\dagger := \mathcal{L}_\dagger\cap\overline{\mathcal{F}}^{\hbar}(\mathcal{E})$. If $\mathcal{L}$ does not cross $\mathcal{E}_2$, then we simply set $\mathcal{F}^{\hbar,2}=\mathcal{F}^{\hbar}_2$ with the consequences: $\mathcal{F}^{\hbar}(\mathcal{E})=\mathcal{F}^{\hbar}_{\dagger}$ and $\mathcal{L}_\dagger=\text{Ł}_\dagger$.

In general, the line $\text{Ł}_\dagger$ separates each of the domains $\mathcal{F}^{\hbar}_1,\mathcal{F}^{\hbar,2}$ in two parts, the parts adjacent to $\partial\mathcal{F}$ are denoted by Π_1,Π_2, respectively. Obviously

$$\mathcal{I}_\alpha = \int_{\mathcal{F}_\alpha}\frac{v_F^2}{x_1}\,dx_1dx_2 \le \sum_{k=1,2}\int_{\Pi_k}\frac{v_F^2}{x_1}\,dx_1dx_2 \qquad (3.68)$$

and for each integral in the right part Lemma 3.71 allows to obtain bounds similar to (3.58). For this reason

$$\mathcal{I}_\alpha \le \int_{\text{Ł}_\dagger} v_F^2 ds\,,$$

where s is the element of the line $\text{Ł}_\dagger$. Further transition from this bound to the bound (3.60) and then to the bounds (3.66), (3.67) are quite similar to the presented above. Thus summarizing, we obtain

$$\int_{\mathcal{F}}\int_{\mathcal{B}_{12}} \frac{v_F^2(y)}{|x-y|^3}\, ds(x)\, ds(y) \le c\,(1+\log^2 \hbar)\|v\|^2_{1,\tau_{0,j}}\,. \tag{3.69}$$

If the part $\mathcal{L}$ of the line $\mathcal{L}_\infty := \{x = (c_\dagger \hbar, x_2, 0)\}$ with $c_\dagger$ defined by the condition

$$\mathcal{L}_\infty \cap \overline{\mathcal{F}}^\hbar_{\mathcal{E}} = \mathcal{L}_\infty \cap \overline{\mathcal{F}}^\hbar$$

crosses several domains $\overline{\mathcal{F}}_{\mathcal{E}_k},\ k = 1, 2, \dots, \Bbbk_+$, then first of all we redefine $\mathcal{L}$. We consider the domain

$$\overline{\mathcal{F}}^\hbar_\dagger = \cup_{k=1}^{\Bbbk_+} \overline{\mathcal{F}}^\hbar_{\mathcal{E}_k}$$

and subdivide it in $\Bbbk_+$ the nonoverlapping subdomains $\mathcal{F}^\hbar_k$ by the edges $\mathcal{S}_k,\ k = 1, 2, \dots, \Bbbk_+ - 1$. The broken line $\mathcal{L}_\dagger$ is defined by the conditions:

α) $\mathcal{L}_\dagger \cap \overline{\mathcal{F}}^\hbar_k$ is a straight line, $k = 1, 2, \dots, \Bbbk_+$, parallel to the edge $\mathcal{E}_k$,

β) $\text{dist}[\mathcal{L}_\dagger, \mathcal{F}^\hbar] \ge c_\dagger \hbar$ for some $c_\dagger = \text{const} > 0$.

The right border of the domain $\mathcal{F}^{\hbar,\Bbbk_+}$ is defined in the same way as for $\Bbbk_+ = 2$, and it is adopted

$$\overline{\mathcal{F}}^\hbar(\mathcal{E}) = (\cup_{k=1}^{\Bbbk_+ -1} \overline{\mathcal{F}}^\hbar_k) \cup \overline{\mathcal{F}}^{\hbar,\Bbbk_+}\,.$$

Further considerations toward the bound (3.67) also repeat the way used for the case $\Bbbk_+ = 2$, except that one has $\Bbbk_+$ integrals instead two in the right part of (3.68), each of which is bounded by Lemma 3.5 for the first step.

Clearly, the case when the line $x_1 \equiv c_\dagger \hbar$ crosses domains $\mathcal{F}^\hbar_{\mathcal{E}_k}$ on the both sides of $\mathcal{F}^\hbar_{\mathcal{E}}$ introduces minor differences in the proof.

It is left to derive the bound

$$\int_{\mathcal{F}}\int_{\mathcal{B}_{11}} \frac{v_F^2(y)}{|x-y|^3}\, ds(x)\, ds(y) \le c\,(1+\log^2 \hbar)\|v\|^2_{1,\tau_{0,j}}\,, \tag{3.70}$$

but since (3.56) holds for $\mathcal{F}' \subset \mathcal{B}_{11}$, the proof closely repeats the proof of (3.69). Indeed technically, it is a part of the proof of (3.69).

Relations (3.51), (3.52), (3.69), (3.70) and the definition of the norm ${}_{00}\|\cdot\|_{1/2,\mathcal{F}}$ show that

$${}_{00}\|v_F\|_{1/2,\mathcal{F}} \le c\,(1+\log^2 \hbar)\|u\|^2_{1,\tau_{0,j}}\,.$$

Since this bound holds also for discrete harmonic functions, for which $\|v\|_{1,\tau_{0,j}} \le \|v\|_{1/2,\partial\tau_{0,j}}$, we have completed the proof. □

Now we turn to Lemma 3.5 used in the proof of Lemma 3.4.

We consider the domain $\Pi_\hbar$ of the quasiuniform triangulation $\Im_\hbar$ with the part of the boundary $\mathcal{E} = \{x : x_1 \equiv 0,\ x_2 \in (a, b),\ -\infty < a < b < \infty\}$ and triangles δ, having on $\overline{\mathcal{E}}$ a vertex or an edge. Two additional conditions are assumed fulfilled. 1) For some constant $c_\dagger > 0$, the intersection Ł of the line $x_1 \equiv c_\dagger \hbar$ is connected and crosses all triangles $\overline{\delta}$. 2) If $\overline{\delta}$ has a single vertex on $\overline{\mathcal{E}}$ and E_δ is the edge opposite to this vertex, than $x_1 \ge c_\dagger \hbar$ for $\forall\, x \in E_\delta$. Note that 2) is fulfilled automatically, if the vertex is in $\mathcal{E}$, however with other conditions fulfilled it can be violated for the triangles having vertices at the ends of $\mathcal{E}$.

Lemma 3.5. *Let* $\Pi_{\hbar,\mathcal{E}} = \{x \in \Pi_\hbar : x_1 \le c_\dagger \hbar\}$*. Let also* v *be any function continuous on* $\Pi_\hbar$ *and linear on each triangle* $\delta \in \Im_\hbar$ *and satisfying the condition* $v\big|_{\mathcal{E}} = 0$*. Then*

$$\int_{\Pi_{\hbar,\mathcal{E}}} \frac{v^2(x)}{x_1}\, dx_2 \le c \int_{Ł} v^2(x)\, dx_2\,, \tag{3.71}$$

with the constant c*, depending only on the quasiuniformity conditions.*

Proof. Let $\overline{\mathcal{E}}'$ be the part of $\partial\Pi'_\hbar$, $\Pi'_\hbar = \Pi_\hbar \backslash \Pi_{\hbar,\mathcal{E}}$, containing only whole edges of triangles $\delta \in \Im_\hbar$, so that $\partial\Pi_\hbar \setminus (\mathcal{E} \cup \mathcal{E}')$ contains only two edges of two triangles. Obviously, on all adjacent δ, having a common vertex on $\overline{\mathcal{E}}'$ and edges on $\overline{\mathcal{E}}$, the function v is linear. Therefore, every such cluster of triangles can be considered as one triangle δ'', and instead of the triangulation $\Im_\hbar$ one can consider the rarefied triangulation $\Im'_\hbar$ consisting of the triangles δ'' and triangles $\delta' = \delta$ having edges on $\mathcal{E}$. At that each triangle δ'', which does not have edges in $\partial\Pi_\hbar \setminus (\mathcal{E} \cup \mathcal{E}')$, has two adjacent triangles δ' and vice versa. Conditions of quasiuniformity for the triangulation $\Im_\hbar$ can be written in the similar to (2.1) form

$$\beta^{(1)}\hbar \le h^{(r)}_{i,j} \le \beta^{(2)}\hbar\,, \qquad \theta \le \theta^{(r)}_i \le \pi - 2\theta\,, \qquad 0 < \beta^{(1)}, \beta^{(2)}, \theta = \text{const}\,, \tag{3.72}$$

where $h^{(r)}_{i,j}$ are lengths of edges and $\theta^{(r)}_i$ angles at the vertices of triangles of the triangulation $\Im_\hbar$. It is easy to see that they are retained for the triangulation $\Im'_\hbar$, in general with other constants, for which, however, we retain the same notations.

Considering one triangle δ', we introduce the notations A' for the length of the line $\mathcal{A}' = Ł \cap \overline{\delta}'$, v_L and v_R for the values of v at the "left" and the "right" ends of $\mathcal{A}'$, respectively, and $\delta'_A := \delta' \cap \Pi_{\hbar,\mathcal{E}}$. For the values of v at the ends of the intersection $\mathcal{A}'_\circ(x_1)$ of the line $x_1 = \text{const}$ with $\overline{\delta}'$, we use the notations $u_L = u_L(x_1)$ and $u_R = u_R(x_1)$, so that $u_L(c_\dagger\hbar) = v_L$ and $u_R(c_\dagger\hbar) = v_R$.

Let the affine mapping $\eta = \eta(x)$, $x \in \delta'$, does not change positions of the two vertices of the triangle δ', which are on $\overline{\mathcal{E}}'$, but shifts third vertex of δ' along $\overline{\mathcal{E}}$ to such nearest position that the triangle $\Delta' = \eta(\delta')$ becomes rectangular. Obviously,

$$\int_{\delta'_A} \frac{v^2}{x_1}\, dx = \int_{\Delta'_A} \frac{v^2}{\eta_1}\, d\eta\,, \qquad \text{where} \qquad \Delta'_A = \eta(\delta'_A)\,.$$

Hence, without loss of generality, we can adopt that δ' itself is a rectangular triangle with one cathetus crossing the left end of $\mathcal{A}'$, and that $x_2 = 0$ for this end. We have

$$v(x) = u_L(x_1) + \frac{u_R(x_1) - u_L(x_1)}{A_\circ(x_1)}\, x_2\,, \qquad x_2 \in (0, A_\circ(x_1))\,,$$

with $A_\circ(x_1)$ being the length of $\mathcal{A}'_\circ(x_1)$, and as it is known, see [17], [?], for such v there hold the inequalities

$$\underline{c} A_\circ(x_1)(u_L^2(x_1) + u_R^2(x_1)) \le \int_0^{A_\circ(x_1)} v^2(x) dx_2 \le \overline{c} A_\circ(x_1)(u_L^2(x_1) + u_R^2(x_1)) \tag{3.73}$$

with absolute constants $\underline{c}, \overline{c} > 0$. On account of this

$$\mathcal{I}_{\delta'_A} := \int_{\delta'_A} \frac{v^2(x)}{x_1}\, dx = \int_0^{c_\dagger \hbar} \frac{1}{x_1} \int_0^{A_\circ(x_1)} [u_L(x_1) + \frac{u_R(x_1) - u_L(x_1)}{A_\circ(x_1)}\, x_2]^2\, dx_2 dx_1 \leq$$

$$\leq \overline{c} \int_0^{c_\dagger \hbar} \frac{A_\circ(x_1)}{x_1} [u_L^2(x_1) + u_R^2(x_1)] dx_1 \,. \tag{3.74}$$

Besides

$$(u_L(x_1), u_R(x_1), A_\circ(x_1)) = \mu(x_1)(v_L, v_R, A')\,, \qquad \mu(x_1) = \frac{x_1}{c_\dagger \hbar}\,, \tag{3.75}$$

so that (3.73)-(3.75) allow us to write

$$\mathcal{I}_{\delta'_A} \leq \overline{c}\, \frac{A'}{(c_\dagger \hbar)^3} [v_L^2(x_1) + v_R^2] \int_0^{c_\dagger \hbar} x_1^2\, dx_1 \leq \overline{c}\, \frac{A'}{3} [v_L^2(x_1) + v_R^2] \leq \frac{\overline{c}}{3\underline{c}} \int_{\mathcal{A}'} v^2 dx_2 \tag{3.76}$$

Now we will consider an arbitrary triangle δ'', which has an edge on $\mathcal{E}$, the length of which will be denoted by B''. We introduce also notations: $\delta''_A := \delta'' \cap \Pi_{\hbar, \mathcal{E}}$, A'' for the length of the line $\mathcal{A}'' = \text{Ł} \cap \overline{\delta}''$, and $A''_\circ(x_1)$ for the intersection of the line $x_1 = \text{const}$ with $\overline{\delta}''$.

It is clear that on each line $x_1 = \text{const}$, crossing $\overline{\delta}''_A$, the function v has a constant value $v = (v_A / c_\dagger \hbar) x_1$ and

$$A''_\circ(x_1) = A'' + \frac{B'' - A''}{c_\dagger \hbar} x_1 \,.$$

Therefore,

$$\mathcal{I}_{\delta''_A} := \int_{\delta''_A} \frac{v^2(x)}{x_1}\, dx = (\frac{v_A}{c_\dagger \hbar})^2 \int_0^{c_\dagger \hbar} x_1 [A'' + \frac{B'' - A''}{c_\dagger \hbar} x_1] dx_1 =$$

$$= [\frac{A''}{2} + \frac{B'' - A''}{3}] v_A^2 = \frac{1}{2} \int_{\mathcal{A}''} v^2 dx_2 + \frac{B'' - A''}{3} v_A^2 \,, \tag{3.77}$$

As it was noted earlier, there is at least one triangle δ'' adjacent to δ'. We have $x_1 = c_\dagger \hbar$ for all $x \in \mathcal{A}'$, and there are hold the quasiuniformity conditions (3.72). In view of this, for any triangle δ' the value A' is bounded from below and from above according to

$$c_- \hbar \leq A' \leq c_+ \hbar \,, \tag{3.78}$$

with the constants $c_-, c_+ > 0$ depending only on the $\beta^{(1)}, \theta$ from (3.72). We pick up a triangle δ' adjacent to δ'', take into account that $v_A = v_L$ or $v_A = v_R$, and estimate second term in (3.77) as follows:

$$\frac{B'' - A''}{3} v_A^2 \leq \frac{\hbar}{3} [v_L^2 + v_R^2] \leq \frac{A'}{3c_-} [v_L^2 + v_R^2] \leq \frac{1}{3c_- \underline{c}} \int_{\mathcal{A}'} v^2 dx_2 \,, \tag{3.79}$$

where we used (3.72), (3.78) and (3.73). Now it is needed only to combine (3.76), (3.77), (3.79) in order to come to the bound

$$\int_{\delta_A^+} \frac{v^2(x)}{x_1}\, dx \leq \max\,[\frac{1}{2}, \frac{1}{3\underline{c}}(1 + \frac{1}{c_-})] \int_{\mathcal{A}^+} v^2(x)\, dx_2 \,, \qquad \delta_A^+ = \delta'_A \cup \delta''_A \,, \tag{3.80}$$

where $\overline{\mathcal{A}}^+ = \overline{\mathcal{A}}' \cup \overline{\mathcal{A}}''$. If the number n of triangles δ', δ'' in the triangulation $\Im'_\hbar$ is even, then the domain $\Pi_{\hbar,\mathcal{E}}$ is decomposed in $n/2$ nonoverlapping pairs δ_A^+ and the bound (3.71) hold with the constant c the same as in (3.80). In the case of an odd n, $\Pi_{\hbar,\mathcal{E}}$ is decomposed in $(n-1)/2$ nonoverlapping pairs δ_A^+ and one overlapping pair, so that c is increased by the factor 2. $\square$

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