

An Upper Bound Theorem concerning lattice polytopes

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Abstract

R. P. Stanley proved the Upper Bound Conjecture in 1975. We imitate his proof for the Ehrhart rings.

We give some upper bounds for the volume of integrally closed lattice polytopes. We derive some inequalities for the delta-vector of integrally closed lattice polytopes. Finally we apply our results for reflexive integrally closed and order polytopes.

1 Introduction

First we recall here some basic facts about lattice polytopes.

A *lattice polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in \mathbb{Z}^d . We shall assume throughout the paper that P is of maximum dimension, so that $\dim P = d$.

Let $L_P(m) := |mP \cap \mathbb{Z}^d|$ denote the number of lattice points in P dilated by a factor of $m \in \mathbb{Z}_{\geq 0}$.

In general the function L_P is a polynomial of degree d , and is called the *Ehrhart polynomial*. Ehrhart showed that certain coefficients of L_P have natural interpretations in terms of P .

⁰**Keywords.** lattice polytope, integrally closed polytope, Cohen–Macaulay ring, h -vector

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Theorem 1.1 ([Ehr67]) *Let P be a d -dimensional convex lattice polytope with Ehrhart polynomial $L_P(m) = c_d m^d + \dots + c_1 m + c_0$. Then:*

- (i) $c_d = \text{vol}(P)$;
- (ii) $c_{d-1} = (1/2)\text{vol}(\partial P)$;
- (iii) $c_0 = 1$.

Let P° denote the strict interior of P .

In [Sta80] Stanley proved that the generating function for L_P can be written as a rational function

$$\text{Ehr}_P(t) := \sum_{m \geq 0} L_P(m)t^m = \frac{\delta_0 + \delta_1 t + \dots + \delta_d t^d}{(1-t)^{d+1}},$$

where the coefficients δ_i are non-negative. The sequence $(\delta_0, \delta_1, \dots, \delta_d)$ is known as the δ -vector of P . For an elementary proof of this and other relevant results, see [BS07] and [BR07].

In the following let $\delta(P) := (\delta_0, \delta_1, \dots, \delta_d)$ denote the delta-vector of P .

The following corollary is a consequence of Theorem 1.1.

Corollary 1.2 *Let P be a d -dimension convex lattice polytope with δ -vector $(\delta_0, \delta_1, \dots, \delta_d)$. Then:*

- (i) $\delta_0 = 1$;
- (ii) $\delta_1 = |P \cap \mathbb{Z}^d| - d - 1$;
- (iii) $\delta_d = |P^\circ \cap \mathbb{Z}^d|$;
- (iv) $\delta_0 + \dots + \delta_d = d! \text{vol}(P)$.

Hibi proved in [Hib94] the following lower bound on the δ_i , commonly referred to as the *Lower Bound Theorem*:

Theorem 1.3 *Let P be a d -dimensional convex lattice polytope with $|P^\circ \cap \mathbb{Z}^d| > 0$. Then $\delta_1 \leq \delta_i$ for every $2 \leq i \leq d - 1$.*

A convex lattice polytope P is called *reflexive* if the dual polytope

$$P^\vee := \{u \in \mathbb{R}^d \mid \langle u, v \rangle \leq 1 \text{ for all } v \in P\}$$

is also a lattice polytope.

In the following Theorem we give an interesting characterization of reflexive polytopes (see for example the list in [HM06]).

Theorem 1.4 *Let P be a d -dimensional convex lattice polytope with $0 \in P^\circ$. The following are equivalent:*

- (i) P is reflexive;
- (ii) $L_P(m) = L_{\partial P}(m) + L_P(m-1)$ for all $m \in \mathbb{Z}_{>0}$;
- (iii) $d \operatorname{vol}(P) = \operatorname{vol}(\partial P)$;
- (iv) $\delta_i = \delta_{d-i}$ for all $0 \leq i \leq d$.

Theorem 1.4 (iv) is known as *Hibi's Palindromic Theorem* [Hib91] and can be generalised to duals of non-reflexive polytopes [FK08]. It is a consequence of a more general result of Stanley (see [Sta78] Theorem 4.4) concerning Gorenstein rings.

Recall that a polytope P is *integrally closed*, if for each $c \in \mathbb{N}$, $z \in cP \cap \mathbb{Z}^d$ there exist $x_1, \dots, x_c \in P \cap \mathbb{Z}^d$ such that $\sum_i x_i = z$.

If a lattice polytope P is covered by integrally closed polytopes then it is integrally closed as well. Hence in particular if the polytope P possesses an unimodular triangulation, then P is integrally closed.

It is well-known that the *unimodular simplices* are integrally closed. Here the unimodular simplices are the lattice simplices $\Delta = \operatorname{conv}(\underline{x}_1, \dots, \underline{x}_d) \subset \mathbb{R}^m$, $\dim(\Delta) = d-1$, with $x_1 - x_j, \dots, x_{j-1} - x_j, x_{j+1} - x_j, \dots, x_k - x_j$ a part of a basis of \mathbb{Z}^d for some j .

The main contribution of the paper is an upper bound for the volume of integrally closed lattice polytopes. We prove:

Corollary 1.5 *Let P be a d -dimensional integrally closed lattice polytope with $n := |P \cap \mathbb{Z}^d|$. Then*

$$d! \operatorname{vol}(P) \leq \binom{n-1}{d}.$$

As a consequence we derive the following upper bound for the volume of reflexive integrally closed lattice polytopes:

Theorem 1.6 *Let P be a d -dimensional reflexive integrally closed lattice polytope with $n := |P \cap \mathbb{Z}^d|$. Then*

$$d! \operatorname{vol}(P) \leq f_{d-1}(C(n, d)),$$

where $f_{d-1}C(n, d)$ denotes the number of facets of the d -dimensional cyclic polytopes on n points.

In our second main contribution we give a sufficient condition for the unimodality of the delta-vector of integrally closed reflexive lattice polytopes:

Theorem 1.7 *Let P be an d -dimensional integrally closed reflexive lattice polytope such that $n \leq d + 4$, where $n := |P \cap \mathbb{Z}^d|$. Then the delta-vector $(\delta_0, \dots, \delta_d)$ of P will be unimodal.*

Finally we derive here some inequalities for the delta-vector of integrally closed lattice polytopes:

Theorem 1.8 *Let P be a d -dimensional integrally closed lattice polytope. Let $(\delta_0, \dots, \delta_s)$ be the delta-vector of P . Let $m \geq 0$ and $n \geq 1$ with $m+n < s$. Then*

$$\delta_1 + \dots + \delta_n \leq \delta_{m+1} + \delta_{m+2} + \dots + \delta_{m+n}.$$

The structure of the paper is the following. In Section 2 we recall some basic facts concerning graded algebras and the Ehrhart ring of lattice polytopes. In Section 3 we present our main results and we apply our results for reflexive integrally closed and order polytopes.

2 Graded algebras

2.1 Cohen–Macaulay graded algebras

A *graded algebra* over a field K is a commutative K -algebra R with identity, together with a vector space direct sum decomposition

$$R = \coprod_{i \geq 0} R_i$$

such that (a) $R_i R_j \subseteq R_{i+j}$, (b) $R_0 = K$ and (c) R is finitely generated as a K -algebra. R is *standard*, if R is generated as a K -algebra by R_1 .

The *Hilbert function* $H(R, \cdot)$ of R is defined by

$$H(R, i) := \dim_K R_i.$$

for each $i \geq 0$. The *Hilbert series* of R is given by

$$F(R, \lambda) := \sum_{i \geq 0} H(R, i) \lambda^i.$$

By Hilbert–Serre Theorem we can write the Hilbert series of R into the form

$$F(R, \lambda) = \frac{h_0 + h_1 \lambda + \dots + h_s \lambda^s}{(1 - \lambda)^d},$$

where d is the Krull dimension of the algebra R . Here $\sum h_i \neq 0$ and $h_s \neq 0$. We call the vector $h(R) := (h_0, \dots, h_s)$ the *h -vector* of R .

We call a finite or infinite sequence of (k_0, k_1, \dots) nonnegative integers an *O -sequence* if there exists an order ideal M of monomials in variables y_1, \dots, y_n such that

$$k_n = |\{u \in M : \deg(u) = n\}|.$$

If h and i are positive integers then h can be written uniquely in the following form

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \dots + \binom{n_j}{j},$$

where $1 \leq j \leq n_j < \dots < n_i$. This expression for h is called the *i -binomial expansion* of h . Then we can define

$$h^{<i>} := \binom{n_i + 1}{i + 1} + \binom{n_{i-1} + 1}{i} + \dots + \binom{n_j + 1}{j + 1}$$

Also let $0^{<i>} := 0$.

The following Theorem is well-known (see [Sta78] Theorem 2.2)

Theorem 2.1 *Let $H : \mathbb{N} \rightarrow \mathbb{N}$ and let K be a field. The following two conditions are equivalent.*

(i) $(H(0), H(1), \dots)$ is an *O -sequence*,

(ii) $H(0) = 1$ and for all $n \geq 1$, $H(n+1) \leq H(n)^{\langle n \rangle}$.

We need for the following theorem about Cohen–Macaulay standard graded algebras. Stanley proved the Upper Bound Conjecture concerning convex polytopes using this result.

We prove here for the reader’s convenience.

Theorem 2.2 *Let R be a Cohen–Macaulay standard graded algebra over a field K , which is generated by n elements. Suppose that $\dim_K R = D$. We can write the Hilbert series of R into the form*

$$F(R, \lambda) = \frac{h_0 + h_1\lambda + \dots + h_s\lambda^s}{(1 - \lambda)^D}.$$

Then

$$h_i \leq \binom{h_1 + i - 1}{i}$$

for each $i \geq 1$.

Proof.

We use here the following result (see [Sta78] Corollary 3.11).

Theorem 2.3 *Let $H(n)$ be a function from \mathbb{N} to \mathbb{N} , and let $D \in \mathbb{N}$. Let K be a field. The following two statements are equivalent.*

- (i) *There exists a Cohen–Macaulay standard graded algebra R with $R_0 = K$, with $\dim R = D$ and with Hilbert function H .*
- (ii) *The power series $(1 - \lambda)^D \sum_{n=0}^{\infty} H(n)\lambda^n$ is a polynomial in λ , say $h_0 + h_1\lambda + \dots + h_s\lambda^s$. Moreover (h_0, \dots, h_s) is an O -sequence.*

The following nice property of O -sequences is well-known (see [Sta77]).

Proposition 2.4 *Let (h_0, \dots, h_s) be an arbitrary O -sequence. Then*

$$0 \leq h_i \leq \binom{h_1 + i - 1}{i}$$

for each $i \geq 1$.

□

Theorem 2.2 follows from Theorem 2.3 and Proposition 2.4. □

Let $S = k[x_1, \dots, x_n]$ denote the polynomial ring in n variables over a field K of characteristic zero, and $m = (x_1, \dots, x_n)$ be the maximal ideal of S . Suppose that we wrote $A = R/I$ for an Artinian K -algebra, then we define the *socle* of $A = R/I$ as

$$\text{Soc}(A) := \text{ann}(A) = (I : m)/I \subset A.$$

Then $A = \bigoplus_{i=0}^s A_i$ is a *level algebra*, if $\text{Soc}(A) = \langle A_s \rangle$. In this case s is called the *socle degree* of A and the vector space dimension of A_s is called the *type* of A . The h -sequence of a level ring is the *level sequence*.

A standard algebra $A = R/I$ is a *Gorenstein algebra* if A is a level algebra of type 1.

The next Proposition is a well-known result of Stanley (see [Sta77] Theorem 2).

Proposition 2.5 *Let $\underline{h} = (h_0, \dots, h_s)$ be a level sequence, with $h_s \neq 0$. If i and j are non-negative integers with $i + j \leq s$, then $h_i \leq h_j h_{i+j}$.*

We call a sequence (h_0, \dots, h_s) with $h_s \neq 0$, which satisfies $h_i = H(R, i)$ for some 0-dimensional standard Gorenstein algebra R , a *Gorenstein sequence*. Stanley proved the following result in [Sta78] Theorem 4.2.

Theorem 2.6 *Let $\underline{h} := (h_0, \dots, h_s)$ be a sequence of nonnegative integers with $h_1 \leq 3$ and $h_s \neq 0$. Then \underline{h} is a Gorenstein sequence if and only if the following two conditions are satisfied:*

- (i) $h_i = h_{s-i}$ for each $0 \leq i \leq s$, and
- (ii) $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_t - h_{t-1})$ is an O -sequence, where $t = \lfloor s/2 \rfloor$.

Finally Stanley proved in [Sta91] Proposition 3.4 the following Proposition.

Proposition 2.7 *Let R be a standard graded Cohen–Macaulay domain of Krull dimension $d \geq 2$ over a field K of characteristic 0. Let $h(R) = (h_0, \dots, h_s)$, where $h_s \neq 0$. Let $m \geq 0$ and $n \geq 1$, with $m + n < s$. Then*

$$h_1 + \dots + h_n \leq h_{m+1} + h_{m+2} + \dots + h_{m+n}.$$

Let $\{b_i\}$, $i \geq 0$ be an O -sequence. Then $\{b_i\}$ is *differentiable*, if the difference sequence, c_i , $c_i = b_i - b_{i-1}$, is again a O -sequence.

In [GMR83] A.V. Geramita, P. Maroscia and L. G. Roberts characterized the Hilbert functions of standard graded reduced algebras. They proved that

Proposition 2.8 *If R is a standard graded reduced algebra, then the Hilbert function of R is a differentiable O -sequence.*

2.2 The Ehrhart ring

Let P be a d -dimensional convex lattice polytope in \mathbb{R}^d . Let $R(P)$ denote the subalgebra of the algebra

$$K[x_1, \dots, x_d, x_1^{-1}, \dots, x_d^{-1}, y]$$

generated by all monomials

$$x_1^{a_1} \dots x_d^{a_d} y^b$$

where $b \geq 1$ and $(a_1, \dots, a_d) \in bP$. This $R(P)$ is the *Ehrhart ring* of the polytope.

In fact, the ring $R(P)$ has a basis consisting of these monomials together with 1. Define a grading on $R(P)$ by setting $\deg(x_1^{a_1} \dots x_d^{a_d} y^b) := b$.

Then the Hilbert function $H(R(P), j)$ is equal to the number of points $\underline{a} \in jP$ satisfying $\underline{a} \in \mathbb{Z}^d$, or in other words

$$H(R(P), j) = |jP \cap \mathbb{Z}^d|.$$

Hence $H(R(P), j)$ is precisely the Ehrhart polynomial of P .

Since $\deg(H(R(P), j)) = d$, thus we get that $\dim R(P) = d + 1$. Moreover, it is easy to see that $R(P)$ is normal, hence it follows from a theorem of Hochster [Ho72] that $R(P)$ is a Cohen–Macaulay ring. Clearly $R(P)$ is a domain. It is also clear, that $K[(R(P))_1]$ contains the monomials $\deg(x_1^{a_1} \dots x_d^{a_d} y^b) = b$ for which (a_1, \dots, a_d) is a vertex of P .

The following Proposition is well-known:

Proposition 2.9 *Let P be a reflexive polytope. Then the Ehrhart ring $R(P)$ is a Gorenstein algebra.*

3 The main results

3.1 Integrally closed polytopes

We connect here the results concerning Cohen–Macaulay standard graded algebras to Ehrhart theory.

Proposition 3.1 *Let P be a lattice polytope. The Ehrhart algebra $R(P)$ is a standard graded algebra if and only if the lattice polytope P is integrally closed.*

Proof. This is clear from the definition. □

Corollary 3.2 *Let $P \subseteq \mathbb{R}^d$ be a d -dimensional integrally closed lattice polytope. Then*

$$\delta_i \leq \binom{\delta_1 + i - 1}{i}$$

for each $i \geq 1$.

Proof.

Consider the Ehrhart ring $R(P)$. Since P is an integrally closed lattice polytope, hence it follows from Proposition 3.1 that $R(P)$ is a Cohen–Macaulay standard graded algebra. Finally we get from Theorem 2.2 that

$$\delta_i \leq \binom{\delta_1 + i - 1}{i}$$

for each $i \geq 1$, because the Hilbert series of the Ehrhart ring $R(P)$ is the Ehrhart series of the polytope P . □

Corollary 3.3 *Let $P \subseteq \mathbb{R}^d$ be a d -dimensional integrally closed lattice polytope. Then the delta-vector $(\delta_0, \dots, \delta_s)$ of P is an O -sequence.*

Proof.

This follows from the proof of Theorem 2.2. \square

Remark. Hibi proved in [Hib89] Corollary 1.3 the following nice property of pure O -sequences.

Proposition 3.4 *Let (h_0, \dots, h_s) be a pure O -sequence with $h_s \neq 0$. Then $h_i \leq h_j$ for each $0 \leq i \leq j \leq s - i$ and consequently $h_0 \leq h_1 \leq \dots \leq h_{\lfloor \frac{s}{2} \rfloor}$.*

We proved that the delta-vector $(\delta_0, \dots, \delta_d)$ of any integrally closed lattice polytope P is an O -sequence. Hence if P is a reflexive integrally closed polytope and we can prove that the delta-vector $(\delta_0, \dots, \delta_d)$ of P is a *pure* O -sequence, then the delta-vector will be unimodal.

Corollary 3.5 *Let P be a d -dimensional integrally closed lattice polytope with $n := |P \cap \mathbb{Z}^d|$. Then*

$$d! \text{vol}(P) \leq \binom{n-1}{d}.$$

Proof. It follows from Corollary 3.2 and Corollary 1.2 (iii) and (iv) that

$$d! \text{vol}(P) = \sum_{i=0}^d \delta_i \leq \sum_{i=0}^d \binom{\delta_1 + i - 1}{i} = \sum_{i=0}^d \binom{n - d + i - 2}{i} = \binom{n-1}{d}.$$

\square

Theorem 3.6 *Let P be a d -dimensional integrally closed lattice polytope. Let $(\delta_0, \dots, \delta_s)$ be the delta-vector of P . Let $m \geq 0$ and $n \geq 1$ with $m+n < s$. Then*

$$\delta_1 + \dots + \delta_n \leq \delta_{m+1} + \delta_{m+2} + \dots + \delta_{m+n}.$$

Proof. This follows easily from Proposition 2.7. \square

Corollary 3.7 *Let P be a d -dimensional integrally closed lattice polytope. Let $(\delta_0, \dots, \delta_s)$ be the delta-vector of P with $\delta_s \neq 0$. Then*

$$2 + (s - 1)(|P \cap \mathbb{Z}^d| - d + 1) \leq d! \text{vol}(P).$$

We have equality if and only if the δ -vector of P equals

$$(1, |P \cap \mathbb{Z}^d| - d - 1, |P \cap \mathbb{Z}^d| - d - 1, \dots, |P \cap \mathbb{Z}^d| - d - 1, 1).$$

Proof. An obvious consequence of Theorem 3.6 that $\delta_1 \leq \delta_i$ for each $1 \leq i \leq s - 1$. Hence by Corollary 1.2 (iv)

$$2 + (s - 1)(|P \cap \mathbb{Z}^d| - d + 1) = 2 + \delta_1(s - 1) \leq 2 + \sum_{i=1}^{s-1} \delta_i \leq \sum_{i=0}^s \delta_i = d! \text{vol}(P).$$

□

Theorem 3.8 *Let P be a d -dimensional integrally closed lattice polytope. The sequence $L_P(m)$, $m \geq 1$, is a differentiable O -sequence.*

Proof. This follows easily from Proposition 2.8, if we apply for $R := R(P)$.

□

3.2 Reflexive integrally closed polytopes

Now we specialize our results to integrally closed reflexive lattice polytopes.

Theorem 3.9 *Let P be an d -dimensional integrally closed reflexive lattice polytope such that $n \leq d + 4$, where $n := |P \cap \mathbb{Z}^d|$. Then the delta-vector $(\delta_0, \dots, \delta_d)$ of P will be unimodal.*

Proof. This follows from Theorem 2.6 and Proposition 2.9. □ Let $d \geq 2$

and $n \geq d + 1$ be integers. Consider the convex hull of any n distinct points on the moment curve $\{(t, t^2, \dots, t^d) : t \in \mathbb{R}\}$. Let us denote this polytope by $C(n, d)$.

It can be shown that the combinatorial structure of the simplicial d -polytope $C(n, d)$ is independent of the actual choice of the points, and this polytope is the cyclic d -polytope with n vertices. It can be shown that

Theorem 3.10

$$f_{d-1}(C(n, d)) = \binom{n - \lfloor (d+1)/2 \rfloor}{n-d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n-d}.$$

Theorem 3.11 *Let P be a d -dimensional reflexive integrally closed lattice polytope with $n := |P \cap \mathbb{Z}^d|$. Then*

$$d! \text{vol}(P) \leq f_{d-1}(C(n, d)).$$

Proof. Since P is reflexive, hence by Theorem 1.4 (iv) $\delta_i = \delta_{d-i}$ for each $0 \leq i \leq d/2$. Now we can apply Corollary 3.2. \square

Theorem 3.12 *Let P be a d -dimensional reflexive integrally closed lattice polytope. Let $(\delta_0, \dots, \delta_s)$ be the delta-vector of P . If i and j are non-negative integers such that $i + j \leq d$, then*

$$\delta_i \leq \delta_j \delta_{i+j}.$$

Proof.

By Proposition 2.9 the Ehrhart ring $R(P)$ is a Gorenstein algebra, since P is a reflexive polytope. Consequently $R(P)$ is a level algebra of type 1 and we can apply Proposition 2.5. \square

Theorem 3.13 *Let P be a d -dimensional integrally closed lattice polytope. The sequence $L_{\partial P}(m)$, $m \geq 1$, is a differentiable O -sequence.*

Proof. This follows from Theorem 3.8 and Theorem 1.4 (ii). \square

3.3 Order polytopes

Let P be a finite partially ordered set with $n := |P|$. Then the vertices of the order polytope $O(P)$ are the characteristic vectors of the order ideals of P , so in particular $O(P)$ is an n -dimensional lattice polytope. Hibi and Ohsugi proved in [HO01] Example 1.3(b) that $O(P)$ is a compressed polytope. This means that all of its pulling triangulations are unimodular, consequently $O(P)$ is integrally closed.

Corollary 3.14 *Let P be a finite partially ordered set with $n := |P|$. Let (w_0, \dots, w_s) denote the delta-vector of the order polytope $O(P)$. Then (w_0, \dots, w_s) is an O -sequence and consequently*

$$w_i \leq \binom{w_1 + i - 1}{i}$$

for each $i \geq 1$. Further let $m \geq 0$ and $n \geq 1$ with $m + n < s$. Then

$$w_1 + \dots + w_n \leq w_{m+1} + w_{m+2} + \dots + w_{m+n}.$$

Proof. Hibi and Ohsugi proved in [HO01] Example 1.3(b) that $O(P)$ is compressed polytope. This means that all of its pulling triangulations are unimodular, consequently $O(P)$ is integrally closed. Hence we can apply Corollary 3.2, Corollary 3.3 and Theorem 3.6 for the order polytope $O(P)$. \square

Remark. The delta-vector of the order polytope $O(P)$ encodes also the Ω -Eulerian polynomial (see [Sta97], Chapter 4).

4 Concluding remarks

Recall that a lattice polytope $P \subseteq \mathbb{R}^d$ is *smooth* if the primitive edge vectors at every vertex of P define a part of a basis of \mathbb{Z}^d . It is well-known that smooth polytopes correspond to projective embeddings of smooth projective toric varieties. Oda asked following famous conjecture:

Conjecture 1 *All smooth polytopes are integrally closed.*

Our next conjecture would follow from Theorem 1.5 and Oda's conjecture:

Conjecture 2 *Let P be a d -dimensional smooth lattice polytope with $n := |P \cap \mathbb{Z}^d|$. Then*

$$d! \operatorname{vol}(P) \leq \binom{n-1}{d}.$$

J. Schepers determined completely the extremal polytopes appearing in Theorem 1.5:

Proposition 4.1 *Let P be a d -dimensional integrally closed lattice polytope with $n := |P \cap \mathbb{Z}^d|$. Then*

$$d! \operatorname{vol}(P) = \binom{n-1}{d}.$$

if and only if

- (i) $d = 1$;
- (ii) P is an unimodular simplex;
- (iii) P is lattice isomorphic to the reflexive polytope with vertices e_1, \dots, e_d and $-e_1 - \dots - e_d$, where e_1, \dots, e_d denote the standard basis vectors.

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