ÖAW AUSTRIAN ACADEMY OF SCIENCES

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FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Betti numbers of Stanley–Reisner rings with pure resolutions

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RICAM-Report 2011-12

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May 9, 2011

Abstract

Let Δ be simplicial complex and let $k[\Delta]$ denote the Stanley–Reisner ring corresponding to Δ. Suppose that $k[\Delta]$ has a pure free resolution. Then we describe the Betti numbers and the Hilbert–Samuel multiplicity of $k[\Delta]$ in terms of the h–vector of Δ. As an application, we derive a linear equation system and some inequalities for the components of the h–vector of the clique complex of an arbitrary chordal graph. As an other application, we derive a linear equation system and some inequalities for the components of the h–vector of Cohen–Macaulay simplicial complexes.

1 Introduction

Let k denote an arbitrary field. Let R be the graded ring $k[x_1, \dots, x_n]$. The vector space $R_s = k[x_1, \dots, x_n]_s$ consists of the homogeneous polynomials of total degree s, together with 0.

Let Δ be a simplicial complex. A facet F is called a *leaf*, if either F is the only facet of Δ, or there exists an other facet G, $G \neq F$ such that $H \cap F \subset G \cap F$ for each facet H with $H \neq F$. A facet G with this property is called a *branch* of F.

[0]**Keywords.** Betti number, Hilbert function, Stanley-Reisner ring
2000 Mathematics Subject Classification. 05E40, 13D02, 13D40

In [16] J. Herzog, T. Hibi, S. Murai, N. V. Trung and X. Zheng call the simplicial complex Δ a *quasi–forest* if there exists a labeling $F_1, \ldots, F_m$ of the facets such that for all i the facet F_i is a leaf of the subcomplex $\langle F_1, \ldots, F_i \rangle$. We call such a labeling a *leaf order*.

A graph is called *chordal* if each cycle of length > 3 has a chord.

J. Herzog, T. Hibi, S. Murai, N. V. Trung and X. Zheng proved the following equivalent characterization of quasi–forests in [16] Theorem 1.1:

Theorem 1.1 *Given a finite sequence $(f_{-1}, \ldots, f_{d-1})$ of integers with each $f_i > 0$, the following conditions are equivalent:*
(i) There is a quasi–forest Δ of dimension $d-1$ with $f(\Delta) := (f_{-1}, \ldots, f_{d-1})$.
(ii) The sequence $(c_1, \ldots, c_d)$, defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^i = \sum_{i=0}^{d} c_i x^i,$$

where $f_{-1} = 1$, satisfies $\sum_{i=0}^{d} c_i > 0$ for each $1 \leq k \leq d$.
(iii) The sequence $(b_1, \ldots, b_d)$ defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^i = \sum_{i=0}^{d} b_i x^{i-1},$$

is positive, i.e., $b_i > 0$ for each $1 \leq i \leq d$.

Let G be a finite graph on $[n]$. A *clique* of G is a subset F of $[n]$ such that $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$.

We write $\Delta(G)$ for the simplicial complex on $[n]$ whose faces are the cliques of G. This $\Delta(G)$ is the *clique complex* of the graph G.

We recall here for the famous Dirac's Theorem (see [4]).

Theorem 1.2 *(Dirac) A finite graph G on $[n]$ is a chordal graph iff G is the 1–skeleton of a quasi–forest*

Let Δ be a simplicial complex. Our main results are some explicit formulas for the Betti numbers of the Stanley–Reisner ring $k[\Delta]$ such that $k[\Delta]$ has a pure free resolution in terms of the h–vector of Δ. We prove:

Theorem 1.3 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a pure free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \qquad (1)$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \quad (2)$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\beta_i = \sum_{\ell=0}^{d_i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{d_i-\ell}$$

for each $0 \leq i \leq p$.

Remark. Clearly $h_i = 0$ for each $i > d$.

Remark. J. Herzog and M. Kühl proved similar formulas for the Betti number in [17] Theorem 1. Here we did not assume that the Stanley–Reisner ring $k[\Delta]$ with pure resolution is Cohen–Macaulay.

As an application we describe some linear equations for the components of the h–vector of the clique complex of an arbitrary chordal graph:

Corollary 1.4 *Let G be an arbitrary chordal graph. Let $\Delta := \Delta(G)$ denote the clique complex of G and $d := dim(\Delta)+1$. Let $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ denote the h-vector of the complex Δ. Let p be the projective dimension of the Stanley–Reisner ring $k[\Delta]$. Then*

$$\sum_{\ell=0}^{j} (-1)^{\ell} h_{j-\ell} \binom{n-d}{\ell} = 0$$

for each $p+2 < j \leq n$.

Remark. We derived here that

$$\sum_{i=0}^{j} (-1)^{i} f_{i-1} \binom{n-i}{j-i} = 0$$

for each $p+2 < j \leq n$.

In Section 2 we collected some basic results about simplicial complices, free resolutions, Hilbert fuctions and Hilbert series. We present our main results in Section 3. We give some applications for Cohen–Macaulay squarefree monomial ideals and chordal graphs in Section 4.

2 Preliminaries

2.1 Free resolutions

Recall that for every finitely generated graded module M over R we can associate to M a *minimal graded free resolution*

$$0 \longrightarrow \bigoplus_{i=1}^{\beta_p} R(-d_{p,i}) \longrightarrow \bigoplus_{i=1}^{\beta_{p-1}} R(-d_{p-1,i}) \longrightarrow \ldots \longrightarrow \bigoplus_{i=1}^{\beta_0} R(-d_{0,i}) \longrightarrow M \longrightarrow 0,$$

where $p \leq n$ and $R(-j)$ is the free R-module obtained by shifting the degrees of R by j.

Here the natural number β_k is the k'th *total Betti number* of M and p is the projective dimension of M.

The module M has a *pure resolution* if there are constants $d_0 < \ldots < d_p$ such that

$$d_{0,i} = d_0, \ldots, d_{p,i} = d_p$$

for all i. If in addition

$$d_i = d_0 + i,$$

for all $1 \leq i \leq p$, then we call the minimal free resolution to be d_0*–linear*.

Let G be a simple graph. We may think of an edge E of a graph as a squarefree monomial $x^E := \prod_{j \in E} x_j$ in R.

The *edge ideal* $I(G)$ is the ideal $\langle x^E : \ E \in E(G) \rangle$, which is generated by the edges of G.

The edge ideal was first introduced by R. Villareal in [22]. Later edge ideals have been studied very widely, see for instance [5, 6, 8, 9, 10, 12, 13, 20, 22, 24, 25].

In [10] R. Fröberg characterized the graphs G such that G has a linear free resolution. He proved:

Theorem 2.1 *Let G be a simple graph on n vertices. Then $R/I(G)$ has linear free resolution precisely when $\overline{G}$, the complementary graph of G is chordal.*

In [6] E. Emtander generalized Theorem 2.1 for generalized chordal hypergraphs.

In [21] Theorem 2.7 the following bound for the Betti numbers was proved.

Theorem 2.2 *Let M be an R–module having a pure resolution of type $(d_0, \dots, d_p)$ and Betti numbers $\beta_0, \dots, \beta_p$, where p is the projective dimension of M. Then*

$$\beta_i \geq \binom{p}{i} \tag{3}$$

for each $0 \leq i \leq p$.

2.2 Hilbert–Serre Theorem

Let $M = \bigoplus_{i\geq 0} M_i$ be a finitely generated nonnegatively graded module over the polynomial ring R. We call the formal power series

$$H_M(z) := \sum_{i=0}^{\infty} h_M(i) z^i$$

the *Hilbert–series* of the module M.

The Theorem of Hilbert–Serre states that there exists a (unique) polynomial $P_M(z) \in \mathbb{Q}[z]$, the so-called *Hilbert polynomial* of M, such that $h_M(i) = P_M(i)$ for each $i >> 0$. Moreover, P_M has degree dim $M - 1$ and (dim $M - 1$)! times the leading coefficient of P_M is the *Hilbert–Samuel multiplicity* of M, denoted here by $e(M)$.

Hence there exist integers $m_0, \dots, m_{d-1}$ such that $h_M(z) = m_0 \cdot \binom{z}{d-1} + m_1 \cdot \binom{z}{d-2} + \dots + m_{d-1}$, where $\binom{z}{r} = \frac{1}{r!} z(z-1)\dots(z-r+1)$ and $d := \dim M$. Clearly $m_0 = e(M)$.

We can summarize the Hilbert-Serre theorem as follows:

Theorem 2.3 *(Hilbert–Serre) Let M be a finitely generated nonnegatively graded R–module of dimension d, then the following statements hold:*
(a) There exists a (unique) polynomial $P(z) \in \mathbb{Z}[z]$ such that the Hilbert–series $H_M(z)$ of M may be written as

$$H_M(z) = \frac{P(z)}{(1-z)^d}$$

(b) d is the least integer for which $(1-z)^d H_M(z)$ is a polynomial.

2.3 Simplicial complexes and Stanley–Reisner rings

We say that $\Delta \subseteq 2^{[n]}$ is a *simplicial complex* on the vertex set $[n] = \{1, 2, \ldots, n\}$, if Δ is a set of subsets of $[n]$ such that Δ is a down–set, that is, $G \in \Delta$ and $F \subseteq G$ implies that $F \in \Delta$, and $\{i\} \in \Delta$ for all i.

The elements of Δ are called *faces* and the *dimension* of a face is one less than its cardinality. An r-face is an abbreviation for an r-dimensional face. The dimension of Δ is the dimension of a maximal face. We use the notation $\dim(\Delta)$ for the dimension of Δ.

If $\dim(\Delta) = d-1$, then the $(d+1)$–tuple $(f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ is called the *f-vector* of Δ, where $f_i(\Delta)$ denotes the number of i–dimensional faces of Δ.

Let Δ be an arbitrary simplicial complex on $[n]$. The *Stanley–Reisner ring* $k[\Delta] := R/I(\Delta)$ of Δ is the quotient of the ring R by the *Stanley–Reisner ideal*

$$I(\Delta) := \langle x^F : \ F \notin \Delta \rangle,$$

generated by the non–faces of Δ.

Proposition 2.4 *Let Δ be a $(d-1)$–dimensional simplicial complex. Then*

$$e(k[\Delta]) = f_{d-1}.$$

Proof. It follows from [1] Proposition 4.1.9 and (4) that

$$e(k[\Delta]) = \left(\sum_{i=0}^{d} h_i z^i \right) |_{z=1} = \sum_{i=0}^{d} h_i = f_{d-1}.$$

The following Theorem was proved in [1] Theorem 5.1.7. □

Theorem 2.5 *Let Δ be a $d-1$–dimensional simplicial complex with f-vector $f(\Delta) := (f_{-1}, \ldots, f_{d-1})$. Then the Hilbert–series of the Stanley–Reisner ring $k[\Delta]$ is*

$$H_{k[\Delta]}(z) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

□

Recall from Theorem 2.3 that a homogeneous k-algebra M of dimension d has a Hilbert series of the form

$$H_M(z) = \frac{P(z)}{(1-z)^d}$$

where $P(z) \in \mathbb{Z}[z]$. Let Δ be a $(d-1)$–dimensional simplicial complex and write

$$H_{k[\Delta]}(z) = \frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^d}. \tag{4}$$

The following Lemma was proved in [1] Lemma 5.1.8.

Lemma 2.6 *The f-vector and the h-vector of a $(d-1)$–dimensional simplicial complex Δ are related by*

$$\sum_i h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}.$$

In particular, the h-vector has length at most d, and

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}, \quad f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i$$

for each $j = 0, \dots, d$.

□

Let Γ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Gamma]$ has a pure free resolution

$$\mathcal{F}_\Gamma : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \dots \longrightarrow \tag{5}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow k[\Gamma] \longrightarrow 0. \tag{6}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

It was proved in [15] Corollary 6.1.7. that

$$(-1)^{n-d}(n-d)!e(k[\Gamma]) = \sum_{i=0}^{p}(-1)^i \beta_i (d_i)^{n-d}. \qquad (7)$$

The following Theorem was proved in [7] Theorem 3.

Theorem 2.7 *(Eagon–Reiner) Let k be an arbitrary field and Δ be a simplicial complex on $[n]$. Then $I(\Delta^*)$ has a q–linear resolution if and only if the Stanley–Reisner ring $k[\Delta]$ is Cohen–Macaulay of dimension $n-q$..*

2.4 Alexander dual

Let $\mathcal{F} \subseteq 2^{[n]}$ be an arbitrary set system. Define the complement of $\mathcal{F}$ as

$$\mathcal{F}' := 2^{[n]} \setminus \mathcal{F}.$$

Consider the following set system

$$\text{co}(\mathcal{F}) := \{[n] \setminus F : \ F \in \mathcal{F}\}.$$

We denote by $\mathcal{F}^*$ the *Alexander dual of $\mathcal{F}$*

$$\mathcal{F}^* := \text{co}(\mathcal{F}') = (\text{co}(\mathcal{F}))' \subseteq 2^{[n]}.$$

Let Δ^* denote the Alexander dual of the simplicial complex Δ. We can easily compute $f(\Delta^*)$, the f-vector of Δ^*:

Lemma 2.8 *Let $f(\Delta) = (f_{-1}(\Delta), \dots, f_{d-1}(\Delta))$ be the f–vector of a $(d-1)$–dimensional simplicial complex Δ. Then the f–vector of the simplicial complex Δ^* is:*

$$f(\Delta^*) = [\underbrace{1}_{f^*_{-1}}, \underbrace{\binom{n}{1}}_{f^*_0}, \dots, \underbrace{\binom{n}{n-d-1}}_{f^*_{n-d-1}}, \underbrace{\binom{n}{n-d} - f_{d-1}}_{f^*_{n-d}}, \dots, \underbrace{\binom{n}{2} - f_1}_{f^*_{n-2}}]. \qquad (8)$$

Hence

$$f^*_i = \binom{n}{i+1} - f_{n-i-1}$$

for each $-1 \le i \le n-1$.

Corollary 2.9 *Let Δ be any simplicial complex. Let k^* denote the positive integer such that $\binom{n}{k^*} \neq f_{k^*-1}$, but $\binom{n}{k^*-1} = f_{k^*-2}$. Then $d^* = dim(k[\Delta^*]) = n - k^*$.*

3 Our main result

3.1 Pure resolutions

In the following first we prove our main result (Theorem 1.3).

Proof of Theorem 1.3:
Let $M := k[\Delta]$ denote the Stanley–Reisner ring of Δ. Then we infer from Theorem 2.5 that

$$H_M(z) = \frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^d}. \tag{9}$$

Since the Hilbert–series is additive on short exact sequences, and since

$$H_R(z) = \frac{1}{(1-z)^n},$$

and consequently

$$H_{R(-s)}(z) = \frac{z^s}{(1-z)^n},$$

the pure resolution

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{10}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow M \longrightarrow 0. \tag{11}$$

yields to

$$H_M(z) = \frac{1}{(1-z)^n} + \sum_{i=0}^{p} (-1)^{i+1} \beta_i \frac{z^{d_i}}{(1-z)^n}, \tag{12}$$

where $p = pdim(M)$.

Write $d := \dim M$, and let $m := \operatorname{codim}(M) = n - d$. It follows from the Auslander–Buchbaum formula that $m \leq p$.

Comparing the two expressions (12) and (9) for H_M, we find

$$(1-z)^m \left(\sum_{i=0}^{d} h_i z^i \right) = \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{d_i} + 1 \tag{13}$$

Using the binomial Theorem we get that

$$\left(\sum_{j=0}^{n-d} (-1)^j \binom{n-d}{j} z^j \right) \left(\sum_{i=0}^{d} h_i z^i \right) = \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{d_i} + 1 \tag{14}$$

Comparing the coefficients on the two sides of (14), we get the result.
□

Corollary 3.1 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a pure free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{15}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{16}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$. Let $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ denote the h-vector of the complex Δ. Let s be a positive integer such that $s \neq d_i$ for each $0 \leq i \leq p$. Then

$$\sum_{\ell=0}^{s} (-1)^\ell h_{s-\ell} \binom{n-d}{\ell} = 0.$$

Proof. Define

$$P(z) := 1 + \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{t+i} \in \mathbb{Q}[z].$$

Comparing the coefficients of both side of (14), we get the result. □

Corollary 3.2 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a pure free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{17}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{18}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$. Then

$$\sum_{\ell=0}^{d_i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{d_i-\ell} \geq \binom{p}{i} \tag{19}$$

for each $0 \leq i \leq p$.

Proof. This follows easily from Theorem 2.2 and Theorem 1.3. □

3.2 t–linear free resolutions

In the following we specialize our results to t–linear resolutions. Since any t–linear free resolution will be a pure resolution, so we can apply Theorem 1.3, Corollary 3.1 and Corollary 3.2.

Corollary 3.3 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a t–linear free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-t-p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{20}$$

$$\longrightarrow R(-t-1)^{\beta_1} \longrightarrow R(-t)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{21}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\sum_{\ell=0}^{j} (-1)^\ell h_{j-\ell} \binom{n-d}{\ell} = 0. \tag{22}$$

for each $p+t < j \leq n$ and $0 < j < t$.

Remark. Since

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1},$$

for each $0 \leq j \leq d$, hence

$$h_{j-\ell} = \sum_{i=0}^{j-\ell} (-1)^{j-\ell-i} \binom{d-i}{j-\ell-i} f_{i-1}$$

for each $0 \leq \ell \leq j$. Substituting these expressions into (22) we get that

$$\sum_{\ell=0}^{j} (-1)^\ell h_{j-\ell} \binom{n-d}{\ell} = \sum_{\ell=0}^{j} (-1)^\ell \binom{n-d}{\ell} \left(\sum_{i=0}^{j-\ell} (-1)^{j-\ell-i} \binom{d-i}{j-\ell-i} f_{i-1} \right)$$

$$= (-1)^j \sum_{i=0}^{j} (-1)^i f_{i-1} \left(\sum_{\ell=0}^{j-i} \binom{n-d}{\ell} \binom{d-i}{j-\ell-i} \right).$$

Using the Vandermonde identities (see [11], 169–170) we get that

$$\sum_{\ell=0}^{j}(-1)^{\ell}h_{j-\ell}\binom{n-d}{\ell} = (-1)^{j}\sum_{i=0}^{j}(-1)^{i}f_{i-1}\binom{n-i}{j-i} = 0$$

for each $p+t < j \leq n$ and $0 < j < t$.

Corollary 3.4 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a t–linear free resolution*

$$\mathcal{F}_{\Delta} : 0 \longrightarrow R(-t-p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{23}$$

$$\longrightarrow R(-t-1)^{\beta_1} \longrightarrow R(-t)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{24}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$. If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\beta_i = \sum_{\ell=0}^{t+i}(-1)^{\ell+i+1}h_{t+i-\ell}\binom{n-d}{\ell}$$

for each $0 \leq i \leq p$.

Remark. Hence we get that

$$\beta_i = (-1)^{t+i}\sum_{\ell=0}^{t+i}(-1)^{\ell}f_{\ell-1}\binom{n-\ell}{t+i-\ell}$$

for each $0 \leq i \leq p$.

Corollary 3.5 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a t–linear free resolution*

$$\mathcal{F}_{\Delta} : 0 \longrightarrow R(-t-p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{25}$$

$$\longrightarrow R(-t-1)^{\beta_1} \longrightarrow R(-t)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{26}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$. If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\sum_{\ell=0}^{t+i}(-1)^{\ell+i+1}\binom{n-d}{\ell}h_{t+i-\ell} \geq \binom{p}{i} \tag{27}$$

for each $0 \leq i \leq p$.

Remark. We derived that

$$(-1)^{t+i}\sum_{\ell=0}^{t+i}(-1)^{\ell}f_{\ell-1}\binom{n-\ell}{t+i-\ell}\geq\binom{p}{i}$$

for each $0\leq i\leq p$.

4 Applications

4.1 Cohen–Macaulay squarefree monomial ideals

Cohen–Macaulay simplicial complexes are very important in combinatorial commutative algebra. It is known that the boundary complex of simplicial polytopes and the order complex of a bounded, locally upper semimodular poset are Cohen–Macaulay.

Now we apply our main results for Cohen–Macaulay simplicial complexes.

Corollary 4.1 *Let Δ be a $(d-1)$–dimensional Cohen–Macaulay simplicial complex. Consider the t–linear free resolution*

$$\mathcal{F}_{\Delta^*}:0\longrightarrow R(-t-p)^{\beta_p^*}\longrightarrow\ldots\longrightarrow \tag{28}$$

$$\longrightarrow R(-t-1)^{\beta_1^*}\longrightarrow R(-t)^{\beta_0^*}\longrightarrow R\longrightarrow k[\Delta^*]\longrightarrow 0. \tag{29}$$

Here $t=n-d$. Let k^* denote the positive integer such that $\binom{n}{k^*}\neq f_{k^*-1}$, but $\binom{n}{k^*-1}=f_{k^*-2}$. Then

$$\sum_{\ell=0}^{j}(-1)^{\ell}h_{j-\ell}^*\binom{k^*}{\ell}=0. \tag{30}$$

for each $p+t<j\leq n$ and $0<j<t$.

Proof. This follows easily from Corollary 2.9 and Corollary 3.3. □

Corollary 4.2 *Let Δ be a $(d-1)$–dimensional Cohen–Macaulay simplicial complex. Consider the t–linear free resolution*

$$\mathcal{F}_{\Delta^*}:0\longrightarrow R(-t-p)^{\beta_p^*}\longrightarrow\ldots\longrightarrow \tag{31}$$

$$\longrightarrow R(-t-1)^{\beta_1^*}\longrightarrow R(-t)^{\beta_0^*}\longrightarrow R\longrightarrow k[\Delta^*]\longrightarrow 0. \tag{32}$$

Here $t = n - d$. Let p denote the projective dimension of $k[\Delta^*]$. Let k^* denote the positive integer such that $\binom{n}{k^*} \neq f_{k^*-1}$, but $\binom{n}{k^*-1} = f_{k^*-2}$. Let $h(\Delta^*) := (h_0^*(\Delta), \ldots, h_{n-2}^*(\Delta))$ be the h-vector of the complex Δ^*, Then

$$\beta_i^* = \sum_{\ell=0}^{t+i} (-1)^{\ell+i+1} h_{t+i-\ell}^* \binom{k^*}{\ell} \tag{33}$$

for each $0 \leq i \leq p$.

Remark. Eagon and Reiner proved similar formulas for the Betti numbers β_i^* in [7] Theorem 4. They described the folowing equation:

$$\sum_{i \geq 1} \beta_i^* t^{i-1} = \sum_{i=0}^{d} h_i(\Delta)(t+1)^i$$

Proof. It follows from the Eagon–Reiner Theorem 2.7 that the free resolution (32) exists. Now we can apply Corollary 3.4 for the t–linear free resolution (32). Since $d^* = n - k^*$ by Corollary 2.9, hence the result follows. □

Corollary 4.3 *Let Δ be a $(d-1)$–dimensional Cohen–Macaulay simplicial complex. Consider the t–linear free resolution*

$$\mathcal{F}_{\Delta^*} : 0 \longrightarrow R(-t-p)^{\beta_p^*} \longrightarrow \ldots \longrightarrow \tag{34}$$

$$\longrightarrow R(-t-1)^{\beta_1^*} \longrightarrow R(-t)^{\beta_0^*} \longrightarrow R \longrightarrow k[\Delta^*] \longrightarrow 0. \tag{35}$$

Here $t = n - d$. Let k^* denote the positive integer such that $\binom{n}{k^*} \neq f_{k^*-1}$, but $\binom{n}{k^*-1} = f_{k^*-2}$. Then

$$e(k[\Delta^*]) = \frac{(-1)^{k^*}}{(k^*)!} \sum_{i=0}^{p} (-1)^i \beta_i^* (n-d+i)^{k^*} \tag{36}$$

where p is the projective dimension of $k[\Delta^*]$.

Proof. Let Δ be a $(d-1)$–dimensional simplicial complex. Let $\Gamma := \Delta^*$. Then using Eagon–Reiner Theorem 2.7 and (7), we get our result. □

Corollary 4.4 *Let Δ be a $(d-1)$–dimensional Cohen–Macaulay simplicial complex. Consider the t–linear free resolution*

$$\mathcal{F}_{\Delta^*} : 0 \longrightarrow R(-t-p)^{\beta_p^*} \longrightarrow \ldots \longrightarrow \tag{37}$$

$$\longrightarrow R(-t-1)^{\beta_1^*} \longrightarrow R(-t)^{\beta_0^*} \longrightarrow R \longrightarrow k[\Delta^*] \longrightarrow 0. \tag{38}$$

Here $t = n-d$. Let k^* denote the positive integer such that $\binom{n}{k^*} \neq f_{k^*-1}$, but $\binom{n}{k^*-1} = f_{k^*-2}$. Then

$$\binom{n}{k^*} - f_{k^*-1} = \frac{(-1)^{k^*}}{(k^*)!} \sum_{i=0}^{p} (-1)^i \beta_i^* (n-d+i)^{k^*}. \tag{39}$$

Proof. Since by Lemma 2.4

$$e(k[\Delta^*]) = f^*_{d^*-1} = \binom{n}{k^*} - f_{k^*-1} \tag{40}$$

and we infer from Corollary 4.3 that

$$e(k[\Delta^*]) = \frac{(-1)^{k^*}}{(k^*)!} \sum_{i=0}^{p} (-1)^i \beta_i^* (t+i)^{k^*} = \frac{(-1)^{k^*}}{(k^*)!} \sum_{i=0}^{p} (-1)^i \beta_i^* (n-d+i)^{k^*}, \tag{41}$$

we get the result. □

Corollary 4.5 *Let Δ be a $(d-1)$–dimensional Cohen–Macaulay simplicial complex. Consider the t–linear free resolution*

$$\mathcal{F}_{\Delta^*} : 0 \longrightarrow R(-t-p)^{\beta_p^*} \longrightarrow \ldots \longrightarrow \tag{42}$$

$$\longrightarrow R(-t-1)^{\beta_1^*} \longrightarrow R(-t)^{\beta_0^*} \longrightarrow R \longrightarrow k[\Delta^*] \longrightarrow 0. \tag{43}$$

Here $t = n-d$. Let k^* denote the positive integer such that $\binom{n}{k^*} \neq f_{k^*-1}$, but $\binom{n}{k^*-1} = f_{k^*-2}$. If $h(\Delta^*) := (h_0(\Delta^*), \ldots, h_d(\Delta^*))$ is the h-vector of the complex Δ^*, then

$$\sum_{\ell=0}^{t+i} (-1)^{\ell+i+1} \binom{k^*}{\ell} h^*_{t+i-\ell} \geq \binom{p}{i} \tag{44}$$

for each $0 \leq i \leq p$.

Proof. This follows easily from Theorem 2.7 and Corollary 3.5. □

4.2 Chordal graphs

We can specialize our results for chordal graphs. First we prove our main result.

Proof of Theorem 1.4:

Let H denote the complementary graph of G. Then the ideal $I(H)$ has a 2–linear free resolution by Theorem 2.1. So we can apply Corollary 3.3. □

Corollary 4.6 *Let G be an arbitrary chordal graph. Let $\Delta := \Delta(G)$ denote the clique complex of G and $d := dim(\Delta) + 1$. Then the Stanley–Reisner ring $k[\Delta]$ has an 2–linear free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-2-p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{45}$$

$$\longrightarrow R(-3)^{\beta_1} \longrightarrow R(-2)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{46}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$. If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex $\Delta(G)$, then

$$\beta_i = \sum_{\ell=0}^{2+i} (-1)^{\ell+i+1} h_{2+i-\ell} \binom{n-d}{\ell}$$

for each $0 \leq i \leq p$.

Remark. It comes out that

$$\beta_i = (-1)^i \sum_{\ell=0}^{2+i} (-1)^{\ell} f_{\ell-1} \binom{n-d}{2+i-\ell}$$

for each $0 \leq i \leq p$.

Proof. Let H denote the complementary graph of G. Then the ideal $I(H)$ has a 2–linear free resolution by Theorem 2.1. Corollary 3.4 gives the result.

Corollary 4.7 *Let G be an arbitrary chordal graph. Let $\Delta := \Delta(G)$ denote the clique complex of G and $d := dim(\Delta) + 1$. Let p be the projective dimension of the Stanley–Reisner ring $k[\Delta]$. If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then*

$$\sum_{\ell=0}^{2+i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{2+i-\ell} \geq \binom{p}{i} \tag{47}$$

for each $0 \leq i \leq p$.

Remark. This yields to

$$(-1)^i \sum_{\ell=0}^{2+i} (-1)^\ell f_{\ell-1} \binom{n-d}{2+i-\ell} \geq \binom{p}{i}$$

for each $0 \leq i \leq p$.

Proof. Let H denote the complementary graph of G. Then the ideal $I(H)$ has a 2–linear free resolution by Theorem 2.1. The result follows from Corollary 3.5

Acknowledgements. I am indebted to Josef Schicho, Russ Woodroofe and Lajos Rónyai for their useful remarks.

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