On shape sensitivity analysis of the cost functional without shape sensitivity of the state variable

H. Kasumba, K. Kunisch

RICAM-Report 2011-07

On Shape Sensitivity Analysis of the Cost Functional Without Shape Sensitivity of the State Variable

H. Kasumba* and K. Kunisch †

Abstract

A general framework for calculating shape derivatives for domain optimization problems with partial differential equations as constraints is presented. The first order approximation of the cost with respect to the geometry perturbation is arranged in an efficient manner that allows the computation of the shape derivative of the cost without the necessity to involve the shape derivative of the state variable. In doing so, the state variable is only required to be Lipschitz continuous with respect to geometry perturbations. Application to shape optimization with the Navier-Stokes equations as PDE constraint is given.

Key words: Shape derivative, Navier-Stokes equations, Cost functional.

1 Introduction

In this paper, we consider the problem of finding a domain Ω (in a class of admissible domains $\mathscr{U}_{ad}$) minimizing the functional

$$J(u,\Omega) \equiv \int_{\Omega} j_1(C_\gamma u)\, dx \tag{1}$$

subject to a constraint

$$E(u,\Omega) = 0, \;\; u \in X. \tag{2}$$

Here $E(u,\Omega) = 0$, represents a partial differential equation posed on Ω with boundary $\partial\Omega$, u is the state variable and $X \subset L^2(\Omega)^l$, $l \in \mathbb{N}$, is a Hilbert space with a dual X^*. The class of admissible domains $\mathscr{U}_{ad}$ does not admit a vector space structure, making the application of traditional optimization methods difficult. This difficulty is bypassed by describing shapes by means of transformations. Due to lack of closed form solutions to $E(u,\Omega) = 0$, problem (1-2) is usually solved numerically using iterative methods, e.g., the gradient descent method. For such methods, one needs to compute the derivative of the cost with respect to Ω. Rigorous derivations of shape derivative of J can be found in literature, see e.g [7], [2]. In [7], the approach taken involves differentiation of the state equation with respect to the domain. The state variable lives in a Hilbert space X which depends on the geometry with respect to which optimization is carried out. To obtain sensitivity information of $\Omega \mapsto \hat{J}(\Omega) = J(\Omega, u(\Omega))$, a chain rule approach involving the shape derivative of $\Omega \mapsto u(\Omega)$ is chosen. Other techniques presented in, e.g., [2]

*Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria, email: henry.kasumba@oeaw.at.

†Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstraße 36, A-8010 Graz, Austria; email: karl.kunisch@uni-graz.at.

use function space parametrization and function space embedding methods. The latter depends strongly on sophisticated differentiability properties of saddle point problems. In this paper, we present a computation of the shape derivative of J under minimal regularity assumptions. The technique we employ was first suggested in [5], and then used in [3], and allows to compute the shape derivative of the mapping $\Omega \mapsto \hat{J}(\Omega)$ without using the shape derivative of the state variable with respect to the geometry. In [5], a cost functional $J : X \mapsto \mathbb{R}$ of the form $J(u,\Omega) = \int_\Omega j_1(u)\, dx$ was considered. However, in many applications such as vortex control in fluids, cost functionals are typically of the form (1), where $C_\gamma : X \mapsto H$, H a Banach space, is either a linear operator, e.g., $C_\gamma u = \text{curl } u$ or generally a non-linear operator, e.g., $C_\gamma u = \det \nabla u$. In addition, we note that cost functionals of the form (1) can be expressed in the form

$$J(u,\Omega) = G(F(u)), \tag{3}$$

where the mappings

$$F : X \mapsto H, \;\; G : H \mapsto \mathbb{R},$$

are defined as $F(u) = C_\gamma u$ and $G(v) = \int_\Omega j_1(v)\, dx$, respectively. In this work, we specifically address this composite structure of the cost functionals of the form (1), where C_γ is an affine operator

$$C_\gamma : u(\cdot) \mapsto Cu(\cdot) + \gamma(\cdot) \qquad \gamma \in L^2(D),$$

D, an open and bounded hold all domain to be specified later, and $C \in \mathscr{L}(X, L^2(\Omega))$ is a linear operator. An application involving a cost functional with a non-linear operator C in the integrand is also presented. The approach that we use can be summarized as follows: The difference quotient of the cost J with respect to the geometry perturbation is arranged in an efficient manner so that computation of the shape derivative of the state can be bypassed. In doing so, the existence of the material derivative of the state u can be replaced by Hölder continuity with exponent greater than or equal to $\frac{1}{2}$ of u with respect to the geometric data. The constraint $E(u,\Omega) = 0$ is observed by introducing an appropriately defined adjoint equation. Furthermore, well known results from the method of mapping and the differentiation of functionals with respect to geometric quantities are utilized on a technical level.

The rest of the paper is organized as follows. In Section 2 we present the proposed general framework to compute the shape derivative for (1-2). The application of the general theory to shape optimization problems with the Navier Stokes equations as equality state constraints is presented in Section 3.

2 Shape derivative

In this section we focus on sensitivity analysis for the shape optimization problem (1)-(2). To describe the class of admissible domains $\mathscr{U}_{ad}$, let $D \subset \mathbb{R}^d$, $d = 2,3$ be a fixed bounded domain with a C^2 boundary ∂D and let S be a domain with a C^2 boundary $\Gamma := \partial S$ satisfying $\bar{S} \subset D$ (see Figure (1)). For the reference domain, either of the following three cases is admitted

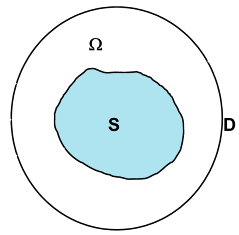

Figure 1: Domain

(i) $\Omega = S$,

(ii) $\Omega = D$,

(iii) $\Omega = D \setminus \bar{S}$.

Then the boundary $\partial\Omega$ for the three cases is given by

(i) $\partial\Omega = \Gamma$,

(ii) $\partial\Omega = \partial D$,

(iii) $\partial\Omega = \Gamma \cup \partial D$.

Shapes are difficult entities to be dealt with directly, so we manipulate them by means of transformations. If Ω is the initial admissible shape, and Ω_t is the shape at time t, one considers transformations $T_t : \Omega \mapsto \Omega_t$. Such transformations can be constructed, for instance, by perturbation of the identity [2]. To construct an admissible class of these transformations, let $\Omega \subset \bar{D}$ be a bounded domain and

$$\mathscr{H} = \{\mathbf{h} \in C^2(\bar{D}) : \mathbf{h}|_{\partial D} = 0\}$$

be the space of deformation fields. The fields $\mathbf{h} \in \mathscr{H}$ define for $t > 0$, a perturbation of Ω by

$$\begin{aligned} T_t : \Omega &\mapsto \Omega_t, \\ x &\mapsto T_t(x) = x + t\mathbf{h}(x). \end{aligned}$$

For each $\mathbf{h} \in \mathscr{H}$, there exists $\tilde{\tau} > 0$ such that $T_t(D) = D$ and $\{T_t\}$ is a family of C^2-diffeomorphisms for $|t| < \tilde{\tau}$ [2]. For each $t \in \mathbb{R}$ with $|t| < \tilde{\tau}$, we set $\Omega_t = T_t(\Omega),\ \Gamma_t = T_t(\Gamma)$. Thus $\Omega_0 = \Omega,\ \Gamma_0 = \Gamma,\ \Omega_t \subset D$.

2.1 Notation

In what follows, the following notation will be used:

$$I_t = \det DT_t, \quad A(t) = I_t(DT_t)^{-1}(DT_t)^{-T}, \tag{4}$$

and ∇u shall stand for $(Du)^T$ where u is either a scalar or vector valued function (if u is bold faced, i.e., $\mathbf{u}$). In (4), $(DT_t)^{-T}$ takes the meaning of transpose of the inverse matrix $(DT_t)^{-1}$. Furthermore, two notations for the inner product in $\mathbb{R}^d$ shall be used, namely (x,y) and $x \cdot y$, respectively. The latter shall be used in case of nested inner products. In addition, throughout this work, unless specified otherwise, the following parenthesis $(\cdot,\cdot)_\Omega$, $(\cdot,\cdot)_{\partial\Omega}$ shall denote the $L^2(\Omega)$, $L^2(\partial\Omega)$ inner products, respectively. In some cases, the subscript Ω may be omitted, but the meaning will remain clear in the given context. The scalar product and the norm in the Hilbert space X will be denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, and the duality pairing between X^* and X is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$. The curl of a vector field $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$, denoted by curl $\mathbf{u}$, is defined as

$$\text{curl } \mathbf{u} := \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y},$$

while the curl of a scalar field u in the case $d = 2$, denoted by **curl** u, is defined as

$$\mathbf{curl}\ u := (\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}).$$

The determinant of the velocity gradient tensor of a vector field $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$, denoted by $\det \nabla \mathbf{u}(x)$, is defined as

$$\det \nabla \mathbf{u}(x) := \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_2}{\partial x}\frac{\partial u_1}{\partial y}. \tag{5}$$

The unit outward normal and tangential vectors to the boundary $\partial\Omega$ shall be denoted by $\mathbf{n} = (n_x,n_y)$ and $\tau = (-n_y,n_x)$, respectively. We denote by $H^m(\mathscr{S})$, $m \in \mathbb{R}$, the standard Sobolev space of order m defined by

$$H^m(\mathscr{S}) := \left\{ u \in L^2(\mathscr{S}) \mid D^\alpha u \in L^2(\mathscr{S}),\ \text{ for } 0 \le |\alpha| \le m \right\},$$

where D^α is the weak (or distributional) partial derivative, and α is a multi-index. Here $\mathscr{S}$, which is either the flow domain Ω, or its boundary $\partial\Omega$, or part of its boundary. The norm $||\cdot||_{H^m(\mathscr{S})}$ associated with $H^m(\mathscr{S})$ is given by

$$||u||^2_{H^m(\mathscr{S})} = \sum_{|\alpha|\le m} \int_{\mathscr{S}} |D^\alpha u|^2 \, dx.$$

Note that $H^0(\mathscr{S}) = L^2(\mathscr{S})$ and $||\cdot||_{H^0(\mathscr{S})} = ||\cdot||_{L^2(\mathscr{S})}$. For the vector valued functions, we define the Sobolev space $\mathbf{H}^m(\mathscr{S})$ by

$$\mathbf{H}^m(\mathscr{S}) := \{\mathbf{u} = (u_1,u_2) \mid u_i \in H^m(\mathscr{S}),\ \text{for } i = 1,2\},$$

and its associated norm

$$||\mathbf{u}||^2_{\mathbf{H}^m(\mathscr{S})} = \sum_{i=1}^{2} ||u_i||^2_{H^m(\mathscr{S})}.$$

2.2 Properties of $\mathbf{T_t}$

Let $\mathscr{J} = [0, \tau_0]$ with τ_0 sufficiently small. Then the following regularity properties of the transformation T_t can be shown, see for example ([5], [7], [2, Chapter 7]):

$$\begin{aligned}
&T_0 = id && t \mapsto T_t \in C^1(\mathscr{J}, C^1(\bar{D};\mathbb{R}^d))\\
&t \mapsto T_t^{-1} \in C^1(\mathscr{J}, C^1(\bar{D};\mathbb{R}^d)) && t \mapsto I_t \in C^1(\mathscr{J}, C(\bar{D}))\\
&t \mapsto (DT_t)^{-T} \in C(\mathscr{J}, C^1(\bar{D};\mathbb{R}^{d\times d})) && \tfrac{d}{dt} T_t|_{t=0} = \mathbf{h}\\
&\tfrac{d}{dt} T_t^{-1}|_{t=0} = -\mathbf{h} && \tfrac{d}{dt} DT_t|_{t=0} = D\mathbf{h}\\
&\tfrac{d}{dt} DT_t^{-1}|_{t=0} = -D\mathbf{h} && \tfrac{d}{dt} I_t|_{t=0} = \operatorname{div} \mathbf{h}\\
&I_t|_{t=0} = 1 && I_t^{-1}|_{t=0} = 1.
\end{aligned} \tag{6}$$

The limits defining the derivatives at $t = 0$ exist uniformly in $x \in \bar{D}$. We shall also make use of the surface divergence, denoted by $\operatorname{div}_\Gamma$, which is defined for $\varphi \in C^1(\bar{D}, \mathbb{R}^d)$ by

$$\operatorname{div}_\Gamma \varphi := \operatorname{div}\varphi|_\Gamma - (D\varphi\ \mathbf{n}) \cdot \mathbf{n}. \tag{7}$$

2.3 The Eulerian derivative

Definition 2.1. *For given $\mathbf{h} \in \mathscr{H}$, the Eulerian derivative of J at Ω in the direction $\mathbf{h}$ is defined as*

$$dJ(u,\Omega)\mathbf{h} = \lim_{t\to 0} \frac{J(u_t, \Omega_t) - J(u,\Omega)}{t}, \tag{8}$$

where u_t satisfies

$$E(u_t, \Omega_t) = 0. \tag{9}$$

The functional J is said to be shape differentiable at Ω if $dJ(\Omega,u)\mathbf{h}$ exists for all $\mathbf{h} \in \mathscr{H}$ and the mapping $\mathbf{h} \mapsto dJ(\Omega,u)\mathbf{h}$ is linear and continuous on $\mathscr{H}$.

Under suitable regularity assumptions one can furthermore show that $dJ(u,\Omega)\mathbf{h}$ only depends on the normal component of the deformation field $\mathbf{h}$ on $\partial\Omega$ and can be represented as

$$dJ(u,\Omega)\mathbf{h} = \int_{\partial\Omega} G_\Omega \mathbf{h} \cdot \mathbf{n}\, ds, \tag{10}$$

where the kernel G_Ω does not involve the shape derivative of u with respect to Ω. This is the main result of the Zolesio-Hadamard structure theorem [2, Pg. 348]. Let $\{X_t\}_{t\geq 0}$ be a family of functional spaces defined over the domains Ω_t. Then the variational form of (9) is given by : Find $u_t \in X_t$ such that

$$\langle E(u_t, \Omega_t), \psi_t \rangle_{X_t^* \times X_t} = 0, \tag{11}$$

holds for all $\psi_t \in X_t$. Throughout we choose $X_t = T_t(X)$ and we assume that equation (11) has a unique solution u_t, for all t sufficiently small. Using the method of mappings, equation (11) represents the weak form of the reference problem (2) given by

$$\langle E(u,\Omega), \psi \rangle_{X^* \times X} = 0, \quad \text{for all } \psi \in X \tag{12}$$

for $t = 0$. The adjoint state $p \in X$ to this equation is defined as the solution to

$$\langle E_u(u,\Omega)\psi, p\rangle_{X^*\times X} = (C^* j_1'(C_\gamma u), \psi). \tag{13}$$

Any function $u_t : \Omega_t \mapsto \mathbb{R}^l$, for $l \in \mathbb{N}$, can be mapped back to the reference domain by

$$u^t = u_t \circ T_t : \Omega \mapsto \mathbb{R}^l. \tag{14}$$

From the chain rule it follows that the gradients of u_t and u^t are related by

$$(\nabla u_t) \circ T_t = (DT_t)^{-T} \nabla u^t, \tag{15}$$

(see [7] Prop: 2.29). Moreover $u^t : \Omega \mapsto \mathbb{R}^l$ satisfies an equation on the reference domain which we express as

$$\tilde{E}(u^t, t) = 0, \quad |t| < \tilde{\tau}.$$

Because $T_0 = id$, one obtains $u^0 = u$ and

$$\tilde{E}(u^0, 0) = E(u, \Omega).$$

In order to circumvent the computation of the derivative of u with respect to Ω, the following assumptions (H1-H4) were imposed on $\tilde{E}$ and E in [5].

(H1) There is a C^1-function $\tilde{E} : X \times (-\tilde{\tau}, \tilde{\tau}) \mapsto X^*$ such that $E(u_t, \Omega_t) = 0$ is equivalent to

$$\tilde{E}(u^t, t) = 0 \ \text{ in } X^*,$$

with $\tilde{E}(u, 0) = E(u, \Omega)$ for all $u \in X$.

(H2) There exists $0 < \tau_0 \leq \tilde{\tau}$ such that for $|t| < \tau_0$, there exists a unique solution $u^t \in X$ to $\tilde{E}(u^t, t) = 0$ and

$$\lim_{t\to 0} \frac{||u^t - u^0||_X}{|t|^{\frac{1}{2}}} = 0.$$

(H3) $E_u(u, \Omega) \in \mathscr{L}(X, X^*)$ satisfies

$$\langle E(v,\Omega) - E(u,\Omega) - E_u(u,\Omega)(v-u), \psi\rangle_{X^*\times X} = \mathscr{O}(||v-u||_X^2).$$

(H4) $\tilde{E}$ and E satisfy

$$\lim_{t\to 0} \frac{1}{t} \left\langle \tilde{E}(u^t, t) - \tilde{E}(u, t) - \big(E(u^t, \Omega) - E(u, \Omega)\big), \psi \right\rangle_{X^*\times X} = 0.$$

Additionally we need that the following assumptions on j_1 and C_γ hold.

(H5) We assume that $\int_\Omega j_1(C_\gamma u)\, dx,\ \int_\Omega (j_1'(C_\gamma u))^2\, dx$ exists for all $u \in X$ and

$$\Big| \int_\Omega I_t \Big[j_1(C_\gamma u^t) - j_1(C_\gamma u) - \Big(j_1'(C_\gamma u), C(u^t - u) \Big) \Big]\, dx \Big| \leq K ||u^t - u||_X^2,$$

where $K > 0$ does not depend on $t \in \mathscr{J}$.

To compute the Eulerian derivative of $J(u,\Omega)$ in (8), we need to transform the value of $J(u_t,\Omega_t) = \int_{\Omega_t} j_1(C_\gamma u_t)\, dx_t$ back to Ω. This is done by using the relation

$$J(u_t,\Omega_t) = \int_{\Omega_t} j_1(C_\gamma u_t)\, dx_t = \int_\Omega j_1((C_\gamma u_t)\circ T_t) I_t\, dx.$$

The transformation of $(C_\gamma u_t)\circ T_t$ back to Ω induces some matrix A_t that we shall require to satisfy:

$$\text{(H6)}\quad \begin{cases} \text{There exists a matrix } A_t \text{ such that } t \mapsto A_t \in C(\mathscr{J}, C(\bar{D},\mathbb{R}^{d\times d})) \text{ and} \\ (C_\gamma u_t)\circ T_t = A_t C u^t + t\mathscr{G} + \gamma,\ \ \mathscr{G} \in L^2(\Omega), \\ C_\gamma(u\circ T_t^{-1}) = (A_t C u + t\mathscr{G} + \gamma)\circ T_t^{-1}, \\ \lim_{t\to 0} \frac{A_t - I}{t} \text{ exists, } A_t|_{t=0} = I. \end{cases}$$

(H7) Let $\mathscr{M}(t) = \int_\Omega I_t \left[j_1(A_t C u^t + t\mathscr{G} + \gamma) - j_1(A_t C u + t\mathscr{G} + \gamma) + j_1(C_\gamma u) - j_1(C_\gamma u^t)\right] dx$. Then we shall require $\mathscr{M}$ to satisfy

$$\lim_{t\to 0} \frac{\mathscr{M}(t)}{t} = 0. \tag{16}$$

Some illustrative examples for (H6) and a remark on (H7) are given next. If $C_\gamma = C = \nabla$, i.e., $\gamma = 0$, then (15) gives $A_t = DT_t^{-T}$ and $\mathscr{G} = 0$. This gives the first relation in (H6). By applying the chain rule on $\nabla(u\circ T_t^{-1})$, we obtain

$$\nabla(u\circ T_t^{-1}) = DT_t^{-T} \nabla u \circ T_t^{-1}. \tag{17}$$

This gives the second relation in (H6). The third relation in (H6) is satisfied by A_t since $\lim\limits_{t\to 0} \dfrac{DT_t^{-T} - I}{t} = -Dh^T$, and $\lim\limits_{t\to 0} DT_t^{-T} = I$.

In the next example, we consider the case where $C_\gamma = C = \text{div}$, i.e., $\gamma = 0$. For this purpose, we derive the transformation of the divergence operator in the following lemma.

Lemma 2.1. *Suppose $\mathbf{u}_t$ and $\mathbf{u}^t$ are related by (14), then*

$$(div\ \mathbf{u}_t)\circ T_t = I_t^{-1}(div\ \mathbf{u}^t) + t\mathscr{G}, \tag{18}$$

where

$$div\ \mathbf{u}^t = (\partial_x u_1^t + \partial_y u_2^t) \text{ and } \mathscr{G} = I_t^{-1}(h_{2,y}\partial_x u_1^t + h_{1,x}\partial_y u_2^t - h_{2,x}\partial_y u_1^t - h_{1,y}\partial_x u_2^t). \tag{19}$$

Proof. By definition

$$(\text{div}\ \mathbf{u})\circ T_t = (\partial_x u_1 + \partial_y u_2)\circ T_t = (\partial_x u_1)\circ T_t + (\partial_y u_2)\circ T_t.$$

Using (15) we have

$$\begin{pmatrix} \partial_x u_{t,1} & \partial_x u_{t,2} \\ \partial_y u_{t,1} & \partial_y u_{t,2} \end{pmatrix} \circ T_t = \frac{1}{I_t} \begin{pmatrix} 1 + th_{2,y} & -th_{2,x} \\ -th_{1,y} & 1 + th_{1,x} \end{pmatrix} \begin{pmatrix} \partial_x u_1^t & \partial_x u_2^t \\ \partial_y u_1^t & \partial_y u_2^t \end{pmatrix}. \tag{20}$$

From (20) we have for the diagonal components

$$I_t(\partial_x u_{t,1})\circ T_t = (1+th_{2,y})\partial_x u_1^t - th_{2,x}\partial_y u_1^t,$$
$$I_t(\partial_y u_{t,2})\circ T_t = -th_{1,y}\partial_x u_2^t + (1+th_{1,x})\partial_y u_2^t,$$

from which upon addition of both terms on the right hand side, one obtains

$$\begin{aligned} I_t(\text{div }\mathbf{u}_t)\circ T_t &= (1+th_{2,y})\partial_x u_1^t - th_{2,x}\partial_y u_1^t - th_{1,y}\partial_x u_2^t + (1+th_{1,x})\partial_y u_2^t \\ &= \text{div }\mathbf{u}^t + t(h_{2,y}\partial_x u_1^t + h_{1,x}\partial_y u_2^t - h_{2,x}\partial_y u_1^t - h_{1,y}\partial_x u_2^t). \end{aligned}$$

□

From Lemma 2.1, we note that A_t from (H6) is given by $A_t = I_t^{-1}I$. For $\mathbf{u}\in X$, div$\mathbf{u}\in L^2(\Omega)$ by assumption, hence $\mathscr{G}$ given in (19) is in $L^2(\Omega)$. Moreover by (6), we have that $\lim_{t\to 0} A_t - I = 0$ and $\lim_{t\to 0}\dfrac{A_t-I}{t} = -\text{div }\mathbf{h}$ holds in $L^\infty(\Omega)$. Since $u_t = u^t\circ T_t^{-1}$, one obtains div $(\mathbf{u}^t\circ T_t^{-1}) = \Big(I_t^{-1}(\text{div }\mathbf{u}^t) + t\mathscr{G}\Big)\circ T_t^{-1}$ from Lemma 2.1. Thus all conditions of assumption (H6) are satisfied by this transformation.

We now provide a remark on assumption (H7).

Remark 2.1. *If we suppose that either $\gamma = 0$ in (H6) and $j_1(t) = |t|^2$ or $\gamma\neq 0$ in (H6) and $j_1(t) = |t-\gamma|^2$, then*

$$\mathscr{M}(t) = \int_\Omega I_t\Big[|(A_tCu^t + t\mathscr{G})|^2 - |(A_tCu + t\mathscr{G}|^2 + |Cu|^2 - |Cu^t|^2\Big]\,dx,\ \ \mathscr{M}(0) = 0.$$

Using $(a^2-b^2) = (a+b)(a-b)$, we can express $\mathscr{M}$ such that

$$\frac{\mathscr{M}(t)}{t} = \int_\Omega I_t\Big[\frac{(A_t-I)}{t}(A_t+I)C(u^t+u) + 2A_t\mathscr{G}\Big]C(u^t-u)\,dx.$$

Note that I_t and I_t^{-1} can be expressed as

$$I_t = I + t\,div\,\mathbf{h} + t^2 detD\mathbf{h}\ \ and\ \ I_t^{-1} = I - t\,div\,\mathbf{h} + t^2 detD\mathbf{h},$$

respectively. Hence for $t\in\mathscr{J}$, I_t, A_t+I, and $\frac{A_t-I}{t}$ are bounded in $L^\infty(\Omega)$. Moreover

$$\Big|\frac{\mathscr{M}(t)}{t}\Big| \le \underbrace{\int_\Omega\Big|I_t\frac{(A_t-I)}{t}(A_t+I)C(u^t+u)C(u^t-u)\Big|\,dx}_{E_1(t)} + \underbrace{2\int_\Omega\Big|I_tA_t\mathscr{G}C(u^t-u)\Big|\,dx}_{E_2(t)},$$

and

$$E_1(t) \le K_1||I_t||_{L^\infty}||\frac{(A_t-I)}{t}||_{L^\infty}||(A_t+I)||_{L^\infty}||u^t+u||_X||u^t-u||_X,$$
$$E_2(t) \le K_2||I_t||_{L^\infty}||A_t||_{L^\infty}||\mathscr{G}||_{L^2}||u^t-u||_X,$$

for some generic constants K_1 and K_2. Hence by H2, we obtain $lim_{t\to 0}\,E_i(t) = 0$, $i = 1,2$ and this leads to (16).

In what follows, the following lemmas shall be utilized.

Lemma 2.2. *[5]*

(1) Let $f \in C(\mathscr{J}, W^{1,1}(D))$, and assume that $f_t(0)$ exists in $L^1(D)$. Then

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)\, dx_t|_{t=0} = \int_{\Omega} f_t(0,x)\, dx + \int_{\Gamma} f(0,x)\mathbf{h}\cdot\mathbf{n}\, ds.$$

(2) Let $f \in C(\mathscr{J}, W^{2,1}(D))$, and assume that $f_t(0)$ exists in $W^{1,1}(D)$. Then

$$\frac{d}{dt}\int_{\Gamma_t} f(t,x)\, dx_t|_{t=0} = \int_{\Gamma} f_t(0,x)\, ds + \int_{\Gamma}\Big(\frac{\partial f(0,x)}{\partial \mathbf{n}} + \kappa f(0,x)\Big)\mathbf{h}\cdot\mathbf{n}\, ds,$$

where κ stands for the mean curvature of Γ.

The assumptions of Lemma 2.2 can be verified using the following Lemma

Lemma 2.3. *[7, Chapter.2]*

(1) If $u \in L^p(D)$, then $t \mapsto u\circ T_t^{-1} \in C(\mathscr{J}, L^p(D)), 1 \le p < \infty$.

(2) If $u \in H^2(D)$, then $t \mapsto u\circ T_t^{-1} \in C(\mathscr{J}, H^2(D))$.

(3) If $u \in H^2(D)$, then $\frac{d}{dt}(u\circ T_t^{-1})|_{t=0}$ exists in $H^1(D)$and is given by

$$\frac{d}{dt}(u\circ T_t^{-1})|_{t=0} = -(Du)\mathbf{h}.$$

Note 2.1. *As a consequence of Lemma 2.3, we note that $\frac{d}{dt}\nabla(u\circ T_t^{-1})\Big|_{t=0}$ exists in $L^2(D)$ and is given by*

$$\frac{d}{dt}\nabla(u\circ T_t^{-1})\Big|_{t=0} = -\nabla(Du\mathbf{h}).$$

For the transformation of domain integrals, the following well known fact will be used repeatedly.

Lemma 2.4. *Let $\phi_t \in L^1(\Omega_t)$, then $\phi_t\circ T_t \in L^1(\Omega)$ and*

$$\int_{\Omega_t} \phi_t\, dx_t = \int_{\Omega} \phi_t\circ T_t\, I_t\, dx.$$

As a main result, we now formulate the representation of the Eulerian derivative of J in the following theorem.

Theorem 2.1. *If (H1-H7) hold, and $j_1(C_\gamma u) \in W^{1,1}(\Omega)$, then the Eulerian derivative of J in the direction $\mathbf{h} \in \mathscr{H}$ exists and is given by the expression*

$$d\hat{J}(\Omega)\mathbf{h} = -\frac{d}{dt}\langle \tilde{E}(u,t), p\rangle_{X^*\times X}|_{t=0} + \int_{\partial\Omega} j_1(C_\gamma u)\mathbf{h}\cdot\mathbf{n}\, ds - \int_{\Omega} j_1'(C_\gamma u)C_\gamma(\nabla u^T\cdot\mathbf{h})\, dx. \tag{21}$$

Proof. The Eulerian derivative of a cost functional $J(u,\Omega)$ is defined by (8). Using Lemma 2.4 we obtain

$$J(u_t,\Omega_t) - J(u,\Omega) = \int_\Omega j_1((C\gamma u_t)\circ T_t)I_t - j_1(C\gamma u)\, dx,$$

and by (H6)

$$\begin{aligned} J(u_t,\Omega_t) - J(u,\Omega) = &\int_\Omega I_t\Big(j_1(A_t C u^t + t\mathscr{G} + \gamma) - j_1(C\gamma u^t)\Big)\, dx \\ &+ \int_\Omega \Big(I_t j_1(C\gamma u^t) - j_1(C\gamma u)\Big)\, dx. \end{aligned}$$

The following estimate is obtained along the lines of [5]. We set

$$\begin{aligned} R(t) &= \int_\Omega \Big(I_t j_1(C\gamma u^t) - j_1(C\gamma u)\Big)\, dx, \;\; R(0)=0, \\ S(t) &= \int_\Omega I_t\Big(j_1(A_t C u^t + t\mathscr{G} + \gamma) - j_1(C\gamma u^t)\Big)\, dx, \;\; S(0)=0. \end{aligned}$$

Since C is a bounded linear operator, we have

$$\begin{aligned} R(t) = &\int_\Omega I_t\Big[j_1(C\gamma u^t) - j_1(C\gamma u) - \Big(j_1'(C\gamma u), C(u^t-u)\Big)\Big]\, dx + \\ &\int_\Omega (I_t-1)\Big(j_1'(C\gamma u), C(u^t-u)\Big)\, dx + \int_\Omega \Big(j_1'(C\gamma u), C(u^t-u)\Big)\, dx \\ &+ \int_\Omega (I_t-1) j_1(C\gamma u)\, dx. \end{aligned}$$

We express $R(t) = R_1(t)+R_2(t)+R_3(t)+R_4(t)$. Using (H2) and (H5), we have that $\lim_{t\to 0}\frac{1}{t}R_1(t) = 0$. Moreover using H5 and similar arguments as in Remark 2.1, we have

$$\Big|\frac{R_2(t)}{t}\Big| \le ||\frac{(I_t-I)}{t}||_{L^\infty}||j_1'(C\gamma u)||_{L^2}||u^t-u||_X.$$

Therefore, by (H2) and (6), one obtains $\lim_{t\to 0}\, |\frac{1}{t}R_2(t)| = 0$. Next observe that using (13) with $\psi = u^t - u \in X$, we have that

$$\begin{aligned} R_3(t) = (j_1'(C\gamma u), C(u^t-u)) &= (C^* j_1'(C\gamma u), (u^t-u)) \\ &= \langle E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X}. \end{aligned} \tag{22}$$

In order to bypass the computation of the shape derivative of u, we arrange terms on the right hand side of (22) in an efficient manner to obtain

$$\begin{aligned} \langle E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X} = &-\langle \tilde{E}(u,t) - \tilde{E}(u,0), p\rangle_{X^*\times X} \\ &- \langle E(u^t,\Omega) - E(u,\Omega) - E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X} \\ &- \langle \tilde{E}(u^t,t) - \tilde{E}(u,t) - E(u^t,\Omega) + E(u,\Omega), p\rangle_{X^*\times X}. \end{aligned}$$

By using assumptions (H2), (H3) and (H4), we have that

$$-\lim_{t\to 0}\frac{1}{t}\langle E(u^t,\Omega) - E(u,\Omega) - E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X} = 0,$$

and

$$-\lim_{t\to 0}\frac{1}{t}\langle \tilde{E}(u^t,t)-\tilde{E}(u,t)-E(u^t,\Omega)+E(u,\Omega),p\rangle_{X^*\times X}=0.$$

Consequently utilizing (H1), we obtain

$$\lim_{t\to 0}\frac{R_3(t)}{t}=-\frac{d}{dt}\langle \tilde{E}(u,t),p\rangle_{X^*\times X}|_{t=0}. \tag{23}$$

We shall turn our attention to $R_4(t)$ later. Now let us focus on

$$S(t)=\int_\Omega I_t\Big(j_1(A_tCu^t+t\mathscr{G}+\gamma)-j_1(C_\gamma u^t)\Big)\,dx,$$

and consider the expression

$$j_1(A_tCu^t+t\mathscr{G}+\gamma)-j_1(C_\gamma u^t).$$

This can be written as

$$j_1(A_tCu^t+t\mathscr{G}+\gamma)-j_1(A_tCu+t\mathscr{G}+\gamma)+j_1(C_\gamma u)-j_1(C_\gamma u^t)+ \\ j_1(A_tCu+t\mathscr{G}+\gamma)-j_1(C_\gamma u).$$

Observe that

$$S(t)=\int_\Omega I_t\Big(j_1(A_tCu+t\mathscr{G}+\gamma)-j_1(C_\gamma u)\Big)\,dx+\mathscr{M}(t). \tag{24}$$

Expressing $S(t)=S_1(t)+\mathscr{M}(t)$, where $S_1(t)$ is the first term in (24). Using (H7), we have that $\lim\limits_{t\to 0}\dfrac{\mathscr{M}(t)}{t}=0$. Therefore collecting the remaining terms, i.e., $R_4(t)$ and $S_1(t)$ into $S_5(t):=R_4(t)+S_1(t)$, we have that

$$S_5(t)=\int_\Omega I_t j_1(A_tCu+t\mathscr{G}+\gamma)-j_1(C_\gamma u)\,dx,\;\; S_5(0)=0.$$

Using Lemma 2.4, we can express S_5 as

$$S_5(t)=\int_{\Omega_t} j_1\big([A_tCu+t\mathscr{G}+\gamma]\circ T_t\big)\,dx_t-\int_\Omega j_1\big(C_\gamma u\big)\,dx. \tag{25}$$

By H6, (25) can further be expressed as

$$S_5(t)=\int_{\Omega_t} j_1\big(C_\gamma(u\circ T_t^{-1})\big)\,dx_t-\int_\Omega j_1(C_\gamma u)\,dx. \tag{26}$$

By definition of Eulerian derivative, we have that

$$\lim_{t\to 0}\frac{S_5(t)}{t}=\frac{d}{dt}\int_{\Omega_t} j_1\big(C_\gamma(u\circ T_t^{-1})\big)\,dx_t\Big|_{t=0}.$$

Since by assumption $j_1(C_\gamma u) \in W^{1,1}(\Omega)$, $\frac{d}{dt}\Big[j_1(C_\gamma(u \circ T_t^{-1}))\Big]_{t=0}$ exists in $L^1(\Omega)$ [7]. Therefore, using Lemma 2.2 and Lemma 2.3, we have that

$$\begin{aligned}
\lim_{t\to 0} \frac{S_5(t)}{t} &= \int_\Omega \frac{d}{dt}\Big[j_1(C_\gamma(u \circ T_t^{-1}))\Big]_{t=0} \, dx + \int_{\partial\Omega} j_1(C_\gamma u)\mathbf{h}\cdot\mathbf{n} \, ds \\
&= \int_{\partial\Omega} j_1(C_\gamma u)\mathbf{h}\cdot\mathbf{n} \, ds - \int_\Omega j_1'(C_\gamma u) C_\gamma(\nabla u^T \cdot \mathbf{h}) \, dx.
\end{aligned}$$

Hence

$$\begin{aligned}
dJ(u,\Omega)\mathbf{h} = \lim_{t\to 0} \frac{R(t)+S(t)}{t} = -\frac{d}{dt}\langle \tilde{E}(u,t), p\rangle_{X^*\times X}|_{t=0} + \int_{\partial\Omega} j_1(C_\gamma u)\mathbf{h}\cdot\mathbf{n} \, ds \\
- \int_\Omega j_1'(C_\gamma u) C_\gamma(\nabla u^T \cdot \mathbf{h}) \, dx.
\end{aligned}$$

□

3 Examples

As an application of the general theory developed in the previous section, we derive the shape derivatives of cost functional used for vortex reduction in fluid dynamics. Here we restrict ourselves to the 2D case. Typical cost functionals used for this purpose, are based on minimization of the curl of the velocity field or tracking-type functionals, [1], i.e.,

$$\begin{aligned}
J_1(\mathbf{u},\Omega) &= \frac{1}{2}\int_\Omega |\text{curl } \mathbf{u}(x)|^2 \, dx, \\
J_2(\mathbf{u},\Omega) &= \frac{1}{2}\int_\Omega |A\mathbf{u}(x) - \mathbf{u}_d(x)|^2 \, dx, \quad A \in \mathbb{R}^{2\times 2},
\end{aligned} \tag{27}$$

where $\mathbf{u}_d$ stands for a given desired flow field which contains some of the expected features of the controlled flow field without the undesired vortices. Furthermore,

$$J_3(\mathbf{u},\Omega) = \int_\Omega g_3(\det \nabla\mathbf{u}) \, dx, \tag{28}$$

where

$$g_3(t) = \begin{cases} 0 & t \le 0, \\ \frac{t^3}{t^2+1} & t > 0, \end{cases}$$

penalizes the complex eigen values of $\nabla\mathbf{u}$ which are responsible for the swirling motion in a given flow (see , e.g.,[4] and references there in). The rigorous characterization of the shape derivative of this functional has not been done before, and therefore it is of a big interest in this work. In (27-28), $\mathbf{u}$ represents the state variable that solves the Navier-Stokes equations

$$\begin{cases} -\eta\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \\ \text{div } \mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u} = 0 \text{ on } \partial\Omega. \end{cases} \tag{29}$$

Here $\eta > 0$, denotes the kinematic viscosity of the fluid, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is the external body force, p the pressure, and with reference to Figure 1, $\Omega = S$ with $\Gamma = \partial S$. Using the notation of the previous section, we observe that $E(\mathbf{u},\Omega) = 0$ is given by system (29). We define the following functional spaces for velocity and pressure, respectively:

$$\begin{aligned} \mathbf{H}_0^1(\Omega) &= \{\psi \in \mathbf{H}^1(\Omega) \mid \psi = \mathbf{0} \text{ on } \Gamma\}, \\ L_0^2(\Omega) &= \{q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0\}. \end{aligned}$$

The variational formulation of (29) is given by: Find $(\mathbf{u},p) \in X \equiv \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} \langle E((\mathbf{u},p),\Omega),(\psi,\xi)\rangle_{X^*\times X} \equiv \eta(\nabla\mathbf{u},\nabla\psi)_\Omega + ((\mathbf{u}\cdot\nabla)\mathbf{u},\psi)_\Omega - (p,\text{div } \psi)_\Omega \\ -(\mathbf{f},\psi)_\Omega - (\text{div } \mathbf{u},\xi)_\Omega = 0, \end{aligned} \tag{30}$$

holds for all $(\psi,\xi) \in X$. It is well known that for sufficiently large values of η or for small values of $\mathbf{f}$, there exists a unique solution $(\mathbf{u},p)$ to (30) in X. Moreover since $\partial\Omega \in C^2$, $(\mathbf{u},p) \in \left(\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\right) \times \left(H^1(\Omega)\cap L_0^2(\Omega)\right)$ [8]. On Ω_t the perturbed weak formulation of (29) reads:
Find $(\mathbf{u}_t,p_t) \in X_t \equiv \mathbf{H}_0^1(\Omega_t) \times L_0^2(\Omega_t)$ such that

$$\begin{aligned} \langle E((\mathbf{u}_t,p_t),\Omega_t),(\psi_t,\xi_t)\rangle_{X_t^*\times X_t} \equiv \eta(\nabla\mathbf{u}_t,\nabla\psi_t)_{\Omega_t} + ((\mathbf{u}_t\cdot\nabla)\mathbf{u}_t,\psi_t)_{\Omega_t} \\ -(p_t,\text{div } \psi_t)_{\Omega_t} - (\mathbf{f}_t,\psi_t)_{\Omega_t} - (\text{div } \mathbf{u}_t,\xi_t)_{\Omega_t} = 0, \end{aligned} \tag{31}$$

holds for all $(\psi_t,\xi_t) \in X_t$. Using the summation convention, the transformation of the divergence [5] is given by

$$(\text{div } \psi_t)\circ T_t = D\psi_i^t A_t^T e_i = (A_t)_i \nabla\psi_i^t,$$

where e_i stands for the i-th canonical basis vector in $\mathbb{R}^d$ and $(A_t)_i$ denotes the i-th row of $A_t = (DT_t)^{-T}$. Thus using (15) the transformation of (31) back to Ω becomes,

$$\begin{aligned} \langle \tilde{E}((\mathbf{u}^t,p^t),t),(\psi,\xi)\rangle_{X^*\times X} \equiv \eta(I_t A_t\nabla\mathbf{u}^t, A_t\nabla\psi)_\Omega + ((\mathbf{u}^t\cdot A_t\nabla)\mathbf{u}^t, I_t\psi)_\Omega \\ -(p^t, I_t(A_t)_k\nabla\psi_k^t)_\Omega - (\mathbf{f}^t I_t,\psi)_\Omega - (I_t(A_t)_k\nabla u_k^t,\xi)_\Omega = 0 \quad \text{for all} \quad (\psi,\xi)\in X. \end{aligned} \tag{32}$$

3.1 The Eulerian derivative of cost functional J_1

For this cost functional, the operator $C_\gamma = C = (\text{curl}, 0)$ and $C_\gamma^* = (\text{curl}, 0)$ with $\gamma = 0$. Moreover it is easy to check that $C \in \mathscr{L}(X,L^2)$. Furthermore, since $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ we have that curl $\mathbf{u} \in L^2(\Omega,\mathbb{R})$ and therefore

$$\mathbf{u} \in H(\text{curl},\Omega) := \left\{\mathbf{u} \in L^2(\Omega,\mathbb{R}^2) \ : \ \text{curl } \mathbf{u} \in L^2(\Omega,\mathbb{R})\right\}.$$

Hence the cost functional $J_1(\mathbf{u},\Omega)$ is well defined. The adjoint state $(\lambda,q) \in X$ is given as a solution to

$$\langle E'((\mathbf{u},p),\Omega)(\psi,\xi),(\lambda,q)\rangle_{X^*\times X} = (\mathbf{curl}(\text{curl } \mathbf{u}),\psi)_\Omega,$$

with right hand side $\mathbf{curl}(\text{curl}\ \mathbf{u}) = -\Delta\mathbf{u}$, which amounts to

$$\eta(\nabla\psi,\nabla\lambda)_\Omega + ((\psi\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\psi,\lambda)_\Omega - (\xi,\text{div }\lambda)_\Omega - (\text{div }\psi,q)_\Omega = (-\Delta\mathbf{u},\psi)_\Omega. \tag{33}$$

Integrating $\big((\mathbf{u}\cdot\nabla)\psi,\lambda\big)_\Omega$ by parts, one obtains the strong form of the adjoint equation in (33), that we express as

$$\begin{cases} -\eta\Delta\lambda+(\nabla\mathbf{u})\cdot\lambda-(\mathbf{u}\cdot\nabla)\lambda+\nabla q = -\Delta\mathbf{u} \ \text{ in } \ \Omega, \\ \text{div }\lambda = 0 \ \text{ in } \ \Omega, \\ \lambda = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{34}$$

where the first equation holds in $\mathbf{L}^2(\Omega)$ and the second one in $L^2(\Omega)$. It is well known that there exists a unique solution $(\lambda,q)\in X$. Moreover, since $\partial\Omega\in C^2$, $(\lambda,q)\in \big(\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\big)\times\big(H^1(\Omega)\cap L_0^2(\Omega)\big)$, (see e.g [5] and references there in). In view of Theorem 2.1 we have to compute $\frac{d}{dt}\langle\tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X}|_{t=0}$, for which we use the representation on Ω_t of (32). This writes

$$\begin{aligned} \langle\tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X} \equiv \eta(\nabla\mathbf{u}\circ T_t^{-1},\nabla\lambda\circ T_t^{-1})_{\Omega_t}+ \\ ((\mathbf{u}\circ T_t^{-1}\cdot\nabla)\mathbf{u}\circ T_t^{-1},\lambda\circ T_t^{-1})_{\Omega_t} \\ -(p\circ T_t^{-1},\text{div }\lambda\circ T_t^{-1})_{\Omega_t}-(\mathbf{f},\lambda\circ T_t^{-1})_{\Omega_t} \\ -(\text{div }\mathbf{u}\circ T_t^{-1},q\circ T_t^{-1})_{\Omega_t}=0, \end{aligned} \tag{35}$$

where $(\mathbf{u},p),(\lambda,q)\in X$ are solutions of (29) and (33), respectively. The computation of $\frac{d}{dt}\langle\tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X}|_{t=0}$, results in

$$\begin{aligned} \frac{d}{dt}\langle\tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X}|_{t=0} = (-\eta\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\mathbf{f},\psi_\lambda)_\Omega+ \\ \eta(\nabla\mathbf{u}\cdot\mathbf{n},\psi_\lambda)_{\partial\Omega}-(p\psi_\lambda,\mathbf{n})_{\partial\Omega}+(-\eta\Delta\lambda+(\nabla\mathbf{u})\lambda-(\mathbf{u}\cdot\nabla)\lambda+\nabla q,\psi_u)_\Omega \\ +\eta\big(\psi_u,\nabla\lambda\cdot\mathbf{n}\big)_{\partial\Omega}-(q\cdot\mathbf{n},\psi_u)_{\partial\Omega}+\eta\int_{\partial\Omega}(\nabla\mathbf{u},\nabla\lambda)\mathbf{h}\cdot\mathbf{n}\,ds, \end{aligned} \tag{36}$$

where $\psi_u = -\nabla\mathbf{u}^T\cdot\mathbf{h}\in\mathbf{H}^1(\Omega)$ and $\psi_\lambda = -\nabla\lambda^T\cdot\mathbf{h}\in\mathbf{H}^1(\Omega)$, with $h\in\mathscr{H}$. By using (34), the expression on the right hand side of (36) can further be simplified to obtain (37), (see [5] for more details).

$$\begin{aligned} \frac{d}{dt}\langle\tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X}|_{t=0} = -\int_{\partial\Omega}\left[\eta\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\frac{\partial\lambda}{\partial\mathbf{n}}\right]\mathbf{h}\cdot\mathbf{n}\,ds \\ +\int_{\partial\Omega}\Big(p(\frac{\partial\lambda}{\partial\mathbf{n}},\mathbf{n})+q(\frac{\partial\mathbf{u}}{\partial\mathbf{n}},\mathbf{n})\Big)\mathbf{h}\cdot\mathbf{n}\,ds+\int_\Omega(\Delta\mathbf{u})\nabla\mathbf{u}^T\cdot\mathbf{h}\,dx. \end{aligned} \tag{37}$$

Using the definition of tangential divergence (7), we have that:

$$p(\frac{\partial\lambda}{\partial\mathbf{n}},\mathbf{n}) = p(\nabla\lambda^T\cdot\mathbf{n})\cdot\mathbf{n} = p\ \text{div }\lambda|_{\partial\Omega}-p\ \text{div}_{\partial\Omega}\,\lambda. \tag{38}$$

Since $\lambda = 0$ on $\partial\Omega$, the last term in (38) vanishes (see [7] Page 82 for details). Furthermore div $\lambda = 0$ which renders this expression to be zero. Analogously, $q(\frac{\partial \mathbf{u}}{\partial \mathbf{n}}, \mathbf{n}) = 0$. Thus

$$\frac{d}{dt}\langle \tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X}|_{t=0} = -\int_{\partial\Omega} \eta \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\frac{\partial \lambda}{\partial \mathbf{n}}\mathbf{h}\cdot\mathbf{n}\, ds + \int_{\Omega}(\Delta \mathbf{u})\nabla \mathbf{u}^T \cdot \mathbf{h}\, dx. \quad (39)$$

In view of Theorem 2.1, we further need to show that assumptions (H1-H7) hold, and moreover that $|\text{curl } \mathbf{u}|^2 \in W^{1,1}(\Omega)$. Assumptions (H1-H4) were verified in [5]. To check (H5), note that

$$\frac{1}{2}|\text{curl } \mathbf{u}^t|^2 - \frac{1}{2}|\text{curl } \mathbf{u}|^2 - \big(\text{curl } \mathbf{u}, \text{curl}(\mathbf{u}^t - \mathbf{u})\big) = \frac{1}{2}(\text{curl}(\mathbf{u}^t - \mathbf{u}))^2.$$

Hence

$$\int_\Omega I_t\Big[\frac{1}{2}|\text{curl } \mathbf{u}^t|^2 - \frac{1}{2}|\text{curl } \mathbf{u}|^2 - \big(\text{curl } \mathbf{u}, \text{curl}(\mathbf{u}^t - \mathbf{u})\big)\Big]\, dx = \int_\Omega \frac{I_t}{2}(\text{curl}(\mathbf{u}^t - \mathbf{u}))^2\, dx.$$

Consequently by Young's inequality, we have

$$\Big|\int_\Omega \frac{I_t}{2}(\text{curl}(\mathbf{u}^t - \mathbf{u}))^2\, dx\Big| \le \max_{t\in[0,\tau_0]} ||I_t||_{L^\infty}||\mathbf{u}^t - \mathbf{u}||_X^2,$$

for τ_0 sufficiently small. Hence (H5) is satisfied with $K = \max_{t\in[0,\tau_0]} ||I_t||_{L^\infty}$.

Condition (H6) is checked next. It illustrates the choice of C_γ for the present example.

Lemma 3.1. *Suppose* $\mathbf{u}_t$ *and* $\mathbf{u}^t$ *are related by (14), then*

$$(\mathit{curl}\ \mathbf{u}_t)\circ T_t = I_t^{-1}(\mathit{curl}\ \mathbf{u}^t) + t\mathscr{G}, \quad (40)$$

where

$$\mathscr{G} = I_t^{-1}(h_{2,y}\partial_x u_2^t - h_{2,x}\partial_y u_2^t + h_{1,y}\partial_x u_1^t - h_{1,x}\partial_y u_1^t). \quad (41)$$

Proof. By definition

$$(\text{curl } \mathbf{u})\circ T_t = (\partial_x u_2 - \partial_y u_1)\circ T_t = (\partial_x u_2)\circ T_t - (\partial_y u_1)\circ T_t.$$

From (20) we have for the non-diagonal components

$$\begin{aligned} I_t(\partial_x u_{t,2})\circ T_t &= (1 + t h_{2,y})\partial_x u_2^t - t h_{2,x}\partial_y u_2^t,\\ I_t(\partial_y u_{t,1})\circ T_t &= -t h_{1,y}\partial_x u_1^t + (1 + t h_{1,x})\partial_y u_1^t,\end{aligned}$$

from which we obtain that

$$\begin{aligned} I_t(\text{curl } \mathbf{u}_t)\circ T_t &= (1 + t h_{2,y})\partial_x u_2^t - t h_{2,x}\partial_y u_2^t + t h_{1,y}\partial_x u_1^t - (1 + t h_{1,x})\partial_y u_1^t\\ &= \text{curl } \mathbf{u}^t + t(h_{2,y}\partial_x u_2^t - h_{2,x}\partial_y u_2^t + h_{1,y}\partial_x u_1^t - h_{1,x}\partial_y u_1^t).\end{aligned}$$

Thus $(\text{curl } \mathbf{u}_t)\circ T_t = I_t^{-1}(\text{curl } \mathbf{u}^t) + t\mathscr{G}$. $\square$

From Lemma 3.1, we observe that A_t from (H6) is given by $A_t = I_t^{-1} I$. Since $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $\mathscr{G}$ given in (41) belongs to the Sobolev space $L^2(\Omega)$. Moreover by (6), we have that $\lim_{t\to 0} A_t - I = 0$ and $\lim_{t\to 0} \dfrac{A_t - I}{t} = -\text{div}\,\mathbf{h}$. Since $\mathbf{u}_t = \mathbf{u}^t \circ T_t^{-1}$, one obtains curl $(\mathbf{u}^t \circ T_t^{-1}) = \left(I_t^{-1}(\text{curl}\,\mathbf{u}^t) + t\mathscr{G}\right) \circ T_t^{-1}$ from Lemma 3.1. Thus all conditions of assumption (H6) are satisfied by this transformation.
Cost functional J_1 satisfies the conditions of Remark 2.1 and therefore (H7) holds. In addition, since $\mathbf{u} \in \mathbf{H}^2(\Omega)$, it follows that $\nabla\text{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)$. Therefore, we infer that $\nabla|\text{curl}\,\mathbf{u}|^2 = 2\text{curl}\,\mathbf{u}\,\nabla\text{curl}\,\mathbf{u} \in \mathbf{L}^1(\Omega)$. Consequently $|\text{curl}\,\mathbf{u}|^2 \in W^{1,1}(\Omega)$. Since all assumptions of Theorem 2.1 are satisfied, using (21) and (39), we can express the Eulerian derivative of J_1 as

$$
\begin{aligned}
dJ_1(\mathbf{u},\Omega)\mathbf{h} = \int_{\partial\Omega} \left[\eta \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \frac{\partial \lambda}{\partial \mathbf{n}}\right] \mathbf{h}\cdot\mathbf{n}\, ds - \int_{\Omega} (\Delta \mathbf{u})\nabla \mathbf{u}^T \cdot \mathbf{h}\, dx + \\
\frac{1}{2}\int_{\partial\Omega} |\text{curl}\,\mathbf{u}|^2\, \mathbf{h}\cdot\mathbf{n}\, ds - \int_{\Omega} \text{curl}\,\mathbf{u}\ \text{curl}\,(\nabla \mathbf{u}^T \cdot \mathbf{h})\, dx.
\end{aligned}
\tag{42}
$$

We want to express (42) into the form required by the Zolesio-Hadamard structure theorem (10). With this in mind, sufficient regularity of $\mathbf{u}$ together with Greens formula for the curl, i.e.,

$$
\int_{\Omega} \left[\text{curl}\,\mathbf{u}\ \text{curl}\,(\nabla \mathbf{u}^T \cdot \mathbf{h}) - (\Delta \mathbf{u})\nabla \mathbf{u}^T \cdot \mathbf{h}\right] dx = \int_{\partial\Omega} (\text{curl}\,\mathbf{u})\tau \cdot (\nabla \mathbf{u}^T \cdot \mathbf{h})\, ds,
$$

see, e.g., [6, Pg. 58], leads to

$$
dJ_1(\mathbf{u},\Omega)\mathbf{h} = \int_{\partial\Omega} \left[\eta \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \frac{\partial \lambda}{\partial \mathbf{n}} + \frac{1}{2}|\text{curl}\,\mathbf{u}|^2 - (\text{curl}\,\mathbf{u})\tau \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right] \mathbf{h}\cdot\mathbf{n}\, ds. \tag{43}
$$

3.2 The Eulerian derivative of cost functional J_2

In this example we define the operator $C_\gamma : \mathbf{u}(x) \mapsto A\mathbf{u} - \mathbf{u}_d \in \mathbf{L}^2(\Omega)$ with $\gamma = -\mathbf{u}_d \in \mathbf{L}^2(D)$. The linear operator $C \in \mathscr{L}(X, \mathbf{L}^2(\Omega))$ is such that $C : \mathbf{u}(\cdot) \mapsto A\mathbf{u}(\cdot)$. Furthermore, since $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ we have that $A\mathbf{u} - \mathbf{u}_d \in \mathbf{L}^2(\Omega)$. Hence the cost functional $J_2(\mathbf{u},\Omega)$ is well defined. For this case the adjoint state $(\lambda, q) \in X$, is given as a solution to

$$
\langle E'((\mathbf{u},p),\Omega)(\psi,\xi),(\lambda,q)\rangle_{X^*\times X} = (A\mathbf{u} - \mathbf{u}_d, \psi)_\Omega,
$$

which amounts to

$$
\begin{cases}
-\eta\Delta\lambda + (\nabla \mathbf{u})\cdot\lambda - (\mathbf{u}\cdot\nabla)\lambda + \nabla q = (A\mathbf{u} - \mathbf{u}_d), \text{ in } \Omega, \\
\text{div}\,\lambda = 0 \text{ in } \Omega, \\
\lambda = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{44}
$$

where the first equation in (44) holds in $\mathbf{L}^2(\Omega)$ and the second one in $L^2(\Omega)$.

Theorem 3.1. *The shape derivative of the cost functional $J_2(\mathbf{u},\Omega)$ can be expressed as*

$$
dJ_2(\mathbf{u},\Omega)\mathbf{h} = \int_{\partial\Omega} \left[\eta \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \frac{\partial \lambda}{\partial \mathbf{n}} + \frac{1}{2}(A\mathbf{u} - \mathbf{u}_d)^2\right] \mathbf{h}\cdot\mathbf{n}\, ds. \tag{45}
$$

Proof. We want to make use of Theorem 2.1 to derive (45). For this purpose, we remark that by using (36), (38) and (44) , one obtains

$$\frac{d}{dt}\langle \tilde{E},(\lambda,q)\rangle_{X^*\times X}|_{t=0} = -\int_{\partial\Omega}\eta\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\frac{\partial\lambda}{\partial\mathbf{n}}\,ds - \int_{\Omega}(A\mathbf{u}-\mathbf{u}_d)A^T(\nabla\mathbf{u}^T)\mathbf{h}\,dx,$$

where (λ,q) solves (44). Furthermore, we need to show that assumptions (H1-H7) of Theorem 2.1 hold, and moreover that $|A\mathbf{u}-\mathbf{u}_d|^2\in W^{1,1}(\Omega)$. As stated earlier, assumptions (H1-H4) were verified in [5]. To check (H5), note that

$$\frac{1}{2}|A\mathbf{u}^t-\mathbf{u}_d|^2-\frac{1}{2}|A\mathbf{u}-\mathbf{u}_d|^2-\big(A\mathbf{u}-\mathbf{u}_d,\,A(\mathbf{u}^t-\mathbf{u})\big)=\frac{1}{2}(A(\mathbf{u}^t-\mathbf{u}))^2.$$

Consequently by Young's inequality, we have

$$\Big|\int_{\Omega}\frac{I_t}{2}(A(\mathbf{u}^t-\mathbf{u}))^2\,dx\Big|\le 4\tilde{a}\max_{t\in[0,\tau_0]}||I_t||_{L^\infty}||\mathbf{u}^t-\mathbf{u}||_X^2,\quad \tilde{a}=\max_{i,j}|a_{i,j}|,\;\; i,j=1,2.$$

Hence (H5) is satisfied with $K=4\tilde{a}\max_{t\in[0,\tau_0]}||I_t||_{L^\infty}$, for τ_0 sufficiently small.
Note that the transformation $(C_\gamma\mathbf{u}_t)\circ T_t = C\mathbf{u}^t-\mathbf{u}_d$ implies that $\mathscr{G}=0$ in (H6). Furthermore A_t is given by $A_t=I$ and $\lim_{t\to 0}\frac{A_t-I}{t}=0$. Moreover $C_\gamma(\mathbf{u}^t\circ T_t^{-1})=(C\mathbf{u}^t-\mathbf{u}_d)\circ T_t^{-1}$. Hence all conditions of (H6) are satisfied. Note that J_2 satisfies conditions of Remark 2.1 and hence (H7) holds. It is also clear that $|A\mathbf{u}-\mathbf{u}_d|^2\in W^{1,1}$ since $\mathbf{u}\in\mathbf{H}^2(\Omega)$. The preceding discussion shows that assumptions (H1-H7) are satisfied. Therefore using Theorem 2.1 together with the fact that $(j_1'(C\mathbf{u}),\,C(\nabla\mathbf{u}^T h))_\Omega = (A\mathbf{u}-\mathbf{u}_d,\,A^T(\nabla\mathbf{u}^T)\mathbf{h})_\Omega$, one obtains

$$dJ_2(\mathbf{u},\Omega)\mathbf{h}=\int_{\partial\Omega}\left[\eta\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\frac{\partial\lambda}{\partial\mathbf{n}}+\frac{1}{2}(A\mathbf{u}-\mathbf{u}_d)^2\right]\mathbf{h}\cdot\mathbf{n}\,ds. \tag{46}$$

□

3.3 The Eulerian derivative of cost functional J_3

First note that $J_3(\mathbf{u},\Omega)$ is well defined. In fact, for $\mathbf{u}\in\mathbf{H}_0^1(\Omega)$, we have $\det\nabla\mathbf{u}\in L^1(\Omega)$. Moreover $0\le\frac{t^3}{t^2+1}\le t$ for $t\ge 0$, hence $g_3(\det\nabla\mathbf{u})$ is integrable. Furthermore for $\delta\mathbf{u}\in\mathbf{H}_0^1(\Omega)$, there exists the directional derivative $J_3'(\mathbf{u},\Omega)(\delta\mathbf{u})$ given by

$$J_3'(\mathbf{u},\Omega)(\delta\mathbf{u})=\int_\Omega g_3'(\det\nabla\mathbf{u})(\det\nabla\mathbf{u})'\delta\mathbf{u}\,dx, \tag{47}$$

where

$$(\det\nabla\mathbf{u})'\delta\mathbf{u}=(u_x^1\delta u_y^2+\delta u_x^1u_y^2-u_x^2\delta u_y^1-u_y^1\delta u_x^2)\text{ and } g_3'(t)=\begin{cases}0 & t\le 0,\\ \frac{t^4+3t^2}{t^4+2t^2+1} & t>0.\end{cases}$$

Where appropriate, we shall use the short form notation g_3' to represent $g_3'(\det\nabla\mathbf{u})$ in what follows.

Lemma 3.2. *The directional derivative $J_3'(\mathbf{u},\Omega)(\delta\mathbf{u})$ can be expressed in the form*

$$J_3'(\mathbf{u},\Omega)(\delta\mathbf{u}) = \int_\Omega T(\mathbf{u})(\delta\mathbf{u})\,dx + \int_{\partial\Omega} P(\mathbf{u})(\delta\mathbf{u})\,ds,$$

where

$$T(\mathbf{u}) = \begin{pmatrix} -curl\big(g_3'\nabla u_2\big) \\ curl\big(g_3'\nabla u_1\big) \end{pmatrix} \text{ and } P(\mathbf{u}) = \begin{pmatrix} g_3'(det\,\nabla\mathbf{u})\left(\frac{\partial u_2}{\partial y}n_x - \frac{\partial u_2}{\partial x}n_y\right) \\ g_3'(det\,\nabla\mathbf{u})\left(\frac{\partial u_1}{\partial x}n_y - \frac{\partial u_1}{\partial y}n_x\right) \end{pmatrix}.$$

Proof. Integrating each term in (47) by parts, we obtain

$$\begin{aligned}
\int_\Omega g_3'\frac{\partial u_1}{\partial x}\frac{\partial(\delta u_2)}{\partial y}\,dx &= \int_{\partial\Omega} g_3'\frac{\partial u_1}{\partial x}(\delta u_2)n_y\,ds - \int_\Omega \frac{\partial}{\partial y}\Big(g_3'\frac{\partial u_1}{\partial x}\Big)\delta u_2\,dx,\\
\int_\Omega g_3'\frac{\partial(\delta u_1)}{\partial x}\frac{\partial u_2}{\partial y}\,dx &= \int_{\partial\Omega} g_3'\frac{\partial u_2}{\partial y}(\delta u_1)n_x\,ds - \int_\Omega \frac{\partial}{\partial x}\Big(g_3'\frac{\partial u_2}{\partial y}\Big)\delta u_1\,dx,\\
\int_\Omega -g_3'\frac{\partial(\delta u_2)}{\partial x}\frac{\partial u_1}{\partial y}\,dx &= -\int_{\partial\Omega} g_3'\frac{\partial u_1}{\partial y}(\delta u_2)n_x\,ds + \int_\Omega \frac{\partial}{\partial x}\Big(g_3'\frac{\partial u_1}{\partial y}\Big)\delta u_2\,dx,\\
\int_\Omega -g_3'\frac{\partial(\delta u_1)}{\partial y}\frac{\partial u_2}{\partial x}\,dx &= -\int_{\partial\Omega} g_3'\frac{\partial u_2}{\partial x}(\delta u_1)n_y\,ds + \int_\Omega \frac{\partial}{\partial y}\Big(g_3'\frac{\partial u_2}{\partial x}\Big)\delta u_1\,dx.
\end{aligned}$$

Summing up the right hand sides of the terms in the above expressions gives the desired result. □

The adjoint state $(\lambda,q)\in X$ is given as a solution to

$$\langle E'((\mathbf{u},p),\Omega)(\psi,\xi),(\lambda,q)\rangle_{X^*\times X} = (g_3'(\det\nabla\mathbf{u}),\ (\det\nabla\mathbf{u})'\psi)_\Omega, \tag{48}$$

which by Lemma 3.2 amounts to

$$\begin{cases} -\eta\Delta\lambda + (\nabla\mathbf{u})\cdot\lambda - (\mathbf{u}\cdot\nabla)\lambda + \nabla q = T(\mathbf{u}), \text{ in } \Omega,\\ \operatorname{div}\lambda = 0, \text{ in } \Omega,\\ \lambda = 0, \text{ on } \partial\Omega, \end{cases} \tag{49}$$

where the first equation in (49) hold in $\mathbf{L}^2(\Omega)$ and the second one in $L^2(\Omega)$. Moreover, since $\partial\Omega\in C^2$, $(\lambda,q)\in\big(\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\big)\times\big(H^1(\Omega)\cap L_0^2(\Omega)\big)$. Let us note that Theorem 2.1 is not directly applicable to compute the shape derivative of J_3 since the operator " det ∇ " in the functional J_3 in (28) is not affine. We therefore give an independent proof following the lines of the proof of Theorem 2.1. Firstly, we state and prove the following lemma that will become important in what follows.

Lemma 3.3. *Suppose $\mathbf{u}_t$ and $\mathbf{u}^t$ are related by (14). Then*

$$(det\,\nabla\mathbf{u}_t)\circ T_t = I_t^{-1}(det\,\nabla\mathbf{u}^t) + t\mathscr{G}_1 + t^2\mathscr{G}_2, \tag{50}$$

where

$$\begin{aligned}
det\,\nabla\mathbf{u}^t &= (\partial_x u_1^t\partial_y u_2^t - \partial_x u_2^t\partial_y u_1^t),\\
\mathscr{G}_1 &= I_t^{-1}(E_2\partial_x u_1^t + E_1\partial_y u_2^t - E_4\partial_x u_2^t - E_3\partial_y u_1^t)\in L^1(\Omega),\\
\mathscr{G}_2 &= I_t^{-1}(E_1E_2 - E_3E_4)\in L^1(\Omega),
\end{aligned}$$

and $E_1 = h_{2,y}\partial_x u_2^t - h_{2,x}\partial_y u_2^t$, $E_2 = h_{1,x}\partial_y u_1^t - h_{1,y}\partial_x u_1^t$, $E_3 = h_{2,y}\partial_x u_1^t - h_{2,x}\partial_y u_1^t$, *and* $E_4 = h_{1,x}\partial_y u_2^t - h_{1,y}\partial_x u_2^t$.

Proof. By definition

$$(\det \nabla \mathbf{u})\circ T_t = (\partial_x u_1 \partial_y u_2 - \partial_x u_2 \partial_y u_1)\circ T_t.$$

From (20) we have

$$\begin{aligned}(\partial_x u_{t,2})\circ T_t &= I_t^{-1}\partial_x u_2^t + tI_t^{-1}E_1,\ (\partial_y u_{t,1})\circ T_t = I_t^{-1}\partial_y u_1^t + tI_t^{-1}E_2,\\ (\partial_x u_{t,1})\circ T_t &= I_t^{-1}\partial_x u_1^t + tI_t^{-1}E_3,\ (\partial_y u_{t,2})\circ T_t = I_t^{-1}\partial_y u_2^t + tI_t^{-1}E_4.\end{aligned}$$

From the above equations, we obtain

$$(\det \nabla \mathbf{u}_t)\circ T_t = I_t^{-1}(\det \nabla \mathbf{u}^t) + t\mathscr{G}_1 + t^2\mathscr{G}_2.$$

Note that for $\mathbf{u}\in X$, $\mathscr{G}_1,\mathscr{G}_2,\in L^1(\Omega)$, and this concludes the proof. □

Proposition 3.1. *Assume that* $\mathbf{f}\in \mathbf{L}^p(\Omega)$, $p > 2$ = *dimension. If (H1-H4) hold, and* $g_3(\det \nabla \mathbf{u}) \in W^{1,1}(\Omega)$, *then the Eulerian derivative of* $J_3(\mathbf{u},\Omega)$ *exists and is given by the expression*

$$dJ_3(\mathbf{u},\Omega)\mathbf{h} = \int_{\partial\Omega}\Big(\eta \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\frac{\partial \lambda}{\partial \mathbf{n}} + g_3(\det \nabla \mathbf{u}) - P(\mathbf{u})\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\Big)\mathbf{h}\cdot\mathbf{n}\, ds. \tag{51}$$

Proof. As stated earlier, assumptions (H1-H4) were verified in [5]. Using (8), we have

$$\begin{aligned}J_3(\mathbf{u}_t,\Omega_t) - J_3(\mathbf{u},\Omega) &= \int_{\Omega_t} g_3(\det \nabla \mathbf{u}_t)\, dx - \int_\Omega g_3(\det \nabla \mathbf{u})\, dx,\\ &= \int_\Omega I_t g_3((\det \nabla \mathbf{u}_t)\circ T_t)\, dx - \int_\Omega g_3(\det \nabla \mathbf{u})\, dx.\end{aligned} \tag{52}$$

Let $\mathscr{G}_3 = \mathscr{G}_1 + t\mathscr{G}_2$. Using equation (50), we can express (52) as

$$J_3(\mathbf{u}_t,\Omega_t) - J_3(\mathbf{u},\Omega) = \int_\Omega I_t g_3(I_t^{-1}(\det \nabla \mathbf{u}^t) + t\mathscr{G}_3)\, dx - \int_\Omega g_3(\det \nabla \mathbf{u})\, dx. \tag{53}$$

The right hand side of (53) can be written as $R(t)+S(t)$, where

$$\begin{aligned}R(t) &= \int_\Omega I_t g_3(\det \nabla \mathbf{u}^t) - g_3(\det \nabla \mathbf{u})\, dx,\ \ R(0)=0,\\ S(t) &= \int_\Omega I_t\Big(g_3(I_t^{-1}(\det \nabla \mathbf{u}^t) + t\mathscr{G}_3) - g_3(\det \nabla \mathbf{u}^t)\Big)\, dx,\ \ S(0)=0.\end{aligned}$$

$R(t)$ can be re-written as

$$\begin{aligned}R(t) &= \int_\Omega I_t\Big(g_3(\det \nabla \mathbf{u}^t) - g_3(\det \nabla \mathbf{u}) - g_3'(\det \nabla \mathbf{u})(\det \nabla \mathbf{u})'(\mathbf{u}^t - \mathbf{u})\Big)\, dx +\\ &\int_\Omega (I_t - 1)g_3'(\det \nabla \mathbf{u})(\det \nabla \mathbf{u})'(\mathbf{u}^t - \mathbf{u})\, dx + \int_\Omega g_3'(\det \nabla \mathbf{u})(\det \nabla \mathbf{u})'(\mathbf{u}^t - \mathbf{u})\, dx\\ &+ \int_\Omega (I_t - 1)g_3(\det \nabla \mathbf{u})\, dx.\end{aligned}$$

We express $R(t) = R_1(t) + R_2(t) + R_3(t) + R_4(t)$. From Lemma A.1 (see Appendix), we have

$$(\det \nabla \mathbf{u})'(\mathbf{u}^t - \mathbf{u}) = \det \nabla \mathbf{u}^t - \det \nabla \mathbf{u} - \det \nabla (\mathbf{u}^t - \mathbf{u}).$$

Consequently $R_1(t)$ can be rewritten as

$$R_1(t) = \int_\Omega I_t \Big(g_3(\det \nabla \mathbf{u}^t) - g_3(\det \nabla \mathbf{u}) - (g_3'(\det \nabla \mathbf{u}),\, \det \nabla \mathbf{u}^t - \det \nabla \mathbf{u}) \Big)\, dx + \int_\Omega I_t g_3'(\det \nabla \mathbf{u}) \det \nabla (\mathbf{u}^t - \mathbf{u})\, dx.$$

Let $s = \det \nabla \mathbf{u}$ and $q = \det \nabla \mathbf{u}^{\mathbf{t}}$. Then $R_1(t)$ can further be rewritten as

$$\begin{aligned} R_1(t) = \int_\Omega I_t \Big\{ \int_0^1 \big[g_3'(s + \gamma(q-s)) - g_3'(s) \big] (q-s)\, d\gamma \Big\}\, dx \\ + \int_\Omega I_t g_3'(\det \nabla \mathbf{u}) \det \nabla (\mathbf{u}^t - \mathbf{u})\, dx. \end{aligned} \tag{54}$$

Note that the functions $g_3(t)$ and $g_3'(t)$ are globally Lipschitz with constant $3/2$, i.e.,

$$\begin{aligned} |g_3(t) - g_3(s)| &\le \frac{3}{2}|t-s|, \\ |g_3'(t) - g_3'(s)| &\le \frac{3}{2}|t-s|, \quad 0 \le t, s \in \mathbb{R}. \end{aligned} \tag{55}$$

Furthermore, Young's inequality implies that

$$|\det \nabla\, \mathbf{u}| \le \frac{1}{2}\Big[\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_2}{\partial x_2}\Big)^2 + \Big(\frac{\partial u_2}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_1}{\partial x_2}\Big)^2 \Big] = \frac{1}{2}(\nabla \mathbf{u} : \nabla \mathbf{u}).$$

Hence the second term $R_{1,2}(t)$ in (54) can be estimated as follows

$$|R_{1,2}(t)| = \Big| \int_\Omega I_t g_3'(\det \nabla \mathbf{u}) \det \nabla (\mathbf{u}^t - \mathbf{u})\, dx \Big| \le \frac{3}{4} \max_{t \in [0,\tau_0]} ||I_t||_{L^\infty} \int_\Omega |\nabla (\mathbf{u}^t - \mathbf{u})|^2\, dx,$$

for τ_0 sufficiently small.

Consequently $\lim\limits_{t\to 0} \dfrac{|R_{1,2}(t)|}{t} \le \dfrac{3}{4} \max\limits_{t\in[0,\tau_0]} ||I_t||_{L^\infty} \lim\limits_{t\to 0} \dfrac{||\mathbf{u}^t - \mathbf{u}||_{H_1}^2}{t} = 0$, by (H2). Similarly the first term can be estimated as

$$\Big| \int_\Omega I_t \Big\{ \int_0^1 \big[g_3'(s + \gamma(q-s)) - g_3'(s) \big] (q-s)\, d\gamma \Big\}\, dx \Big| \le \frac{3}{2} \max_{t\in[0,\tau_0]} ||I_t||_{L^\infty} \int_\Omega |(q-s)|^2\, dx.$$

Note that $\int_\Omega |(q-s)|^2\, dx = \int_\Omega |A+B|^2\, dx \le \int_\Omega |A|^2 + 2|AB| + |B|^2\, dx$, where $A = \frac{\partial u_1^t}{\partial x_1}\frac{\partial u_2^t}{\partial x_2} - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}$ and $B = \frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2^t}{\partial x_1}\frac{\partial u_1^t}{\partial x_2}$. Furthermore, let $a = \frac{\partial u_1}{\partial x_1}$, $b = \frac{\partial u_2}{\partial x_2}$, $c = \frac{\partial u_2}{\partial x_1}$, and $d = \frac{\partial u_1}{\partial x_2}$. Then $A = a^t b^t - ab = a^t(b^t - b) + b(a^t - a)$, and $B = cd - c^t d^t = d(c - c^t) + c^t(d - d^t)$. Since $\partial\Omega \in C^2$ and $\mathbf{f} \in \mathbf{L}^p(\Omega)$ for $p > 2$ = dimension, $\mathbf{u} \in$

$W^{2,p}(\Omega),\ p>2$. Consequently, $a,\ a^t,\ b,\ b^t,\ c,\ c^t,\ d,\ d^t \in W^{1,p}(\Omega) \hookrightarrow C(\Omega)$ for $p>2$, and

$$\begin{aligned}\int_\Omega |A|^2\,dx &= \int_\Omega (a^t)^2(b^t-b)^2 + 2a^t b(b^t-b)(a^t-a) + b^2(a^t-a)\,dx\\ &\le ||a^t||^2_{L^\infty}\int_\Omega (b^t-b)^2\,dx + ||a^t||_{L^\infty}||b||_{L^\infty}\int_\Omega (a^t-a)^2+(b^t-b)^2\,dx\\ &+||b||^2_{L^\infty}\int_\Omega (a^t-a)^2\,dx.\end{aligned}$$

Hence since (H2) is satisfied, $\lim_{t\to 0}\int_\Omega \frac{|A|^2}{t}\,dx = 0$ follows. Analogously we can show that $\lim_{t\to 0}\int_\Omega \frac{|AB|}{t}\,dx=0$ and $\lim_{t\to 0}\int_\Omega \frac{|B|^2}{t}\,dx = 0$. Therefore $\lim_{t\to 0}\frac{|R_1(t)|}{t}=0$. Furthermore, $\lim_{t\to 0}\dfrac{R_2(t)}{t} \le \lim_{t\to 0}\dfrac{3}{2}||\dfrac{I_t-1}{t}||_{L^\infty}\int_\Omega |(\det \nabla\mathbf{u})'(\mathbf{u}^t-\mathbf{u})|\,dx = 0$, by (H2). Using (48) with $\psi = \mathbf{u}^t-\mathbf{u} \in \mathbf{H}^1_0(\Omega)$, $\xi \in L^2(\Omega)$, we have

$$R_3(t) = \int_\Omega (g_3'(\det \nabla\mathbf{u}),(\det\nabla\mathbf{u})'\psi)\,dx = \langle E'((\mathbf{u},p),\Omega)(\psi,\xi),(\lambda,q)\rangle_{X^*\times X}. \tag{56}$$

Proceeding as in the proof of Theorem 2.1, the term on the right hand side of (56) is arranged in an efficient manner so that (23) holds. Consequently, by using the computation that led to (39), it follows that

$$\begin{aligned}\lim_{t\to 0}\frac{R_3(t)}{t} = -\frac{d}{dt}\langle \tilde{E}((\mathbf{u},p),t),(\lambda,q)\rangle_{X^*\times X}|_{t=0} = \int_{\partial\Omega}\eta\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\frac{\partial\lambda}{\partial\mathbf{n}}\mathbf{h}\cdot\mathbf{n}\,ds+\\ \int_\Omega T(\mathbf{u})\nabla\mathbf{u}^T\cdot\mathbf{h}\,dx,\end{aligned} \tag{57}$$

where (λ,q) solves (49). We shall turn our attention to the last term $R_4(t)$ later. Let us now look at

$$S(t) = \int_\Omega I_t\Big(g_3(I_t^{-1}(\det\nabla\mathbf{u}^t)+t\mathscr{G}_3) - g_3(\det\nabla\mathbf{u}^t)\Big)\,dx.$$

The expression $g_3(I_t^{-1}(\det\nabla\mathbf{u}^t)+t\mathscr{G}_3) - g_3(\det\nabla\mathbf{u}^t)$, can be written as

$$\begin{aligned}g_3(I_t^{-1}(\det\nabla\mathbf{u}^t)+t\mathscr{G}_3) - g_3(I_t^{-1}(\det\nabla\mathbf{u})+t\mathscr{G}_3) + g_3(\det\nabla\mathbf{u}) - g_3(\det\nabla\mathbf{u}^t)+\\ g_3(I_t^{-1}(\det\nabla\mathbf{u})+t\mathscr{G}_3) - g_3(\det\nabla\mathbf{u}).\end{aligned}$$

Observe that the function $g_3(r)$ can be expressed as $g_3(r) = r - \dfrac{r}{r^2+1}$. Let $s = \det\nabla\mathbf{u}$, $q = \det\nabla\mathbf{u^t}$, and $\mathscr{A} = g_3(I_t^{-1}q+t\mathscr{G}_3) - g_3(I_t^{-1}s+t\mathscr{G}_3) + g_3(s) - g_3(q)$. Then

$$S(t) = S_1(t)+S_2(t) = \int_\Omega I_t\mathscr{A}\,dx + \int_\Omega I_t\Big(g_3(I_t^{-1}(\det\nabla\mathbf{u})+t\mathscr{G}_3) - g_3(\det\nabla\mathbf{u})\Big)\,dx. \tag{58}$$

Note that $\mathscr{A}$ can be expressed as

$$\mathscr{A} = (I_t^{-1}-1)(q-s) + \mathscr{W}(q) - \mathscr{W}(s), \tag{59}$$

where $\mathscr{W}(r) = \frac{r}{r^2+1} - \frac{I_t^{-1}r+t\mathscr{G}_3}{(I_t^{-1}r+t\mathscr{G}_3)^2+1}$. The difference $\mathscr{D} = \mathscr{W}(q) - \mathscr{W}(s)$ can be expressed as

$$\mathscr{D} = \frac{(t\mathscr{G}_3 + (I_t^{-1}-1)q)\left(I_t^{-1}q^2 + qt\mathscr{G}_3 - 1\right)}{(q^2+1)\left((I_t^{-1}q + t\mathscr{G}_3)^2+1\right)} - \frac{(t\mathscr{G}_3 + (I_t^{-1}-1)s)\left(I_t^{-1}s^2 + st\mathscr{G}_3 - 1\right)}{(s^2+1)\left((I_t^{-1}s + t\mathscr{G}_3)^2+1\right)}.$$

Let $\vartheta_1 = t\mathscr{G}_3 + (I_t^{-1}-1)q,\ \vartheta_2 = t\mathscr{G}_3 + (I_t^{-1}-1)s,\ r_1 = I_t^{-1}q^2 + qt\mathscr{G}_3 - 1$,
$r_2 = I_t^{-1}s^2 + qt\mathscr{G}_3 - 1$, $n_1 = \left(q^2+1\right)\left((I_t^{-1}q + t\mathscr{G}_3)^2+1\right)$,
$n_2 = \left(s^2+1\right)\left((I_t^{-1}q + t\mathscr{G}_3)^2+1\right)$, $\beta := \frac{n_2 r_1}{n_1 n_2}$, and $\rho := \frac{n_1 r_2}{n_2 r_1}$. Then

$$\mathscr{D} = \frac{\vartheta_1 r_1}{n_1} - \frac{\vartheta_2 r_2}{n_2} = \frac{n_2 r_1\left(\vartheta_1 - \frac{n_1}{n_2}\frac{r_2}{r_1}\vartheta_2\right)}{n_1 n_2} = \beta[(1-\rho)\vartheta_1 + \rho(\vartheta_1 - \vartheta_2)].$$

Note that $(\vartheta_1 - \vartheta_2) = (I_t^{-1}-1)(q-s)$ and

$$\frac{\mathscr{A}}{t} = \frac{(I_t^{-1}-1)}{t}(q-s) + \beta[(1-\rho)\frac{\vartheta_1}{t} + \rho\frac{(I_t^{-1}-1)}{t}(q-s)].$$

Consequently the estimate for $S_1(t)/t$ reads

$$|S_1(t)/t| \le \max_{t\in[0,\tau_0]} ||I_t||_{L^\infty} ||\frac{I_t^{-1}-1}{t}||_{L^\infty}(1 + ||\rho||_{L^\infty}||\beta||_{L^\infty})\,||\det\nabla\mathbf{u}^t - \det\nabla\mathbf{u})||_{L^1}$$
$$+ \max_{t\in[0,\tau_0]} ||I_t||_{L^\infty}||\beta||_{L^\infty}||1-\rho||_{L^1}||\frac{\vartheta_1}{t}||_{L^\infty},$$

for τ_0 sufficiently small. Note that since $\mathbf{u} \in W^{2,p},\ p>2,\ \beta,\ \rho$ and $\frac{\vartheta_1}{t}$ are bounded in $L^\infty(\Omega)$. Furthermore $\frac{n_1}{n_2} \to 1$ in $L^1(\Omega)$, $\frac{r_2}{r_1} \to 1$ in $L^1(\Omega)$, and $\rho \to 1$ in $L^1(\Omega)$. By (H2) it follows that $\lim_{t\to 0}\frac{|S_1(t)|}{t} = 0$. Therefore collecting the remaining terms into $S_5(t) := R_4(t) + S_2(t)$, we have that

$$S_5(t) = \int_\Omega I_t g_3(I_t^{-1}\det\nabla\mathbf{u} + t\mathscr{G}_3)) - g_3(\det\nabla\mathbf{u})\,dx.$$

Observe that $g_3'(\det\nabla\mathbf{u}) \in L^\infty(\Omega)$ and since $\mathbf{u} \in \mathbf{H}^2(\Omega)$, we have $\nabla(\det\nabla\mathbf{u}) \in \mathbf{L}^1(\Omega)$ and $\nabla(g_3(\det\nabla\mathbf{u})) = g_3'(\det\nabla\mathbf{u})\nabla(\det\nabla\mathbf{u}) \in \mathbf{L}^1(\Omega)$. Consequently $g_3(\det\nabla\mathbf{u}) \in W^{1,1}(\Omega)$. This implies that $\frac{d}{dt}\Big[g_3(\det\nabla(\mathbf{u}\circ T_t^{-1}))\Big]_{t=0}$ exists in $L^1(\Omega)$, [7]. Hence using (50) and Lemma 2.2, we have that

$$
\begin{aligned}
\lim_{t\to 0}\frac{S_5(t)}{t} &= \lim_{t\to 0}\frac{\int_\Omega I_t g_3(I_t^{-1}\det \nabla\mathbf{u}+t\mathscr{G}_3)) - g_3(\det \nabla\mathbf{u})\, dx}{t},\\
&= \lim_{t\to 0}\frac{\int_{\Omega_t} g_3([I_t^{-1}\det \nabla\mathbf{u}+t\mathscr{G}_3]\circ T_t^{-1}) - \int_\Omega g_3(\det \nabla\mathbf{u})\, dx}{t},\\
&= \frac{\lim_{t\to 0}\int_{\Omega_t} g_3(\det \nabla(\mathbf{u}\circ T_t^{-1})) - \int_\Omega g_3(\det \nabla\mathbf{u})\, dx}{t},\\
&= \frac{d}{dt}\int_{\Omega_t} g_3(\det \nabla(\mathbf{u}\circ T_t^{-1}))\,\Big|_{t=0}\, dx_t,\\
&= \int_{\partial\Omega} g_3(\det \nabla\mathbf{u}))\mathbf{h}\cdot\mathbf{n}\, ds + \int_\Omega g_3'(\det \nabla\mathbf{u})\frac{d}{dt}(\det \nabla(\mathbf{u}\circ T_t^{-1})\Big|_{t=0}\, dx.
\end{aligned}
\tag{60}
$$

The second term on the right hand side in (60) can be simplified using Lemma 2.3 and integration by parts leading to

$$
\int_\Omega g_3'\frac{d}{dt}(\det \nabla(\mathbf{u}\circ T_t^{-1}))\big|_{t=0}\, dx = -\int_\Omega T(\mathbf{u})D\mathbf{u}\cdot\mathbf{h}\, dx - \int_{\partial\Omega} P(\mathbf{u})D\mathbf{u}\cdot\mathbf{h}\, ds.
$$

Therefore

$$
\lim_{t\to 0}\frac{S_5(t)}{t} = \int_{\partial\Omega}\Big(g_3(\det \nabla\mathbf{u}) - P(\mathbf{u})\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\Big)\mathbf{h}\cdot\mathbf{n}\, ds - \int_\Omega T(\mathbf{u})D\mathbf{u}\cdot\mathbf{h}\, dx. \tag{61}
$$

Finally, using (57) and (61), we obtain

$$
dJ_3(\mathbf{u},\Omega)\mathbf{h} = \int_{\partial\Omega}\Big(\eta\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\frac{\partial\lambda}{\partial\mathbf{n}} + g_3(\det \nabla\mathbf{u}) - P(\mathbf{u})\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\Big)\mathbf{h}\cdot\mathbf{n}\, ds. \tag{62}
$$

□

Expressions for $dJ_i(\mathbf{u},\Omega)\mathbf{h}$ in (43), (46), and (62) are linear and continuous in $\mathbf{h}$, and hence the cost functionals J_1, J_2, and J_3 are shape differentiable.

References

[1] F. Abergel and R. Temam. On some control problems in fluid mechanics. *Theoretical and Computational Fluid Dynamics*, 1(6):303–325, 1990.

[2] M. C. Delfour and J. P. Zolésio. *Shapes and geometries: analysis, differential calculus, and optimization*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2001.

[3] J. Haslinger, K. Ito, T. Kozubek, K. Kunisch, and G. Peichl. On the shape derivative for problems of bernoulli type. *Interfaces and Free Boundaries*, 11(2):317–330, 2009.

[4] M. Hintermüller, K. Kunisch, Y. Spasov, and S. Volkwein. Dynamical systems based optimal control of incompressible fluids. *Int. J. Numer. Methods in Fluids*, 4:345–359, 2004.

[5] K. Ito, K. Kunisch, and G. Peichl. Variational approach to shape derivatives. *ESAIM: Control, Optimisation and Calculus of Variations*, 14:517–539, 2008.

[6] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford University Press, 2003.

[7] J. Sokolowski and J. P. Zolésio. *Introduction to Shape Optimization. Shape Sensitivity Analysis*. Springer-Verlag, 1992.

[8] R. Temam. *Navier Stokes Equations: Theory and Numerical Analysis (Studies in Mathematics and its Applications 2)*. North-Holland, 1977.

A Appendix

Lemma A.1. *Let $\psi = \mathbf{u}^t - \mathbf{u}$, then for a 2D vector field $\mathbf{u}$, the following relation holds*

$$det\ \nabla \mathbf{u}^t - det\ \nabla \mathbf{u} - det\ \nabla \psi = (det\ \nabla \mathbf{u})'(\mathbf{u}^t - \mathbf{u}),$$

where

$$(det\ \nabla \mathbf{u})'(\psi) = \frac{\partial u_1}{\partial x}\frac{\partial (u_2^t - u_2)}{\partial y} + \frac{\partial u_2}{\partial y}\frac{\partial (u_1^t - u_1)}{\partial x} - \frac{\partial u_1}{\partial y}\frac{\partial (u_2^t - u_2)}{\partial x} - \frac{\partial u_2}{\partial x}\frac{\partial (u_1^t - u_1)}{\partial y}.$$

Proof. Using (5), we have $\det \nabla \psi = \dfrac{\partial (u_1^t - u_1)}{\partial x}\dfrac{\partial (u_2^t - u_2)}{\partial y} - \dfrac{\partial (u_2^t - u_2)}{\partial x}\dfrac{\partial (u_1^t - u_1)}{\partial y}$. Expansion of the differential terms leads to

$$\det \nabla \psi = \frac{\partial u_1^t}{\partial x}\frac{\partial (u_2^t - u_2)}{\partial y} - \frac{\partial u_1}{\partial x}\frac{\partial (u_2^t - u_2)}{\partial y} - \frac{\partial u_2^t}{\partial x}\frac{\partial (u_1^t - u_1)}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial (u_1^t - u_1)}{\partial y}.$$

On the other hand

$$\begin{aligned}\det \nabla \mathbf{u}^t - \det \nabla \mathbf{u} &= \Big(\frac{\partial u_1^t}{\partial x}\frac{\partial u_2^t}{\partial y} - \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}\Big) - \Big(\frac{\partial u_2^t}{\partial x}\frac{\partial u_1^t}{\partial y} - \frac{\partial u_2}{\partial x}\frac{\partial u_1}{\partial y}\Big) \\ &= \frac{\partial u_1^t}{\partial x}\frac{\partial (u_2^t - u_2)}{\partial y} + \frac{\partial u_2}{\partial y}\frac{\partial (u_1^t - u_1)}{\partial x} - \frac{\partial u_2^t}{\partial x}\frac{\partial (u_1^t - u_1)}{\partial y} \\ &\quad - \frac{\partial u_1}{\partial y}\frac{\partial (u_2^t - u_2)}{\partial x}.\end{aligned}$$

Thus

$$\begin{aligned}\det \nabla \mathbf{u}^t - \det \nabla \mathbf{u} - \det \nabla \psi = &\frac{\partial u_1}{\partial x}\frac{\partial (u_2^t - u_2)}{\partial y} + \frac{\partial u_2}{\partial y}\frac{\partial (u_1^t - u_1)}{\partial x} - \frac{\partial u_1}{\partial y}\frac{\partial (u_2^t - u_2)}{\partial x} \\ &- \frac{\partial u_2}{\partial x}\frac{\partial (u_1^t - u_1)}{\partial y}.\end{aligned}$$

□