On shape sensitivity analysis of the cost functional without shape sensitivity of the state variable

H. Kasumba, K. Kunisch

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H. Kasumba* and K. Kunisch †

Abstract

A general framework for calculating shape derivatives for domain optimization problems with partial differential equations as constraints is presented. The first order approximation of the cost with respect to the geometry perturbation is arranged in an efficient manner that allows the computation of the shape derivative of the cost without the necessity to involve the shape derivative of the state variable. In doing so, the state variable is only required to be Lipschitz continuous with respect to geometry perturbations. Application to shape optimization with the Navier-Stokes equations as PDE constraint is given.

Key words: Shape derivative, Navier-Stokes equations, Cost functional.

1 Introduction

In this paper, we consider the problem of finding a domain \(\Omega\) (in a class of admissible domains \(\mathcal{U}_{ad}\)) minimizing the functional

\[
J(u, \Omega) \equiv \int_\Omega j_1(C_\gamma u) \, dx
\]

subject to a constraint

\[
E(u, \Omega) = 0, \quad u \in X.
\]

Here \(E(u, \Omega) = 0\), represents a partial differential equation posed on \(\Omega\) with boundary \(\partial \Omega\), \(u\) is the state variable and \(X \subset L^2(\Omega)^l\), \(l \in \mathbb{N}\), is a Hilbert space with a dual \(X^*\). The class of admissible domains \(\mathcal{U}_{ad}\) does not admit a vector space structure, making the application of traditional optimization methods difficult. This difficulty is bypassed by describing shapes by means of transformations. Due to lack of closed form solutions to \(E(u, \Omega) = 0\), problem (1-2) is usually solved numerically using iterative methods, e.g., the gradient descent method. For such methods, one needs to compute the derivative of the cost with respect to \(\Omega\). Rigorous derivations of shape derivative of \(J\) can be found in literature, see e.g [7], [2]. In [7], the approach taken involves differentiation of the state equation with respect to the domain. The state variable lives in a Hilbert space \(X\) which depends on the geometry with respect to which optimization is carried out. To obtain sensitivity information of \(\Omega \mapsto J(\Omega) = J(\Omega, u(\Omega))\), a chain rule approach involving the shape derivative of \(\Omega \mapsto u(\Omega)\) is chosen. Other techniques presented in, e.g., [2]

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use function space parametrization and function space embedding methods. The latter depends strongly on sophisticated differentiability properties of saddle point problems. In this paper, we present a computation of the shape derivative of $J$ under minimal regularity assumptions. The technique we employ was first suggested in [5], and then used in [3], and allows to compute the shape derivative of the state variable with respect to the geometry. In [5], a cost functional $J : X \mapsto \mathbb{R}$ of the form $J(u, \Omega) = \int_\Omega j_1(u) \, dx$ was considered. However, in many applications such as vortex control in fluids, cost functionals are typically of the form (1), where $C_\gamma : H \mapsto H$, a Banach space, is either a linear operator, e.g., $C_\gamma u = \text{curl } u$ or generally a non-linear operator, e.g., $C_\gamma u = \text{det } \nabla u$. In addition, we note that cost functionals of the form (1) can be expressed in the form

$$J(u, \Omega) = G(F(u)),$$

where the mappings

$$F : X \mapsto H, \quad G : H \mapsto \mathbb{R},$$

are defined as $F(u) = C_\gamma u$ and $G(v) = \int_\Omega j_1(v) \, dx$, respectively. In this work, we specifically address this composite structure of the cost functionals of the form (1), where $C_\gamma$ is an affine operator

$$C_\gamma : u(\cdot) \mapsto C u(\cdot) + \gamma(\cdot) \quad \gamma \in L^2(D),$$

$D$, an open and bounded hold all domain to be specified later, and $C \in \mathscr{L}(X, L^2(\Omega))$ is a linear operator. An application involving a cost functional with a non-linear operator $C$ in the integrand is also presented. The approach that we use can be summarized as follows: The difference quotient of the cost $J$ with respect to the geometry perturbation is arranged in an efficient manner so that computation of the shape derivative of the state can be bypassed. In doing so, the existence of the material derivative of the state $u$ can be replaced by Hölder continuity with exponent greater than or equal to $\frac{1}{2}$ of $u$ with respect to the geometric data. The constraint $E(u, \Omega) = 0$ is observed by introducing an appropriately defined adjoint equation. Furthermore, well known results from the method of mapping and the differentiation of functionals with respect to geometric quantities are utilized on a technical level.

The rest of the paper is organized as follows. In Section 2 we present the proposed general framework to compute the shape derivative for (1)-(2). The application of the general theory to shape optimization problems with the Navier Stokes equations as equality state constraints is presented in Section 3.

2 Shape derivative

In this section we focus on sensitivity analysis for the shape optimization problem (1)-(2). To describe the class of admissible domains $\mathscr{U}_{ad}$, let $D \subset \mathbb{R}^d$, $d = 2, 3$ be a fixed bounded domain with a $C^2$ boundary $\partial D$ and let $S$ be a domain with a $C^2$ boundary $\Gamma := \partial S$ satisfying $\bar{S} \subset D$ (see Figure (1)). For the reference domain, either of the following three cases is admitted
(i) \( \Omega = S \),
(ii) \( \Omega = D \),
(iii) \( \Omega = D \setminus \bar{S} \).

Then the boundary \( \partial \Omega \) for the three cases is given by

(i) \( \partial \Omega = \Gamma \),
(ii) \( \partial \Omega = \partial D \),
(iii) \( \partial \Omega = \Gamma \cup \partial D \).

Shapes are difficult entities to be dealt with directly, so we manipulate them by means of transformations. If \( \Omega \) is the initial admissible shape, and \( \Omega_t \) is the shape at time \( t \), one considers transformations \( T_t : \Omega \mapsto \Omega_t \). Such transformations can be constructed, for instance, by perturbation of the identity [2]. To construct an admissible class of these transformations, let \( \Omega \subset \bar{D} \) be a bounded domain and

\[ \mathcal{H} = \{ h \in C^2(\bar{D}) : h|_{\partial D} = 0 \} \]

be the space of deformation fields. The fields \( h \in \mathcal{H} \) define for \( t > 0 \), a perturbation of \( \Omega \) by

\[ T_t : \Omega \mapsto \Omega_t, \quad x \mapsto T_t(x) = x + th(x). \]

For each \( h \in \mathcal{H} \), there exists \( \tilde{t} > 0 \) such that \( T_t(D) = D \) and \( \{ T_t \} \) is a family of \( C^2 \)-diffeomorphisms for \( |t| < \tilde{t} \) [2]. For each \( t \in \mathbb{R} \) with \( |t| < \tilde{t} \), we set \( \Omega_t = T_t(\Omega) \), \( \Gamma_t = T_t(\Gamma) \). Thus \( \Omega_0 = \Omega \), \( \Gamma_0 = \Gamma \), \( \Omega_t \subset D \).

### 2.1 Notation

In what follows, the following notation will be used:

\[ l_t = \det DT_t, \quad A(t) = l_t(DT_t)^{-1}(DT_t)^{-T}, \quad (4) \]
and $\nabla u$ stand for $(Du)^T$ where $u$ is either a scalar or vector valued function (if $u$ is bold faced, i.e., $\mathbf{u}$). In (4), $(DT)^{-T}$ takes the meaning of transpose of the inverse matrix $(DT)_i^{-1}$. Furthermore, two notations for the inner product in $\mathbb{R}^d$ shall be used, namely $(x,y)$ and $x \cdot y$, respectively. The latter shall be used in case of nested inner products. In addition, throughout this work, unless specified otherwise, the following parenthesis $(\cdot,\cdot)_\Omega$, $(\cdot,\cdot)_{\partial \Omega}$ shall denote the $L^2(\Omega)$, $L^2(\partial \Omega)$ inner products, respectively. In some cases, the subscript $\Omega$ may be omitted, but the meaning will remain clear in the given context. The scalar product and the norm in the Hilbert space $X$ will be denoted by $(\cdot,\cdot)_X$ and $||\cdot||_X$, respectively, and the duality pairing between $X^*$ and $X$ is denoted by $\langle \cdot , \cdot \rangle_{X^* \times X}$. The curl of a vector field $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$, denoted by $\text{curl} \, \mathbf{u}$, is defined as

$$\text{curl} \, \mathbf{u} := \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y},$$

while the curl of a scalar field $u$ in the case $d = 2$, denoted by $\text{curl} \, u$, is defined as

$$\text{curl} \, u := (\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}).$$

The determinant of the velocity gradient tensor of a vector field $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$, denoted by $\det \nabla u(x)$, is defined as

$$\det \nabla u(x) := \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} - \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial y}. \quad (5)$$

The unit outward normal and tangential vectors to the boundary $\partial \Omega$ shall be denoted by $\mathbf{n} = (n_x,n_y)$ and $\tau = (-n_y,n_x)$, respectively. We denote by $H^m(\mathcal{S})$, $m \in \mathbb{R}$, the standard Sobolev space of order $m$ defined by

$$H^m(\mathcal{S}) := \{ u \in L^2(\mathcal{S}) \mid D^\alpha u \in L^2(\mathcal{S}), \text{ for } 0 \leq |\alpha| \leq m \},$$

where $D^\alpha$ is the weak (or distributional) partial derivative, and $\alpha$ is a multi-index. Here $\mathcal{S}$, which is either the flow domain $\Omega$, or its boundary $\partial \Omega$, or part of its boundary. The norm $|| \cdot ||_{H^m(\mathcal{S})}$ associated with $H^m(\mathcal{S})$ is given by

$$||u||^2_{H^m(\mathcal{S})} = \sum_{|\alpha| \leq m} \int_{\mathcal{S}} |D^\alpha u|^2 \, dx.$$

Note that $H^0(\mathcal{S}) = L^2(\mathcal{S})$ and $|| \cdot ||_{H^0(\mathcal{S})} = || \cdot ||_{L^2(\mathcal{S})}$. For the vector valued functions, we define the Sobolev space $H^m(\mathcal{S})$ by

$$H^m(\mathcal{S}) := \{ \mathbf{u} = (u_1,u_2) \mid u_i \in H^m(\mathcal{S}), \text{ for } i = 1,2 \},$$

and its associated norm

$$||\mathbf{u}||^2_{H^m(\mathcal{S})} = \sum_{i=1}^2 ||u_i||^2_{H^m(\mathcal{S})}.$$
\section*{2.2 Properties of $T_t$}

Let $\mathcal{J} = [0, \tau_0]$ with $\tau_0$ sufficiently small. Then the following regularity properties of the transformation $T_t$ can be shown, see for example ([5], [7], [2, Chapter 7]):

\begin{align*}
T_0 &= \mathrm{id} \quad t \mapsto T_t \in C^1(\mathcal{J}, C^1(\mathcal{D}; \mathbb{R}^d)) \\
T_t^{-1} &\in C^1(\mathcal{J}, C^1(\mathcal{D}; \mathbb{R}^d)) \quad t \mapsto I_t \in C(\mathcal{J}, C(\mathcal{D})) \\
\frac{d}{dt} T_t^{-1} |_{t=0} &= -h \quad \frac{d}{dt} I_t |_{t=0} = D h \\
\frac{d}{dt} DT_t^{-1} |_{t=0} &= -D h \quad \frac{d}{dt} I_t |_{t=0} = \text{div } h \\
I_t |_{t=0} &= 1 \quad I_t^{-1} |_{t=0} = 1.
\end{align*}

The limits defining the derivatives at $t = 0$ exist uniformly in $x \in \mathcal{D}$. We shall also make use of the surface divergence, denoted by $\text{div}_\Gamma$, which is defined for $\varphi \in C^1(\bar{\mathcal{D}}, \mathbb{R}^d)$ by

$$\text{div}_\Gamma \varphi := \text{div}\varphi |_\Gamma - (D\varphi \cdot \mathbf{n}).$$

\section*{2.3 The Eulerian derivative}

\textbf{Definition 2.1.} For given $h \in \mathcal{H}$, the Eulerian derivative of $J$ at $\Omega$ in the direction $h$ is defined as

$$dJ(u, \Omega)h = \lim_{t \to 0} \frac{J(u_t, \Omega_t) - J(u, \Omega)}{t},$$

where $u_t$ satisfies

$$E(u_t, \Omega_t) = 0.$$  \hfill (8)

The functional $J$ is said to be shape differentiable at $\Omega$ if $dJ(\Omega, u)h$ exists for all $h \in \mathcal{H}$ and the mapping $h \mapsto dJ(\Omega, u)h$ is linear and continuous on $\mathcal{H}$.

Under suitable regularity assumptions one can furthermore show that $dJ(u, \Omega)h$ only depends on the normal component of the deformation field $h$ on $\partial \Omega$ and can be represented as

$$dJ(u, \Omega)h = \int_{\partial \Omega} G_\Omega h \cdot \mathbf{n} \, ds,$$

where the kernel $G_\Omega$ does not involve the shape derivative of $u$ with respect to $\Omega$. This is the main result of the Zolesio-Hadamard structure theorem [2, Pg. 348]. Let $\{X_t\}_{t \geq 0}$ be a family of functional spaces defined over the domains $\Omega_t$. Then the variational form of (9) is given by: Find $u_t \in X_t$ such that

$$\langle E(u_t, \Omega_t), \psi_t \rangle_{X_t^* \times X_t} = 0,$$  \hfill (11)

holds for all $\psi_t \in X_t$. Throughout we choose $X_t = T_t(X)$ and we assume that equation (11) has a unique solution $u_t$, for all $t$ sufficiently small. Using the method of mappings, equation (11) represents the weak form of the reference problem (2) given by

$$\langle E(u, \Omega), \psi \rangle_{X^* \times X} = 0, \quad \text{for all } \psi \in X$$  \hfill (12)
for $t = 0$. The adjoint state $p \in X$ to this equation is defined as the solution to
\[ (E_u(u, \Omega) \psi, p)_{X^* \times X} = (C^* j_1'(C_p u), \psi). \]
(13)

Any function $u_t : \Omega_t \mapsto \mathbb{R}^l$, for $l \in \mathbb{N}$, can be mapped back to the reference domain by \( u' = u_t \circ T_t : \Omega \mapsto \mathbb{R}^l \).
(14)

From the chain rule it follows that the gradients of $u_t$ and $u'$ are related by \[ (\nabla u_t) \circ T_t = (DT_t)^{-T} \nabla u', \]
(see [7] Prop. 2.29). Moreover $u' : \Omega \mapsto \mathbb{R}^l$ satisfies an equation on the reference domain which we express as \[ E(u', t) = 0, \ |t| < \tau. \]

Because $T_0 = id$, one obtains $u^0 = u$ and \[ E(u^0, 0) = E(u, \Omega). \]

In order to circumvent the computation of the derivative of $u$ with respect to $\Omega$, the following assumptions (H1-H4) were imposed on $E$ and $E$ in [5].

(H1) There is a $C^1$-function $\tilde{E} : X \times (-\tau, \tau) \mapsto X^*$ such that $E(u_t, \Omega_t) = 0$ is equivalent to $\tilde{E}(u', t) = 0$ in $X^*$, with $\tilde{E}(u, 0) = E(u, \Omega)$ for all $u \in X$.

(H2) There exists $0 < \tau_0 \leq \tau$ such that for $|t| < \tau_0$, there exists a unique solution $u' \in X$ to $\tilde{E}(u', t) = 0$ and
\[ \lim_{t \to 0} \frac{|u' - u^0||x|}{|t|^\frac{3}{2}} = 0. \]

(H3) $E_u(u, \Omega) \in \mathscr{L}(X, X^*)$ satisfies
\[ \langle E(v, \Omega) - E(u, \Omega) - E_u(u, \Omega)(v-u), \psi \rangle_{X^* \times X} = o(||v-u||_X^2). \]

(H4) $\tilde{E}$ and $E$ satisfy
\[ \lim_{t \to 0} \frac{1}{t} \langle \tilde{E}(u', t) - \tilde{E}(u, t) - (E(u', \Omega) - E(u, \Omega)), \psi \rangle_{X^* \times X} = 0. \]

Additionally we need the following assumptions on $j_1$ and $C_p$ hold.

(H5) We assume that $\int_{\Omega} j_1(C_p u) \, dx$, $\int_{\Omega} (j_1'(C_p u))^2 \, dx$ exists for all $u \in X$ and
\[ \left| \int_{\Omega} L \left[ j_1(C_p u') - j_1(C_p u) - \left(j_1'(C_p u), C(u' - u) \right) \right] \, dx \right| \leq K ||u' - u||_X^2, \]
where $K > 0$ does not depend on $t \in \mathscr{J}$. 

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To compute the Eulerian derivative of \( J(u, \Omega) \) in (8), we need to transform the value of \( J(u, \Omega) = \int_{\Omega} j_1(C_T u_t) \, dx \) back to \( \Omega \). This is done by using the relation

\[
J(u_t, \Omega) = \int_{\Omega} j_1(C_T u_t) \, dx = \int_{\Omega} j_1((C_T u_t) \circ T_t) \, dx.
\]

The transformation of \((C_T u_t) \circ T_t\) back to \( \Omega \) induces some matrix \( A_t \) that we shall require to satisfy:

\(\text{(H6)}\) There exists a matrix \( A_t \) such that \( t \mapsto A_t \in C(\mathcal{F}, C(\mathcal{D}, \mathbb{R}^{d \times d})) \) and

\[
(C_T u_t) \circ T_t = A_t C u_t + t \mathcal{G} + \gamma, \quad \mathcal{G} \in L^2(\Omega),
\]

\[
C_T (u \circ T_t^{-1}) = (A_t C u + t \mathcal{G} + \gamma) \circ T_t^{-1},
\]

\[
\lim_{t \to 0} \frac{A_t - I}{t} \exists, \quad A_t|_{t=0} = I.
\]

\(\text{(H7)}\) Let \( \mathcal{M}(t) = \int_{\Omega} L_t [j_1(A_t C u' + t \mathcal{G} + \gamma) - j_1(A_t C u + t \mathcal{G} + \gamma) + j_1(C_T u) - j_1(C_T u')] \, dx \).

Then we shall require \( \mathcal{M} \) to satisfy

\[
\lim_{t \to 0} \frac{\mathcal{M}(t)}{t} = 0. \tag{16}
\]

Some illustrative examples for (H6) and a remark on (H7) are given next. If \( C_T = C = \nabla \), i.e., \( \gamma = 0 \), then (15) gives \( A_t = DT_t^{-T} \) and \( \mathcal{G} = 0 \). This gives the first relation in (H6). By applying the chain rule on \( \nabla (u \circ T_t^{-1}) \), we obtain

\[
\nabla (u \circ T_t^{-1}) = DT_t^{-T} \nabla u \circ T_t^{-1}. \tag{17}
\]

This gives the second relation in (H6). The third relation in (H6) is satisfied by \( A_t \) since

\[
\lim_{t \to 0} \frac{DT_t^{-T} - I}{t} = -Dh^T, \quad \text{and} \quad \lim_{t \to 0} DT_t^{-T} = I.
\]

In the next example, we consider the case where \( C_T = C = \text{div} \), i.e., \( \gamma = 0 \). For this purpose, we derive the transformation of the divergence operator in the following lemma.

**Lemma 2.1.** Suppose \( u_t \) and \( u'_t \) are related by (14), then

\[
(div u_t) \circ T_t = I_t^{-1}(div u'_t) + t \mathcal{G},
\]

where

\[
div u'_t = (\partial_x u'_1 + \partial_y u'_2) \text{ and } \mathcal{G} = I_t^{-1}(h_{2,y} \partial_x u'_1 + h_{1,z} \partial_x u'_2 - h_{2,z} \partial_y u'_1 - h_{1,y} \partial_y u'_2). \tag{19}
\]

**Proof.** By definition

\[
(div u) \circ T_t = (\partial_x u_1 + \partial_y u_2) \circ T_t = (\partial_x u_1) \circ T_t + (\partial_y u_2) \circ T_t.
\]

Using (15) we have

\[
\begin{pmatrix}
\partial_x u_{1,1} & \partial_x u_{1,2} \\
\partial_y u_{1,1} & \partial_y u_{1,2}
\end{pmatrix} \circ T_t = \frac{1}{I_t} \begin{pmatrix}
1 + th_{2,y} & -th_{2,x} \\
-th_{1,y} & 1 + th_{1,x}
\end{pmatrix} \begin{pmatrix}
\partial_x u'_1 & \partial_y u'_1 \\
\partial_x u'_2 & \partial_y u'_2
\end{pmatrix}. \tag{20}
\]
From (20) we have for the diagonal components

\[
I_i(\partial_i u_{i1}) \circ T = (1 + th_{2,y}) \partial_i u_{i1}' - th_{2,y} \partial_i u_{i1}',
\]
\[
I_i(\partial_i u_{i2}) \circ T = -th_{1,y} \partial_i u_{i2}' + (1 + th_{1,x}) \partial_i u_{i2}'.
\]

from which upon addition of both terms on the right hand side, one obtains

\[
I_i(\text{div } u) \circ T = (1 + th_{2,y}) \partial_i u_{i1}' - th_{2,y} \partial_i u_{i1}' - th_{1,y} \partial_i u_{i2}' + (1 + th_{1,x}) \partial_i u_{i2}' = \text{div } u' + t(h_{2,y} \partial_i u_{i1}' + h_{1,x} \partial_i u_{i2}' - h_{2,y} \partial_i u_{i1}' - h_{1,x} \partial_i u_{i2}').
\]

From Lemma 2.1, we note that \( A_t \) from (H6) is given by \( A_t = I_t^{-1} I \). For \( u \in X \), \( \text{div } u \in L^2(\Omega) \) by assumption, hence \( \mathcal{G} \) given in (19) is in \( L^2(\Omega) \). Moreover by (6), we have that \( \lim_{t \to 0} A_t - I = 0 \) and \( \lim_{t \to 0} \frac{A_t - I}{t} = -\text{div } h \) holds in \( L^\infty(\Omega) \). Since \( u_t = u' \circ T_t^{-1} \), one obtains \( \text{div } (u' \circ T_t^{-1}) = (I_t^{-1}(\text{div } u') + t \mathcal{G}) \circ T_t^{-1} \) from Lemma 2.1. Thus all conditions of assumption (H6) are satisfied by this transformation.

We now provide a remark on assumption (H7).

**Remark 2.1.** If we suppose that either \( \gamma = 0 \) in (H6) and \( j_1(t) = |t|^2 \) or \( \gamma \neq 0 \) in (H6) and \( j_1(t) = |t - \gamma|^2 \), then

\[
\mathcal{M}(t) = \int_\Omega I_t \left[ |(A_t C u' + t \mathcal{G})|^2 - |A_t C u + t \mathcal{G}|^2 + |C u|^2 - |C u'|^2 \right] dx, \quad \mathcal{M}(0) = 0.
\]

Using \( (a^2 - b^2) = (a + b)(a - b) \), we can express \( \mathcal{M} \) such that

\[
\frac{\mathcal{M}(t)}{t} = \int_\Omega I_t \left[ \frac{(A_t - I)}{t} (A_t + I) C(u' + u) + 2A_t \mathcal{G} \right] C(u' - u) \ dx.
\]

Note that \( I_t \) and \( I_t^{-1} \) can be expressed as

\[
I_t = I + t \text{div } h + t^2 \text{det } Dh \quad \text{and} \quad I_t^{-1} = I - t \text{div } h + t^2 \text{det } Dh,
\]

respectively. Hence for \( t \in \mathcal{J} \), \( A_t + I \) and \( \frac{A_t - I}{t} \) are bounded in \( L^\infty(\Omega) \). Moreover

\[
\left| \frac{\mathcal{M}(t)}{t} \right| \leq \int_\Omega \left[ I_t \left( \frac{(A_t - I)}{t} (A_t + I) C(u' + u) C(u' - u) \right) \ dx + 2 \int_\Omega I_t A_t \mathcal{G} C(u' - u) \ dx, \right.
\]

and

\[
E_1(t) \leq K_1 \left| I_t |\mathcal{L}|| (A_t - I) ||\mathcal{L}|| (A_t + I) ||\mathcal{L}|| |u' + u| \ |u' - u| \ |x|,
\]
\[
E_2(t) \leq K_2 \left| I_t |\mathcal{L}|| |\mathcal{L}|| A_t ||\mathcal{G}|| |\mathcal{G}|| |u' - u| \ |x|,
\]

for some generic constants \( K_1 \) and \( K_2 \). Hence by H2, we obtain \( \lim_{t \to 0} -E_i(t) = 0, i = 1, 2 \) and this leads to (16).
In what follows, the following lemmas shall be utilized.

**Lemma 2.2.** [5]

1. Let \( f \in C(\mathcal{J}, W^{1,1}(D)) \), and assume that \( f_t(0) \) exists in \( L^1(D) \). Then
   \[
   \frac{d}{dt} \int_{\Omega} f(t,x) \, dx|_{t=0} = \int_{\Omega} f_t(0,x) \, dx + \int_{\Gamma} f(0,x) h \cdot n \, ds.
   \]

2. Let \( f \in C(\mathcal{J}, W^{2,1}(D)) \), and assume that \( f_t(0) \) exists in \( W^{1,1}(D) \). Then
   \[
   \frac{d}{dt} \int_{\Gamma} f(t,x) \, dx|_{t=0} = \int_{\Gamma} f_t(0,x) \, ds + \int_{\Gamma} \left( \frac{\partial f(0,x)}{\partial n} + \kappa f(0,x) \right) h \cdot n \, ds,
   \]
   where \( \kappa \) stands for the mean curvature of \( \Gamma \).

The assumptions of Lemma 2.2 can be verified using the following Lemma

**Lemma 2.3.** [7, Chapter 2]

1. If \( u \in L^p(D) \), then \( t \mapsto u \circ T_t^{-1} \in C(\mathcal{J}, L^p(D)) \), \( 1 \leq p < \infty \).
2. If \( u \in H^2(D) \), then \( t \mapsto u \circ T_t^{-1} \in C(\mathcal{J}, H^2(D)) \).
3. If \( u \in H^2(D) \), then \( \frac{d}{dt}(u \circ T_t^{-1})|_{t=0} \) exists in \( H^1(D) \) and is given by
   \[
   \frac{d}{dt}(u \circ T_t^{-1})|_{t=0} = -(Du)h.
   \]

**Note 2.1.** As a consequence of Lemma 2.3, we note that \( \frac{d}{dt}(u \circ T_t^{-1})|_{t=0} \) exists in \( L^2(D) \) and is given by
   \[
   \frac{d}{dt}(u \circ T_t^{-1})|_{t=0} = -\nabla(Duh).
   \]

For the transformation of domain integrals, the following well known fact will be used repeatedly.

**Lemma 2.4.** Let \( \phi \in L^1(\Omega_t) \), then \( \phi \circ T_t \in L^1(\Omega) \) and
   \[
   \int_{\Omega_t} \phi_t \, dx_t = \int_{\Omega} (\phi \circ T_t) \, dx.
   \]

As a main result, we now formulate the representation of the Eulerian derivative of \( J \) in the following theorem.

**Theorem 2.1.** If \((H1-H7)\) hold, and \( j_1(C\gamma u) \in W^{1,1}(\Omega) \), then the Eulerian derivative of \( J \) in the direction \( h \in \mathcal{H} \) exists and is given by the expression
   \[
   dJ(\Omega)h = -\frac{d}{dt} \left( E(u,t), p \right)_{X \times X} |_{t=0} + \int_{\partial \Omega} j_1(C\gamma u)h \cdot n \, ds - \int_{\Omega} j_1(C\gamma u)C\gamma(\nabla u^T \cdot h) \, dx.
   \]

(21)
Proof. The Eulerian derivative of a cost functional \( J(u, \Omega) \) is defined by (8). Using Lemma 2.4 we obtain
\[
J(u_t, \Omega_t) - J(u, \Omega) = \int_\Omega j_1((C_p u_t) \circ T_t) - j_1(C_p u) \, dx,
\]
and by (H6)
\[
J(u_t, \Omega_t) - J(u, \Omega) = \int_\Omega I_t \left( j_1(A_t C u' + t \mathcal{G} + \gamma) - j_1(C_p u') \right) \, dx
+ \int_\Omega \left( I_t j_1(C_p u') - j_1(C_p u) \right) \, dx.
\]
The following estimate is obtained along the lines of [5]. We set
\[
R(t) = \int_\Omega I_t \left( j_1(C_p u') - j_1(C_p u) \right) \, dx, \quad R(0) = 0,
\]
\[
S(t) = \int_\Omega I_t \left( j_1(A_t C u' + t \mathcal{G} + \gamma) - j_1(C_p u') \right) \, dx, \quad S(0) = 0.
\]
Since \( C \) is a bounded linear operator, we have
\[
R(t) = \int_\Omega I_t \left[ j_1(C_p u') - j_1(C_p u) - \left( j_1'(C_p u), C(u' - u) \right) \right] \, dx +
\int_\Omega (I_t - 1) \left( j_1'(C_p u), C(u' - u) \right) \, dx + \int_\Omega \left( j_1'(C_p u), C(u' - u) \right) \, dx
+ \int_\Omega (I_t - 1) j_1(C_p u) \, dx.
\]
We express \( R(t) = R_1(t) + R_2(t) + R_3(t) + R_4(t) \). Using (H2) and (H5), we have that
\[
\lim_{t \to 0} \frac{1}{t} R_1(t) = 0.\]
Moreover using H5 and similar arguments as in Remark 2.1, we have
\[
\left| \frac{R_2(t)}{t} \right| \leq \left\| \frac{I_t - 1}{t} \right\|_{L^2} \left\| j_1'(C_p u) \right\|_{L^2} \left\| u' - u \right\|_X.
\]
Therefore, by (H2) and (6), one obtains \( \lim_{t \to 0} \frac{1}{t} R_2(t) = 0 \). Next observe that using (13) with \( \psi = u' - u \in X \), we have that
\[
R_3(t) = \left( j_1'(C_p u), C(u' - u) \right) = \left( C^* j_1'(C_p u), (u' - u) \right) = \langle E_u(u, \Omega)(u' - u), p \rangle_{X^* \times X}.
\]
In order to bypass the computation of the shape derivative of \( u \), we arrange terms on the right hand side of (22) in an efficient manner to obtain
\[
\langle E_u(u, \Omega)(u' - u), p \rangle_{X^* \times X} = -\langle E(u, t) - E(u, 0), p \rangle_{X^* \times X}
- \langle E(u', \Omega) - E(u, \Omega) - E_u(u, \Omega)(u' - u), p \rangle_{X^* \times X}
- \langle \hat{E}(u', t) - \hat{E}(u, t) - E(u', \Omega) + E(u, \Omega), p \rangle_{X^* \times X}.
\]
By using assumptions (H2), (H3) and (H4), we have that
\[
-\lim_{t \to 0} \frac{1}{t} \langle E(u', \Omega) - E(u, \Omega) - E_u(u, \Omega)(u' - u), p \rangle_{X^* \times X} = 0.
\]
and

\[- \lim_{t \to 0} \frac{1}{t} (E(u', t) - E(u, t) - E(u', \Omega) + E(u, \Omega), p)_{X^* \times X} = 0.\]

Consequently utilizing (H1), we obtain

\[
\lim_{t \to 0} \frac{R_3(t)}{t} = - \frac{d}{dt} (E(u, t), p)_{X^* \times X}|_{t=0}. \tag{23}
\]

We shall turn our attention to \( R_4(t) \) later. Now let us focus on

\[
S(t) = \int_{\Omega} I_t \left( j_1 (A_t Cu + t\mathcal{G} + \gamma) - j_1 (C_t \mu') \right) dx,
\]

and consider the expression

\[
j_1 (A_t Cu + t\mathcal{G} + \gamma) - j_1 (C_t \mu').
\]

This can be written as

\[
j_1 (A_t Cu + t\mathcal{G} + \gamma) - j_1 (A_t Cu + t\mathcal{G} + \gamma) + j_1 (C_t \mu) - j_1 (C_t \mu') + j_1 (A_t Cu + t\mathcal{G} + \gamma) - j_1 (C_t \mu).
\]

Observe that

\[
S(t) = \int_{\Omega} I_t \left( j_1 (A_t Cu + t\mathcal{G} + \gamma) - j_1 (C_t \mu) \right) dx + \mathcal{M}(t). \tag{24}
\]

Expressing \( S(t) = S_1(t) + \mathcal{M}(t) \), where \( S_1(t) \) is the first term in (24). Using (H7), we have that \( \lim_{t \to 0} \frac{\mathcal{M}(t)}{t} = 0 \). Therefore collecting the remaining terms, i.e., \( R_4(t) \) and \( S_1(t) \) into \( S_5(t) := R_4(t) + S_1(t) \), we have that

\[
S_5(t) = \int_{\Omega} I_t j_1 (A_t Cu + t\mathcal{G} + \gamma) - j_1 (C_t \mu) dx, \quad S_5(0) = 0.
\]

Using Lemma 2.4, we can express \( S_5 \) as

\[
S_5(t) = \int_{\Omega} j_1 (A_t Cu + t\mathcal{G} + \gamma) \circ T_t \) dx - \int_{\Omega} j_1 (C_t \mu) dx. \tag{25}
\]

By H6, (25) can further be expressed as

\[
S_5(t) = \int_{\Omega} j_1 (C_t (u \circ T_t^{-1})) \) dx - \int_{\Omega} j_1 (C_t \mu) dx. \tag{26}
\]

By definition of Eulerian derivative, we have that

\[
\lim_{t \to 0} \frac{S_5(t)}{t} = \frac{d}{dt} \int_{\Omega} j_1 (C_t (u \circ T_t^{-1})) \) dx|_{t=0}.
\]
Since by assumption \( j_1(C_\gamma u) \in W^{1,1}(\Omega) \), \( \frac{d}{dt} \left[ j_1(C_\gamma(u \circ T_{-t}^{-1})) \right] \big|_{t=0} \) exists in \( L^1(\Omega) \) [7]. Therefore, using Lemma 2.2 and Lemma 2.3, we have that
\[
\lim_{t \to 0} \frac{S_5(t)}{t} = \int_{\Omega} j_1(C_\gamma u) \cdot \nabla u \, dx + \int_{\partial \Omega} j_1(C_\gamma u) \cdot \mathbf{n} \, ds.
\]
Hence
\[
dJ(u, \Omega) = \lim_{t \to 0} \frac{R(t) + S(t)}{t} = -\frac{d}{dt} \langle E(u, t), p \rangle_{X^* \times X} \big|_{t=0} + \int_{\partial \Omega} j_1(C_\gamma u) \cdot \mathbf{n} \, ds
\]
\[
-\int_{\Omega} j_1'(C_\gamma u) C_\gamma \nabla u \cdot \mathbf{h} \, dx.
\]

\section{Examples}

As an application of the general theory developed in the previous section, we derive the shape derivatives of cost functionals used for vortex reduction in fluid dynamics. Here we restrict ourselves to the 2D case. Typical cost functionals used for this purpose, are based on minimization of the curl of the velocity field or tracking-type functionals, [1], i.e.,
\[
J_1(u, \Omega) = \frac{1}{2} \int_{\Omega} |\nabla \times u(x)|^2 \, dx,
\]
\[
J_2(u, \Omega) = \frac{1}{2} \int_{\Omega} |Au(x) - u_d(x)|^2 \, dx, \quad A \in \mathbb{R}^{2 \times 2}, \tag{27}
\]
where \( u_d \) stands for a given desired flow field which contains some of the expected features of the controlled flow field without the undesired vortices. Furthermore,
\[
J_3(u, \Omega) = \int_{\Omega} g_3(\det \nabla u) \, dx, \tag{28}
\]
where
\[
g_3(t) = \begin{cases} 0 & t \leq 0, \\ \frac{t^3}{t^2 + 1} & t > 0, \end{cases}
\]
penalizes the complex eigen values of \( \nabla u \) which are responsible for the swirling motion in a given flow (see, e.g.,[4] and references there in). The rigorous characterization of the shape derivative of this functional has not been done before, and therefore it is of a big interest in this work. In (27-28), \( u \) represents the state variable that solves the Navier-Stokes equations
\[
\begin{cases}
-\eta \Delta u + (u \cdot \nabla) u + \nabla p = f \text{ in } \Omega, \\
\text{div } u = 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{cases} \tag{29}
\]
Here $\eta > 0$, denotes the kinematic viscosity of the fluid, $f \in L^2(\Omega)$ is the external body force, $p$ the pressure, and with reference to Figure 1, $\Omega = S$ with $\Gamma = \partial S$. Using the notation of the previous section, we observe that $E(u, \Omega) = 0$ is given by system (29).

We define the following functional spaces for velocity and pressure, respectively:

$$H_0^1(\Omega) = \{ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma \},$$

$$L_0^2(\Omega) = \{ q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0 \}.$$

The variational formulation of (29) is given by: Find $(u, p) \in X \equiv H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\langle E((u, p), \Omega), (\psi, \xi) \rangle_{X^* \times X} \equiv \eta(\nabla u, \nabla \psi)_{\Omega} + ((u \cdot \nabla) u, \psi)_{\Omega} - (p, \nabla \psi)_{\Omega} - (f, \psi)_{\Omega} - (\nabla \psi_0, \psi)_{\Omega} = 0, \quad (30)$$

holds for all $(\psi, \xi) \in X$. It is well known that for sufficiently large values of $\eta$ or for small values of $f$, there exists a unique solution $(u, p)$ to (30) in $X$. Moreover, since $\partial \Omega \in C^2$, $(u, p) \in \left( H^2(\Omega) \cap H_0^1(\Omega) \right) \times \left( H^1(\Omega) \cap L_0^2(\Omega) \right)$ [8]. On $\Omega$, the perturbed weak formulation of (29) reads:

Find $(u_t, p_t) \in X_t \equiv H_0^1(\Omega_t) \times L_0^2(\Omega_t)$ such that

$$\langle E((u_t, p_t), \Omega_t), (\psi_t, \xi_t) \rangle_{X_t^* \times X_t} \equiv \eta(\nabla u_t, \nabla \psi_t)_{\Omega_t} + ((u_t \cdot \nabla) u_t, \psi_t)_{\Omega_t} - (p_t, \nabla \psi_t)_{\Omega_t} - (f_t, \psi_t)_{\Omega_t} - (\nabla \psi_0, \psi_t)_{\Omega_t} = 0, \quad (31)$$

holds for all $(\psi_t, \xi_t) \in X_t$. Using the summation convention, the transformation of the divergence [5] is given by

$$(\nabla \psi_t) \circ T_t = Dp_t^\top e_i = (A_t)_i \nabla \psi_t,$$

where $e_i$ stands for the $i$-th canonical basis vector in $\mathbb{R}^d$ and $(A_t)_i$ denotes the $i$-th row of $A_t = (DT_t)^{-T}$. Thus using (15) the transformation of (31) back to $\Omega$ becomes,

$$\langle E((u', p'), t), (\psi, \xi) \rangle_{X^* \times X} \equiv \eta(l_i A_t \nabla' u', A_t \nabla' \psi)_{\Omega} + ((u' \cdot A_t \nabla) u', l_i \psi)_{\Omega} - (p', l_i A_t \nabla' \psi)_{\Omega} - (f', l_i \psi)_{\Omega} - (l_i (A_t)_k \nabla' \psi_0, \xi)_{\Omega} = 0 \text{ for all } (\psi, \xi) \in X.$$  

### 3.1 The Eulerian derivative of cost functional $J_1$

For this cost functional, the operator $C_\gamma = C = (\text{curl}, 0)$ and $C_\gamma' = (\text{curl}, 0)$ with $\gamma = 0$. Moreover it is easy to check that $C \in \mathcal{L}(X, L^2)$. Furthermore, since $u \in H_0^1(\Omega)$ we have that $\text{curl } u \in L^2(\Omega, \mathbb{R}^2)$ and therefore

$$u \in H(\text{curl}, \Omega) := \{ u \in L^2(\Omega, \mathbb{R}^2) : \text{curl } u \in L^2(\Omega, \mathbb{R}) \}.$$

Hence the cost functional $J_1(u, \Omega)$ is well defined. The adjoint state $(\lambda, q) \in X$ is given as a solution to

$$\langle E'(((u, p), \Omega))(\psi, \xi), (\lambda, q) \rangle_{X^* \times X} = (\text{curl}(\text{curl } u), \psi)_{\Omega}.$$
with right hand side $\text{curl}(\text{curl } u) = -\Delta u$, which amounts to

$$\eta(\nabla \psi, \nabla \lambda) + ((\psi \cdot \nabla) u + (u \cdot \nabla) \psi, \lambda)_{\Omega} - (\xi, \text{div } \lambda)_{\Omega} - (\text{div } \psi, q)_{\Omega} = (-\Delta u, \psi)_{\Omega}. \tag{33}$$

Integrating $((u \cdot \nabla) \psi, \lambda)_{\Omega}$ by parts, one obtains the strong form of the adjoint equation in (33), that we express as

$$\begin{cases}
-\eta \Delta \lambda + (\nabla u) \cdot \lambda - (u \cdot \nabla) \lambda + \nabla q = -\Delta u & \text{in } \Omega, \\
\text{div } \lambda = 0 & \text{in } \Omega, \\
\lambda = 0 & \text{on } \partial \Omega, 
\end{cases} \tag{34}$$

where the first equation holds in $L^2(\Omega)$ and the second one in $L^2(\Omega)$. It is well known that there exists a unique solution $(\lambda, q) \in X$. Moreover, since $\partial \Omega \in C^2$, $(\lambda, q) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^1(\Omega) \cap L^2(\Omega))$, (see e.g [5] and references there in). In view of Theorem 2.1 we have to compute $\frac{d}{dt}(E((u, p), t), (\lambda, q)|_{X \times X}|_{t=0}$, for which we use the representation on $\Omega_t$ of (32). This writes

$$\langle E((u, p), t), (\lambda, q) \rangle_{X \times X} \equiv \eta(\nabla u \circ T^{-1}_t, \nabla \lambda \circ T^{-1}_t)_{\Omega_t} + (u \circ T^{-1}_t, \nabla \lambda \circ T^{-1}_t)_{\Omega_t} - (p \circ T^{-1}_t, \text{div } \lambda \circ T^{-1}_t)_{\Omega_t} - (\lambda \circ T^{-1}_t, q \circ T^{-1}_t)_{\Omega_t}, \tag{35}$$

where $(u, p), (\lambda, q) \in X$ are solutions of (29) and (33), respectively. The computation of $\frac{d}{dt}(E((u, p), t), (\lambda, q)|_{X \times X}|_{t=0}$, results in

$$\frac{d}{dt}(E((u, p), t), (\lambda, q))_{X \times X}|_{t=0} = (-\eta \Delta u + (u \cdot \nabla) u + \nabla p - f, \psi_\lambda)_{\Omega_t} + \eta(\nabla u \cdot n, \psi_\lambda)_{\partial \Omega_t} - (p \psi_\lambda \cdot n)_{\partial \Omega_t} + (-\eta \Delta \lambda + (\nabla u) \lambda - (u \cdot \nabla) \lambda + \nabla q, \psi_\mu)_{\Omega_t} + \eta(\psi_\mu, \nabla \lambda \cdot n)_{\partial \Omega_t} - (q \cdot n, \psi_\mu)_{\partial \Omega_t} + \eta \int_{\partial \Omega_t} (\nabla u, \nabla \lambda) h \cdot n \, ds, \tag{36}$$

where $\psi_\mu = -\nabla u^T \cdot h \in H^1(\Omega)$ and $\psi_\lambda = -\nabla \lambda^T \cdot h \in H^1(\Omega)$, with $h \in \mathcal{H}$. By using (34), the expression on the right hand side of (36) can further be simplified to obtain (37), (see [5] for more details).

$$\frac{d}{dt}(E((u, p), t), (\lambda, q))_{X \times X}|_{t=0} = -\int_{\partial \Omega} \left[ \eta \frac{\partial u}{\partial n} \frac{\partial \lambda}{\partial n} \right] h \cdot n \, ds + \int_{\partial \Omega} \left( p \frac{\partial \lambda}{\partial n} \cdot n + q \frac{\partial u}{\partial n} \cdot n \right) h \cdot n \, ds + \int_\Omega (\Delta u) \nabla u^T \cdot h \, dx. \tag{37}$$

Using the definition of tangential divergence (7), we have that:

$$p(\frac{\partial \lambda}{\partial n} \cdot n) = p(\nabla \lambda^T \cdot n) \cdot n = p \text{ div } \lambda|_{\partial \Omega} - p \text{ div } \lambda|_{\partial \Omega}. \tag{38}$$
Since \( \lambda = 0 \) on \( \partial \Omega \), the last term in (38) vanishes (see [7] Page 82 for details). Furthermore \( \text{div} \ \lambda = 0 \) which renders this expression to be zero. Analogously, \( q(\frac{\partial u}{\partial n}, n) = 0 \). Thus

\[
\frac{d}{dt} \langle E((u, p), t), (\lambda, q) \rangle_{X', X} = - \int_{\partial \Omega} \frac{\partial u}{\partial n} \frac{\partial \lambda}{\partial n} h \cdot n \, ds + \int_{\Omega} (\Delta u) \nabla u' \cdot h \, dx. \tag{39}
\]

In view of Theorem 2.1, we further need to show that assumptions (H1-H7) hold, and moreover that \( |\text{curl } u|^2 \in W^{1,1}(\Omega) \). Assumptions (H1-H4) were verified in [5]. To check (H5), note that

\[
\frac{1}{2} |\text{curl } u'|^2 - \frac{1}{2} |\text{curl } u|^2 - (\text{curl } u, \text{curl}(u' - u)) = \frac{1}{2} (\text{curl}(u' - u))^2.
\]

Hence

\[
\int \frac{1}{2} |\text{curl } u'|^2 - \frac{1}{2} |\text{curl } u|^2 - (\text{curl } u, \text{curl}(u' - u)) \, dx = \int \frac{1}{2} (\text{curl}(u' - u))^2 \, dx.
\]

Consequently by Young’s inequality, we have

\[
\left| \int \frac{1}{2} (\text{curl}(u' - u))^2 \, dx \right| \leq \max_{t_0 \in [0, T_0]} ||I_t||_{L^2} ||u' - u||^2_{H^2},
\]

for \( t_0 \) sufficiently small. Hence (H5) is satisfied with \( K = \max_{t_0 \in [0, T_0]} ||I_t||_{L^2} \).

Condition (H6) is checked next. It illustrates the choice of \( C_T \) for the present example.

**Lemma 3.1.** Suppose \( u \) and \( u' \) are related by (14), then

\[
(\text{curl } u) \circ T_t = I_t^{-1}(\text{curl } u') + t\mathcal{G},
\]

where

\[
\mathcal{G} = I_t^{-1}(h_{2,y} \partial_y u_2' - h_{2,x} \partial_x u_2' + h_{1,y} \partial_y u_1' - h_{1,x} \partial_x u_1'). \tag{41}
\]

**Proof.** By definition

\[
(\text{curl } u) \circ T_t = (\partial_y u_2 - \partial_x u_1) \circ T_t = (\partial_y u_2) \circ T_t - (\partial_x u_1) \circ T_t.
\]

From (20) we have for the non-diagonal components

\[
\begin{align*}
I_t(\partial_y u_{2,y}) \circ T_t &= (1 + th_{2,y}) \partial_y u_2' - th_{2,y} \partial_x u_2', \\
I_t(\partial_y u_{2,x}) \circ T_t &= -th_{2,y} \partial_y u_2' + (1 + th_{1,x}) \partial_x u_2',
\end{align*}
\]

from which we obtain that

\[
\begin{align*}
I_t(\text{curl } u_t) \circ T_t &= (1 + th_{2,y}) \partial_y u_2' - th_{2,y} \partial_x u_2' + th_{1,y} \partial_y u_1' - (1 + th_{1,x}) \partial_x u_1' \\
&= \text{curl } u' + t(h_{2,y} \partial_y u_2' - h_{2,x} \partial_x u_2' + h_{1,y} \partial_y u_1' - h_{1,x} \partial_x u_1').
\end{align*}
\]

Thus \( (\text{curl } u_t) \circ T_t = I_t^{-1}(\text{curl } u') + t\mathcal{G} \). \( \square \)
From Lemma 3.1, we observe that \( A_t \) from (H6) is given by \( A_t = I - I_t^{-1} I \). Since \( \mathbf{u} \in H^s_0(\Omega) \), \( H \) given in (41) belongs to the Sobolev space \( L^2(\Omega) \). Moreover by (6), we have that \( \lim_{t \to 0} A_t - I = 0 \) and \( \lim_{t \to 0} \frac{A_t - I}{t} = -\text{div} \mathbf{h} \). Since \( \mathbf{u}_t = \mathbf{u}' \circ T_t^{-1} \), one obtains \( \text{curl} (\mathbf{u}' \circ T_t^{-1}) = \left( T_t^{-1}(\text{curl} \mathbf{u}') + tH \right) \circ T_t^{-1} \) from Lemma 3.1. Thus all conditions of assumption (H6) are satisfied by this transformation.

Cost functional \( J_1 \) satisfies the conditions of Remark 2.1 and therefore (H7) holds. In addition, since \( \mathbf{u} \in H^2(\Omega) \), it follows that \( \nabla \text{curl} \mathbf{u} \in L^2(\Omega) \). Therefore, we infer that \( \nabla \text{curl} \mathbf{u}^2 = 2\text{curl} \mathbf{u} \nabla \text{curl} \mathbf{u} \in L^1(\Omega) \). Consequently \( \text{curl} \mathbf{u}^2 \in W^{1,1}(\Omega) \). Since all assumptions of Theorem 2.1 are satisfied, using (21) and (39), we can express the Eulerian derivative of \( J_1 \) as

\[
dJ_1(\mathbf{u}, \Omega)\mathbf{h} = \int_{\partial \Omega} \left[ \eta \frac{\partial \mathbf{u}}{\partial n} \frac{\partial \lambda}{\partial n} \right] \mathbf{n} \cdot \mathbf{h} \, ds - \int_{\Omega} (\Delta \mathbf{u}) \nabla (\mathbf{u}^T \cdot \mathbf{h}) \, dx + \frac{1}{2} \int_{\partial \Omega} |\text{curl} \mathbf{u}|^2 \mathbf{h} \cdot \mathbf{n} \, ds - \int_{\Omega} \text{curl} \mathbf{u} \nabla (\mathbf{u}^T \cdot \mathbf{h}) \, dx.
\]

We want to express (42) into the form required by the Zolesio-Hadamard structure theorem (10). With this in mind, sufficient regularity of \( \mathbf{u} \) together with Greens formula for the curl, i.e.,

\[
\int_{\Omega} \left[ \text{curl} \mathbf{u} \text{curl} (\nabla \mathbf{u}^T \cdot \mathbf{h}) - (\Delta \mathbf{u}) \nabla (\mathbf{u}^T \cdot \mathbf{h}) \right] \, dx = \int_{\partial \Omega} (\text{curl} \mathbf{u}) \tau \cdot (\nabla \mathbf{u}^T \cdot \mathbf{h}) \, ds,
\]

leads to

\[
dJ_1(\mathbf{u}, \Omega)\mathbf{h} = \int_{\partial \Omega} \left[ \eta \frac{\partial \mathbf{u}}{\partial n} \frac{\partial \lambda}{\partial n} + \frac{1}{2} |\text{curl} \mathbf{u}|^2 - (\text{curl} \mathbf{u}) \tau \cdot \frac{\partial \mathbf{u}}{\partial n} \right] \mathbf{n} \cdot \mathbf{h} \, ds.
\]

### 3.2 The Eulerian derivative of cost functional \( J_2 \)

In this example we define the operator \( C_\gamma : \mathbf{u}(x) \mapsto A \mathbf{u} - \mathbf{u}_d \in L^2(\Omega) \) with \( \gamma = -\mathbf{u}_d \in L^2(\Omega) \). The linear operator \( C \in \mathcal{L}(X, L^2(\Omega)) \) is such that \( C : \mathbf{u}(\cdot) \mapsto A \mathbf{u}(\cdot) \). Furthermore, since \( \mathbf{u} \in H^s_0(\Omega) \), we have that \( A \mathbf{u} - \mathbf{u}_d \in L^2(\Omega) \). Hence the cost functional \( J_2(\mathbf{u}, \Omega) \) is well defined. For this case the adjoint state \((\lambda, q) \in X\), is given as a solution to

\[
\langle E'(\mathbf{u}, p) \Omega | (\psi, \xi), (\lambda, q) \rangle_{X^* \times X} = \langle (A \mathbf{u} - \mathbf{u}_d), \psi \rangle_{\Omega},
\]

which amounts to

\[
\begin{align*}
-\eta \Delta \lambda + (\nabla \mathbf{u}) \cdot \lambda - (\mathbf{u} \cdot \nabla) \lambda + \nabla q &= (A \mathbf{u} - \mathbf{u}_d), & \text{in } \Omega, \\
\text{div} \lambda &= 0 & \text{in } \Omega, \\
\lambda &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where the first equation in (44) holds in \( L^2(\Omega) \) and the second one in \( L^2(\Omega) \).

**Theorem 3.1.** The shape derivative of the cost functional \( J_2(\mathbf{u}, \Omega) \) can be expressed as

\[
dJ_2(\mathbf{u}, \Omega)\mathbf{h} = \int_{\partial \Omega} \left[ \eta \frac{\partial \mathbf{u}}{\partial n} \frac{\partial \lambda}{\partial n} + \frac{1}{2} (A \mathbf{u} - \mathbf{u}_d)^2 \right] \mathbf{n} \cdot \mathbf{h} \, ds.
\]
Proof. We want to make use of Theorem 2.1 to derive (45). For this purpose, we remark that by using (36), (38) and (44), one obtains

$$\frac{d}{dt} \langle \tilde{E}, (\lambda, q) \rangle_{X \times X} \big|_{t=0} = -\int_\Omega \eta \frac{\partial u}{\partial n} \cdot \frac{\partial \lambda}{\partial n} \, ds - \int_\Omega (Au - ud) A^T (\nabla u^T) h \, dx,$$

where \((\lambda, q)\) solves (44). Furthermore, we need to show that assumptions (H1-H7) of Theorem 2.1 hold, and moreover that \(|Au - ud|^2 \in W^{1,1}(\Omega)\). As stated earlier, assumptions (H1-H4) were verified in [5]. To check (H5), note that

$$\frac{1}{2} |Au - ud|^2 - \frac{1}{2} |Au - ud|^2 - (Au - ud, A(u' - u)) = \frac{1}{2} (A(u' - u))^2.$$ 

Consequently by Young’s inequality, we have

$$\left| \int_\Omega \frac{1}{2} (A(u' - u))^2 \, dx \right| \leq 4a \max_{t \in [0, t_0]} |I_t| |||u'||||_2^2, \quad a = \max_{i,j} |a_{i,j}|, \; i, j = 1, 2.$$

Hence (H5) is satisfied with \(K = 4a \max_{t \in [0, t_0]} |I_t| |||u'||||_2^2\), for \(t_0\) sufficiently small.

Note that the transformation \((C_y u_t) \circ T_t = Cu - ud\) implies that \(G = 0\) in (H6). Furthermore \(A_t\) is given by \(A_t = I\) and \(\lim_{t \to 0} \frac{A_t - I}{t} = 0\). Moreover \(C_y (u' \circ T_t^{-1}) = (Cu - ud) \circ T_t^{-1}\). Hence all conditions of (H6) are satisfied. Note that \(J_2\) satisfies conditions of Remark 2.1 and hence (H7) holds. It is also clear that \(|Au - ud|^2 \in W^{1,1}\) since \(u \in H^2(\Omega)\). The preceding discussion shows that assumptions (H1-H7) are satisfied. Therefore using Theorem 2.1 together with the fact that \(J_2(C_u, C(\nabla u^T h))_{\Omega} = (Au - ud, A^T (\nabla u^T) h)_{\Omega}\), one obtains

$$dJ_2(u, \Omega) h = \int_{\partial \Omega} \left[ \eta \frac{\partial u}{\partial n} \frac{\partial \lambda}{\partial n} + \frac{1}{2} (Au - ud)^2 \right] h \cdot n \, ds.$$  \hspace{1cm} (46)

\[\square\]

3.3 The Eulerian derivative of cost functional \(J_3\)

First note that \(J_3(u, \Omega)\) is well defined. In fact, for \(u \in H_0^1(\Omega)\), we have \(\det \nabla u \in L^1(\Omega)\). Moreover \(0 \leq \frac{t^2}{t^2 + 1} \leq t\) for \(t \geq 0\), hence \(g_3(\det \nabla u)\) is integrable. Furthermore for \(\delta u \in H_0^1(\Omega)\), there exists the directional derivative \(J_3'(u, \Omega)(\delta u)\) given by

$$J_3'(u, \Omega)(\delta u) = \int_{\Omega} g_3'(\det \nabla u)(\det \nabla u)' \delta u \, dx,$$  \hspace{1cm} (47)

where

$$\langle \det \nabla u \rangle ' \delta u = \left( u_i^1 \delta u_i^1 + \delta u_i^1 u_i^1 - u_i^2 \delta u_i^2 - u_i^1 \delta u_i^2 \right) \text{ and } g_3'(t) = \begin{cases} 0 & t \leq 0, \\ \frac{t^2 + 3t^2}{t^2 + 2t + 1} & t > 0. \end{cases}$$

Where appropriate, we shall use the short form notation \(g_3'(\det \nabla u)\) to represent \(g_3'(\det \nabla u)\) in what follows.
Lemma 3.2. The directional derivative $J'_3(\mathbf{u}, \Omega)(\delta \mathbf{u})$ can be expressed in the form

$$J'_3(\mathbf{u}, \Omega)(\delta \mathbf{u}) = \int_{\Omega} T(\mathbf{u})(\delta \mathbf{u}) \, dx + \int_{\partial \Omega} P(\mathbf{u})(\delta \mathbf{u}) \, ds,$$

where

$$T(\mathbf{u}) = \begin{pmatrix} -\operatorname{curl}(g_3' \nabla u_2) \\ \operatorname{curl}(g_3' \nabla u_1) \end{pmatrix} \quad \text{and} \quad P(\mathbf{u}) = \begin{pmatrix} g_3'(\operatorname{det} \nabla \mathbf{u}) \left( \frac{\partial u_2}{\partial y} n_x - \frac{\partial u_1}{\partial x} n_y \right) \\ g_3'(\operatorname{det} \nabla \mathbf{u}) \left( \frac{\partial u_2}{\partial y} n_x - \frac{\partial u_1}{\partial x} n_y \right) \end{pmatrix}.$$

Proof. Integrating each term in (47) by parts, we obtain

$$\int_{\Omega} g_3' \frac{\partial u_1}{\partial x} \frac{\partial (\delta u_2)}{\partial y} dx = -\int_{\Omega} g_3' \frac{\partial u_2}{\partial x} (\delta u_1)n_x \, dx - \int_{\Omega} \frac{\partial}{\partial x} \left( g_3' \frac{\partial u_1}{\partial y} \right) \delta u_2 dx,$$

$$\int_{\Omega} g_3' \frac{\partial (\delta u_2)}{\partial x} \frac{\partial u_1}{\partial y} dx = \int_{\Omega} g_3' \frac{\partial u_2}{\partial x} (\delta u_1)n_x \, dx - \int_{\Omega} \frac{\partial}{\partial y} \left( g_3' \frac{\partial u_1}{\partial y} \right) \delta u_2 dx,$$

$$\int_{\Omega} -g_3' \frac{\partial (\delta u_2)}{\partial x} \frac{\partial u_2}{\partial y} dx = -\int_{\Omega} g_3' \frac{\partial u_2}{\partial x} (\delta u_1)n_x \, dx + \int_{\Omega} \frac{\partial}{\partial y} \left( g_3' \frac{\partial u_1}{\partial y} \right) \delta u_2 dx,$$

Summing up the right hand sides of the terms in the above expressions gives the desired result.

The adjoint state $(\lambda, q) \in X$ is given as a solution to

$$(E'(\mathbf{u}, p), \Omega)(\psi, \xi), (\lambda, q))_{X' \times X} = (g_3'(\operatorname{det} \nabla \mathbf{u}), (\det \nabla \mathbf{u})'\psi)_\Omega, \quad (48)$$

which by Lemma 3.2 amounts to

$$\begin{cases} -\eta \Delta \lambda + (\nabla \mathbf{u}) \cdot \lambda - (\mathbf{u} \cdot \nabla) \lambda + \nabla q = T(\mathbf{u}), & \text{in } \Omega, \\
\operatorname{div} \lambda = 0, & \text{in } \Omega, \\
\lambda = 0, & \text{on } \partial \Omega, \end{cases} \quad (49)$$

where the first equation in (49) hold in $L^2(\Omega)$ and the second one in $L^2(\Omega)$. Moreover, since $\partial \Omega \in C^2$, $(\lambda, q) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times (H^1(\Omega) \cap L^2(\Omega))$. Let us note that Theorem 2.1 is not directly applicable to compute the shape derivative of $J_3$ since the operator ” det $\nabla$ ” in the functional $J_3$ in (28) is not affine. We therefore give an independent proof following the lines of the proof of Theorem 2.1. Firstly, we state and prove the following lemma that will become important in what follows.

Lemma 3.3. Suppose $\mathbf{u}$ and $\mathbf{u}'$ are related by (14). Then

$$(\det \nabla \mathbf{u}) \circ T = I_1^{-1}(\det \nabla \mathbf{u}') + t\mathcal{G}_1 + t^2\mathcal{G}_2, \quad (50)$$

where

$$\det \nabla \mathbf{u}' = (\partial_i u'_j \partial_j u'_2 - \partial_i u'_2 \partial_j u'_1),$$

$$\mathcal{G}_1 = I_1^{-1}(E_2 \partial_i u'_1 + E_i \partial_j u'_2 - E_4 \partial_i u'_2 - E_3 \partial_j u'_1) \in L^1(\Omega),$$

$$\mathcal{G}_2 = I_1^{-1}(E_4 E_2 - E_3 E_4) \in L^1(\Omega),$$
and $E_1 = h_{2,2} \partial_3 u_2 - h_{2,2} \partial_3 u_2$, $E_2 = h_{1,1} \partial_3 u_1 - h_{1,1} \partial_3 u_1$, $E_3 = h_{2,2} \partial_3 u_1 - h_{2,2} \partial_3 u_1$, and $E_4 = h_{1,1} \partial_3 u_2 - h_{1,1} \partial_3 u_2$.

Proof. By definition
\[
(\det \nabla u) \circ T_t = (\partial_x u_1 \partial_y u_2 - \partial_x u_2 \partial_y u_1) \circ T_t.
\]

From (20) we have
\[
(\partial_x u_{1,2}) \circ T_t = I_t^{-1} \partial_x u_2 + t I_t^{-1} E_1, \quad (\partial_x u_{1,1}) \circ T_t = I_t^{-1} \partial_x u_1 + t I_t^{-1} E_2,
\]
\[
(\partial_x u_{2,1}) \circ T_t = I_t^{-1} \partial_x u_1 + t I_t^{-1} E_3, \quad (\partial_x u_{2,2}) \circ T_t = I_t^{-1} \partial_x u_2 + t I_t^{-1} E_4.
\]

From the above equations, we obtain
\[
(\det \nabla u) \circ T_t = I_t^{-1} (\det \nabla u') + t \mathcal{G}_1 + t^2 \mathcal{G}_2.
\]

Note that for $u \in X$, $\mathcal{G}_1, \mathcal{G}_2 \in L^1(\Omega)$, and this concludes the proof.

Proposition 3.1. Assume that $f \in L^p(\Omega)$, $p > 2 = \text{dimension}$. If (H1-H4) hold, and $g_3(\det \nabla u) \in W^{1,1}(\Omega)$, then the Eulerian derivative of $J_3(u, \Omega)$ exists and is given by the expression
\[
dJ_3(u, \Omega) = \int_{\partial \Omega} \left( n \frac{\partial u}{\partial n} \frac{\partial \lambda}{\partial n} + g_3(\det \nabla u) - P(u) \frac{\partial u}{\partial n} \right) h \cdot n \ ds.
\]

Proof. As stated earlier, assumptions (H1-H4) were verified in [5]. Using (8), we have
\[
J_3(u, \Omega) = J_3(u, \Omega) = \int_{\Omega} g_3(\det \nabla u') \ dx - \int_{\Omega} g_3(\det \nabla u) \ dx,
\]
\[
= \int_{\Omega} I_t g_3((\det \nabla u) \circ T_t) \ dx - \int_{\Omega} g_3(\det \nabla u) \ dx.
\]

Let $\mathcal{G}_3 = \mathcal{G}_1 + t \mathcal{G}_2$. Using equation (50), we can express (52) as
\[
J_3(u, \Omega) - J_3(u, \Omega) = \int_{\Omega} I_t g_3(I_t^{-1} (\det \nabla u') + t \mathcal{G}_3) \ dx - \int_{\Omega} g_3(\det \nabla u) \ dx.
\]

The right hand side of (53) can be written as $R(t) + S(t)$, where
\[
R(t) = \int_{\Omega} I_t g_3(I_t^{-1} (\det \nabla u') - g_3(\det \nabla u) \ dx, \quad R(0) = 0,
\]
\[
S(t) = \int_{\Omega} I_t \left( g_3(I_t^{-1} (\det \nabla u') + t \mathcal{G}_3) - g_3(\det \nabla u') \right) \ dx, \quad S(0) = 0.
\]

$R(t)$ can be re-written as
\[
R(t) = \int_{\Omega} I_t \left( g_3(\det \nabla u') - g_3(\det \nabla u) - g_3(\det \nabla u'(u' - u)) \right) \ dx + \int_{\Omega} (I_t - 1) g_3(\det \nabla u)(\det \nabla u'(u' - u)) \ dx + \int_{\Omega} g_3(\det \nabla u)(\det \nabla u'(u' - u)) \ dx.
\]
We express $R(t) = R_1(t) + R_2(t) + R_3(t) + R_4(t)$. From Lemma A.1 (see Appendix), we have
\[
\det \nabla u' u' - u = \det \nabla u' - \det \nabla (u' - u).
\]
Consequently $R_1(t)$ can be rewritten as
\[
R_1(t) = \int_\Omega L \left( g_3(\det \nabla u') - g_3(\det \nabla u) - (g_3'(\det \nabla u), \det \nabla u' - \det \nabla u) dx \right.
\]
\[
+ \int_\Omega L g_3'(\det \nabla u) \det \nabla (u' - u) dx.
\]
Let $s = \det \nabla u$ and $q = \det \nabla u^t$. Then $R_1(t)$ can further be rewritten as
\[
R_1(t) = \int_\Omega L \left\{ g_3(s + \gamma(q - s)) - g_3(s) \right\} (q - s) \, d\gamma \, dx
\]
\[
+ \int_\Omega L g_3'(\det \nabla u) \det \nabla (u' - u) dx.
\]
Note that the functions $g_3(s)$ and $g_3'(s)$ are globally Lipschitz with constant $3/2$, i.e.,
\[
|g_3(s) - g_3(t)| \leq \frac{3}{2}|s - t|,
\]
\[
|g_3'(s) - g_3'(t)| \leq \frac{3}{2}|s - t|, \quad 0 \leq t, s \in \mathbb{R}.
\]
Furthermore, Young’s inequality implies that
\[
|\det \nabla u| \leq \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_2} \right)^2 \right] = \frac{1}{2} (\nabla u : \nabla u).
\]
Hence the second term $R_{1,2}(t)$ in (54) can be estimated as follows
\[
|R_{1,2}(t)| = \int_\Omega L g_3'(\det \nabla u) \det \nabla (u' - u) \, dx \leq \frac{3}{4} \max_{t \in [0, t_0]} ||L||_{L^\infty(\Omega)} \int_\Omega |\nabla (u' - u)|^2 \, dx,
\]
for $t_0$ sufficiently small. Consequently
\[
\lim_{t \to 0} \frac{|R_{1,2}(t)|}{t} \leq \frac{3}{4} \max_{t \in [0, t_0]} ||L||_{L^\infty(\Omega)} \lim_{t \to 0} \frac{|u' - u|_{H^1}^2}{t} = 0, \text{ by (H2).}
\]
Similarly the first term can be estimated as
\[
\left| \int_\Omega L \left\{ \int_0^1 \left[ g_3'(s + \gamma(q - s)) - g_3'(s) \right] (q - s) \, d\gamma \right\} \, dx \right| \leq \frac{3}{2} \max_{t \in [0, t_0]} ||L||_{L^\infty(\Omega)} \int_\Omega |(q - s)|^2 \, dx.
\]
Note that
\[
\int_\Omega |(q - s)|^2 \, dx = \int_\Omega |A + B|^2 \, dx \leq \int_\Omega |A|^2 + 2|AB| + |B|^2 \, dx,
\]
where
\[
A = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}, \quad B = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}, \quad C = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}, \quad D = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2},
\]
and $d = \frac{\partial u}{\partial x_2}$, $\gamma = \frac{\partial u}{\partial x_1}$. Then $A = d' b' - a b = d'(b' - b) + b(d' - a)$, and $B = c d' + c' d' = d(c - c') + c'(d - d')$. Since $\partial \Omega \in C^2$ and $f \in L^p(\Omega)$ for $p > 2$, dimension, $u \in$
\[ W^{2,p}(\Omega), \ p > 2. \] Consequently, \( a, a', b, b', c, c', d, d' \in W^{1,p}(\Omega) \rightarrow C(\Omega) \) for \( p > 2 \), and
\[
\int_{\Omega} |A|^2 \, dx = \int_{\Omega} \left( (a')^2(b' - b)^2 + 2a'b(b' - b)(a' - a) + b^2(a' - a) \right) \, dx \\
\leq ||a'||_{L^2} \int_{\Omega} (b' - b)^2 \, dx + ||a'||_{L^\infty} ||b||_{L^2} \int_{\Omega} (a' - a)^2 + (b' - b)^2 \, dx \\
+ ||b||_{L^2}^2 \int_{\Omega} (a' - a)^2 \, dx.
\]

Hence since (H2) is satisfied, \( \lim_{t \to 0} \int_{\Omega} |A|^2 \, dx = 0 \) follows. Analogously we can show that \( \lim_{t \to 0} \int_{\Omega} |Aa|^2 \, dx = 0 \) and \( \lim_{t \to 0} \int_{\Omega} |b|^2 \, dx = 0 \). Therefore \( \lim_{t \to 0} |R_{2(t)}(\cdot)| = 0 \).

Furthermore, \( \lim_{t \to 0} \frac{R_{2(t)}}{t} \leq \lim_{t \to 0} \frac{\frac{3}{2} \int_{\Omega} (\det \nabla u)'(u' - u) \, dx}{t} = 0 \), by (H2).

Using (48) with \( \psi = u' - u \in H_0^1(\Omega) \), \( \xi \in L^2(\Omega) \), we have
\[
R_{3(t)} = \int_{\Omega} (g^3(\det \nabla u), (\det \nabla u)'(u')) \, dx = (E'(u, p), \Omega(\psi, t), (\lambda, q))_{\lambda' \times \lambda}. \quad (56)
\]

Proceeding as in the proof of Theorem 2.1, the term on the right hand side of (56) is arranged in an efficient manner so that (23) holds. Consequently, by using the computation that led to (39), it follows that
\[
\lim_{t \to 0} \frac{R_{3(t)}}{t} = -\frac{d}{dt} (E((u, p), t), (\lambda, q))_{\lambda' \times \lambda} \big|_{t=0} = \int_{\partial \Omega} \frac{\partial u}{\partial n} \frac{\partial \lambda}{\partial n} \mathbf{h} \cdot \mathbf{n} \, ds + \int_{\Omega} T(u) \nabla u^T \cdot \mathbf{h} \, dx, \quad (57)
\]

where \( (\lambda, q) \) solves (49). We shall turn our attention to the last term \( R_{4(t)} \) later. Let us now look at
\[
S(t) = \int_{\Omega} I_1 \left( g_3(I_1^{-1}(\det \nabla u') + t\mathcal{G}_3) - g_3(\det \nabla u') \right) \, dx.
\]

The expression \( g_3(I_1^{-1}(\det \nabla u') + t\mathcal{G}_3) - g_3(\det \nabla u') \), can be written as
\[
g_3(I_1^{-1}(\det \nabla u') + t\mathcal{G}_3) - g_3(I_1^{-1}(\det \nabla u) + t\mathcal{G}_3) + g_3(\det \nabla u) - g_3(\det \nabla u') + g_3(I_1^{-1}(\det \nabla u) + t\mathcal{G}_3) - g_3(\det \nabla u).
\]

Observe that the function \( g_3(r) \) can be expressed as \( g_3(r) = r - \frac{r}{r^2 + 1} \). Let
\[ s = \det \nabla u, \quad q = \det \nabla u', \quad \mathcal{A} = g_3(I_1^{-1}(q + t\mathcal{G}_3) - g_3(I_1^{-1}q + t\mathcal{G}_3) + g_3(s) - g_3(q).
\]

Then
\[
S(t) = S_1(t) + S_2(t) = \int_{\Omega} I_1 \mathcal{A} \, dx + \int_{\Omega} I_1 \left( g_3(I_1^{-1}(\det \nabla u) + t\mathcal{G}_3) - g_3(\det \nabla u) \right) \, dx.
\]

Note that \( \mathcal{A} \) can be expressed as
\[
\mathcal{A} = (I_1^{-1} - 1)(q - s) + \mathcal{W}(q) - \mathcal{W}(s), \quad (59)
\]
where \( W(r) = \frac{r}{r+1} - \frac{r^{-1}+\tau\theta_3}{(r^{-1}+\tau\theta_3)^2+1} \). The difference \( \mathcal{D} = W(q) - W(s) \) can be expressed as

\[
\mathcal{D} = \frac{(r\theta_3 + (I_t^{-1} - 1)s)}{(q^2 + 1)((I_t^{-1} + r\theta_3)^2 + 1)} - \frac{(r\theta_3 + (I_t^{-1} - 1)s)}{(s^2 + 1)((I_t^{-1} + s\theta_3)^2 + 1)}.
\]

Let \( \vartheta_1 = r\theta_3 + (I_t^{-1} - 1)s \), \( \vartheta_2 = r\theta_3 + (I_t^{-1} - 1)s \), \( r_1 = (I_t^{-1}q + q\theta_3 - 1) \), \( r_2 = (I_t^{-1}q + q\theta_3 - 1) \), \( n_1 = (q^2 + 1)((I_t^{-1}q + r\theta_3)^2 + 1) \), \( n_2 = (s^2 + 1)((I_t^{-1}q + s\theta_3)^2 + 1) \), \( \beta := \frac{n_1r_1}{n_2r_2} \), and \( \rho := \frac{n_1r_1}{n_2r_2} \). Then

\[
\mathcal{D} = n_2r_1 \left( \frac{\vartheta_1 - \frac{n_1r_1}{n_2r_2} \vartheta_2}{n_1n_2} \right) = \beta [((1 - \rho) \vartheta_1 + \rho (\vartheta_1 - \vartheta_2)].
\]

Note that \( (\vartheta_1 - \vartheta_2) = (I_t^{-1} - 1)(q - s) \) and

\[
\frac{\mathcal{D}}{t} = \frac{(I_t^{-1} - 1)}{t}(q - s) + \beta [(1 - \rho) \frac{\vartheta_1}{t} + \rho (I_t^{-1} - 1) \frac{(q - s)}{t}].
\]

Consequently the estimate for \( S_1(t)/t \) reads

\[
|S_1(t)/t| \leq \max_{t \in [0, \tau_0]} ||u||_{L^\infty} \left| \frac{I_t^{-1} - 1}{t} \right| ||u||_{L^\infty} (1 + ||u||_{L^\infty} ||u||_{L^\infty}) ||\text{det } \nabla u - \text{det } \nabla u||_{L^1} \\
+ \max_{t \in [0, \tau_0]} ||u||_{L^\infty} ||\text{det } \nabla u||_{L^1} (1 - \rho) \frac{\vartheta_1}{t} ||\text{det } \nabla u||_{L^\infty},
\]

for \( \tau_0 \) sufficiently small. Note that since \( u \in W^{2,p} \), \( p > 2 \), \( \beta \), \( \rho \) and \( \frac{\partial}{\partial t} \) are bounded in \( L^{\infty}(\Omega) \). Furthermore \( \frac{\partial}{\partial t} \to 1 \) in \( L^1(\Omega) \), \( \frac{\partial}{\partial t} \to 1 \) in \( L^1(\Omega) \), and \( \rho \to 1 \) in \( L^1(\Omega) \).

By (H2) it follows that \( \lim_{t \to 0} \frac{|S_1(t)|}{t} = 0 \). Therefore collecting the remaining terms into \( S_3(t) := R_3(t) + S_2(t) \), we have that

\[
S_3(t) = \int_\Omega I_k g_3(\text{det } \nabla u + t\theta_3)) - g_3(\text{det } \nabla u) dx.
\]

Observe that \( g_3(\text{det } \nabla u) \in L^{\infty}(\Omega) \) and since \( u \in H^2(\Omega) \), we have \( \nabla(\text{det } \nabla u) \in L^1(\Omega) \) and \( \nabla(\text{det } \nabla u) \in L^1(\Omega) \). Consequently \( g_3(\text{det } \nabla u) \in W^{1,1}(\Omega) \). This implies that \( g_3(\text{det } \nabla(u \circ T_t^{-1})) \) exists in \( L^1(\Omega) \). [7]. Hence
using (50) and Lemma 2.2, we have that
\[
\lim_{t \to 0} \frac{S_5(t)}{t} = \lim_{t \to 0} \frac{\int_{\Omega} g_3 \left((I - T_t^{-1}) \det \nabla u + t \mathcal{G}_3\right) - g_3(\det \nabla u) \, dx}{t},
\]
\[
= \lim_{t \to 0} \frac{\int_{\Omega} g_3 \left((I - T_t^{-1}) \det \nabla u + t \mathcal{G}_3\right) \circ T_t^{-1} - \int_{\Omega} g_3(\det \nabla u) \, dx}{t},
\]
\[
= \lim_{t \to 0} \int_{\Omega} g_3(\det (u \circ T_t^{-1})) - \int_{\Omega} g_3(\det \nabla u) \, dx,
\]
\[
= \frac{d}{dt} \int_{\Omega} g_3(\det (u \circ T_t^{-1})) \bigg|_{t=0} \, dx,
\]
\[
= \int_{\partial \Omega} g_3(\det \nabla u) \mathbf{h} \cdot \mathbf{n} \, ds + \int_{\Omega} g_3'(\det \nabla u) \frac{d}{dt} (\det (u \circ T_t^{-1})) \bigg|_{t=0} \, dx.
\]
(60)

The second term on the right hand side in (60) can be simplified using Lemma 2.3 and integration by parts leading to
\[
\int_{\Omega} g_3' \frac{d}{dt} (\det (u \circ T_t^{-1})) \bigg|_{t=0} \, dx = - \int_{\Omega} T(u) D u \cdot \mathbf{h} \, dx - \int_{\partial \Omega} P(u) D u \cdot \mathbf{h} \, ds.
\]

Therefore
\[
\lim_{t \to 0} \frac{S_5(t)}{t} = \int_{\partial \Omega} \left( g_3(\det \nabla u) - P(u) \frac{\partial u}{\partial \mathbf{n}} \right) \mathbf{h} \cdot \mathbf{n} \, ds - \int_{\Omega} T(u) D u \cdot \mathbf{h} \, dx.
\]
(61)

Finally, using (57) and (61), we obtain
\[
dJ_3(u, \Omega) \mathbf{h} = \int_{\partial \Omega} \left( \mathbf{n} \frac{\partial u}{\partial \mathbf{n}} \frac{\partial \lambda}{\partial \mathbf{n}} + g_3(\det \nabla u) - P(u) \frac{\partial u}{\partial \mathbf{n}} \right) \mathbf{h} \cdot \mathbf{n} \, ds.
\]
(62)

Expressions for $dJ_i(u, \Omega) \mathbf{h}$ in (43), (46), and (62) are linear and continuous in $\mathbf{h}$, and hence the cost functionals $J_1, J_2,$ and $J_3$ are shape differentiable.

References


Lemma A.1. Let $\psi = u' - u$, then for a 2D vector field $u$, the following relation holds
\[
\det \nabla u' - \det \nabla u - \det \nabla \psi = (\det \nabla u)'(u' - u),
\]
where
\[
(\det \nabla u)'(\psi) = \frac{\partial u_1}{\partial x} \frac{\partial (u'_2 - u_2)}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial (u'_1 - u_1)}{\partial y} - \frac{\partial u_1}{\partial y} \frac{\partial (u'_2 - u_2)}{\partial x} - \frac{\partial u_2}{\partial y} \frac{\partial (u'_1 - u_1)}{\partial x}.
\]

Proof. Using (5), we have
\[
\det \nabla \psi = \frac{\partial (u'_1 - u_1)}{\partial x} \frac{\partial (u'_2 - u_2)}{\partial y} - \frac{\partial (u'_2 - u_2)}{\partial x} \frac{\partial (u'_1 - u_1)}{\partial y}.
\]
Expansion of the differential terms leads to
\[
\det \nabla \psi = \frac{\partial u_1}{\partial x} \frac{\partial (u'_2 - u_2)}{\partial y} \frac{\partial u_2}{\partial y} \frac{\partial (u'_1 - u_1)}{\partial x} + \frac{\partial u_2}{\partial x} \frac{\partial (u'_1 - u_1)}{\partial y} \frac{\partial u_1}{\partial y} \frac{\partial (u'_2 - u_2)}{\partial x}.
\]
On the other hand
\[
\det \nabla u' - \det \nabla u = \left( \frac{\partial u'_1}{\partial x} \frac{\partial u'_2}{\partial y} - \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} \right) - \left( \frac{\partial u'_2}{\partial x} \frac{\partial u'_1}{\partial y} - \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial y} \right)
\]
\[
= \frac{\partial u'_1}{\partial x} \frac{\partial (u'_2 - u_2)}{\partial y} + \frac{\partial u'_2}{\partial x} \frac{\partial (u'_1 - u_1)}{\partial y} - \frac{\partial u_1}{\partial x} \frac{\partial (u'_2 - u_2)}{\partial y} - \frac{\partial u_2}{\partial x} \frac{\partial (u'_1 - u_1)}{\partial y}.
\]
Thus
\[
\det \nabla u' - \det \nabla u - \det \nabla \psi = \frac{\partial u_1}{\partial x} \frac{\partial (u'_2 - u_2)}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial (u'_1 - u_1)}{\partial y} - \frac{\partial u_1}{\partial y} \frac{\partial (u'_2 - u_2)}{\partial x} - \frac{\partial u_2}{\partial y} \frac{\partial (u'_1 - u_1)}{\partial x}.
\]