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**Abstract.** Moreau-Yosida based approximation techniques for optimal control of variational inequalities are investigated. Properties of the path generated by solutions to the regularized equations are analyzed. Combined with a semi-smooth Newton method for the regularized problems these lead to an efficient numerical technique.

**Keywords.** Variational inequalities, optimal control, regularization, sensitivity equation, path-following, sufficient optimality conditions, semi-smooth Newton method.

# 1 Introduction, problem statement, regularization

Here we continue our investigations from [9] concerning the optimal control of stationary variational inequalities. The specific problem under investigation is given by

$$\text{(P)} \qquad \begin{cases} \min \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ under the variational inequality constraint} \\ a(y, \phi - y) \geq (u, \phi - y), \quad y \in K, \text{ for all } \phi \in K, \end{cases}$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a bilinear form satisfying

$$\nu_1 \|v\|_{H_0^1}^2 \leq a(v,v), \text{ and } a(v,w) \leq \nu_2 \|v\|_{H_0^1} \|w\|_{H_0^1}, \tag{1.1}$$

with $0 < \nu_1 \leq \nu_2$, and

$$K = \{v \in H_0^1(\Omega) : v \leq \psi\}, \tag{1.2}$$

where $\psi \in H^1(\Omega)$ with $\psi \geq 0$ on the boundary of $\Omega$. Throughout it will be convenient to alternatively use the operator representation of the bilinear form, i.e.

$$a(v_1, v_2) = \langle Av_1, v_2 \rangle, \text{ for } v_1, v_2 \in H_0^1(\Omega).$$

Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It is well known that under the conditions to be specified below the variational inequality in (P) can equivalently be expressed as

$$Ay + \lambda = u, \quad y \leq \psi, \ \lambda \geq 0, \ \langle \lambda, y - \psi \rangle = 0, \tag{1.3}$$

where $\lambda \in H^{-1}$, and $\lambda \geq 0$ is short for $\langle \lambda, v \rangle \geq 0$ for all $v \in H_0^1(\Omega),$ with $v \geq 0$. In this way (P) is an optimization problem subject to a complementary condition constraint. If $\lambda \in L^2(\Omega)$, then the complementarity condition in (1.3) can equivalently be expressed as

$$\lambda = \max(0, \lambda + c(y - \psi)), \tag{1.4}$$

for any $c > 0$.

In [9] we investigated first order necessary and second order sufficient optimality conditions for (P). In particular, under the standing assumptions, which are specified at the end of this section, we have:

**First order necessary condition**

Let $(y^*, u^*)$ be a locally optimal pair for the optimal control problem (P). Further let $\lambda^* \in L^2(\Omega)$ be the Lagrange multiplier associated to the lower level problem, which is the variational inequality in (P). Then there exist uniquely determined adjoint states $p^* \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\mu^* \in H^{-1}(\Omega) \cap (L^\infty(\Omega))^*$ such that in addition to (1.3) we have

$$A^* p^* + \mu^* + g'(y^*) = 0 \quad \text{and} \quad p^* \geq 0 \text{ where } y^* = \psi, \tag{1.5}$$

$$\lambda^* p^* = 0 \text{ a.e. on } \Omega, \quad \text{and } \langle \mu^*, p^* \rangle \geq 0 \tag{1.6}$$

$$\langle \mu^*, \varphi(y^* - \psi) \rangle = 0 \text{ for all } \varphi \in C^1(\bar{\Omega}) \text{ such that } \varphi\psi|_\Gamma = 0, \tag{1.7}$$

(1.8)

$$\langle \mu^*, \phi \rangle \geq 0 \quad \text{for all } \phi \in H_0^1(\Omega) \text{ with } \phi \geq 0 \text{ on } \{y^* = \psi\} \text{ and } \langle \lambda^*, \phi \rangle = 0,$$

$$j'(u^*) - p^* = 0. \tag{1.9}$$

Moreover, we have the following sign condition for $\mu^*$ on the biactive set $B = \{\lambda^* = 0, \ y = \psi\}$:

$$\langle \mu^*, \phi \rangle \geq 0 \quad \text{for all } \phi \in H_0^1(\Omega), \text{ with } \phi \geq 0 \text{ on } B, \ \phi = 0 \text{ on } \Omega \setminus B. \tag{1.10}$$

First order conditions for optimal control of variational inequalities have a longstanding history. We refer to [1, 2, 4, 7] in this respect. Due to the appearance of multipliers, which are only measures, and as a consequence of the complementarity conditions (1.3), (1.7), (1.8), these conditions are not well-suited for numerical realisation.

To overcome this difficulty we introduce a regularization of the original problem. It will render the second Lagrange multiplier $\mu$ to be a pointwise a.e. well-defined function rather than a measure. Moreover, regularization has the effect that the complementarity conditions in the first order necessary conditions are replaced by nonlinear equations.

The regularized problems that will be utilized are given by

$$(\mathrm{P}_c) \qquad \begin{cases} \min \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ subject to} \\ Ay + \max_c(\bar{\lambda} + c(y-\psi)) = u, \end{cases}$$

where $\bar{\lambda} \geq 0$, $\bar{\lambda} \in L^\infty(\Omega)$, is given, and $\max_c$ is a $C^2$-approximation of $x \to \max(0,x)$, where the precise assumptions imposed on $\max_c$ are specified below. For properly chosen $\bar{\lambda}$ the solutions $y_c$ to $(\mathrm{P}_c)$ are feasible, i.e. $y_c \leq \psi$. If $g$ and $j$ are $C^1$-regular, then the first order optimality system for $(\mathrm{P}_c)$ is given by

$$Ay_c + \max_c\left(\bar{\lambda} + c(y_c - \psi)\right) = u_c, \tag{1.11a}$$

$$A^* p_c + c\max_c'\left(\bar{\lambda} + c(y_c-\psi)\right) p_c + g'(y_c) = 0, \tag{1.11b}$$

$$j'(u_c) - p_c = 0. \tag{1.11c}$$

In [8] and [9] existence of solutions $(y_c, u_c) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ to (1.11) was established and subsequential convergence for $c \to \infty$ to a solution $(y^*, u^*)$ of the unregularized problem was argued. For the same subsequence we have

$$\lambda_c \to \lambda^* \text{ and } \mu_c \rightharpoonup \mu^* \text{ in } H^{-1}(\Omega),\ p_c \rightharpoonup p^* \text{ in } H_0^1(\Omega),$$

where $\to$ and $\rightharpoonup$ denote strong and weak convergence respectively, and

$$\lambda_c = \max_c(\bar{\lambda} + c(y_c - \psi)), \qquad \mu_c = c\max_c'(\bar{\lambda} + c(y_c - \psi)).$$

The focus of the present paper is the behavior of the solution to $(\mathrm{P}_c)$ and (1.11) as a function of $c$. We denote by

$$V(c) = J(y_c, u_c),$$

the value function, which is the value of the objective functional along local solutions $(y_c, u_c)$ of $(\mathrm{P}_c)$. Further $c \to (y_c, u_c, p_c)$, for $c \in (0, \infty)$, is referred to as the path associated with (1.11). Due to the fact that $(\mathrm{P}_c)$ is not convex the solutions to $(\mathrm{P}_c)$ are not unique and hence special care and additional assumptions are needed to make the concept of a path associated to (1.11) precise.

Concerning the choice of $\bar{\lambda}$ we refer to $\bar{\lambda} = 0$ as the *infeasible case*, since the iterates will in general not satisfy $y_c \leq \psi$. On the other hand, if $\bar{\lambda}$ is sufficiently large, we have that $y_c$ is *feasible*, i.e. $y_c \leq \psi$, for each $c \geq \underline{c}$. A precise statement is given in Proposition 1.2 below.

Our interest in properties of the value function and the path, besides its own inherent value, lies in the use for numerical realisations. For moderate values of $c$, while being yet distant to the 'true' solution, one solves a problem with smooth approximations to $\lambda$ and $\mu$. As $c$ is increased the regularity of $\mu$ progressively decreases. Once the existence of the path is established, the value function can be evaluated along this path and its properties can be used advantageously to devise rules for updating the parameter $c$ given current information on $(y_c, u_c, p_c)$.

The notion of path is well-established for interior point methods for finite dimensional optimization problems, we refer e.g. to [15] and the references given there. Much fewer investigations were carried out for path-following in the context of interior point methods in function spaces. We refer, however, to the investigations in [14] and [11] which are carried out for optimal control problems with control and state constraints for finite dimensional dynamical systems and for optimal control with constraints on the state for stationary diffusion systems, respectively. A detailed regularity analysis of the central path for interior point methods for bilaterally control-constrained optimal control problems for elliptic equations is given in [13].

Path following for Moreau-Yosida type regularization as in $(\mathrm{P}_c)$ involving $\max_c$ was first analyzed in [6] for solving obstacle problems by regularized semi-smooth Newton methods. This technique was also used in [12] for the solution of contact problems in linear elasticity. Central for the use of a path evaluated along the minimal value function is the fact that it can be approximated quite accurately by a low-parametric curve, the model function. The model function serves an important guideline for the update strategy of the regularisation parameter.

Let us briefly outline the contents of this paper. Sections 2 and 3 are devoted to deriving assumptions which guarantee continuity and Lipschitz continuity of the path. These assumptions are small modifications of second order sufficient optimality conditions for the regularized problems ($\mathrm{P}_c$). The results can be used to introduce directional differentiability of the path. In Section 4 qualitative properties of the value-function are developed. In particular this concerns the asymptotic behavior of $V(c)$ as $c \to \infty$ and monotonicity and concavity/convexity properties of $V$. It will be shown that, inspite of the fact that the constraint is not convex, the value function is 'almost' monotonically increasing and concave in the infeasible case and that it is monotonically decreasing for $c$ sufficiently large and 'almost' convex in the feasible case. The concluding section contains numerical experiments and demonstrates the use of the value function.

## Standing assumptions

Throughout the paper we rely on the following regularity assumptions.

(A.i) The domain $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ is bounded and either convex and polygonal or of the class $C^{1,1}$.

(A.ii) The operator $A$ is an elliptic differential operator defined by

$$(Ay)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}y(x)\right) + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}y(x) + a_0(x)y(x)$$

with functions $a_{ij} \in C^{0,1}(\bar{\Omega})$, $a_j, \frac{\partial}{\partial x_j}a_j, a_0 \in L^\infty(\Omega)$ satisfying the conditions $a_{ij}(x) = a_{ji}(x)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta_0|\xi|^2 \qquad \text{a.e. on } \Omega \text{ for all } \xi \in \mathbb{R}^n$$

with some $\delta_0 > 0$. Additionally, we require $a_0(x) \geq \delta_1 \geq 0$ with $\delta_1$ sufficiently large such that the bilinear form $a(\cdot,\cdot)$ induced by $A$ fulfills the coercivity condition (1.1).

(A.iii) The obstacle $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ fulfills $\psi \geq 0$ on $\Gamma$ and $A\psi \in L^\infty(\Omega)$.

The functions $g, j$ satisfy:

(A.iv) $g : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable and bounded from below, (A.v) $j : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable, bounded from below, radially unbounded, and

$$u_n \rightharpoonup u \text{ in } L^2(\Omega), \text{ and } \; j(u_n) \to j(u) \; \text{ implies } \; u_n \to u \text{ in } L^2(\Omega).$$

The Assumptions (A.iv) and (A.v) are satisfied for instance for the quadratic cost functional

$$J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2} + \frac{\alpha}{2}\|u\|^2_{L^2}$$

with $\alpha > 0$ and $y_d \in L^2(\Omega)$.
Let us introduce the adjoint operator $A^*$ to $A$ by

$$(A^*p)(x) = -\sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\sum_{i=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}p(x) + a_j(x)p(x)\right) + a_0(x)p(x).$$

Due to the assumptions on the coefficients, the equations $Ay = f$ and $A^*p = g$ admit solutions in $H^2(\Omega)$ for right-hand sides $f, g \in L^2(\Omega)$.

## Assumptions on the smooth approximation of max

We assume that the function $\max_c$ admits the following properties:

(B.i) $\max_c : \; (c,x) \mapsto \max_c(x)$, $(c,x) \in (0,+\infty) \times \mathbb{R}$, is twice continuously differentiable with respect to $(c,x)$,

(B.ii) $\max_c(x) = \max(0,x)$ for all $x$ with $|x| \ge 1/2c$.

We will denote the derivatives with respect to $x$ by $\max_c'$, $\max_c''$, whereas the derivatives with respect to $c$ are denoted by $\frac{\partial}{\partial c}\max_c$.

In addition we assume that there is a constant $M > 0$ such that the following inequalities are satisfied for all $x, x'$:

(B.iii) $0 \le \max_c'(x) \le 1$,

(B.iv) $0 \le \max_c''(x) \le Mc$,

(B.v) $|\max_c''(x) - \max_c''(x')| \le Mc^2|x - x'|$,

(B.vi) $\left|\frac{\partial}{\partial c}\max_c(x)\right| \le \frac{M}{c^2}$ ,

(B.vii) $\left|\frac{\partial}{\partial c}\max_c'(x)\right| \le \frac{M}{c}$ , (B.viii) $\left|\frac{\partial^2}{\partial c^2}\max_c(x)\right| \leq \frac{M}{c^3}$.

Note, that the function

$$m_c(x) := \begin{cases} \max(0, x) & \text{if } |x| \geq \frac{1}{2c}, \\ \frac{c^3}{2}\left(x + \frac{1}{2c}\right)^3 \left(\frac{3}{2c} - x\right) & \text{if } |x| < \frac{1}{2c}, \end{cases} \tag{1.12}$$

satisfies the requirements above. In fact, the derivatives of $m_c$ with respect to $c > 0$ and $x$ with $|x| < 1/2c$, together with their maximum over $x \in \mathbb{R}$ are given by the following expressions.

$$\begin{aligned}
m_c'(x) &= 2c^3\left(x + \frac{1}{2c}\right)^2\left(\frac{1}{c} - x\right), & \|m_c'\|_{L^\infty(\mathbb{R})} &= 1, \\
m_c''(x) &= 6c^3\left(x + \frac{1}{2c}\right)\left(\frac{1}{2c} - x\right), & \|m_c''\|_{L^\infty(\mathbb{R})} &= \frac{3}{2}c, \\
\frac{\partial}{\partial c}m_c(x) &= -\frac{3}{2}c^2\left(x + \frac{1}{2c}\right)^2\left(\frac{1}{2c} - x\right)^2, & \left\|\frac{\partial}{\partial c}m_c\right\|_{L^\infty(\mathbb{R})} &= \frac{3}{32}\cdot\frac{1}{c^2}, \\
\frac{\partial}{\partial c}m_c'(x) &= 6c^2x\left(x + \frac{1}{2c}\right)\left(\frac{1}{2c} - x\right), & \left\|\frac{\partial}{\partial c}m_c'\right\|_{L^\infty(\mathbb{R})} &= \frac{1}{2\sqrt{3}}\cdot\frac{1}{c}, \\
\frac{\partial^2}{\partial c^2}m_c(x) &= -3c\left(x^4 - \frac{1}{16c^4}\right), & \left\|\frac{\partial^2}{\partial c^2}m_c\right\|_{L^\infty(\mathbb{R})} &= \frac{3}{16}\cdot\frac{1}{c^3}.
\end{aligned}$$

Moreover the second derivative of $m_c''$ w.r.t. $x$ is globally Lipschitz continuous:

$$|m_c''(x) - m_c''(x')| \leq 6c^2|x - x'|.$$

We also have

$$\frac{\partial}{\partial c}m_c(x) \leq 0, \qquad \frac{\partial^2}{\partial c^2}m_c(x) \leq 0. \tag{1.13}$$

Different kind of smooth approximations of $\max(0, \cdot)$ were used by Hintermüller and Kopacka [5].

Throughout the paper we denote by $(y^*, u^*)$ a strict local optimal solution of the original problem (P).

## 1.1 Preliminary results for the regularized problem

Let us briefly recall known results for the regularized state equation

$$Ay + \max_c(\bar{\lambda} + c(y - \psi)) = u, \tag{1.14}$$

with $\bar{\lambda} \in L^\infty(\Omega)$ and $c > 0$. Due to the monotonicity of $\max_c$, it admits for every right-hand side $u \in H^{-1}(\Omega)$ a unique solution $y_c \in H_0^1(\Omega)$. In case the dependence of $y_c$ on $u$ is relevant we write $y_c(u)$. Furthermore we define

$$\lambda_c := \max_c(\bar{\lambda} + c(y_c - \psi)).$$

**Proposition 1.1.** *(a) Let a family of controls $\{u_c\}_{c>0}$ with $u_c \rightharpoonup u$ in $L^2(\Omega)$, for $c \to \infty$, be given. Then the solutions $y_c(u_c)$ converge strongly in $H_0^1(\Omega)$ to the solution $y$ of the variational inequality. Moreover, $\lambda_c$ converges strongly in $H^{-1}(\Omega)$ to $\lambda$. If the solutions $y_c$ are feasible, i.e. $y_c \le \psi$, then $y_c \to y$ in $L^\infty(\Omega)$.*

*(b) Let a sequence of controls $\{u_{c_k}\}$ with $u_c \rightharpoonup u$ in $L^2(\Omega)$, for $c_k \to c$, be given. Then the solutions $y_{c_k}(u_{c_k})$ converge strongly in $H_0^1(\Omega)$ to the solution $y_c$ of* (1.14)*.*

*Proof.* The proof follows along the lines of similar proofs in [5, 7, 9]. □

In [9], we proved the following result regarding feasibility of states associated with locally optimal controls of $(\mathrm{P}_c)$.

**Proposition 1.2.** *If $\|j'(u)\|_{L^\infty(\Omega)} \ge K(\|u\|_{L^\infty(\Omega)} - 1)$ for a constant $K$ independent of $u \in L^\infty(\Omega)$, then for any $\underline{c}$ there exists $\rho_{\underline{c}} > 0$ such that every locally optimal control $u_c$ to $(\mathrm{P}_c)$ satisfies $\|u_c\|_{L^\infty(\Omega)} \le \rho_{\underline{c}}$ for all $c \ge \underline{c}$. Then, if*

$$\bar{\lambda} \ge \max(0, -A\psi + \rho_{\underline{c}})$$

*we have that $y_c$ is* feasible, *i.e. $y_c \le \psi$, for each $c \ge \underline{c}$.*

Let us remark that the condition on $j$ is fulfilled for the choice $j(u) = \frac{\alpha}{2}\|u\|_{L^2}^2$.

# 2 Continuity of the path

For the discussion of path-following algorithms we have to study local properties of the 'path' $(y_c, u_c)$, i.e. we need to consider $c \to (y_c, u_c)$. Due to lack of uniqueness of the optimal solutions to $(\mathrm{P}_c)$ the discussion is based on local solutions.

Let $(y^*, u^*)$ be a strict local optimal solution of the original problem (P). Thus there exists $\rho > 0$ such that $(y^*, u^*)$ satisfies:

$$J(y^*, u^*) < J(y(u), u) \quad \forall u \neq u^* : \ \|u - u^*\|_{L^2} \leq \rho.$$

In the previous work [9] we already showed the existence of a family of local minimizers $(y_c, u_c)$ converging to $(y^*, u^*)$ for $c \to \infty$. More precisely let $(y_c, u_c)$ be defined as global solutions of the auxiliary problems

$$(\mathrm{P}_c^\rho) \qquad \begin{cases} \min \ J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ with } \|u - u^*\|_{L^2} \leq \rho, \text{ subject to} \\ Ay + \max_c\left(\bar\lambda + c(y - \psi)\right) = u. \end{cases}$$

Clearly, the problem $(\mathrm{P}_c^\rho)$ admits global solutions. Moreover, as proven in [9], we have the following convergence result.

**Theorem 2.1.** *Let $(y_c, u_c)$, $c > 0$, be a family of global solutions of $(\mathrm{P}_c^\rho)$. Then there exists $C_0 > 0$ such that for all $c > C_0$, $(y_c, u_c)$ are local solutions of $(\mathrm{P}_c)$. Moreover for $c \to \infty$ we have*

$$u_c \to u^* \text{ in } L^2(\Omega), \quad y_c \to y^* \text{ in } H_0^1(\Omega).$$

That is, continuity of the path at 'infinity' is established and for $c$ sufficiently large $(y_c, u_c)$ are local solutions of $(\mathrm{P}_c)$ satisfying $\|u - u^*\|_{L^2} < \rho$. For the discussion of path-following algorithms we need to analyse the behavior of $c \to (y_c, u_c)$ for $c < \infty$ as well.

In this section sufficient conditions will be given which guarantee that the global solutions $(y_c, u_c)$ of $(\mathrm{P}_c^\rho)$ satisfy:

(i) the value function $c \mapsto J(y_c, u_c)$ is continuous for $c > 0$,

(ii) the path $c \mapsto u_c$ is continuous for $c$ sufficiently large.

We first establish continuity of the value function $J(y_c, u_c)$.

**Theorem 2.2.** *Let $\{y_c, u_c\}$, $c > 0$, be a family of global solutions of $(\mathrm{P}_c^\rho)$. Then the mapping $c \mapsto J(y_c, u_c)$ is continuous for all $c > 0$.*

*Proof.* Let us take a sequence $c_k \to c_0$ with associated global minimizers of $(\mathrm{P}^\rho_{c_k})$ denoted by $u_k := u_{c_k}$ and $y_k := y_{c_k}$.

Due to the construction of $(\mathrm{P}^\rho_c)$, the sequence $\{u_k\}$ is bounded in $L^2(\Omega)$. Thus after extracting a subsequence if necessary, $u_k \rightharpoonup \tilde{u}$ in $L^2(\Omega)$ and $u_k \to \tilde{u}$ in $H^{-1}(\Omega)$. Convergence $y_k \to y_{c_0}(\tilde{u})$ in $H^1_0(\Omega)$ follows from Proposition 1.1 (b).

Since $(y_k, u_k)$ are global solutions of the auxiliary problems $(\mathrm{P}^\rho_{c_k})$, we have $J(y_k, u_k) \le J(y_{c_k}(u_{c_0}), u_{c_0})$, which gives $\limsup J(y_k, u_k) \le J(y_{c_0}, u_{c_0})$, by (A.iv) of the standing assumptions and Proposition 1.1 (b).

Moreover as $(y_{c_0}, u_{c_0})$ is a global solution of $(\mathrm{P}^\rho_{c_0})$, we have $J(y_{c_0}, u_{c_0}) \le J(y_{c_0}(u_k), u_k)$. Hence,

$$\begin{aligned} &J(y_{c_0}, u_{c_0}) \le \liminf J(y_{c_0}(u_k), u_k) = \liminf \big(g(y_{c_0}(u_k)) + j(u_k)\big) \\ &\quad = g(\tilde{y}) + \liminf j(u_k)) = \liminf \big(g(y_{c_k}(u_k)) + j(u_k)\big) = \liminf J(y_k, u_k). \end{aligned}$$

This implies that $J(y_{c_0}, u_{c_0}) = \lim_{k\to\infty} J(y_k, u_k)$. □

The remainder of this section is denoted to proving continuity of $c \mapsto u_c$. We shall rely on a second-order condition imposed at the local solution $(y^*, u^*)$.

**Assumption 1.** *(i) There is $\gamma > 0$ such that*

$$j''(u^*)(h,h) \ge \gamma \|h\|^2_{L^2} \quad \text{for all } h \in L^2(\Omega).$$

*(ii) We assume that*

$$g''(y^*)(z,z) + j''(u^*)(h,h) > 0. \tag{2.1}$$

*holds for all $(z, h, \eta) \in H^1_0(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$, $h \ne 0$, satisfying*

$$j'(u^*)h + g'(y^*)z = 0, \tag{2.2}$$

*and*

$$\begin{aligned} Az + \eta &= h, \\ \langle \eta, z \rangle &\ge 0. \end{aligned} \tag{2.3}$$

For a more detailed discussion of second-order conditions, we refer to our previous work [9]. Note that (2.2)-(2.3) are first order necessary conditions for $(y^*, u^*)$.

As it will turn out, this condition alone is not sufficient for continuity of the path. The reason is that we approximate a non-smooth optimization problem by a smooth one and thus one cannot expect convergence of second derivatives. To overcome this difficulty we make the following assumption.

**Assumption 2.** *There is a radius $\rho_1 \in (0, \rho)$ and a constant $C_1 > 0$ such that for all $c > C_1$ and $\tilde{u}_c$ with $\|u_c - u^*\|_{L^2} < \rho_1$, the solutions $(y_c, p_c)$ of the state and adjoint equations* (1.11a)–(1.11b) *satisfy*

$$p_c \geq 0 \text{ on } \{\max_c''(\bar{\lambda} + c(y_c - \psi)) > 0\}.$$

**Remark 2.3.** Note that $p_c \geq 0$ on all of $\Omega$ if $g'(y_c) \leq 0$. In fact expressing $p_c = p_c^+ - p_c^-$ with $p_c^\pm \geq 0$ we have from (1.11b)

$$-(A^* p_c, p_c^-) - c \max_c'(\bar{\lambda} + c(y_c - \psi) p_c, p^-) = (g'(y_c), p^-) \leq 0.$$

This implies that $\|p_c^-\|_{H_0^1}^2 \leq 0$ and hence $p_c \leq 0$.

**Theorem 2.4.** *Let Assumptions 1 and 2 be satisfied. Then there is a constant $C_2 \in (0, C_1]$ such that the problem* $(\mathrm{P}_c^\rho)$ *is uniquely solvable for all $c > C_2$. Moreover, the path $c \mapsto u_c$ is continuous from $(C_2, \infty]$ to the strong topology in $L^2(\Omega)$.*

*Proof.* Let us assume that there exists a sequence $c_k \to \infty$ such that the problem $(\mathrm{P}_{c_k}^\rho)$ is not uniquely solvable for each $c_k$. Then there are two sequences $(y_{k,1}, u_{k,1})$ and $(y_{k,2}, u_{k,2})$ of global solutions to $(\mathrm{P}_{c_k}^\rho)$ with $u_{k,1} \neq u_{k,2}$.

For any $k = 1, 2 \ldots$ let us consider the objective functional $J(y_{c_k}(u), u)$ on the line segment $\mathcal{L}$ from $u_{k,1}$ to $u_{k,2}$. Then $J$ admits a local maximum $\tilde{u}_k \neq u_{k,1}, u_{k,2}$ on this compact set $\mathcal{L}$ with associated state $\tilde{y}_k$. Since the ball $\{u \in L^2(\Omega) : \|u - u^*\|_{L^2} \leq \rho\}$ is strictly convex, we have $\|\tilde{u}_k - u^*\|_{L^2} < \rho$. Associated to $\tilde{u}_k$ and $\tilde{y}_k := y_{c_k}(\tilde{u}_k)$ there exist adjoint states $\tilde{p}_k$ such that the first-order system

$$\begin{aligned} &A^* \tilde{p}_k + c_k \max_{c_k}' \big(\bar{\lambda} + c_k(\tilde{y}_k - \psi)\big) \tilde{p}_k + g'(\tilde{y}_k) = 0, \\ &(j'(\tilde{u}_k) - \tilde{p}_k,\ u_{k,1} - u_{k,2}) = 0 \end{aligned} \tag{2.4}$$

is satisfied. Since $(\tilde{y}_k, \tilde{u}_k, \tilde{p}_k)$ are the solutions of smooth optimization problems, they fulfill the following second-order necessary optimality condition:

$$j''(\tilde{u}_k)(h_k, h_k) + g''(\tilde{y}_k)(z_k, z_k) + (c_k^2 \max''_{c_k} \big(\bar{\lambda} + c_k(\tilde{y}_k - \psi)\big)\tilde{p}_k z_k, z_k) \leq 0,$$

where $h_k := \frac{u_{k,1} - u_{k,2}}{\|u_{k,1} - u_{k,2}\|_{L^2}}$ and $z_k$ are the solutions of the linearized equations

$$Az + c_k \max'_{c_k} \big(\bar{\lambda} + c_k(\tilde{y}_k - \psi)\big) z = h_k.$$

Since $u_{k,1}$ and $u_{k,2}$ converge to $u^*$, we have $\tilde{u}_k \to u^*$ in $L^2(\Omega)$. Hence by Assumption 2 and (B.iv), we obtain that $\max''_{c_k} \big(\bar{\lambda} + c_k(\tilde{y}_k - \psi)\big)\tilde{p}_k \geq 0$ on $\Omega$, provided that $k$ is large enough. This implies that

$$j''(\tilde{u}_k)(h_k, h_k) + g''(\tilde{y}_k)(z_k, z_k) \leq 0 \tag{2.5}$$

for all $k$ large enough. Now, we can argue as in [9, Theorem 3.14] to obtain a contradiction to Assumption 1: The sequence $\{h_k\}$ is bounded in $L^2(\Omega)$, and after extracting subsequences if necessary, we obtain the weak convergence

$$h_k \rightharpoonup h \text{ in } L^2(\Omega), \quad z_k \rightharpoonup z \text{ in } H_0^1(\Omega),$$
$$\eta_k := c_k \max'_{c_k} \big(\bar{\lambda} + c_k(\tilde{y}_k - \psi)\big) z_k \rightharpoonup \eta \text{ in } H^{-1}(\Omega).$$

These weak limits satisfy $Az + \eta = h$ in $H^{-1}(\Omega)$ and $\langle \eta, z \rangle \geq 0$, where the latter relation follows from non-negativity of $\max'_c$, see (B.iii). Thus the triple $(z, h, \eta)$ fulfills the conditions in (2.3). Moreover by (2.4) we have $j'(\tilde{u}_k)h_k + g'(\tilde{y}_k)z_k = 0$. Passing to the limit $k \to \infty$ implies that $j'(u^*)h + g'(y^*)z = 0$ and hence (2.2) holds.

Now let us pass to the limit in inequality (2.5). By Proposition 1.1, we obtain $\tilde{y}_k \to y^*$ in $H_0^1(\Omega)$, which gives

$$\lim_{k \to \infty} g''(\tilde{y}_k)(z_k, z_k) = g''(y^*)(z, z).$$

Due to the continuity of $j''$ and the lower semicontinuity of $h \mapsto j''(u^*)(h, h)$, see Assumption 1(i), we obtain

$$\liminf_{k \to \infty} j''(\tilde{u}_k)(h_k, h_k) = \liminf_{k \to \infty} j''(u^*)(h_k, h_k) \geq j''(u^*)(h, h),$$

which gives with (2.5)

$$j''(u^*)(h, h) + g''(y^*)(z, z) \leq 0.$$

Hence, Assumption 1(ii) implies that $h = 0$ and $z = 0$. Due to Assumption 1(i) and $\|h_k\|_{L^2(\Omega)} = 1$, we obtain from (2.5)

$$0 < \gamma \le j''(u^*)(h_k, h_k) \le -g''(\tilde{y}_k)(z_k, z_k) + j''(u^*)(h_k, h_k) - j''(\tilde{u}_k)(h_k, h_k)$$

Since $z_k \rightharpoonup 0 = z$ in $H_0^1(\Omega)$ and $\tilde{u}_k \to u^*$ in $L^2(\Omega)$, the right-hand side vanishes for $k \to \infty$, which yields a contradiction.

Thus, the global minimizers of $(\mathrm{P}_c^\rho)$ are uniquely determined for all $c > C_2$ with $C_2$ large enough.

Now, let us take $c_0 > C_2$, which implies that $(\mathrm{P}_c^\rho)$ is uniquely solvable. We will prove that $c \mapsto u_c$ is continuous at $c_0$. For this purpose let $c_k$ denote an arbitrary sequence converging to $c_0$ and set $u_k := u_{c_k}$, $y_k := y_{c_k}(u_k)$. After extracting a subsequence if necessary, we have $u_k \rightharpoonup \tilde{u}$ in $L^2(\Omega)$ and $y_k \to \tilde{y} = y_{c_0}(\tilde{u})$ in $H_0^1(\Omega)$. As in the proof of Theorem 2.2, we obtain $J(\tilde{y}, \tilde{u}) = J(y_{c_0}, u_{c_0})$. Since $(y_{c_0}, u_{c_0})$ is the unique global minimum of $(\mathrm{P}_{c_0}^\rho)$, it follows $\tilde{u} = u_{c_0}$. Hence, $u_c \rightharpoonup u_{c_0}$ in $L^2(\Omega)$ for $c \to c_0$.

We obtain from the convergence $y_k \to \tilde{y}$ and $J(y_k, u_k) \to J(y_{c_0}, u_{c_0})$ that $j(u_k) \to j(u_{c_0})$. Thus by the Assumptions (A.v) on $j$, strong convergence $u_{c_k} \to u_{c_0}$ in $L^2(\Omega)$ follows. □

Let us remark that we proved that the path $c \mapsto u_c$ is continuous at all values $c$, for which $(\mathrm{P}_c^\rho)$ is uniquely solvable.

**Remark 2.5.** Choosing $\bar{\lambda}$ large enough yields feasibility of the solutions of the regularized state equation, see Proposition 1.2 and the discussion in [7, 9]. In this case we can weaken Assumption 1. As in [9] we can prove more properties of the limit $(z, h, \eta)$ of the sequence $(z_k, h_k, \eta_k)$, which allow to shrink the set of test functions in Assumption 1(ii) to those $(z, h, \eta)$ which satisfy

$$\begin{gathered} Az + \eta = h \\ \langle \eta, y^* - \psi \rangle = 0,\ \langle \eta, z \rangle \ge 0,\ \lambda^* z = 0 \text{ a.e. on } \Omega. \end{gathered} \tag{2.3'}$$

# 3 Lipschitz estimates

In the previous section, we established continuity of the path $c \mapsto (y_c, u_c)$. Here we turn to Lipschitz continuity of this mapping.

The following abbreviations will be used:

$$\begin{aligned} f_c(y) &:= \max{}_c \left(\bar{\lambda} + c(y - \psi)\right), \\ f_c'(y) &:= c \max{}_c' \left(\bar{\lambda} + c(y - \psi)\right), \\ f_c''(y) &:= c^2 \max{}_c'' \left(\bar{\lambda} + c(y - \psi)\right). \end{aligned}$$

Let us recall that $(y_c, u_c)$ with associated adjoint state $p_c$ satisfy the optimality system

$$\begin{cases} Ay_c + f_c(y_c) = u_c, \\ A^* p_c + f_c'(y_c)\, p_c + g'(y_c) = 0, \\ j'(u_c) - p_c = 0, \end{cases} \tag{3.1}$$

At first, we investigate local Lipschitz continuity of the regularized equation constraint of ($\mathrm{P}_c$). For this purpose we require the following result.

**Lemma 3.1.** *Let $I$ be a compact subset of $(0, +\infty)$ and $Y$ be a bounded subset of $L^\infty(\Omega)$. Then there exists a constant $K = K(I, Y)$ such that for all $c_1, c_2 \in I$ and $y_1, y_2 \in Y$ it holds*

$$\|f_{c_1}(y_1) - f_{c_2}(y_2)\|_{L^\infty(\Omega)} + \|f_{c_1}'(y_1) - f_{c_2}'(y_2)\|_{L^\infty(\Omega)} \le K(|c_1 - c_2| + \|y_1 - y_2\|_{L^\infty(\Omega)}).$$

*Proof.* Let $c_1, c_2 \in I$ and $y_1, y_2 \in Y$ be given. Let us write

$$f_{c_1}(y_1) - f_{c_2}(y_2) = f_{c_1}(y_1) - f_{c_1}(y_2) + f_{c_1}(y_2) - f_{c_2}(y_2).$$

By (B.iii) we have $\|f_{c_1}(y_1) - f_{c_1}(y_2)\|_{L^\infty(\Omega)} \le (\sup I)\|y_1 - y_2\|_{L^\infty(\Omega)}$. Using

$$\frac{\partial}{\partial c} f_c(y) = \frac{\partial}{\partial c} \max{}_c(\bar{\lambda} + c(y - \psi)) + \max{}_c'(\bar{\lambda} + c(y - \psi))(y - \psi)$$

we can apply (B.vi) and (B.iii) to obtain

$$\|f_{c_1}(y_2) - f_{c_2}(y_2)\|_{L^\infty(\Omega)} \le \left( \frac{M}{(\inf I)^2} + \sup_{y \in Y} \|y - \psi\|_{L^\infty(\Omega)} \right) |c_1 - c_2|. \tag{3.2}$$

This implies the desired estimate for $\|f_{c_1}(y_1) - f_{c_2}(y_2)\|_{L^\infty(\Omega)}$. The estimate involving $f_c'(y)$ can be derived analogously employing (B.iii), (B.iv), and (B.vii). ☐

**Lemma 3.2.** *Let $I$ be a compact subset of $(0,+\infty)$ and $U$ be a bounded subset of $L^2(\Omega)$. Then the set*

$$\{y_c(u) :\ c \in I, u \in U\}$$

*is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$, where the bound depends on $U$ but not on $I$. Moreover, there exists a constant $K = K(I,U)$ such that for all $c_1, c_2 \in I$ and $u_1, u_2 \in U$*

$$\|y_1 - y_2\|_{H^1} + \|y_1 - y_2\|_{L^\infty(\Omega)} \leq K(|c_1 - c_2| + \|u_1 - u_2\|_{L^2}),$$

*where $y_i := y_{c_i}(u_i)$ are solutions of the regularized state equation* (3.1)*(i) for controls $u_i$.*

*Proof.* The boundedness result can be obtained by testing the weak formulation of the state equation by the solution $y_c(u)$ itself. Due to (A.ii) and the monotonicity of $\max_c$, we get $\|y_c(u)\|^2_{H^1(\Omega)} \leq (y_c(u), u)$, which proves the $H_0^1(\Omega)$-boundedness. The boundedness in $L^\infty(\Omega)$ follows by Stampacchia's method.

Now, let us write

$$\begin{aligned} f_{c_1}(y_1) - f_{c_2}(y_2) &= f_{c_1}(y_1) - f_{c_2}(y_1) + f_{c_2}(y_1) - f_{c_2}(y_2) \\ &= f_{c_1}(y_1) - f_{c_2}(y_1) + \int_0^1 f'_{c_2}(y_2 + s(y_1 - y_2))(y_1 - y_2)ds, \end{aligned}$$

where $y_i := y_{c_i}(u_i)$. Then the difference $\delta y = y_1 - y_2$ fulfills the equation

$$A\delta y + d(y_1, y_2)\delta y = u_1 - u_2 - (f_{c_1}(y_1) - f_{c_2}(y_1))$$

with $d(y_1, y_2) = \int_0^1 f'_{c_2}(y_2 + s(y_1 - y_2))ds$. Since $d(y_1, y_2)$ is non-negative by (B.iii), the claim follows using (A.ii) and the Lipschitz continuity result of Lemma 3.1. □

For the solutions of the adjoint equation in the regularized optimality system, we have the following Lipschitz estimate.

**Lemma 3.3.** *Let $I$ be a compact subset of $(0,+\infty)$ and $U$ be a bounded subset of $L^2(\Omega)$. Then there exists a constant $K = K(I,U)$ such that for all $c_1, c_2 \in I$ and $u_1, u_2 \in U$*

$$\|p_1 - p_2\|_{H^1} \leq K(|c_1 - c_2| + \|u_1 - u_2\|_{L^2}),$$

*where $p_i := p_{c_i}(u_i)$ are solutions of the regularized adjoint equation* (3.1)*(ii) corresponding to the states $y_i := y_{c_i}(u_i)$.*

*Proof.* Let $c_1, c_2 \in I$ and $u_1, u_2 \in U$ be given and set $y_i := y_{c_i}(u_i)$, $p_i := p_{c_i}(u_i)$. Let us write

$$f'_{c_1}(y_1)p_1 - f'_{c_2}(y_2)p_2 = \big(f'_{c_1}(y_1) - f'_{c_2}(y_2)\big)p_1 + f'_{c_2}(y_2)(p_1 - p_2). \tag{3.3}$$

The difference $\delta p := p_1 - p_2$ solves the equation

$$A\delta p + f'_{c_2}(y_2)\delta p = -g'(y_1) + g'(y_2) - \big(f'_{c_1}(y_1) - f'_{c_2}(y_2)\big)p_1. \tag{3.4}$$

Let us show that the adjoint states $p_i$ belong to a bounded set in $H^1_0(\Omega)$. Due to Lemma 3.2 the $H^1_0(\Omega)$-norms of $y_c(u)$ are bounded uniformly for $c \in I$, $u \in U$. Since $g'$ is Lipschitz continuous by (A.iv), the $L^2(\Omega)$-norm of $g'(y_c(u))$ is bounded uniformly for $c \in I$, $u \in U$ as well. Due to (A.ii) and the non-negativity of $f'_c$ by (B.iii), there is a constant $M > 0$ independent of $c$, such that $\|p_c(u)\|_{H^1} \le M\|g'(y_c(u))\|_{L^2}$, which proves that the $H^1(\Omega)$-norms of $p_c(u)$ are uniformly bounded for $c \in I, u \in U$.

Combining the results of the previous Lemmata 3.2 and 3.1 to estimate the $L^2(\Omega)$ norm of the right-hand side of (3.4) yields the claim. $\square$

In the proof of the main result of this section below, we will need the following estimate for remainder terms in differences of solutions of the non-linear state equation and the solution of a linearized equation.

**Lemma 3.4.** *Let $I$ be a compact subset of $(0, +\infty)$ and $U$ be a bounded subset of $L^2(\Omega)$. Let $z$ be the solution of the linearized equation*

$$Az + f'_{c_2}(y_{c_2}(u_2))z = u_1 - u_2.$$

*Then there is a constant $K = K(I, U)$ such that for all $c_1, c_2 \in I$ and $u_1, u_2 \in U$*

$$\|z - (y_{c_1}(u_1) - y_{c_2}(u_2))\|_{H^1} \le K(|c_1 - c_2| + r)$$

*with $r = o(|c_1 - c_2| + \|u_1 - u_2\|_{L^2})$, where the convergence $\frac{r}{|c_1-c_2|+\|u_1-u_2\|_{L^2}} \to 0$ for $|c_1 - c_2| + \|u_1 - u_2\|_{L^2} \to 0$ is uniform on $I, U$.*

*Proof.* Let $c_1, c_2 \in I$ and $u_1, u_2 \in U$ be given and set $y_i := y_{c_i}(u_i)$. It holds

$$\begin{aligned} f_{c_1}(y_1) - f_{c_2}(y_2) &= f_{c_1}(y_1) - f_{c_2}(y_1) + f_{c_2}(y_1) - f_{c_2}(y_2) \\ &= f_{c_1}(y_1) - f_{c_2}(y_1) + f'_{c_2}(y_2)(y_1 - y_2) + r_1 \end{aligned}$$

with the remainder term $r_1$ given by

$$r_1 = \int_0^1 \int_0^s f''_{c_2}(y_2 + t(y_1 - y_2))(y_1 - y_2)^2 \, dt \, ds.$$

Here, we apply (B.iv) to estimate

$$\|r_1\|_{L^2(\Omega)} \leq \frac{1}{2} M(\sup I)^3 \|y_1 - y_2\|^2_{L^4(\Omega)} \leq \tilde{K} \|y_1 - y_2\|_{L^\infty(\Omega)}, \tag{3.5}$$

where $\tilde{K} = \tilde{K}(I, U)$, where we use the fact that by Lemma 3.2, the set $\{y_c(u) : \ c \in I, u \in U\}$ is bounded in $L^\infty(\Omega)$.

Subtracting the defining equations for $y_1, y_2, z$, we find that the function $v := z - (y_1 - y_2)$ satisfies

$$Av + f'_{c_2}(y_2)v = f_{c_1}(y_1) - f_{c_2}(y_1) + r_1.$$

Applying inequality (3.2) in the proof of Lemma 3.1 to estimate $f_{c_1}(y_1) - f_{c_2}(y_1)$, and using Lemma 3.2 to bound the remainder $r_1$ implies the desired conclusion. □

In order to prove Lipschitz continuity, we will need that a sufficient optimality condition is fulfilled along the path. In the previous work [9], we showed that this condition is a consequence of the coercivity condition Assumption 1 and the sign condition Assumption 2.

**Lemma 3.5.** *Let Assumptions 1 and 2 be satisfied. Then there exists $\alpha > 0$ and $C_3 > 0$ such that for all $c > C_3$*

$$g''(y_c)(z, z) + j''(u_c)(h, h) + (f''_c(y_c)p_c z, z) \geq \alpha \|h\|^2_{L^2} \tag{3.6}$$

*for all $(z, h) \in H^1_0(\Omega) \times L^2(\Omega)$ satisfying*

$$Az + f'_c(y_c)z = h. \tag{3.7}$$

Applying this sufficient optimality condition, we show Lipschitz continuity of the path.

**Theorem 3.6.** *Let Assumptions 1 and 2 be satisfied. Then the path $c \mapsto (y_c, u_c, p_c)$, $\mathbb{R} \to (H^1_0(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega) \times H^1_0(\Omega)$ is Lipschitz continuous on compact subsets of $(C_4, +\infty)$ with $C_4 = \max(C_2, C_3)$. That is, for any compact set $I \subset (C_4, +\infty)$ there exists a constant $K = K(I) > 0$ such that*

$$\|u_{c_1} - u_{c_2}\|_{L^2} + \|y_{c_1} - y_{c_2}\|_{H^1} + \|y_{c_1} - y_{c_2}\|_{L^\infty} + \|p_{c_1} - p_{c_2}\|_{H^1} \leq K|c_1 - c_2|$$

*for all $c_1, c_2 \in I$.*

*Proof.* Let us take a compact set $I \subset (C_4, +\infty)$ and $c_1, c_2 \in I$. Then the set $\{(y_c, u_c, p_c)\}_{c\in I}$ is bounded in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega)$ by Theorem 2.1 and Lemmata 3.2 and 3.3. Throughout the proof, $K$ denotes generic, positive constant that depends on $I$ and the global bounds of $\{(y_c, u_c, p_c)\}_{c\in I}$ in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega)$. We denote $\delta u := u_{c_1} - u_{c_2}$, $\delta y := y_{c_1} - y_{c_2}$, $\delta p := p_{c_1} - p_{c_2}$.

At first we substract the optimality conditions (3.1)(iii) for $c_1$ and $c_2$ to obtain

$$(\delta p, \delta u) = (j'(u_{c_1}) - j'(u_{c_2}))(\delta u) = j''(u_{c_2})(\delta u, \delta u) + r_1 \tag{3.8}$$

with $r_1 = o(\|\delta u\|_{L^2}^2)$ by the differentiability assumptions (A.v) on $j$.

Testing the regularized state equations (3.1)(i) for $y_{c_1}$ and $y_{c_2}$ by $\delta p$ gives

$$\langle A\delta y, \delta p\rangle + (f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_2}), \delta p) = (\delta u, \delta p).$$

With arguments as in the proof of Lemma 3.4 we have

$$f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_2}) = f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_1}) + f'_{c_2}(y_{c_2})(y_{c_1} - y_{c_2}) + r_2$$

with $\|r_2\|_{L^2} = o(\|\delta y\|_{H^1})$, which satisfies by (3.5)

$$\frac{\|r_2\|_{L^2(\Omega)}}{\|\delta y\|_{H^1}} \to 0 \text{ as } \|\delta y\|_{H^1} \to 0 \text{ uniformly on } I.$$

Hence we can write

$$(\delta u, \delta p) = \langle A\delta y, \delta p\rangle + (f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_1}), \delta p) + (f'_{c_2}(y_{c_2})\delta y, \delta p) + (r_2, \delta p). \tag{3.9}$$

Testing the adjoint equations (3.1)(ii) for $p_{c_1}$ and $p_{c_2}$ with $\delta y$ and substracting them yields

$$\langle A^*\delta p, \delta y\rangle + (f'_{c_1}(y_{c_1})p_{c_1} - f'_{c_2}(y_{c_2})p_{c_2} + g'(y_{c_1}) - g'(y_{c_2}),\ \delta y) = 0.$$

Due to the assumptions (A.iv) on $g$ we find

$$(g'(y_{c_1}) - g'(y_{c_2}),\ \delta y) = g''(y_{c_2})(\delta y, \delta y) + r_3$$

with $r_3 = o(\|\delta y\|_{L^2}^2)$.

Similarly to the splitting in (3.3), let us write

$$f'_{c_1}(y_{c_1})p_{c_1} - f'_{c_2}(y_{c_2})p_{c_2} = (f'_{c_1}(y_{c_1}) - f'_{c_2}(y_{c_1}))p_{c_1} + f'_{c_2}(y_{c_1})\delta p + (f'_{c_2}(y_{c_1}) - f'_{c_2}(y_{c_2}))p_{c_2}.$$

The last addend can be transformed to

$$(f'_{c_2}(y_{c_1}) - f'_{c_2}(y_{c_2}))p_{c_2} = p_{c_2} f''_{c_2}(y_{c_2})\delta y + r_4$$

with the remainder term

$$r_4 = \int_0^1 \left(f''_{c_2}(y_{c_2} + s\delta y) - f''_{c_2}(y_{c_2})\right)\, ds\ \delta y\, p_{c_2},$$

which can be estimated due to (B.v) as

$$\|r_4\|_{L^2} \le \frac{1}{2} M c^5 \|\delta y\|^2_{L^6} \|p_{c_2}\|_{L^6}.$$

Since $\|p_c\|_{H^1}$ is bounded for $c \in I$, we have by the embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$

$$\|r_4\|_{L^2} \le K \|\delta y\|^2_{H^1},$$

with a constant $K = K(I)$.

These computations yield the equation

$$\begin{aligned}(3.10)\quad \langle A^*\delta p, \delta y\rangle = &-((f'_{c_1}(y_{c_1}) - f'_{c_2}(y_{c_1}))p_{c_1}, \delta y) - (f'_{c_2}(y_{c_1})\delta p, \delta y)\\ &- g''(y_{c_2})(\delta y, \delta y) - (f''_{c_2}(y_{c_2})p_{c_2}\, \delta y, \delta y) - r_3 - (r_4, \delta y).\end{aligned}$$

Combining (3.8)–(3.10) implies that

$$\begin{aligned}(3.11)\quad & j''(u_{c_2})(\delta u, \delta u) + g''(y_{c_2})(\delta y, \delta y) + (f''_{c_2}(y_{c_2})p_{c_2}\, \delta y, \delta y)\\ &= -((f'_{c_1}(y_{c_1}) - f'_{c_2}(y_{c_1}))p_{c_1}, \delta y) + (f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_1}), \delta p)\\ &\quad + ((f'_{c_2}(y_{c_2}) - f'_{c_2}(y_{c_1}))\delta y, \delta p) - r_1 + (r_2, \delta p) - r_3 - (r_4, \delta y).\end{aligned}$$

The pair $(\delta u, \delta y)$ is not suitable as test function in inequality (3.6), since $\delta y$ does not satisfy a linearized equation. Let us therefore introduce the function $z$ as the solution of

$$Az + f'_{c_2}(y_{c_2})z = \delta u.$$

Then $(z, \delta u)$ can be used as test function and we have

$$(3.12)\qquad g''(y_{c_2})(z, z) + j''(u_{c_2})(\delta u, \delta u) + (f''_{c_2}(y_{c_2})p_{c_2}\, z, z) \ge \alpha \|\delta u\|^2_{L^2}.$$

We shall use

$$\begin{aligned}(3.13)\quad g''(y_{c_2})(\delta y, \delta y) + (f''_{c_2}(y_{c_2})p_{c_2}\, \delta y, \delta y) &= g''(y_{c_2})(z, z) + (f''_{c_2}(y_{c_2})p_{c_2}\, z, z)\\ &+ g''(y_{c_2})(\delta y - z, \delta y + z) + (f''_{c_2}(y_{c_2})p_{c_2}(\delta y - z), \delta y + z).\end{aligned}$$

Combining (3.11)–(3.12) implies

$$
\begin{aligned}
\alpha\|\delta u\|_{L^2}^2 \leq &-((f'_{c_1}(y_{c_1}) - f'_{c_2}(y_{c_1}))p_{c_1}, \delta y) + (f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_1}), \delta p) \\
&+ ((f'_{c_2}(y_{c_2}) - f'_{c_2}(y_{c_1}))\delta y, \delta p) \\
&- g''(y_{c_2})(\delta y - z, \delta y + z) - (f''_{c_2}(y_{c_2})p_{c_2}(\delta y - z), \delta y + z) \\
&- r_1 + (r_2, \delta p) - r_3 - (r_4, \delta y),
\end{aligned}
\tag{3.14}
$$

where it remains to estimate the right-hand side in terms of $\|\delta u\|_{L^2}$ and $|c_1 - c_2|$.

By Lemma 3.1 we infer the existence of a constant $K = K(I)$, such that it holds

$$
\begin{aligned}
\left|((f'_{c_1}(y_{c_1}) - f'_{c_2}(y_{c_1}))p_{c_1}, \delta y)\right| &\leq K|c_1 - c_2|\|\delta y\|_{H^1}, \\
|(f_{c_1}(y_{c_1}) - f_{c_2}(y_{c_1}), \delta p)| &\leq K|c_1 - c_2|\|\delta p\|_{H^1}, \\
\left|((f'_{c_2}(y_{c_2}) - f'_{c_2}(y_{c_1}))\delta y, \delta p)\right| &\leq K|c_1 - c_2|\|\delta y\|_{H^1}\|\delta p\|_{H^1}.
\end{aligned}
\tag{3.15}
$$

Thanks to Lemmata 3.2 and 3.3, we can estimate

$$
\|\delta y\|_{H^1} + \|\delta y\|_{L^\infty} + \|\delta p\|_{H^1} \leq K(|c_1 - c_2| + \|\delta u\|_{L^2}).
$$

Moreover, we have by Lemma 3.4

$$
\|z - \delta y\|_{H^1} \leq K|c_1 - c_2| + r_5
$$

with $r_5 = o(|c_1 - c_2| + \|\delta u\|_{L^2})$.

Due to compact embeddings, the sets $\{y_c\}_{c\in I}$ and $\{p_c\}_{c\in I}$ are compact in $L^4(\Omega)$. Then by the assumptions on $g$ and $f_c$, we have that the sets $\{g''(y_c)\}_{c\in I}$ and $\{f''_c(y_c)p_c\}_{c\in I}$ are bounded in $L^2(\Omega)\times L^2(\Omega)$ and $H^{-1}(\Omega)$.

This yields

$$
\begin{aligned}
&\left|g''(y_{c_2})(\delta y - z, \delta y + z) + (f''_{c_2}(y_{c_2})p_{c_2}(\delta y - z), \delta y + z)\right| \\
&\qquad\qquad \leq K(|c_1 - c_2| + \|\delta u\|_{L^2})(|c_1 - c_2| + r_5).
\end{aligned}
\tag{3.16}
$$

The remainder terms appearing in (3.14) satisfy

$$
|-r_1 + (r_2, \delta p) - r_3 - (r_4, \delta y)| = o(|c_1 - c_2|^2 + \|\delta u\|_{L^2}^2).
\tag{3.17}
$$

Collecting (3.14)–(3.17) gives the estimate

$$
\|\delta u\|_{L^2} \leq K|c_1 - c_2| + r_6
$$

with $r_6 = o(|c_1 - c_2| + \|\delta u\|_{L^2})$, where the convergence $\frac{r_6}{|c_1-c_2|+\|\delta u\|_{L^2}} \to 0$ as $|c_1 - c_2| + \|\delta u\|_{L^2} \to 0$ is uniform on $I$ and $\{u_c\}_{c\in I}$.

Then we can choose $\rho_1 > 0$ small enough such that $r_6 \le \frac{1}{2}(|c_1 - c_2| + \|\delta u\|_{L^2})$, which yields

$$\|\delta u\|_{L^2} \le K|c_1 - c_2| \ \text{ if } \ |c_1 - c_2| + \|\delta u\|_{L^2} < \rho_1.$$

By Theorem 2.4, the path $c \mapsto u_c$ is continuous from $I$ to $L^2(\Omega)$. Since $I$ is compact, we can choose $\rho$, $0 < \rho < \rho_1/2$ small enough to ensure $\|\delta u\|_{L^2} = \|u_{c_1} - u_{c_2}\|_{L^2} < \rho_1/2$ for $|c_1 - c_2| < \rho$, which gives

$$\|u_{c_1} - u_{c_2}\|_{L^2} = \|\delta u\|_{L^2} \le K|c_1 - c_2| \quad \text{for all } c \text{ satisfying } |c_1 - c_2| < \rho.$$

Utilizing compactness of $I$ the Lipschitz continuity of $c \to u_c$ for $c \in I$ follows. The Lipschitz estimates for states and adjoints can now be shown using Lemmata 3.2 and 3.3. ☐

As can be seen from the proof, one can prove Lipschitz continuity of the path for all values $c$ such where the claim of Lemma 3.5 holds and the path is continuous at $c$.

# 4 Differentiability of the path

Let us take $c_0$ such that the path is locally Lipschitz at $c_0$. Then clearly a weak limit of the difference quotient $\frac{1}{h}(u_{c+h} - u_c)$ exist. We will be able to prove more, i.e. Fréchet differentiability, using implicit function theorems. To this end, consider the linearized and perturbed system

$$Ay + f_c'(y_c)y = u + z_y, \tag{4.1a}$$

$$A^*p + f_c'(y_c)p + f_c''(y_c)p_c y + g''(y_c)y = z_p, \tag{4.1b}$$

$$j''(u_c)u - p = z_u, \tag{4.1c}$$

with perturbation

$$z = (z_y, z_u, z_p) \in H^{-1}(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) =: Z.$$

**Lemma 4.1.** *Let Assumptions 1 and 2 be satisfied. Then for every $c > C_3$, $C_3$ given by Lemma 3.5, and perturbation $z \in Z$ the system 4.1 admits a unique solution $(y_z, u_z, p_z)$. Moreover, the mapping $z \mapsto (y_z, u_z, p_z)$ is linear and continuous from $Z$ to $H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$.*

*Proof.* The system (4.1) is the first-order necessary optimality condition of the constrained optimization problem:

$$\min \frac{1}{2}(f_c''(y_c)\,p_c\,y,y) + \frac{1}{2}g''(y_c)(y,y) + \frac{1}{2}j''(u_c)(u,u) - \langle z_p, y\rangle - (z_u,u)$$

subject to the linearized equation (4.1a). Due to Lemma 3.5 this problem is convex and admits a unique solution. The convexity property also implies that each solution of the system (4.1) is a minimizer. Hence the system is uniquely solvable. Clearly, the mapping $z \mapsto (y_z, u_z, p_z)$ is linear. To prove the continuity, let us test (4.1a) by $p_z$, (4.1b) by $y_z$, and (4.1c) by $u_z$. This yields

$$j''(u_c)(u_z,u_z) + g''(y_c)(y_z,y_z) + (f_c''(y_c)p_c y_z, y_z) = (z_u,u_z) + \langle z_y, p_z\rangle + \langle z_p, y_z\rangle.$$

The coercivity result of Lemma 3.5 together with standard estimates applied to the elliptic equations (4.1a), (4.1b) allow to conclude the proof. □

**Theorem 4.2.** *Let Assumptions 1 and 2 be satisfied. Then for all $c > C_4$, $C_4$ given by Theorem 3.6, the path $c \mapsto (y_c, u_c, p_c)$ is Fréchet-differentiable with values in $H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$. The derivative with respect to $c$, denoted by $(\dot y_c, \dot u_c, \dot p_c)$ is the unique solution of the system*

$$A\dot y_c + f_c'(y_c)\dot y_c + \frac{\partial}{\partial c} f_c(y_c) = \dot u_c, \tag{4.2a}$$

$$A^*\dot p_c + f_c'(y_c)\dot p_c + f_c''(y_c)p_c\dot y_c + \frac{\partial}{\partial c} f_c'(y_c)p_c + g''(y_c)\dot y_c = 0, \tag{4.2b}$$

$$j''(u_c)\dot u_c - \dot p_c = 0, \tag{4.2c}$$

*Proof.* At first, note that due to the requirements specified in (B) the mapping

$$(y,u,p) \mapsto F_c(y,u,p) = \begin{pmatrix} Ay + f_c(y) - u \\ A^*p + f_c'(y)p + g'(y) \\ j'(u) + p \end{pmatrix}$$

is continuously Fréchet-differentiable from $H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)$ to $H^{-1}(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$ for each $c$. Moreover, the derivative with respect to $(y,u,p)$ is continuous in $c$. In addition, the mapping $c \mapsto F_c(y,u,p)$ is Lipschitz continuous uniformly in $(y,u,p)$ and continuously Fréchet-differentiable. In particular, the mapping $(c,y,p) \mapsto \frac{\partial}{\partial c} f_c'(y)p$ is continuous from $\mathbb{R}^+ \times H_0^1(\Omega)\times H_0^1(\Omega)$

to $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, due to (B.iv) and (B.vii). Due to Lemma 4.1 the equation

$$F'_c(y_c, u_c, p_c)(y, u, p) = z$$

admits for all $c > C_4$ a unique solution that depends linearly and continuously on $z$. Hence by the implicit function theorem [3, Theorem 2.1], the mapping $c \mapsto (y_c, u_c, p_c)$ is Fréchet differentiable and the derivative with respect to $c$ is given as the unique solution of (4.2). □

# 5 Path and value-function properties

Here, we will investigate properties of the value function, which is defined as

$$V(c) := J(y_c, u_c).$$

Throughout this section we will assume that Assumptions 1 and 2 are satisfied. Then due to the previous sections, $V$ is continuous and differentiable from $(C_4, +\infty)$ to $\mathbb{R}$. Hence, we work throughout the whole section with values $c > C_4$ without explicitely mentioning.

We further assume throughout this section that

$$j(u) = \frac{\beta}{2}\|u\|^2_{L^2} \quad \text{and} \quad \psi = 0 \text{ on } \partial\Omega.$$

**Lemma 5.1.** *The value function $V$ is continuous and twice differentiable from $(C_4, +\infty)$ to $\mathbb{R}$. Moreover we have*

(5.1)
$$\begin{aligned}\dot V(c) &= \left(\frac{\partial}{\partial c} f_c(y_c),\, p_c\right)\\ &= \left(\frac{\partial}{\partial c}\max{}_c\left(\bar\lambda + c(y_c - \psi)\right) + \max{}'_c(\bar\lambda + c(y_c - \psi))(y_c - \psi),\, p_c\right),\end{aligned}$$

*and*

(5.2)
$$\begin{aligned}\ddot V &= \beta\|\dot u_c\|^2_{L^2} + g''(y_c)(\dot y_c, \dot y_c) + (f''_c(y_c)p_c\dot y_c, \dot y_c)\\ &\qquad + 2(p_c, \frac{\partial}{\partial c} f'_c(y_c)\dot y_c) + (p_c, \frac{\partial^2}{\partial c^2} f_c(y)).\end{aligned}$$

*Proof.* Due to Theorem 4.2, $V$ is differentiable. By application of the chain rule to $V(c)$, we find

$$\dot{V}(c) = (g'(y_c), \dot{y}_c) + (j'(u_c), \dot{u}_c).$$

Using the sensitivity system (4.1), this relation can be transformed to

$$\begin{aligned}
\dot{V}(c) &= (g'(y_c), \dot{y}_c) + (j'(u_c), \dot{u}_c) \\
&= -\langle A^* p_c, \dot{y}_c \rangle - (f_c'(y_c) p_c, \dot{y}_c) + (p_c, \dot{u}_c) \\
&= -\langle A \dot{y}_c, p_c \rangle - (f_c'(y_c) \dot{y}_c, p_c) + (\dot{u}_c, p_c) \\
&= \left( \frac{\partial}{\partial c} f_c(y_c),\, p_c \right).
\end{aligned}$$

Due to the smoothness of $\max_c$ and the continuity and differentiability of the path, $\dot{V}$ is continuous and differentiable for $c > C_4$. Thus, differentiating this expression with respect to $c$, yields the second derivative of $V$:

$$\ddot{V}(c) = (\frac{\partial}{\partial c} f_c(y_c),\, \dot{p}_c) + (\frac{\partial}{\partial c} f_c'(y_c)\dot{y}_c,\, p_c) + (\frac{\partial^2}{\partial c^2} f_c(y_c),\, p_c).$$

Using (4.2a), (4.2c) and (4.2b) we obtain

$$\begin{aligned}
(\frac{\partial}{\partial c} f_c(y_c), \dot{p}_c) &= (\dot{u}_c, \dot{p}_c) - \langle A\dot{y}_c, \dot{p}_c \rangle - (f_c'(y_c)\dot{y}_c, \dot{p}_c) \\
&= \beta \|\dot{u}_c\|_{L^2}^2 - \langle A^* \dot{p}_c, \dot{y}_c \rangle - (f_c'(y_c)\dot{y}_c, \dot{c}) \\
&= \beta \|\dot{u}_c\|_{L^2}^2 + g''(y_c)(\dot{y}_c, \dot{y}_c) + (\frac{\partial}{\partial c} f_c'(y_c) p_c, \dot{y}_c) + (f_c''(y_c) p_c \dot{y}_c, \dot{y}_c).
\end{aligned}$$

Combining these equalities implies the desired result (5.2). □

We will now show asymptotic properties of the value function and distinguish between two cases:

(i) **infeasible case:** the shift parameter $\bar{\lambda}$ is set to zero, $\bar{\lambda} = 0$,

(ii) **feasible case:** the states $y_c$ are assumed to be feasible, i.e. $y_c \le \psi$ for all $c > C_4$.

Let us remark that the feasibility of solutions $y_c$ can be enforced by using a parameter $\bar{\lambda}$ that is large enough, see Proposition 1.2.

We define two family of sets that will play an important role in the analysis

$$\mathcal{N}_c = \left\{ x : |\bar{\lambda}(x) + c(y_c(x) - \psi(x))| < \frac{1}{2c} \right\} \tag{5.3}$$

$$\mathcal{A}_c = \left\{ x : \bar{\lambda}(x) + c(y_c(x) - \psi(x)) \geq \frac{1}{2c} \right\}. \tag{5.4}$$

On the 'strongly active' set $\mathcal{A}_c$ the approximation $\max_c \big(\bar{\lambda} + c(y - \psi)\big)$ coincides with $\max(0, \bar{\lambda} + c(y_c - \psi))$. On the 'strongly inactive' set $\Omega \setminus (\mathcal{N}_c \cup \mathcal{A}_c)$ we have $\max_c \big(\bar{\lambda} + c(y - \psi)\big) = 0$. The set $\mathcal{N}_c$ can be interpreted as an approximation of the biactive set $B = \{y^* = \psi,\, \lambda^* = 0\}$. It causes most of the difficulties in the subsequent analysis.

Let us start the investigation of $V$ with the analysis of the asymptotic behavior of $\dot{V}$ for $c \to \infty$.

**Proposition 5.2.** *(i) infeasible case: $\dot{V}(c) = O(\frac{1}{c})$ for $c \to \infty$,*

*(ii) feasible case: $\dot{V}(c) = o(\frac{1}{c})$ for $c \to \infty$.*

*Proof.* We have

$$\dot{V}(c) = \left( \frac{\partial}{\partial c} \max_c \big(\bar{\lambda} + c(y_c - \psi)\big) + \max_c'(\bar{\lambda} + c(y_c - \psi))(y_c - \psi),\, p_c \right) = I + II. \tag{5.5}$$

Since $\frac{\partial}{\partial c}\max_c$ is zero on $\mathcal{A}_c$, we find for $I$ by (B.vi)

$$|I| \leq \frac{M}{c^2} \int_{\mathcal{N}_s} |p_c| = O\left(\frac{1}{c^2}\right) \text{ for } c \to \infty. \tag{5.6}$$

To estimate $II$, we first give an estimate of $y_c - \psi$ on the active set $\mathcal{A}_c$. We have

$$A(y_c - \psi) + f_c(y_c) - f_c(\psi) = u_c - A\psi - f_c(\psi),$$

and hence by testing this equation with $y - \psi$ and applying (A.ii)

$$\int_{\mathcal{A}_c} (f_c(y_c) - f_c(\psi))(y_c - \psi) \leq M(\|u_c - A\psi - \max_c(\bar{\lambda})\|_{L^2}^2$$

with a constant $M > 0$ independent of $c$. Here, we used $f_c(\psi) = \max_c(\bar{\lambda})$ and the monotonicity of $f_c$ on the set $\mathcal{N}_c$. Moreover, on the set $\mathcal{A}_c$ we have $\max_c \big(\bar{\lambda} + c(y_c - \psi)\big) = \bar{\lambda} + c(y_c - \psi)$, which yields

$$f_c(y_c) - f_c(\psi) = \bar{\lambda} + c(y_c - \psi) - \bar{\lambda} = c(y_c - \psi) \text{ on } \mathcal{A}_c.$$

Using (B.ii) we obtain the estimate

$$\|y_c - \psi\|_{L^2(\mathcal{A}_c)} \le \frac{M}{\sqrt{c}},$$

for a modified constant $M$. Testing the adjoint equation for $p_c$ by $p_c$ itself we obtain

$$a(p_c, p_c) + c(\max_c'(\bar\lambda + c(y_c - \psi))p_c, p_c) = -(g'(y_c), p_c),$$

which implies the existence of $M > 0$ independent of $c \ge 1$ such that

$$c(\max_c'(\bar\lambda + c(y_c - \psi))p_c, p_c) \le M^2, \; c \ge 1.$$

Then we can estimate

$$\begin{aligned} II &= (\max_c'(\bar\lambda + c(y_c - \psi))(y_c - \psi), p_c) \\ &\le \left(\int_\Omega \max_c'(\bar\lambda + c(y_c - \psi))p_c^2\right)^{\frac{1}{2}} \left(\int_\Omega \max_c'(\bar\lambda + c(y_c - \psi))(y_c - \psi)^2\right)^{\frac{1}{2}} \\ &\le \frac{M}{\sqrt{c}} \left(\int_{\mathcal{A}_c} (y_c - \psi)^2 + \int_{\mathcal{N}_c} \left(\frac{1}{2c}\right)^2\right)^{\frac{1}{2}} = O\left(\frac{1}{c}\right). \end{aligned}$$

In the feasible case slightly more can be proved. In fact,

$$cII = (c\max_c'(\bar\lambda + c(y_c - \psi))(y_c - \psi), p_c) = c\int_\Omega \mu_c(y_c - \psi) \to \langle \mu^*, y^* - \psi\rangle = 0$$

for $c \to \infty$, where $\mu_c = c\max_c'\left(\bar\lambda + c(y_c - \psi)\right)p_c$, as was proved in [7]. By (5.6), we obtain $\dot V(c) = o(1/c)$ for $c \to \infty$. □

In the statement of the following results we shall frequently use assumption (B) for $\max_c$. It will be convenient to add an index to the constant $M$ and let $M_4$ refer to the constant that arises in (B.iv), for example.

**Proposition 5.3.** *(Monotonicity of $V$). Assume that*

$$p_c \ge 0 \text{ on } \mathcal{A}_c. \tag{5.7}$$

*Then we have:*

*(i) infeasible case:* $\dot V(c) \ge -\frac{1}{c^2}(M_5 + \frac{1}{2})\,|p_c|_{L^1(\mathcal{N}_c)}$, *on* $(0, \infty)$.

(ii) *feasible case:* $\dot V(c) \le \frac{M_5}{c^2}\,|p_c|_{L^1(\mathcal{N}_c)}$, *for all c sufficiently large. If* $\max_c$ *is defined as in* (1.12), *then* $\dot V(c) \le 0$ *for all c sufficiently large.*

*Proof.* We use (5.5) and turn to the infeasible case first. The properties of $\max_c$ imply that $I \ge -\frac{M_5}{c^2}\,|p_c|_{L^1(\mathcal{N}_c)}$. The expression $II$ is nonnegative on $\mathcal{A}_c^s$ and $\Omega\backslash(\mathcal{A}_c^s \cup \mathcal{N}_c)$. Hence $II$ restricted to $\mathcal{N}_c$, only, needs our attention:

$$\int_{\mathcal{N}_c} p_c \cdot \max\nolimits_c'(c(y_c-\psi))(y_c-\psi) \ge -\frac{1}{2c^2}\int_{\mathcal{N}_c} |p_c|.$$

This gives the desired estimate in the infeasible case. For the feasible case, assume that $c$ is sufficiently large so that $y_c \le \psi$. Then $II \le 0$ by (5.7) and since $\max_c' \ge 0$. Moreover $I \le \frac{M_5}{c^2}|p_c|_{L^1(\mathcal{N}_c)}$ and $I \le 0$ if $\max_c$ is given as in (1.12), see (1.13). □

**Proposition 5.4.** *('Concavity' of $V$, infeasible case). Assume that*

$$\dot p_c \le 0 \text{ on } \mathcal{A}_c \text{ and } p_c \ge 0 \text{ on } \mathcal{N}_c. \tag{5.8}$$

*Then*

$$\ddot V(c) \le -(g''(y_c)\dot y_c,\, \dot y_c) + \frac{M_4+M_7+M_8}{c^3}\,|p_c|_{L^1(\mathcal{N}_c)}$$

*holds for all c sufficiently large.*

Let us remark, that for the choice $\max_c = m_c$ by (1.13) the estimate remains true with $M_8 = 0$.

*Proof.* For $c > 0$ we have

$$\ddot V(c) = (\dot p_c,\, \frac{\partial}{\partial c}\, f_c\,(y_c)) + (p_c,\, \frac{\partial^2}{\partial c^2}\, f_c(y_c)) + (p_c,\, \frac{\partial}{\partial c}\, f'(y_c)\dot y_c),$$

where by the sensitivity equations (4.2b), (4.2a), and (4.2c) for $\dot y_c, \dot p_c, \dot u_c$ we have

$$\begin{aligned}
(p_c,\, \frac{\partial}{\partial c}\, f_c'(y_c)\dot y_c) &= -(g''(y_c)\dot y_c,\, \dot y_c) - (A^*\dot p_c + f_c''(y_c)p_c\,\dot y_c + f_c'(y_c)\dot p_c,\, \dot y_c)\\
&= -(g''(y_c)\dot y_c,\, \dot y_c) - (f_c''(y_c)p_c\,\dot y_c, \dot y_c) - (A\dot y_c + f_c'(y_c)\dot y_c,\, \dot p_c)\\
&= -(g''(y_c)\dot y_c,\, \dot y_c) - (f_c''(y_c)p_c\,\dot y_c, \dot y_c) - (\frac{1}{\beta}\,\dot p_c - \frac{\partial}{\partial c}\, f_c(y_c), \dot p_c).
\end{aligned}$$

Hence we obtain

$$\ddot V(c) = -(g''(y_c)\dot y_c,\, \dot y_c) - \frac{1}{\beta}\,\|\dot p_c\|^2_{L^2} - (f_c''(y_c)p_c\,\dot y_c, \dot y_c) + 2(\dot p_c,\, \frac{\partial}{\partial c}\, f_c(y_c)) + (p_c,\, \frac{\partial^2}{\partial c^2}\, f_c(y_c)) = \tilde I + \ldots + \tilde V. \tag{5.9}$$

Since by assumption $p_c \ge 0$ on $\mathcal{N}_c$ we have by (B.iv)

$$I\tilde I I \le 0. \tag{5.10}$$

Using (B.vi), (B.iii) and the assumption $\dot p_c \le 0$ on $\mathcal{A}_c$ we find

$$\begin{aligned} I\tilde V &= 2\int_\Omega \dot p_c\left( (\frac{\partial}{\partial c}) \max{}_c(c(y_c-\psi)) + \max{}_c'(c(y_c-\psi))(y-\psi)\right) \\ &\le \frac{2M_6+1}{c^2}|\dot p_c|_{L^1(\mathcal{N}_c)}. \end{aligned} \tag{5.11}$$

To estimate $\tilde V$ note that

$$\frac{\partial^2}{\partial c^2}\, f_c(y_c) = (\frac{\partial^2}{\partial c^2}\max{}_c)(c(y_c-\psi)) + 2\,(\frac{\partial}{\partial c}\max{}_c')(c(y_c-\psi))(y_c-\psi) + \max{}_c''(c(y_c-\psi))(y_c-\psi)^2. \tag{5.12}$$

Therefore by (B.viii), (B.vii), and (B.iii)

$$\tilde V \le \frac{M_8+M_7+M_4}{c^3}\,|p_c|_{L^1(\mathcal{N}_c)}. \tag{5.13}$$

Combining (5.9)-(5.13) we obtain for $c > C_4$

$$\ddot V(c) \le -(g''(y_c)\dot y_c,\, \dot y_c) + \left(-\frac{1}{\beta} + \frac{2M_6+1}{c^2}\right)|\dot p_c|^2_{L^2(\Omega)} + \frac{M_8+M_7+M_4}{c^3}\,|p_c|_{L^1(\mathcal{N}_c)}$$

and hence for $c > \max\left(C_4, \sqrt{\frac{2M_6+1}{\beta}}\right)$

$$\ddot V(c) \le -(g''(y_c)\dot y_c,\, \dot y_c) + \frac{M_8+M_7+M_4}{c^3}\,|p_c|_{L^1(\mathcal{N}_c)},$$

which gives the desired estimate. $\square$

**Remark 5.5.** Proposition (5.4) implies in particular that for $c$ sufficiently large, $\ddot{V}(c) \le \frac{K}{c^3}$, where $K = \sup\{|p_c|_{L^1(\Omega)}\}_{c>1} < \infty$ and hence, up to an error of $O(\frac{1}{c^3})$, the function $c \to V(c)$ is concave.

The requirement that $\dot{p}_c \le 0$ on $\mathcal{A}_c^s$ is difficult to check a-priori. It is consistent with the assumption that $p_c \ge 0$ on $\mathcal{A}_c^s \cup \mathcal{N}_c$ and fact that for the solution of the limit problem ($c = \infty$) we have $p^* = 0$ on the strongly active set, where $\lambda > 0$.

**Proposition 5.6.** *('Convexity' of $V$, feasible case). Assume that there exists $C_5 \ge C_4$ such that for $c \ge C_5$ $y_c \le \psi$*

$$\dot{y}_c \ge 0 \quad \text{on } \mathcal{N}_c \cup \mathcal{A}_c \quad \text{and} \quad p_c \ge 0 \text{ on } \mathcal{N}_c \cup \mathcal{A}_c, \tag{5.14}$$

*and*

$$\frac{\partial}{\partial c}\max{}_c'(x) \ge 0, \quad \text{and} \quad \dot{y}_c \ge \frac{1}{c^3} + \frac{2\bar{\lambda}}{c^2} \ge 0 \quad \text{on } \mathcal{N}_c. \tag{5.15}$$

*Then for $c \ge C_5$ we have*

$$\ddot{V}(c) \ge \beta\,\|\dot{u}_c\|_{L^2}^2 + g''(y_c)(\dot{y}_c,\,\dot{y}_c) - \left(\frac{M_8 + M_7}{c^3} + \frac{2M_7|\bar{\lambda}|_{L^\infty(\mathcal{N}_c)}}{c^2}\right)|p_c|_{L^1(\mathcal{N}_c)}.$$

*Proof.* From Lemma 5.1, eq. (5.2), we have

(5.16)
$$\ddot{V}(c) = \beta\,\|\dot{u}_c\|_{L^2}^2 + g''(y_c)(\dot{y}_c,\,\dot{y}_c) + (f_c''(y_c)p_c\,\dot{y}_c,\,\dot{y}_c) + 2(p_c,\,\frac{\partial}{\partial c}f_c'(y_c)\dot{y}_c)$$
$$+ (p_c,\,\frac{\partial^2}{\partial c^2}f_c(y_c)) = \tilde{I} + \ldots + \tilde{V}.$$

Assumptions (5.14) and (5.15) imply that

$$\begin{aligned}
\tilde{IV} &= 2\big(p_c,\,\frac{\partial}{\partial c}\,(c\,\max{}_c'(\bar{\lambda} + c(y_c - \psi)))\,\dot{y}_c\big)\\
&= 2[(\max{}_c'(\bar{\lambda} + c(y_c - \psi))p_c,\,\dot{y}_c)\\
&\quad + c(\max{}_c''(\bar{\lambda} + c(y_c - \psi))(y_c - \psi)p_c,\,\dot{y}_c) + c(\frac{\partial}{\partial c}\,\max{}_c'(\bar{\lambda} + c(y_c - \psi))p_c,\,\dot{y}_c)]\\
&\ge 2c\big(\max{}_c''(\bar{\lambda} + c(y_c - \psi))(y_c - \psi)p_c,\,\dot{y}_c\big)_{\mathcal{N}_c}.
\end{aligned}$$

On $\mathcal{N}_c$ we have $y_c - \psi \geq -\frac{1}{2c^2} - \frac{\bar{\lambda}}{c}$. This leads together with the second inequality in (5.15) to

$$
\begin{aligned}
\tilde{III} + \tilde{IV} &\geq \int_{\mathcal{N}_c} \max''_c(\bar{\lambda} + c(y_c - \psi)) p_c \dot{y}_c (2c(y_c - \psi) + c^2 \dot{y}_c) \\
&\geq \int_{\mathcal{N}_c} \max''_c(\bar{\lambda} + c(y_c - \psi)) p_c \dot{y}_c (c^2 \dot{y}_c - \frac{1}{c} - 2\bar{\lambda}) \geq 0.
\end{aligned}
\tag{5.17}
$$

As in (5.12), expression $\tilde{V}$ can be written as

$$
\begin{aligned}
&\frac{\partial^2}{\partial c^2} f_c(y_c) = (\frac{\partial^2}{\partial c^2} \max_c)(\bar{\lambda} + c(y_c - \psi)) \\
&\quad + 2\,(\frac{\partial}{\partial c} \max'_c)(\bar{\lambda} + c(y_c - \psi))(y_c - \psi) + \max''_c(\bar{\lambda} + c(y_c - \psi))(y_c - \psi)^2.
\end{aligned}
$$

All these terms vanish outside $\mathcal{N}_c$. On the set $\mathcal{N}_c$, $|y_c - \psi| \leq \frac{1}{2c^2} + \frac{\bar{\lambda}}{c}$ holds. Then we can estimate using (B.viii), (B.vii), (B.iv), and (5.14)

$$
\tilde{V} = \int_{\mathcal{N}_c} p_c \frac{\partial^2}{\partial c^2} f_c(y_c) \geq -|p_c|_{L^1(\mathcal{N}_c)} \left( \frac{M_8}{c^3} + \frac{2M_7}{c} \left( \frac{1}{2c^2} + \frac{|\bar{\lambda}|_{L^\infty(\mathcal{N}_c)}}{c} \right) \right).
\tag{5.18}
$$

Combining (5.16)-(5.18) we obtain the desired result. □

**Remark 5.7.** Proposition 5.6 implies that $\ddot{V}(c) \geq -\frac{K}{c^2}$, for a constant independent of $c$, and hence, up to an error of order $O(\frac{1}{c^2})$ the function $c \to V(c)$ is convex. The requirement that $\dot{y}_c \geq 0$ is consistent with the monotone increasing property of $c \to y_c$, if $\bar{\lambda}$ is chosen as in Proposition 1.2. Monotonicity can be shown with the same arguments as feasibility. The first condition in (5.15) holds for $\max = m_c$, see (1.13). The second condition in (5.15) cannot be checked a-priori.

# 6 Path-following algorithm and computational results

## 6.1 Model function

The goal is to compute a solution to the optimal control problem subject to the variational inequality. This corresponds to setting the path parameter to

$c = +\infty$. Here, an efficient strategy to control the parameter to follow the path is needed.

We shall use a path-following algorithm that is based on the previous analysis of the value function. For a given parameter $c_k$ the value function and its derivatives at $c_k$ can be obtained from the formulas in Section 5. We will use these information to extrapolate the behavior of the value function with the help of a model function $M_{c_k}(c) \approx V(c)$. Such an idea was applied successfully to the numerical solution of the obstacle problem in [6].

Let us suppose that for a given parameter $c_k$ approximations of $V(c_k)$, $\dot{V}(c_k)$, and $\ddot{V}(c_k)$ are available. We then choose $M_{c_k}(c)$ such that $M_{c_k}$ matches $V$ at $c_k$ together with its first and second derivatives. As prototype for $M_{c_k}$ we take the parametrization

$$M_{c_k}(c) = a_1 + \frac{a_2}{c + a_3}, \tag{6.1}$$

for real parameters $a_i$, with the constraint $a_3 > -c$, which guarantees that the singularity of $M_{c_k}$ is on the left of $c_k$. The asymptotic behavior of $V$ for the infeasible case can be recovered by choosing $a_2 < 0$, while $a_2 > 0$ reflects the asymptotics of $V$ for the feasible case.

Given $V(c_k)$, $\dot{V}(c_k)$, and $\ddot{V}(c_k)$, the parameters $a_i$ are computed as

$$\begin{aligned} a_3 &= -2\frac{\dot{V}(c_k)}{\ddot{V}(c_k)} - c_k, \\ a_2 &= -\dot{V}(c_k)(c_k + a_3)^2, \\ a_1 &= V(c_k) - \frac{a_2}{a_3 + c_k}. \end{aligned} \tag{6.2}$$

The properties of $V$ imply that $\dot{V}(c)$ and $\ddot{V}(c)$ have different signs for sufficiently large $c$. Hence, the coefficient $a_3$ is greater than $-c_k$ provided that $c_k$ is large enough. In the computations it happened that $\dot{V}(c)$ and $\ddot{V}(c)$ had the same sign for small $c$. Then $a_3$ as given above would yield a model function that would have singularity at some $c > c_k$, which would makes it useless to predict the trajectory of $V$. In this case, we set $a_3 := 0$ and computed $a_2, a_1$ as above, which resulted in a model function that only fitted the first derivative but did not satisfy $\ddot{M}_{c_k}(c_k) = \ddot{V}(c_k)$.

On the basis of the model function $M_{c_k}$ we determined the next parameter value $c_{k+1}$ such that the distance of $V(c_k)$ to $V(+\infty)$ is reduced by a factor

$\theta \in (0,1)$ from one step to the next. That is, $c_{k+1}$ is computed as the solution of

$$(M_{c_k}(c) - a_1) = \theta((M_{c_k}(c_k) - a_1). \tag{6.3}$$

Here we used $a_1$ to approximate $V(+\infty)$.

The algorithm was stopped when the defect functional

$$\eta(y_c, u_c, \lambda_c, p_c, \mu_c) = \|y_c - \psi\|_{L^2(\mathcal{A}_c)} + |\langle \lambda_c, y_c - \psi\rangle| + |\langle \lambda_c, p\rangle| + |\langle \mu_c, y_c - \psi\rangle| + \|p_c^-\|_{\mathcal{N}_c \cup \mathcal{A}_c}$$

dropped below a certain threshold. This functional measures the defect in the optimality system (1.5)–(1.10) of the original problem and hence $\eta(y^*, u^*, \lambda^*, p^*, \mu^*) = 0$. We stopped the algorithm when $\eta(y_c, u_c, \lambda_c, p_c, \mu_c) < \epsilon_{\text{cc}}$ for fixed, small $\epsilon_{\text{cc}}$.

The regularized optimality system (1.11) was solved by means of Newton's method. As initial guess for this inner loop computation the solution from the last iteration was used. This Newton iteration was stopped as soon as the nonlinear residual was small enough, i.e. if

$$\begin{aligned} \delta^j &:= |\langle f_{c_k}(y_{c_k}^j) - f_{c_k}(y_{c_k}^{j-1}) - f_{c_k}'(y_{c_k}^{j-1})(y_{c_k}^j - y_{c_k}^{j-1}),\ p_{c_k}^j\rangle| \\ &+ |\langle f_{c_k}'(y_{c_k}^j)p_{c_k}^j - f_{c_k}'(y_{c_k}^{j-1})p_{c_k}^j - f_{c_k}''(y_{c_k}^{j-1})p_{c_k}^{j-1}(y_{c_k}^j - y_{c_k}^{j-1}),\ y_{c_k}^j - \psi\rangle| < \epsilon_{\text{newt}} \end{aligned} \tag{6.4}$$

for a given tolerance $\epsilon_{\text{newt}}$ was fulfilled.

The overall solution algorithm is described in Algorithm 1. We refer to this method as exact path-following, since the inner loop was executed until convergence. This technique is close to the theoretical considerations, where we assumed that $(y_c, u_c, p_c)$ are local solution of the regularized problem.

We also tested a modified algorithm in which only one iteration of the inner loop was performed, except for the first iteration of the outer loop $(k = 1)$ where the Newton's method was driven to convergence. In contrast to Algorithm 1, we will refer to this procedure as inexact path-following.

**Remark 6.1.** To obtain the model function in the manner described above requires to solve one additional linear system to obtain $\ddot{V}(c_k)$. We also compared with an approach which evaluates the parameters of the model function without the help of the sensitivity system (4.2). The parameters were determined to satisfy $M_{c_k}(c_k) = V(c_k)$, $\dot{M}_{c_k} = \dot{V}(c_k)$, and $M_{c_k}(c_{k-1}) = V(c_{k-1})$. It turned out this method led to a less accurate approximation of $V$ and consequently that the path-following procedure was less effective.

**Algorithm 1** Exact Path-following

Parameters: $c_0$, $\epsilon_{\rm cc}$, $\epsilon_{\rm newt}$, $\theta \in (0,1)$, $k := 0$, initial guess: $(y_{c_0}, u_{c_0}, p_{c_0})$
**repeat** {Path-following}
  Set $k := k+1$
  Set $j := 0$, $(y^0_{c_k}, u^0_{c_k}, p^0_{c_k}) := (y_{c_{k-1}}, u_{c_{k-1}}, p_{c_{k-1}})$
  **repeat** {Newton's method for ($\mathrm{P}_c$)}
    Set $j := j+1$
    Perform Newton step for (1.11) to obtain $(y^j_{c_k}, u^j_{c_k}, p^j_{c_k})$
    Compute $\delta^j$ from (6.4)
  **until** $\|\delta^j\| < \epsilon_{\rm newt}$
  Set $(y_{c_k}, u_{c_k}, p_{c_k}) := (y^j_{c_k}, u^j_{c_k}, p^j_{c_k})$
  {Model function and new step size}
  Solve the sensitivity system (4.2)
  Compute $V(c_k), \dot V(c_k), \ddot V(c_k)$ and $a_0, a_1, a_2$ as in (6.2)
  Compute $c_{k+1}$ from (6.3)
  Compute $\eta_c := \eta(y_c, u_c, \lambda_c, p_c, \mu_c)$
**until** $\eta_c < \epsilon_{\rm cc}$

## 6.2 Numerical experiments

Let us report on the numerical results for the solution of the following problem: Minimize

$$J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2} + \frac{\alpha}{2}\|u\|^2_{L^2}$$

subject to the variational inequality

$$(\nabla y,\ \nabla v - \nabla y) \geq (u, v - y) \quad \forall v \in K$$

with $K$ as in (1.2). As domain we choose $\Omega = (0,1)^2$.

The function $m_c$ given in (1.12) was chosen as smooth approximation of max. Both choices for $\bar\lambda$, the infeasible and the feasible variant were tested. Here we report only on the infeasible case. The two approaches behave quite similarly. The asymptotic behavior $c \to \infty$, for example is the same. For fixed values of $c$ the feasible case tends to take more Newton iterations than the infeasible one.

The underlying partial differential equation was discretized by finite elements. We used $P1$-elements for state and adjoint discretization, and $P0$-elemente for the control. The computational mesh consisted of $80,000$ triangles with maximal diameter $h = 0.0071$.

### 6.2.1 Example 1

The data for our first example are given by

$$y_d(x) = 5x_1 + x_2 - 1, \quad \psi(x) = 4(x_1(x_1 - 1) + x_2(x_2 - 1)) + 1.5.$$

Furthermore, we set $\alpha = 0.01$. With these choices, all the standing assumptions are satisfied.

We started the path-following algorithm with $c_0 = 100$, and set the parameter $\theta = 0.5$. As tolerances we chose $\epsilon_{\text{cc}} = 10^{-8}$, $\epsilon_{\text{newt}} = 10^{-8}$.

The path-following algorithm with exact and inexact inner solve required 32 and 38 outer loop iterations to reach the stopping criterion, while the inner Newton loop in the exact path-following needed about four iterations in each step. We first report on the results for exact-path following. At the end of this section we will compare the results between exact and inexact methods.

The numerical solutions $(y_{c,h}, p_{c,h}, \lambda_{c,h}, \mu_{c,h})$ for $c = c_N \approx 10^{10}$ are depicted in Figures 1 and 2. The discrete control $u_{c,h}$ is not plotted, since it is the $L^2$-projection of $p_{c,h}$ on the space of piecewise constant functions. As one can see, the multipliers $\lambda_{c,h}$ and $\mu_{c,h}$ only have low regularity.

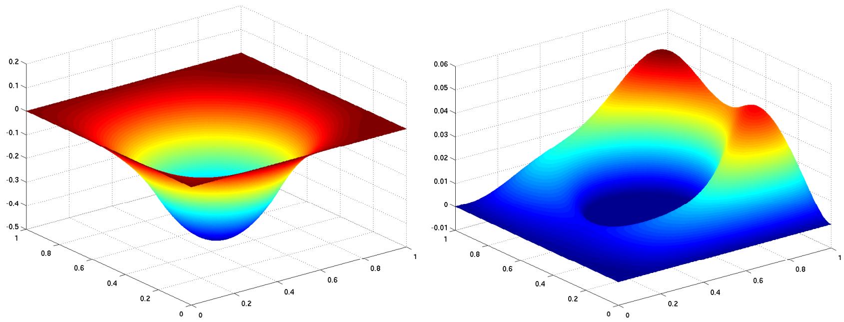

Figure 1: Numerical solution: $y_{c,h}$, $p_{c,h}$

Let us discuss Assumptions 1 and 2 for the present example. The coercivity condition of Assumption 1 is satisfied with $\gamma = \alpha$ due to our choice of $J, j, g$. This does not imply, however, that the original problem (P) is convex, [9]. In [10] it was verified that the condition $\psi \leq y_d$ implies that the optimization problem is convex. This condition is not satisfied in our example. In [9] we argued that $y^* \leq y_d$ is sufficient for local convexity. This condition

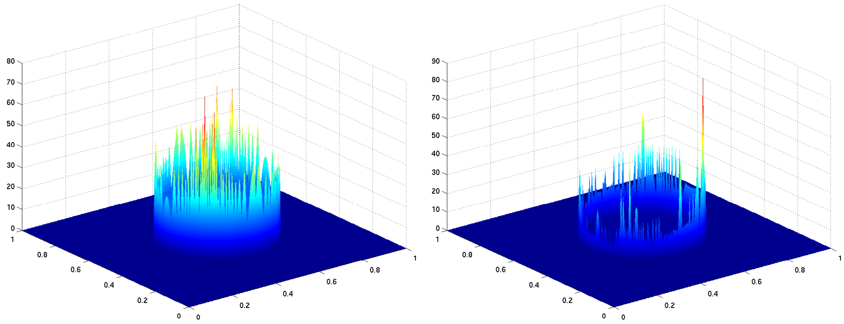

Figure 2: Numerical solution: $\lambda_{c,h}$, $\sqrt{\mu_{c,h}}$

is satisfied for the discrete solution, i.e. $y_{c,h} \leq y_d$. Moreover, the discrete adjoint state $p_{c,h}$ as well as the multiplier $\mu_{c,h}$ are non-negative, as can be seen from Figures 1 and 2. This indicates that Assumption 2 is fulfilled for this example.

As predicted by the theory, in particular by Propositions 5.3 and 5.4, the value function $V$ is monotonically increasing and concave, see Figure 3. The asymptotic behavior of $\dot{V}$ and $\ddot{V}$ is shown in Figure 4. We see that $\dot{V}(c) \sim c^{-2}$ and $\ddot{V}(c) \sim c^{-3}$. Thus, the convergence order is higher than what we were able to prove in Section 5, see e.g. Propositions 5.2, 5.4, and 5.6. This can be the focus of further research.

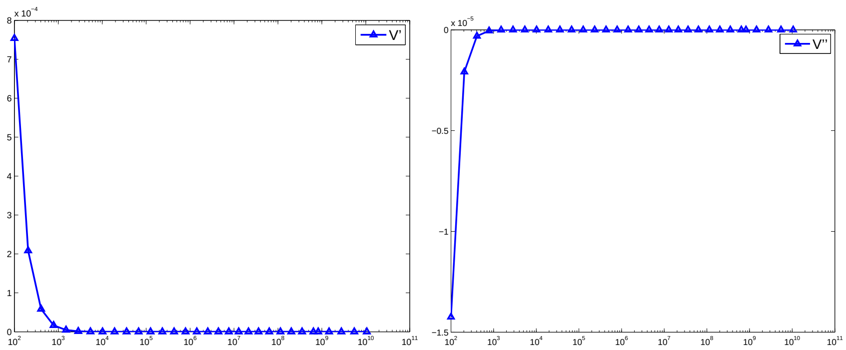


Figure 3: Value function: $\dot{V}(c)$, $\ddot{V}(c)$

On the basis of these numerical observations let us assume that there are

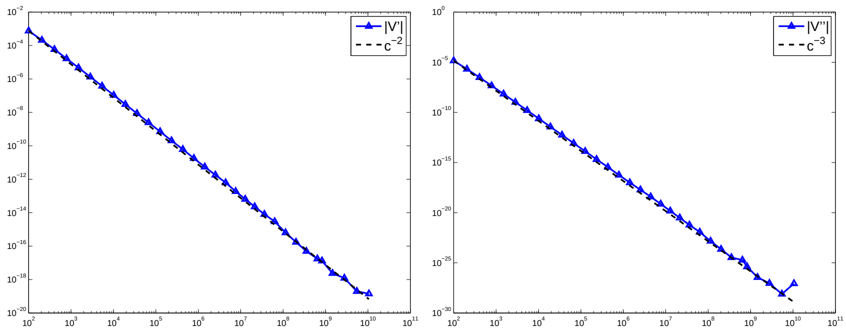


Figure 4: Value function: $\dot{V}(c)$, $\ddot{V}(c)$ - convergence rate

constants $N_1, N_2$ such that

$$|\dot{V}(c)| \sim N_1\, c^{-2}, \qquad |\ddot{V}(c)| \sim N_2\, c^{-3}. \tag{6.5}$$

Then we can show that the steps $c_k$ form a geometric sequence. Indeed, the step $c_{k+1}$ is computed as the solution of (6.3), which is by the construction of the model function (6.1) equivalent to

$$\frac{1}{a_3 + c_{k+1}} = \frac{\theta}{a_3 + c_k}. \tag{6.6}$$

Using (6.2) to get $a_3$ and by assumption (6.5) we find

$$c_{k+1} = c_k - 2(1-\theta)\frac{\dot{V}(c)}{\ddot{V}(c)} \sim \left(1 + 2(1-\theta)\frac{N_1}{N_2}\right) c_k =: \sigma c_k. \tag{6.7}$$

Here, the factor $\sigma$ is larger than one, since $\dot{V}(c)$ and $\ddot{V}(c)$ have different signs for sufficiently large $c$. This means that under assumption (6.5) the path-following algorithm generates a sequence $(c_k)$ that converges to infinity.

With the help of these considerations one can estimate also the difference between the value function $V$ and the model function $M$. In Figure 5 the difference between $V(c_{k+1})$ and the prediction $M_{c_k}(c_{k+1})$ is shown. We observe the asymptotic behavior $|V(c_{k+1}) - M_{c_k}(c_{k+1})| \sim c_{k+1}^{-1}$.

We now estimate the difference $V(c_{k+1}) - M_{c_k}(c_{k+1})$ under assumption

(6.5). Using (6.1) and (6.2) we obtain

$$\begin{aligned}
V(c_{k+1}) - M_{c_k}(c_{k+1}) &= V(c_{k+1}) - a_1 - \frac{a_2}{a_3 + c_{k+1}} \\
&= V(c_{k+1}) - V(c_k) + a_2 \left( \frac{1}{a_3 + c_k} - \frac{1}{a_3 + c_{k+1}} \right) \\
&= V(c_{k+1}) - V(c_k) + \frac{a_2(1-\theta)}{a_3 + c_k} \\
&= V(c_{k+1}) - V(c_k) + 2(1-\theta)\frac{\dot{V}(c_k)^2}{\ddot{V}(c_k)}.
\end{aligned}$$

By (6.5), and (6.7) we have

$$|V(c_{k+1}) - V(c_k)| \sim N_1 \left( \frac{1}{c_k} - \frac{1}{c_{k+1}} \right) = N_1 \frac{\sigma}{(1+\sigma)c_k}.$$

This gives us together with $\sigma$ as defined in (6.7)

$$|V(c_{k+1}) - M_{c_k}(c_{k+1})| \sim \left( N_1 \frac{\sigma}{1+\sigma} + 2(1-\theta)\frac{N_1^2}{N_2} \right) \frac{1}{c_k} = N_1 \left( \frac{\sigma}{1+\sigma} + \sigma - 1 \right) \frac{1}{c_k},$$

which confirms the numerically observed convergence rate.

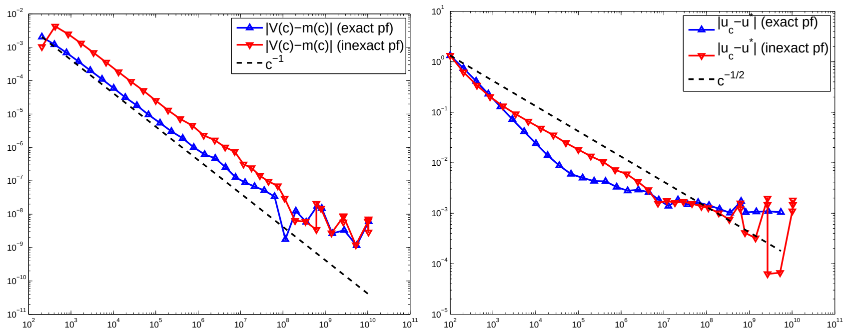


Figure 5: $|V(c_{k+1}) - M_{c_k}(c_{k+1})|$, $\|u_{c,h} - u_h^*\|_{L^2}$ for exact/inexact path-following

To comment on the convergence of $u_{c,h}$ for $c \to \infty$ we depict in Figure 5 the $L^2(\Omega)$-distance between $u_{c_k,h}$ and the last iterate $u_{c_N,h} \approx u_h^*$. This suggests the convergence rate $\|u_c - u^*\|_{L^2} \sim c^{-1/2}$.

Finally we compare the performance of exact and inexact path following. In Figure 5 we plotted for comparison the prediction error $V(c_{k+1}) - M_{c_k}(c_{k+1})$ and the regularization error $u$. As expected the inexact method is somewhat worse in comparison to the exact method. However, the inexact method needed 31 inner steps, while the exact method require 165 Newton steps. For the examples that we tested the inexact method produced results compared favorably with the exact one.

### 6.2.2 Example 2

Here we consider an example that does not fall into the class were Assumption 2 can be guaranteed to hold by a-posteriori inspection.

The data are the same as for Example 1, except for $\alpha = 0.1$ and

$$y_d(x) = 5x_1 + x_2 - 3, \quad \psi(x) = 4(x_1(x_1 - 1) + x_2(x_2 - 1)) + 1.5.$$

Again Assumption 1 is satisfied due to the quadratic nature of the cost functional. However, we cannot judge for this example whether the resulting reduced cost is convex or locally convex. The discrete version of the condition $y^* \leq y_d$ is not satisfied anymore, nor can we guarantee a uniform sign property of $p_{c,h}$ or $\mu_{c,h}$. In fact, the verification of Assumption 2 is much more delicate than in the previous example. It turned out that the discrete adjoint state $p_{c,h}$ is of the order of $10^{-6}$ on the set $\{\max_c''(\bar{\lambda}+c(y_{c,h}-\psi)) > 0\}$, assuming positive and negative values.

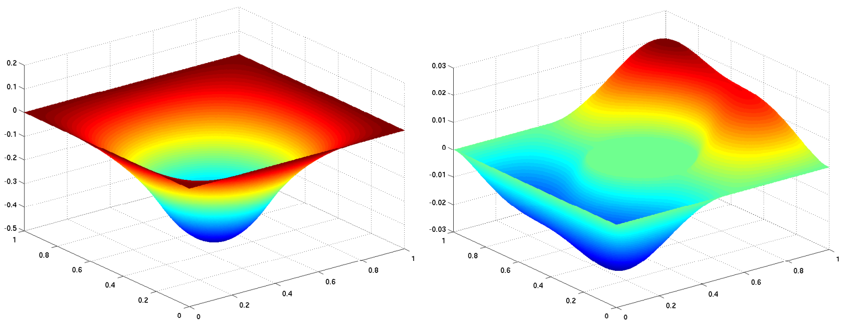


Figure 6: Numerical solution: $y_{c,h}$, $p_{c,h}$

We give the results for the converged values for $y_{c,h}, p_{c,h}, \lambda_{c,h}, \mu_{c,h}$ in Figures 6 and 7. Again the model function approach converges for both the exact and inexact methods, requiring 30 outer iterations for the exact and 31 iterations for the inexact versions to reach the stopping criterion.

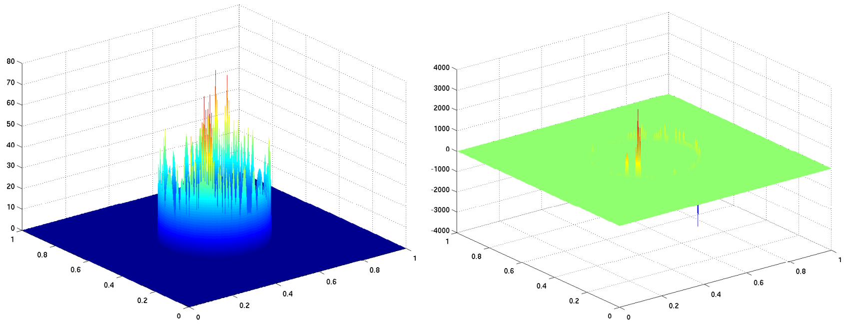

Figure 7: Numerical solution: $\lambda_{c,h}$, $\mu_{c,h}$

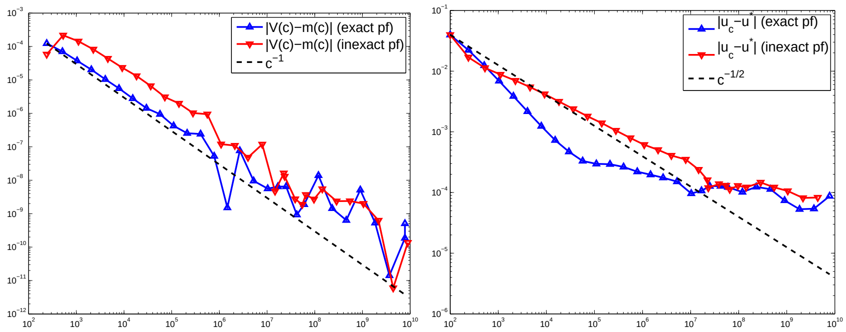


Figure 8: $|V(c_{k+1}) - M_{c_k}(c_{k+1})|$, $\|u_{c,h} - u_h^*\|_{L^2}$ for exact/inexact path-following

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