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Sergei V. Pereverzev [*] Sergei G. Solodky [†] Evgeny A. Volynets [‡]

Abstract

In the paper we discuss a regularization of semi-discrete ill-posed problem appearing as a result of application of a collocation method to Fredholm integral equation of the first kind. In this context we analyse Tikhonov regularization in Sobolev scales and prove error bounds under general source conditions. Moreover, we study an a-posteriori regularization parameter choice by means of the balancing principle.

KEY WORDS: inverse problems in Sobolev scales, Tikhonov regularization, collocation method, a-posteriori parameter choice, error bound

1 Introduction

Let us consider an equation of the first kind

$$Af = g \tag{1}$$

with an integral operator A

$$Af(x) := \int_\Omega k(x,t)f(t)dt, \qquad x \in \Omega.$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary, and kernel $k(x,t) : \Omega \times \Omega \to \mathbb{R}$ is such that A is a compact operator with infinite dimensional range acting from $L_2 = L_2(\Omega)$ into L_2. Without loss of generality we may assume that $\|A\| \leq 1$.

Since the problem (1) is ill-posed to obtain a stable solution some regularization methods should be used. If the equation (1) is discretized properly then a stable approximate solution can be obtained without any additional regularization. Such an approach is sometimes called regularization by discretization, or self-regularization. It

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can be realized only in case when one controls the smallest singular values of corresponding discrete approximations A_n of the operator A, as it was discussed, for example, in [11], [1]. But in general the estimation of the singular values is more difficult than the approximate solving of (1). Therefore, if information about the rate of the decay of singular values is not available, then other techniques should be used for regularizing ill-posed problem.

It is worth to note that the previous study was mainly restricted to the case when one looks for a regularized solution in the whole space $L_2(\Omega)$ (see, e.g. [10]). The application of the Tikhonov regularization scheme in Hilbert scales to discretized ill-posed problems was discussed in [8] and [6]. But only the discretization by projection was considered there. The first result on Tikhonov regularization in Hilbert scales in combination with the discretization by collocation has been obtained recently in [4]. It should be noted that only a-priori selection of regularization parameters has been discussed there.

The present paper is devoted to an a-posteriori regularization parameter choice for Tikhonov method in Hilbert-Sobolev scales applied to an equation (1) discretized by collocation. For this purpose we shall use the balancing principle in the form suggested in [9], [7].

Suppose that we are given a set of pairwise distinct points $X = \{x_1, \ldots, x_n\} \subset \Omega$. Then, within a collocation scheme based on X the original equation (1) is replaced by an operator equation

$$A_X f = \bar{g}, \tag{2}$$

where $\bar{g} = \{g_1, \ldots, g_n\}^T$, $g_j = g(x_j)$, and A_X is considered as an operator in $\mathbb{R}^n$ defined as follows

$$(A_X f)_j = Af(x_j), \quad 1 \leq j \leq n.$$

Another way of writing this is $A_X f = Af|_X$, where $u|_X$ means the restriction of a function $u(x)$, $x \in \Omega$, to the set X.

In the sequel we shall assume that n-dimentional Eucledean space $\mathbb{R}^n$ is equipped with the standard norm

$$\|x\|_n = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

and the corresponding inner product $\langle \cdot, \cdot \rangle_n$.

In practice due to a measurement error only noisy data vectors $\bar{g}^\delta = \{g_1^\delta, \ldots, g_n^\delta\}^T$ are available usually such that $|g_j - g_j^\delta| \leq \delta$, $j = \overline{1, n}$. Then the whole data error can be estimated as $\|\bar{g} - \bar{g}^\delta\|_n \leq \delta\sqrt{n}$.

Our aim is a stable recovery of an unknown solution f from these noisy values $\{g_j^\delta\}$ obtained on the set X.

The paper is organized as follows. In Section 2 we discuss the regularization of equations (2) by means of Tikhonov method in Sobolev scales. To analyse this regularization we need some auxiliary results which are collected in Section 3. The results of our analysis are presented in Section 4. This section specifies some results of [4] and containts a theoretical justification of an a-posteriori parameter choice rule, which can be seen as the main achievement of the present paper.

2 Tikhonov regularization in Sobolev scales

Recall that Sobolev space $\mathcal{H}^\tau(\Omega) = \mathcal{H}^\tau$ is the space of all functions $u \in L_2(\Omega)$ having generalized derivatives $D^i u \in L_2(\Omega)$ for $|i| \leq \tau$. These spaces are equipped with the norms

$$\|u\|_\tau = \|u\|_{\mathcal{H}^\tau(\Omega)} := \left(\sum_{|i|\leq\tau} \|D^i u\|^2_{L_2(\Omega)} \right)^{1/2}.$$

Following [4] we assume that in (1) the operator A acts along scale of Sobolev spaces $\{\mathcal{H}^\tau\}$ with step $\alpha > 0$, which means that there are constants $c'' \geq c' \geq 0$ such that for fixed α and all $\tau \in \mathbb{R}$ it holds

$$c'\|f\|_\tau \leq \|Af\|_{\tau+\alpha} \leq c''\|f\|_\tau. \tag{3}$$

Let f^* be an exact solution of (1). It is easy to see that f^* also solves semi-discrete problem (2) and can be represented in the form

$$f^* = f^+ + v_0,$$

where $f^+ = A_X^+ \bar{g}$, A_X^+ is the Moore-Penrose generalized inverse of A_X, and v_0 belongs to the null space of A_X.

At the same time, it should be realized that only the element f^+ can be reconstructed from equation (2). Therefore, in the sequel we shall be interested only in f^+. In view of (3) the operator A_X defined by $A_X f = Af|_X$ can be considered as an operator from $\mathcal{H}^\tau$ to $\mathbb{R}^n$, $\tau \in \mathbb{R}$. Therefore, f^+ also belongs to $\mathcal{H}^\tau$ for some $\tau > 0$. Moreover, from the paper [5] (see also [2], [3] for a preliminary discussion) it follows that there always exists some continuous increasing (unknown) index function $\varphi(\lambda)$, $\lambda \in [0, 1]$, such that $\varphi(0) = 0$, and

$$f^+ = \varphi(A_X^* A_X)v,$$

where $v \in \mathcal{H}^\tau$, $\|v\|_\tau \leq \rho$, and $A_X^* : \mathbb{R}^n \to \mathcal{H}^\tau$ is the adjoint of A_X.

Recall that the Tikhonov method in Sobolev spaces consists in finding the minimizer of the functional

$$\gamma\|f\|^2_\tau + \|A_X f - \bar{g}^\delta\|^2_n,$$

where γ is a small positive parameter that has to be choosen properly. This minimizer f_γ^δ should solve the equation

$$\gamma f + A_X^* A_X f = A_X^* \bar{g}^\delta$$

and can be represented in the form

$$f_\gamma^\delta = (\gamma I + A_X^* A_X)^{-1} A_X^* \bar{g}^\delta,$$

where $I : \mathcal{H}^\tau \to \mathcal{H}^\tau$ is the identity operator. Moreover, we will also use the notation

$$\begin{aligned} f_\gamma = f_\gamma^0 &= (\gamma I + A_X^* A_X)^{-1} A_X^* \bar{g} \\ &= (\gamma I + A_X^* A_X)^{-1} A_X^* A_X \varphi(A_X^* A_X)v. \end{aligned}$$

3 Auxiliary assertions

In what follows we need the following assertion (see [7, Proposition 1]).

Lemma 1 *Assume that φ is an increasing function such that $\varphi(0) = 0$ and the function $\varphi(t)/t$ is nonincreasing. Then for all $0 < s, t \le 1$ we have*

$$\sup_{0<t\le 1}\left|\frac{s}{s+t}\varphi(t)\right| \le \varphi(s).$$

Lemma 2 *It holds*

$$\|f^+ - f_\gamma\|_\tau \le \rho\varphi(\gamma), \tag{4}$$

$$\|A_X f^+ - A_X f_\gamma\|_n \le \rho\bar{\varphi}(\gamma), \tag{5}$$

where

$$\bar{\varphi}(\gamma) = \gamma,$$

if $\frac{\varphi(\gamma)}{\gamma}$ is nonincreasing function with $\varphi(\gamma) \le \sqrt{\gamma}$, and

$$\bar{\varphi}(\gamma) = \sqrt{\gamma}\varphi(\gamma),$$

if $\frac{\varphi(\gamma)}{\sqrt{\gamma}}$ is nonincreasing function with $\varphi(\gamma) \ge \sqrt{\gamma}$.

Proof. Keeping in mind Lemma 1 we have

$$\begin{aligned}
\|f^+ - f_\gamma\|_\tau &= \|(I - (\gamma I + A_X^* A_X)^{-1} A_X^* A_X)\varphi(A_X^* A_X)v\|_\tau \\
&\le \|v\|_\tau \sup\nolimits_{0\le\lambda\le 1}\left|\varphi(\lambda) - \tfrac{\lambda}{\gamma+\lambda}\varphi(\lambda)\right| \\
&= \|v\|_\tau \sup\nolimits_{0\le\lambda\le 1}\left|\tfrac{\gamma}{\gamma+\lambda}\varphi(\lambda)\right| \\
&\le \|v\|_\tau \varphi(\gamma).
\end{aligned}$$

Thus, the first inequality is proven.

To prove the second inequality we observe that

$$\begin{aligned}
\|A_X f^+ - A_X f_\gamma\|_n &= \|A_X(I - (\gamma I + A_X^* A_X)^{-1} A_X^* A_X)\varphi(A_X^* A_X)v\|_n \\
&\le \|v\|_\tau \sup\nolimits_{0\le\lambda\le 1}\left|\sqrt{\lambda}\left(\varphi(\lambda) - \tfrac{\lambda}{\gamma+\lambda}\varphi(\lambda)\right)\right| \\
&= \|v\|_\tau \sup\nolimits_{0\le\lambda\le 1}\left|\sqrt{\lambda}\tfrac{\gamma}{\gamma+\lambda}\varphi(\lambda)\right| \\
&= \gamma\|v\|_\tau \sup\nolimits_{0\le\lambda\le 1}\left|\tfrac{\sqrt{\lambda}\varphi(\lambda)}{\gamma+\lambda}\right|.
\end{aligned}$$

We need to estimate the last expression. Consider two cases.

1. $\lambda \leq \gamma$. Taking into account that φ is increasing we have

$$\gamma \frac{\sqrt{\lambda}\varphi(\lambda)}{\gamma+\lambda} \leq \gamma \frac{\sqrt{\gamma}\varphi(\gamma)}{\gamma} = \sqrt{\gamma}\varphi(\gamma).$$

2. $\gamma \leq \lambda$. First we consider the case $\varphi(\gamma) \leq \sqrt{\gamma}$. Then

$$\frac{\sqrt{\lambda}\varphi(\lambda)}{\gamma+\lambda} \leq \frac{\lambda}{\gamma+\lambda} \leq 1.$$

   Hence

$$\|A_X f^+ - A_X f_\gamma\|_n \leq \|v\|_\tau \gamma.$$

   Finally consider the case of nonincreasing function $\varphi(\gamma)/\sqrt{\gamma}$ with $\varphi(\gamma) \geq \sqrt{\gamma}$. Then

$$\frac{\sqrt{\lambda}\varphi(\lambda)}{\gamma+\lambda} = \frac{\lambda}{\gamma+\lambda}\frac{\varphi(\lambda)}{\sqrt{\lambda}} \leq \frac{\varphi(\gamma)}{\sqrt{\gamma}},$$

   and hence

$$\|A_X f^+ - A_X f_\gamma\|_n \leq \|v\|_\tau \sqrt{\gamma}\varphi(\gamma).$$

Lemma is completely proven. □

Remark 1 *In the case of moderatelly ill-posed problems the conditions used to prove (5) complement each other. Namely, for* $\varphi(\gamma) = \gamma^\beta$ *the condition of nonincreasing function* $\varphi(\gamma)/\gamma$ *with* $\varphi(\gamma) \leq \sqrt{\gamma}$ *means that* $1/2 \leq \beta \leq 1$, *while the condition of nonincreasing function* $\varphi(\gamma)/\sqrt{\gamma}$ *with* $\varphi(\gamma) \geq \sqrt{\gamma}$ *means the complementary condition:* $0 < \beta \leq 1/2$.

We also need the following estimates proven in [4, Lemmas 4.1, 4.5]

$$\|f_\gamma - f_\gamma^\delta\|_\tau \leq \frac{\delta}{2}\sqrt{\frac{n}{\gamma}}, \tag{6}$$

$$\|A_X f_\gamma - A_X f_\gamma^\delta\|_n \leq \sqrt{n}\delta. \tag{7}$$

As in [4] we define the data density of X in Ω by the fill distance

$$h := \sup_{x\in\Omega} \min_{x_i\in X} \|x - x_i\|_d.$$

Our subsequent analysis is based on the following sampling inequality proven in [4, Theorem 4.7] for any function $u \in \mathcal{H}^\theta = \mathcal{H}^\theta(\Omega)$, $\theta > d/2$, and sufficiently small h

$$\|u\|_\sigma \leq \kappa \left(h^{\theta-\sigma}\|u\|_\theta + h^{\frac{d}{2}-\sigma}\|u|_X\|_n \right), \tag{8}$$

where $\sigma \in [0, \lfloor\theta\rfloor)$, and κ is some constant, which does not depend on u and h.

4 Error bounds

Theorem 1 *Let the condition (3) be satisfied. Then for any discrete set X with sufficiently small data density h there is a constant $c_1 > 0$ such that*

$$\|f^+ - f_\gamma^\delta\|_{L_2} \le c_1 \left(\rho h^\tau \varphi(\gamma) + \frac{\delta}{2} h^\tau \sqrt{n/\gamma} + \rho h^{\frac{d}{2}-\alpha} \bar{\varphi}(\gamma) + h^{\frac{d}{2}-\alpha} \sqrt{n}\delta \right). \tag{9}$$

Proof.
At first we estimate $\|f^+ - f_\gamma^\delta\|_\tau$. In view of (4) and (6) we have

$$\begin{aligned} \|f^+ - f_\gamma^\delta\|_\tau &\le \|f^+ - f_\gamma\|_\tau + \|f_\gamma - f_\gamma^\delta\|_\tau \\ &\le \rho\varphi(\gamma) + \tfrac{\delta}{2}\sqrt{\tfrac{n}{\gamma}}. \end{aligned} \tag{10}$$

Using sampling inequality (8) with $u = A(f^+ - f_\gamma^\delta)$, $\sigma = \alpha$ and $\theta = \tau + \alpha$ we obtain

$$\|A(f^+ - f_\gamma^\delta)\|_\alpha \le \kappa \left(h^\tau \|A(f^+ - f_\gamma^\delta)\|_{\tau+\alpha} + h^{\frac{d}{2}-\alpha} \|A(f^+ - f_\gamma^\delta)|_X\|_n \right).$$

Applying relation (3) to both sides of the last inequality we have

$$c'\|f^+ - f_\gamma^\delta\|_{L_2} \le \kappa \left(c'' h^\tau \|f^+ - f_\gamma^\delta\|_\tau + h^{\frac{d}{2}-\alpha} \|A(f^+ - f_\gamma^\delta)|_X\|_n \right).$$

Keeping in mind that $Af|_X = A_X f$ we find

$$\|f^+ - f_\gamma^\delta\|_{L_2} \le c_1 \left(h^\tau \|f^+ - f_\gamma^\delta\|_\tau + h^{\frac{d}{2}-\alpha} \|A_X(f^+ - f_\gamma^\delta)\|_n \right), \tag{11}$$

where $c_1 = \frac{\kappa}{c'} \max\{1, c''\}$.
This bound together with (5), (7), (10) gives us the statement of the theorem. □

Remark 2 *Let us discuss the case of exactly given data, when $\delta = 0$. This case has been also studied in [4, Theorem 4.8]. In our terms the following error bound has been proven there*

$$\|f^+ - f_\gamma\|_{L_2} \le C \left(h^\tau + h^{\frac{d}{2}-\alpha} \sqrt{\gamma} \right). \tag{12}$$

At the same time, from our theorem it follows that

$$\|f^+ - f_\gamma\|_{L_2} \le c_1 \left(h^\tau \varphi(\gamma) + h^{\frac{d}{2}-\alpha} \bar{\varphi}(\gamma) \right).$$

It can be easily seen that the latter bound is more accurate then (12), since $\varphi(\gamma)$ and $\bar{\varphi}(\gamma)$ tend to zero as $\gamma \to 0$.

Let partition of the set X be quasi-uniform, i.e. $h = \chi n^{-\frac{1}{d}}$ for some constant χ. Then the inequality (9) takes the form

$$\|f^+ - f_\gamma^\delta\|_{L_2} \le c_1 \left(\chi^{\frac{d}{2}-\alpha} \left(n^{\frac{\alpha}{d}} \delta + \rho n^{\frac{\alpha}{d}-\frac{1}{2}} \bar{\varphi}(\gamma) \right) + \chi^\tau \left(\rho n^{-\frac{\tau}{d}} \varphi(\gamma) + \frac{n^{-\frac{\tau}{d}+\frac{1}{2}}}{2} \frac{\delta}{\sqrt{\gamma}} \right) \right). \tag{13}$$

We are going to optimize the right-hand side of (13) selecting γ by means of the balancing principle [9]. Main point of this approach is to choose the regularization parameter γ in such a way that it balances two functions determing final approximation error. In considered case these functions are

$$\Phi(\gamma) := c_1\rho\left(\chi^{\frac{d}{2}-\alpha}n^{\frac{\alpha}{d}-\frac{1}{2}}\bar{\varphi}(\gamma)+\chi^{\tau}n^{-\frac{\tau}{d}}\varphi(\gamma)\right),$$

$$\Psi(\gamma) := c_1\left(\chi^{\frac{d}{2}-\alpha}n^{\frac{\alpha}{d}}\delta+\chi^{\tau}n^{-\frac{\tau}{d}+\frac{1}{2}}\frac{\delta}{\sqrt{\gamma}}\right).$$

Due to monotonicity of φ the function Φ is increasing while the function Ψ is decreasing. Now (13) can be rewritten as

$$\|f^+ - f^\delta_\gamma\|_{L_2} \le \Phi(\gamma)+\Psi(\gamma). \tag{14}$$

In view of (14) the optimal choice of the regularization parameter would be to balance the values of $\Phi(\gamma)$ and $\Psi(\gamma)$ such that for $\gamma=\gamma_{opt}$ we have $\Phi(\gamma_{opt})=\Psi(\gamma_{opt})$, and

$$\|f^+ - f^\delta_{\gamma_{opt}}\|_{L_2} \le 2\Phi(\gamma_{opt}).$$

But if the function φ is unknown such a-priori choice of parameter γ is impossible. Therefore we need to apply some a-posteriori rule for choosing γ. The balancing principle is one of such rules. To apply it we consider the following discrete set of values of the regularization parameter

$$\Delta_N = \left\{\gamma_i = (q^2)^i\gamma_0, \quad i=1,\dots,N\right\}, \quad q>1. \tag{15}$$

Here $\gamma_0 = n\delta^2,\ \ N:\gamma_N \asymp 1$.

Using the balancing principle one operates with the set

$$M^+(\Delta_N) = \left\{\gamma_i\in\Delta_N : \|f^\delta_{\gamma_i} - f^\delta_{\gamma_j}\|_{L_2} \le 4\Psi(\gamma_j), \quad j=1,\dots,i\right\}$$

and chooses the value of the regularization parameter as follows

$$\gamma = \gamma_+ := \max\left\{\gamma\in M^+(\Delta_N)\right\}.$$

Note that such a choice does not require any knowledge of φ describing the smoothness of f^+. Nevertheless, as it will be shown below, $\gamma=\gamma_+$ allows an error bound that only by the factor $3q$ worse than the optimal error bound.

To proove this we consider one more set

$$M(\Delta_N) := \{\gamma_i\in\Delta_N : \quad \Phi(\gamma_i)\le\Psi(\gamma_i)\},$$

and

$$\gamma_* := \max\{\gamma\in M(\Delta_N)\}.$$

Without loss of generality we assume that $M(\Delta_N)\neq\emptyset$ and $\Delta_N\backslash M(\Delta_N)\neq\emptyset$.

Now we can estimate closeness of exact and approximate solutions for the regularization parameter γ_+.

Theorem 2 *Let the set Δ_N be defined by (15). Then the following estimate*

$$\|f^+ - f^\delta_{\gamma_+}\|_{L_2} \leq 6q\Phi(\gamma_{opt}) \tag{16}$$

holds.

Proof. We begin by proving the inequality $\gamma_* \leq \gamma_+$. Due to (14) for $\gamma_j < \gamma_*$ we have

$$\begin{aligned} \|f^\delta_{\gamma_*} - f^\delta_{\gamma_j}\|_{L_2} &\leq \|f^+ - f^\delta_{\gamma_*}\|_{L_2} + \|f^+ - f^\delta_{\gamma_j}\|_{L_2} \\ &\leq \Phi(\gamma_*) + \Psi(\gamma_*) + \Phi(\gamma_j) + \Psi(\gamma_j) \\ &\leq 2\Phi(\gamma_*) + \Psi(\gamma_*) + \Psi(\gamma_j) \\ &\leq 3\Psi(\gamma_*) + \Psi(\gamma_j) \leq 4\Psi(\gamma_j). \end{aligned}$$

Thus, the inclusion $\gamma_* \in M^+(\Delta_N)$ is proven. Then by the very definition it holds $\gamma_* \leq \gamma_+$.

Moreover, using (14) with $\gamma{=}\gamma_*$ we deduce

$$\begin{aligned} \|f^+ - f^\delta_{\gamma_+}\|_{L_2} &\leq \|f^+ - f^\delta_{\gamma_*}\|_{L_2} + \|f^\delta_{\gamma_*} - f^\delta_{\gamma_+}\|_{L_2} \\ &\leq 6\Psi(\gamma_*). \end{aligned} \tag{17}$$

In view of monotonicity of Ψ, it is easy to see that

$$\begin{aligned} \Psi(q^2\gamma_*) &= c_1\left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}}\delta + \chi^\tau n^{-\frac{\tau}{d}+\frac{1}{2}}\frac{\delta}{q\sqrt{\gamma_*}}\right) \\ &\geq \tfrac{c_1}{q}\left(\chi^{\frac{d}{2}-\alpha} n^{\frac{\alpha}{d}}\delta + \chi^\tau n^{-\frac{\tau}{d}+\frac{1}{2}}\frac{\delta}{\sqrt{\gamma_*}}\right) \\ &= \tfrac{1}{q}\Psi(\gamma_*). \end{aligned} \tag{18}$$

On the other hand, it is clear that $\gamma_* \leq \gamma_{opt} \leq q^2\gamma_*$. Together with (17) and (18) it gives us

$$\begin{aligned} \|f^+ - f^\delta_{\gamma_+}\|_{L_2} &\leq 6q\Psi(q^2\gamma_*) \\ &\leq 6q\Psi(\gamma_{opt}) = 6q\Phi(\gamma_{opt}). \end{aligned}$$

The theorem is proven. □

Corollary 1 *If $\varphi(\gamma) \geq \sqrt{\gamma}$, but $\varphi(\gamma)/\sqrt{\gamma}$ is nonincreasing function, we have*

$$\|f^+ - f^\delta_{\gamma_+}\|_{L_2} \leq 6q\Phi\left(\theta^{-1}\Big(\frac{\delta\sqrt{n}}{\rho}\Big)\right)$$

with $\theta(\gamma) = \varphi(\gamma)\sqrt{\gamma}$. In particular, for $\varphi(\gamma) = \gamma^\beta$, $0 < \beta \leq 1/2$, it means that

$$\|f^+ - f^\delta_{\gamma_+}\|_{L_2} \leq 6qc_1\left(\chi^{\frac{d}{2}-\alpha}\delta n^{\frac{\alpha}{d}} + \chi^\tau \rho^{\frac{1}{2\beta+1}} n^{-\frac{\tau}{d}}(\delta\sqrt{n})^{\frac{2\beta}{2\beta+1}}\right). \tag{19}$$

Proof. According to definition of γ_{opt} we have $\Phi(\gamma_{opt}) = \Psi(\gamma_{opt})$. Then by virtue of $\bar{\varphi}(\gamma_{opt}) = \varphi(\gamma_{opt})\sqrt{\gamma_{opt}}$ it holds

$$\rho\sqrt{\gamma_{opt}}\varphi(\gamma_{opt})\left(\chi^{\frac{d}{2}-\alpha}n^{\frac{\alpha}{d}-\frac{1}{2}} + \chi^{\tau}\frac{n^{-\frac{\tau}{d}}}{\sqrt{\gamma_{opt}}}\right) = \delta\sqrt{n}\left(\chi^{\frac{d}{2}-\alpha}n^{\frac{\alpha}{d}-\frac{1}{2}} + \chi^{\tau}\frac{n^{-\frac{\tau}{d}}}{\sqrt{\gamma_{opt}}}\right).$$

Hence, $\gamma_{opt} = \theta^{-1}\left(\frac{\delta\sqrt{n}}{\rho}\right)$.

Substituting the last quality in (16) and keeping in mind that $\theta^{-1}(\gamma) = \gamma^{\frac{2}{2\beta+1}}$ for $\varphi(\gamma) = \gamma^{\beta}$ we obtain the statement. □

Remark 3 *In view of the data error estimation*

$$\|\bar{g} - \bar{g}^{\delta}\|_n \leq \delta\sqrt{n}$$

it is natural to assume that $\delta\sqrt{n} \ll 1$, or, what is the same, $n \ll \delta^{-2}$. If n can be chosen at will, then, as it has been shown in [4, Corollary 4.13], under the condition $\alpha + \tau > d/2$, an optimal choice is $n \simeq \delta^{-\frac{d}{\alpha+\tau}}$. However, it is very often, that the amount of available noisy data is limited such that one should deal with

$$n \ll \delta^{-\frac{d}{\alpha+\tau}}.$$

For such n using a-priori parameter choice $\tilde{\gamma} = \delta n^{-\frac{\alpha+\tau-d}{d}}$ suggested in [4, Corollary 4.11] one has the following error bound

$$\begin{aligned} \|f^{+} - f^{\delta}_{\tilde{\gamma}}\|_{L_2} &\leq \tilde{C}\left(n^{-\frac{\tau}{d}} + \delta n^{\frac{\alpha}{d}} + \sqrt{\delta}n^{\frac{\alpha-\tau}{2d}}\right) \\ &= O(n^{-\frac{\tau}{d}}). \end{aligned}$$

At the same time, from Corollary 1 it follows that a-posteriori parameter choice $\gamma = \gamma_{+}$ allows a higher order error bound. Indeed, keeping in mind that

$$n^{-\frac{\tau}{d}} \gg \delta n^{-\frac{\alpha}{d}}, \quad n^{-\frac{\tau}{d}} \gg \sqrt{\delta}n^{-\frac{\alpha-\tau}{2d}}$$

from (19) we have

$$\|f^{+} - f^{\delta}_{\gamma_{+}}\|_{L_2} \ll n^{-\frac{\tau}{d}}.$$

Remark 4 *Recall that we are looking for the solution f^{+} of a normally solvable problem (2). It is well known (see, for example, [1, Section 3.3]) that in such situation the error bound for direct reconstruction of f^{+} from noisy data is determined by $\frac{\varepsilon}{\lambda_n}$, where ε is a given data error level of the right-hand side and λ_n is the smallest singular value of A_X. In view of the condition (3) it is natural to assume that in our case it holds $\lambda_n \sim n^{-\frac{\alpha}{d}}$. Then, keeping in mind $\varepsilon = \delta\sqrt{n}$ we obtain*

$$\frac{\varepsilon}{\lambda_n} \sim \delta n^{\frac{\alpha}{d}+\frac{1}{2}}. \tag{20}$$

At the same time, from (19) it follows that for $\delta^{-1} \le n^{\frac{1}{2}+\frac{(2\beta+1)(\alpha+\tau)}{d}}$

$$\|f^+ - f^\delta_{\gamma_+}\|_{L_2} \le O(\delta n^{\frac{\alpha}{d}+\frac{1}{2}}). \tag{21}$$

Comparing (20) and (21) one can conclude that, if the amount n *of available discrete data is sufficiently large such that* $n \ll \delta^{-2}$ *but*

$$\delta^{-\frac{2d}{2(2\beta+1)(\alpha+\tau)+d}} \ll n,$$

or (see Remark 3)

$$\delta^{-\frac{2d}{2(2\beta+1)(\alpha+\tau)+d}} \ll n \ll \delta^{-\frac{d}{\alpha+\tau}}$$

then the regularized solution $f^\delta_{\gamma_+}$ *allows a better error bound (in the sense of order) than the direct reconstruction.*

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