

Optimal Dirichlet boundary control of Navier-Stokes equations with state constraint

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Abstract

We investigate optimal boundary control of the steady-state Navier-Stokes equations. The control goal is to increase the lift while maintaining a drag constraint. The resulting problem has control as well as state constraints. We use as control space $L^2(\Gamma)$, which makes it necessary to work with very weak solutions of the Navier-Stokes equations. Moreover, the low regularity of control and state forces to reformulate cost functional and state constraint, which results in a problem with non-linear and mixed control-state constraint. We derive first-order necessary and second-order sufficient optimality conditions for this optimal control problem. Moreover, we report on numerical experiments on the solution of the first order optimality system.

Keywords. Optimal control, Navier-Stokes, very weak solutions, lift maximization, drag constraint, state constraints, necessary optimality conditions, sufficient optimality conditions.

AMS subject classifications. Primary: 49K20, Secondary: 76D55

1 Introduction

In this article, we are investigating the following optimal flow control problem. We consider active control of the flow of a fluid around an aircraft by means of suction and blowing on the wing to influence the resulting lift and drag. This actuation scheme has been proven to be effective in experiments as well as simulations, see e.g. [3, 4, 24]. One major requirement is that the control should act in such a way that the lift of the aircraft is increased while the drag stays beyond a given threshold. We will characterize optimality of control strategies for this setting by necessary and sufficient optimality conditions.

The resulting force in direction \vec{e} of the fluid on a solid body embedded in the fluid is given as the boundary integral

$$F = \int_{\Gamma_b} \left(\nu \frac{\partial y}{\partial \mathbf{n}_b} - p \mathbf{n}_b \right) \cdot \vec{e} \, d\gamma,$$

where Γ_b is the boundary of the body with its outer normal \mathbf{n}_b . Since \mathbf{n}_b points into the fluid, the normal \mathbf{n}_b is the negative of the outer normal of the fluid domain Ω , $\mathbf{n}_b = -\mathbf{n}$. Let us denote vectors \vec{e}_l and \vec{e}_d to indicate the directions

of lift and drag, which will be chosen in the above integral for the direction \vec{e} . Then \vec{e}_l is the normalized vector directed opposite to the gravity, and \vec{e}_d is the normalized vector in the opposing direction of the main flow field.

The optimization problem is then formulated as: Find a control u in $L^2(\Gamma)^n$ that maximizes the lift force

$$- \int_{\Gamma_b} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p \mathbf{n} \right) \cdot \vec{e}_l \, d\gamma \quad (1.1)$$

subject to the maximal drag constraint

$$- \int_{\Gamma_b} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p \mathbf{n} \right) \cdot \vec{e}_d \, d\gamma \leq D_0, \quad (1.2)$$

the steady state Navier-Stokes equations

$$\begin{aligned} -\nu \Delta y + (y \cdot \nabla) y + \nabla p &= 0 & \text{in } \Omega \\ \operatorname{div} y &= 0 & \text{in } \Omega \\ y &= u & \text{on } \Gamma_c, \\ y &= g & \text{on } \Gamma \setminus \Gamma_c, \end{aligned} \quad (1.3)$$

and the convex control constraints

$$u(x) \in U \text{ a.e. on } \Gamma_c. \quad (1.4)$$

Here, Ω is an open bounded domain of \mathbb{R}^n , $n = 2, 3$ with boundary Γ , which is assumed to be sufficiently smooth. The velocity field of the fluid is denoted by y , the pressure by p . The control is a boundary velocity field that is denoted by u . The viscosity parameter $\nu = 1/Re$ is a positive number. The boundary Γ_b is a curve satisfying

$$\int_{\Gamma_b} \mathbf{n} d\gamma = 0.$$

The control acts on a part of the boundary of the body $\Gamma_c \subset \Gamma_b$. On the boundary $\Gamma \setminus \Gamma_c$ homogeneous Dirichlet boundary conditions are prescribed.

The set of admissible controls U is a bounded, convex, closed, and non-empty subset of \mathbb{R}^n . Furthermore, we assume $0 \in U$, which makes the option to turn off the control admissible in the optimization problem. For a more detailed discussion of such convex control constraints, we refer to [29].

Let us shortly review available literature on analysis of optimal control problems for the Navier-Stokes equations. Starting with Abergel and Temam [1] there is an ever growing list of contributions. Let us only mention the work by Gunzburger, Hou, and Svobodny [14], Gunzburger and Manservigi [15], Hinze and Kunisch [16, 17], Kunisch and Reyes [8], Tröltzsch and Roubicek [23] and Wachsmuth [28]. Finite-element error estimates can be found in the work of Casas, Mateos and Raymond [5]. Optimal flow control problems with state constraints were studied by Griesse and Reyes [13], Reyes and Kunisch [21].

The novelty of this work is that it combines the use of very low regular boundary controls, i.e. in $L^2(\Gamma)$, and integral state constraints. There are only a few contributions to optimal control theory using Dirichlet controls in L^2 , see for instance Kunisch and Vexler [18]. In the context of steady-state Navier-Stokes equations this is a new and promising approach, since the use of L^2 -controls yields localizable optimality conditions, whereas the use of for instance

$H^{1/2}(\Gamma)$ -controls yields optimality conditions containing non-local boundary operators.

In view of the low regularity of the controls, the boundary integrals (1.1) and (1.2) are no longer well-defined, since the velocity field y is not regular enough to admit traces on the boundary. To cope with this difficulty, we transform the boundary integrals into volume integrals yielding in case of the drag constraint to a non-standard mixed control-state constraint, see Section 2.4 below.

As it is well-known the steady-state Navier-Stokes equation are solvable. If the data and/or the Reynolds number $1/\nu$ are small enough then the solution will be unique. To judge whether this condition is fulfilled in a concrete application is a delicate issue in particular in the case of inhomogeneous boundary conditions, see the discussion in the monograph of Galdi [11]. Hence, instead of assuming smallness of the data, we assume non-singularity at the optimal control, which is equivalent to unique solvability of a certain linearized equation, see Section 2.3.

By assuming the existence of a linearized Slater point to the state constraint (1.2), we are able to prove first-order necessary optimality conditions, see Section 3.2. For the special case of smooth controls, the resulting optimality system simplifies considerably, see Section 4. Furthermore, we state a second-order sufficient optimality condition for the problem under consideration. The article is complemented by numerical experiments on a high-lift configuration.

2 Preliminary results

2.1 Notation

We assume that the boundary of Ω is of class C^2 . The outer unit normal on Γ is denoted by \mathbf{n} . The boundary Γ is the union of m connected components, $\Gamma = \bigcup_{j=1}^m \Gamma_j$. Furthermore, we define the following spaces on Ω and Γ :

$$\mathbf{H}^s(\Omega) = \{v \in H^s(\Omega)^n : \operatorname{div} v = 0 \text{ on } \Omega, \langle u \cdot \mathbf{n}, 1 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0\}, \quad s \geq 0,$$

$$\mathbf{H}_0^s(\Omega) = \{v \in H^s(\Omega)^n : \operatorname{div} v = 0 \text{ on } \Omega, u = 0 \text{ on } \Gamma\}, \quad s \geq 1/2,$$

$$\mathbf{H}^s(\Gamma) = \{v \in H^s(\Gamma) : \int_{\Gamma_j} u \cdot \mathbf{n} = 0 \quad \forall j = 1 \dots m\}, \quad s \geq 0,$$

$$\mathbf{L}^p(\Omega) = \{v \in L^p(\Omega)^n : \operatorname{div} v = 0\}, \quad p \geq 1,$$

$$\mathbf{L}^p(\Gamma) = L^p(\Gamma)^n, \quad p \geq 1$$

$$\mathbf{H}^{-s}(\Gamma) = (\mathbf{H}^s(\Gamma))', \quad s \geq 0$$

$$\mathbf{H}^{-s}(\Omega) = (\mathbf{H}^s(\Omega) \cap \mathbf{H}_0^1(\Omega))', \quad s \geq 1.$$

2.2 Very weak formulation

Let us recall known results about solvability of the nonlinear Navier-Stokes system with inhomogeneous Dirichlet boundary condition,

$$\begin{aligned} -\nu \Delta y + (y \cdot \nabla) y + \nabla p &= 0 & \text{in } \Omega \\ \operatorname{div} y &= 0 & \text{in } \Omega \\ y &= u & \text{on } \Gamma. \end{aligned} \tag{2.1}$$

Let us specify, in what sense we want to solve the state equation. Here, we have to resort to the notation of very weak solutions, since in general weak solutions in $y \in \mathbf{H}^1(\Omega)$ do not exist due to low regularity of boundary data.

Definition 2.1. *Let $u \in \mathbf{H}^0(\Gamma)$ be given. Then we call $y \in \mathbf{L}^{2n/(n-1)}(\Omega)$ a very weak solution of the state equation (2.1) if for all test functions $v \in \mathbf{H}_0^2(\Omega)$, $\pi \in H^1(\Omega)$ it holds*

$$\int_{\Omega} (y \cdot (-\nu \Delta v) - (y \cdot \nabla) v y) dx + \int_{\Gamma} u \cdot \nu \frac{\partial v}{\partial \mathbf{n}} d\gamma = 0 \quad (2.2a)$$

and

$$\int_{\Omega} \nabla \pi \cdot y dx - \int_{\Gamma} (u \cdot \mathbf{n}) \pi d\gamma = 0 \quad (2.2b)$$

Here, the first equation is obtained by twice partial integrating the Navier-Stokes equation. The second equation is the weak formulation of $\operatorname{div} y = 0$. Moreover as discussed in [10], it incorporates the Dirichlet boundary condition for the normal component of y , since the term $\int_{\Gamma} u \cdot \frac{\partial v}{\partial \mathbf{n}} d\gamma$ acts only on tangential components.

The existence of very weak solutions with inhomogeneous Dirichlet boundary conditions can be found for instance in [9, 10, 12, 19]. For boundary data in $\mathbf{H}^0(\Gamma)$ it holds the following. We remark that it is essential to have $\int_{\Gamma_i} u \cdot \mathbf{n} d\gamma = 0$ for all connected components of Γ to obtain existence of solutions for arbitrary large data.

Theorem 2.2. *For every $u \in \mathbf{H}^0(\Gamma)$ there exists a very weak solution $y \in \mathbf{L}^{2n/(n-1)}(\Omega)$ of (2.1). In the two-dimensional case, this solution belongs to $\mathbf{H}^{1/2}(\Omega)$. If the data is small compared to ν the solution is uniquely determined.*

The existence proof and a quantification of the smallness assumption can be found in [19]. The $\mathbf{H}^{1/2}$ -regularity for the 2d-case result can be proven following the lines of [19]. Unique solvability with respect to less regular data, i.e. in $W^{-1/q,q}(\Gamma)$ is investigated in the articles by Farwig, Galdi, Sohr [9, 10, 12]. Once, existence of a solution is proven, the pressure field p can be reconstructed by means of De Rham's Lemma.

In view of the existence result, let us define for abbreviation the state space

$$\mathbf{Y} := \mathbf{L}^{2n/(n-1)}(\Omega).$$

2.2.1 More regular solutions

Let us briefly show that more regular boundary data in $\mathbf{L}^p(\Gamma)$ yields more regular solutions. In the following considerations, we will split the state y in two parts, $y = y_0 + y_1$. The function y_0 is defined as the unique very weak solution to the Stokes equation with inhomogeneous Dirichlet boundary conditions

$$\begin{aligned} -\nu \Delta y_0 + \nabla p_0 &= 0 & \text{in } \Omega \\ \operatorname{div} y_0 &= 0 & \text{in } \Omega \\ y_0 &= u & \text{on } \Gamma. \end{aligned} \quad (2.3)$$

Then $y_1 = y - y_0$ solves the following equation subject to homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\nu\Delta y_1 + (y \cdot \nabla)y_1 + \nabla p_1 &= -(y \cdot \nabla)y_0 & \text{in } \Omega \\ \operatorname{div} y_1 &= 0 & \text{in } \Omega \\ y_1 &= 0 & \text{on } \Gamma. \end{aligned} \quad (2.4)$$

As one can easily see, both systems are uniquely solvable. At first, let us prove higher regularity of y_0 .

Lemma 2.3. *Let $u \in \mathbf{L}^p(\Gamma) \cap \mathbf{H}^0(\Gamma)$, $p \geq 2$, be given. Then the solution of (2.3) satisfies $y_0 \in \mathbf{L}^q(\Omega)$, where q is given by*

$$q = \begin{cases} \frac{np}{n-1} & \text{if } 2 \leq p < \infty, \\ +\infty & \text{if } p = \infty, n = 3. \end{cases} \quad (2.5)$$

Proof. The mapping $u \mapsto y_0$ is linear and continuous from $\mathbf{W}^{-1/p,p}(\Gamma)$ to $\mathbf{L}^p(\Omega)$ and from $\mathbf{W}^{1-1/p,p}(\Gamma)$ to $\mathbf{W}^{1,p}(\Omega)$, see e.g. [2, 7]. By interpolation arguments, we have continuity of this solution mapping from $\mathbf{L}^p(\Gamma)$ to $\mathbf{W}^{1/p,p}(\Omega)$. The claim follows by the imbedding argument $\mathbf{W}^{1/p,p}(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$. The result for $p = \infty$, $n = 3$ can be found in [25]. \square

Applying this result, we can prove higher regularity of the function y_1 and in consequence of the solution y of the nonlinear system.

Lemma 2.4. *If the boundary data u is in $\mathbf{L}^p(\Gamma) \cap \mathbf{H}^0(\Gamma)$, $p \geq 2$ ($n = 2$) or $p \geq 4$ ($n = 3$), then y belongs to $\mathbf{L}^q(\Omega)$ with q given by (2.5).*

Proof. Let us first consider the 2d-case, $n = 2$. Then we have by Theorem 2.2 $y \in \mathbf{L}^4(\Omega)$. This implies $(y \cdot \nabla)y \in \mathbf{H}^{-1}(\Omega)$, hence $y_1 = y - y_0$ solves

$$\begin{aligned} -\nu\Delta y_1 + \nabla p_1 &= -(y \cdot \nabla)y & \text{in } \Omega \\ \operatorname{div} y_1 &= 0 & \text{in } \Omega \\ y_1 &= 0 & \text{on } \Gamma, \end{aligned} \quad (2.6)$$

and we have $y_1 \in \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^t(\Omega)$ for all $t < \infty$.

In the 3d-case, $n = 3$, Theorem 2.2 gives $y \in \mathbf{L}^3(\Omega)$. Since $p \geq 4$ by assumption, Lemma 2.3 yields $y_0 \in L^{3p/2}(\Omega) \subset L^6(\Omega)$. Then $(y \cdot \nabla)y_0$ belongs to $\mathbf{H}^{-1}(\Omega)$, and the solution $y_1 = y - y_0$ of (2.4) belongs to $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$. Hence $y = y_0 + y_1$ is in $\mathbf{L}^6(\Omega)$ as well. This in turn gives $(y \cdot \nabla)y \in W^{-1,r}(\Omega)$, $r = \frac{6p}{p+4} \geq 4$. By [7], the function y_1 as solution of (2.6) belongs then to $\mathbf{W}^{1,4}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$.

In both cases, $n = 2, 3$, the function y_1 is as regular as y_0 , hence the solution y belongs to the space $\mathbf{L}^q(\Omega)$ with q as in (2.5). \square

2.3 Regularity assumption

It is well known that the stationary Navier-Stokes equation is uniquely solvable if the data is small. Hence, if we want to have a unique response y to each control u we would have to impose restrictions on the control to enforce uniqueness of

solutions. This technique is widely employed in optimal control of the stationary Navier-Stokes equations, see e.g. [13, 21, 22, 23, 28]. We will however proceed without a smallness assumption. Since we allow multiple solutions of the state equation, we have to clarify the meaning of optimality.

Definition 2.5. *A pair (\bar{y}, \bar{u}) is called locally optimal, if there exist $\rho_y, \rho_u > 0$ such that $J(\bar{y}, \bar{u}) \leq J(y, u)$ holds for all admissible pairs (y, u) with $\|y - \bar{y}\|_{\mathbf{Y}} < \rho_y$ and $\|u - \bar{u}\|_{\mathbf{H}^0} < \rho_u$.*

Here, a pair (y, u) is admissible if it satisfies the constraints (1.2)–(1.4).

Instead of enforcing uniqueness of solutions for all controls, we will impose the following regularity condition on an optimal state. A similar assumption is used to derive error estimates for distributed control in [5].

Definition 2.6. *A pair $(\bar{y}, \bar{u}) \in \mathbf{Y} \times \mathbf{H}^0(\Gamma)$ is called non-singular, if the linearized Navier-Stokes equation*

$$\begin{aligned} -\nu\Delta y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p &= f & \text{in } \Omega \\ \operatorname{div} y &= 0 & \text{in } \Omega \\ y &= u & \text{on } \Gamma, \end{aligned} \tag{2.7}$$

admits a unique very weak solution $y \in \mathbf{Y}$ for all $u \in \mathbf{H}^0(\Gamma)$ and $f \in \mathbf{H}^{-1}(\Omega)$. Moreover, we assume that the solution mapping $u \mapsto y$ for $f = 0$ is linear and continuous from $\mathbf{H}^0(\Gamma)$ to $\mathbf{H}^{1/2}(\Omega)$, and the mapping $f \mapsto y$ for $u = 0$ is linear and continuous from $\mathbf{H}^{-1}(\Omega)$ to $\mathbf{H}_0^1(\Omega)$.

This condition is fulfilled of course, if the state \bar{y} is small enough [20, Lemma B.1]. The assumption of non-singularity implies that the state equation can be solved uniquely in a neighborhood of the reference control and state, confer [5, Theorem 2.5] for a proof using an implicit function theorem.

Theorem 2.7. *Let $(\bar{y}, \bar{u}) \in \mathbf{Y} \times \mathbf{H}^0(\Gamma)$ be a non-singular solution of (1.2). Then there exist an open neighborhood $\mathcal{O}(\bar{u})$ of \bar{u} in $\mathbf{H}^0(\Gamma)$, an open neighborhood $\mathcal{O}(\bar{y})$ of \bar{y} in \mathbf{Y} , and a mapping S from $\mathcal{O}(\bar{u})$ to $\mathcal{O}(\bar{y})$ of class C^2 such that, for all $u \in \mathcal{O}(\bar{u})$, $S(u) = y$ is the unique very weak solution in $\mathcal{O}(\bar{y})$ of (1.3).*

Furthermore, the action of the first Fréchet derivative $y = S'(\bar{u})u$ is given as the unique very weak solution of the linearized equation (2.7).

2.4 Reformulation of the boundary integrals

Since we will work with very weak solutions $y \in \mathbf{Y}$, we have to clarify the meaning of the boundary integrals in the objective functional (1.1) and the state constraint (1.2). These would be well-defined if the regularity $\frac{\partial y}{\partial \mathbf{n}} \in L^1(\Gamma)$ could be guaranteed. This is not fulfilled for very weak solutions from $\mathbf{H}^{1/2}(\Omega)$ or \mathbf{Y} . We will thus extend the linear functionals in (1.1) and (1.2) to the larger space \mathbf{Y} .

Let us assume for a while that $y \in C^2(\Omega)^n, p \in C^1(\Omega)^n$ are a classical solution of (1.3) to the control u . Multiplying the state equation (1.3) with a

function $\varphi_i \in \mathbf{H}^2(\Omega)$, we obtain by partial integration

$$\begin{aligned}
0 &= (-\nu\Delta y + (y \cdot \nabla)y + \nabla p, \varphi_i) \\
&= \int_{\Omega} (\nu\nabla y \cdot \nabla \varphi_i + (y \cdot \nabla)y\varphi_i) dx - \int_{\Gamma} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p\mathbf{n} \right) \varphi_i d\gamma \\
&= \int_{\Omega} (-\nu y \cdot \Delta \varphi_i - (y \cdot \nabla)\varphi_i y) dx + \int_{\Gamma} \left(\nu y \frac{\partial \varphi_i}{\partial \mathbf{n}} + (y \cdot \mathbf{n})(\varphi_i \cdot y) \right) d\gamma \\
&\quad - \int_{\Gamma} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p\mathbf{n} \right) \varphi_i d\gamma.
\end{aligned} \tag{2.8}$$

In order to represent the functionals in (1.1), (1.2), let us introduce functions φ_i , $i \in \{d, l\}$, that take the right boundary values. Let φ_i denote the weak solutions of

$$\begin{aligned}
-\Delta \varphi_i + \nabla \pi_i &= 0 \quad \text{in } \Omega \\
\operatorname{div} \varphi_i &= 0 \quad \text{in } \Omega
\end{aligned} \tag{2.9}$$

with the boundary values

$$\varphi_i = \begin{cases} \vec{e}_i & \text{on } \Gamma_b \\ 0 & \text{on } \Gamma \setminus \Gamma_b \end{cases}, \quad \varphi_d = \begin{cases} \vec{e}_d & \text{on } \Gamma_b \\ 0 & \text{on } \Gamma \setminus \Gamma_b \end{cases}.$$

Using the transformation above, we can write for $i \in \{d, l\}$.

$$\begin{aligned}
- \int_{\Gamma_b} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p\mathbf{n} \right) \vec{e}_i d\gamma &= - \int_{\Gamma} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p\mathbf{n} \right) \varphi_i d\gamma \\
&= \int_{\Omega} (\nu y \cdot \Delta \varphi_i + (y \cdot \nabla)\varphi_i y) dx \\
&\quad - \int_{\Gamma} \left(\nu y \frac{\partial \varphi_i}{\partial \mathbf{n}} + (y \cdot \mathbf{n})(\varphi_i \cdot y) \right) d\gamma.
\end{aligned}$$

Taking the right-hand side of this expression, we define the functional

$$\tilde{f}_i(y) = \int_{\Omega} (\nu y \cdot \Delta \varphi_i + (y \cdot \nabla)\varphi_i y) dx - \int_{\Gamma} \left(\nu y \frac{\partial \varphi_i}{\partial \mathbf{n}} + (y \cdot \mathbf{n})(\varphi_i \cdot y) \right) d\gamma.$$

This function is well-defined for $y \in \mathbf{H}^s(\Omega)$, $s > 1/2$, but not for $y \in \mathbf{Y}$. Substituting $y = u$ on the boundary yields

$$f_i(y, u) = \int_{\Omega} (\nu y \cdot \Delta \varphi_i + (y \cdot \nabla)\varphi_i y) dx - \int_{\Gamma} \left(\nu u \frac{\partial \varphi_i}{\partial \mathbf{n}} + (u \cdot \mathbf{n})(\varphi_i \cdot u) \right) d\gamma. \tag{2.10}$$

In contrast to (1.1) and (1.2), the function f_i , $i \in \{d, l\}$, is well-defined for states $y \in \mathbf{Y}$ and controls $u \in \mathbf{H}^0(\Gamma)$, since the functions φ are very regular in comparison to the very weak solutions in \mathbf{Y} . Their boundary values are in fact a constant vector, thus, the regularity of φ_i is only influenced by the regularity of the boundary Γ . The following result can be deduced from [11, Vol. 1, Thm. IV.6.1]:

Lemma 2.8. *The functions φ_i , $i \in \{d, l\}$, belong to $\mathbf{H}^2(\Omega) \cap W^{2,p}(\Omega)^n$ for all $p < \infty$.*

Then we can prove the continuity and differentiability statement for f_i .

Lemma 2.9. *The functions f_i are continuous and two-times Fréchet-differentiable from $\mathbf{Y} \times \mathbf{H}^0(\Gamma)$ to \mathbb{R} . Moreover, for given $y \in L^q(\Omega)^n$ and $u \in L^p(\Gamma)^n$, $p, q \in (1, +\infty)$, it holds $f'_{i,y}(y, u) \in L^q(\Omega)^n$ and $f'_{i,u}(y, u) \in L^p(\Omega)^n$.*

Proof. We get the continuity and the Fréchet-derivatives by standard arguments. Because of Lemma 2.8, φ_i , $i \in \{d, l\}$ possesses enough regularity to give $f'_{i,y}(y, u) \in L^q(\Omega)^n$ and $f'_{i,u}(y, u) \in L^p(\Omega)^n$. \square

3 The optimal control problem

Now, we can reformulate the original optimal control problem (1.1)–(1.4), where we use the very weak solutions and the extended functionals f_d, f_l (2.10). We will denote the following optimal control problem (3.1)–(3.4) by **(P)**: Minimize

$$J(y, u) := -f_l(y, u) + \frac{\alpha}{2} \|u\|_{\mathbf{H}^0}^2 \quad (3.1)$$

subject to the very weak form of

$$\begin{aligned} -\nu \Delta y + (y \cdot \nabla) y + \nabla p &= 0 & \text{in } \Omega \\ \operatorname{div} y &= 0 & \text{in } \Omega \\ y &= u & \text{on } \Gamma_c, \\ y &= 0 & \text{on } \Gamma \setminus \Gamma_c, \end{aligned} \quad (3.2)$$

the control constraints

$$u \in U_{ad} := \{u \in \mathbf{H}^0(\Gamma) : u(x) \in U \text{ a.e. on } \Gamma_c, u(x) = 0 \text{ a.e. on } \Gamma \setminus \Gamma_c\} \quad (3.3)$$

and the integral control-state constraint

$$f_d(y, u) \leq D_0. \quad (3.4)$$

Here, we introduced an additional regularization term $\frac{\alpha}{2} \|u\|_{\mathbf{H}^0}^2$, which measures the cost of the control. The parameter α is supposed to be non-negative.

3.1 Existence of solutions

Now, we would like to prove existence of solutions of problem **(P)**. Unfortunately, we can not prove that the objective functional is bounded from below. This is due to the absence of a uniqueness result for the state equation for large data. Moreover, bounds on the state of the kind $\|y\|_{\mathbf{Y}} \leq C \|u\|_{\mathbf{H}^0}$ can only be derived for small data. This is different to the distributed control problems for Navier-Stokes, where we can test the state equation with the state itself and obtain an a-priori bound without smallness assumptions.

Since we can not prove the existence of solutions for the problem **(P)**, we will introduce a modification. We will instead consider the minimization of

$$\tilde{J}(y, u) := g(-f_l(y, u)) + \frac{\tilde{\alpha}}{2} \|y\|_{\mathbf{H}^{1/2}}^2 + \frac{\alpha}{2} \|u\|_{\mathbf{H}^0}^2. \quad (3.5)$$

Here, $\tilde{\alpha}$ is a positive and small parameter. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous, monotonely increasing, and bounded from below, e.g. $g(r) \geq$

g_{min} for all $r \in \mathbb{R}$. Furthermore, we will impose the following control constraint. Let \tilde{U}_{ad} be a closed and convex set such that

$$\tilde{U}_{ad} \subset \left\{ u \in U_{ad} : \int_{\Gamma_b} (u \cdot \mathbf{n})(\varphi_i \cdot u) d\gamma = 0, i \in \{d, l\} \right\}. \quad (3.6)$$

If the control boundary is not part of the observation boundary, i.e. $\Gamma_c \cap \Gamma_o = \emptyset$, one can choose $\tilde{U}_{ad} = U_{ad}$. This choice is also valid in the case of pure tangential controls $u \cdot \mathbf{n} = 0$.

Let us denote the modified minimization (3.5)–(3.6) problem by $(\tilde{\mathbf{P}})$.

Theorem 3.1. *If there is an admissible pair $(y^0, u^0) \in \mathbf{H}^{1/2}(\Omega) \times \tilde{U}_{ad}$, which satisfies (3.2)–(3.4) and the control constraint (3.6), then the problem $(\tilde{\mathbf{P}})$ admits at least one solution.*

Proof. The objective functional \tilde{J} is bounded from below by construction. We can restrict the optimization problem to the set of all admissible pairs (y, u) with $\tilde{J}(y, u) \leq \tilde{J}(y^0, u^0)$ without changing the set of global minimizers. Let us take such an admissible pair. We then obtain

$$\frac{\tilde{\alpha}}{2} \|y\|_{\mathbf{H}^{1/2}}^2 + \frac{\alpha}{2} \|u\|_{\mathbf{H}^0}^2 \leq -g(-f_l(y, u)) + J(y^0, u^0) \leq -g_{min} + J(y^0, u^0), \quad (3.7)$$

which implies that the set of admissible pairs with smaller value of the objective than $J(y^0, u^0)$ is bounded.

Since \tilde{J} is bounded from below, there exists a minimizing sequence $(y_n, u_n) \in \mathbf{H}^{1/2}(\Omega) \times \mathbf{H}^0(\Gamma)$. In view of (3.7), this sequence is bounded and we can extract a weakly converging subsequence, which is again denoted by (y_n, u_n) , i.e. $y_n \rightharpoonup \bar{y}$ in $\mathbf{H}^{1/2}(\Omega)$ and $u_n \rightharpoonup \bar{u}$ in $\mathbf{H}^0(\Gamma)$. By compact embeddings, we have $y_n \rightarrow y$ in $L^p(\Omega)^n$ for all $p < 3$ after extracting another subsequence.

The set \tilde{U}_{ad} is weakly closed by construction, which implies $\bar{u} \in \tilde{U}_{ad}$. Together with the control constraint $u_n \in \tilde{U}_{ad}$, this allows us to pass to the limit in the functions f_i , $\lim f_i(y_n, u_n) = f_i(\bar{y}, \bar{u})$.

The pass to the limit in the very weak solution is straight-forward, which implies that \bar{y} is a very weak solution to \bar{u} . Now, standard arguments using lower semi-continuity of norms conclude the proof. \square

Let us summarize the difficulties in proving existence of solutions:

1. The functional (3.1) is not bounded from below, since there is no a-priori bound $\|y\|_Y \leq C\|u\|_{\mathbf{H}^0}$.
2. If a minimizing sequence would exist, the sequence y_n is not necessarily bounded in \mathbf{Y} .
3. The functions f_i are not weakly continuous on $\mathbf{Y} \times \mathbf{H}^0(\Gamma)$.

Then the modification of the objective and the control constraint were made to cope with this points.

1. The function g makes the objective function bounded from below.
2. The regularization term $\|y\|_{\mathbf{H}^{1/2}}^2$ gives boundedness of y_n in $\mathbf{H}^{1/2}$. By compact embeddings, it allows furthermore to pass to the limit in the part of f_i that involves the state y .

3. The control constraint $\int_{\Gamma} (u \cdot \mathbf{n})(\varphi_i \cdot u) d\gamma = 0$ makes it possible to pass to the limit in the non-linear part of f_i that involves the control.

Of course, there are several other possibilities to enforce existence of solutions. For instance, one could add regularization with respect stronger norms. Another popular change would be to explicitly impose a smallness condition on the controls.

3.2 First order necessary optimality conditions

Let us return to problem **(P)** stated at the beginning of Section 3. We will now derive necessary optimality conditions to characterize local optimal solutions. Here, we will follow the presentation in [27, Section 6.1.2] of the regularity theory of [30]. Let now $(\bar{y}, \bar{u}) \in \mathbf{Y} \times U_{ad}$ be a non-singular and locally optimal pair for **(P)**. Let us define the operator

$$G = (G_1, G_2) : \mathcal{O}(\bar{y}) \times \mathcal{O}(\bar{u}) \rightarrow \mathbf{Y} \times \mathbb{R} \quad (3.8)$$

$$(y, u) \mapsto \begin{pmatrix} S(u) - y \\ f_d(y, u) - D_0 \end{pmatrix}$$

and the cone

$$K = \{0\} \times (-\infty, 0] \subset \mathbf{Y} \times \mathbb{R}.$$

Here, K induces a partial ordering $<_K$ on $\mathbf{Y} \times \mathbb{R}$ by: $x <_K 0 \Leftrightarrow x \in K$. It is easy to show using Theorem 2.7 that the mapping G is twice Fréchet differentiable. Then (\bar{y}, \bar{u}) is a local solution of the minimization problem

$$\min_{y \in \mathcal{O}(\bar{y}), u \in \mathcal{O}(\bar{u})} J(y, u) \text{ subject to } G(y, u) <_K 0, u \in U_{ad}. \quad (3.9)$$

Let us define the Lagrangian associated with this optimization problem

$$L(y, u; \theta, \xi) = J(y, u) + \langle S(u) - y, \theta \rangle_{\mathbf{Y}', \mathbf{Y}} + \xi(f_d(y, u) - D_0), \quad \theta \in \mathbf{Y}', \xi \in \mathbb{R}.$$

To show existence of Lagrange multipliers, we will assume the following regularity condition: there exists $\tilde{u} \in U_{ad} \cap \mathcal{O}(\bar{u})$ such that

$$G'_1(\bar{y}, \bar{u})(\tilde{y}, \tilde{u}) = 0, \quad G'_2(\bar{y}, \bar{u}) + G'_2(\bar{y}, \bar{u})(\tilde{y} - \bar{y}, \tilde{u} - \bar{u}) <_K 0. \quad (3.10)$$

This condition is sufficient for the Zowe-Kurcyusz regularity assumption [30], see e.g. [26].

3.2.1 The optimality system

Under these assumptions, the existence of Lagrange multipliers follows by known results, see e.g. [26, 27, 30].

Theorem 3.2. *Let (\bar{y}, \bar{u}) a non-singular local optimal solution for (P). Let us assume that there $\tilde{u} \in U_{ad} \cap \mathcal{O}(\bar{u})$ such that the linearized Slater condition (3.10) is satisfied. Then there exists $\theta \in \mathbf{Y}'$ and $\xi \geq 0$ such that the equation*

$$\theta = -f'_{l,y}(\bar{y}, \bar{u}) + \xi f'_{d,y}(\bar{y}, \bar{u}), \quad (3.11a)$$

the variational inequality

$$(\alpha \bar{u} - f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u}) + S'(\bar{u})^* \theta, u - \bar{u})_{L^2(\Gamma_c)} \geq 0 \quad \forall u \in U_{ad}, \quad (3.11b)$$

and the complementarity condition

$$\xi(f_d(\bar{y}, \bar{u}) - D_0) = 0, \xi \geq 0, f_d(\bar{y}, \bar{u}) \leq D_0 \quad (3.11c)$$

hold.

Here, the adjoint operator $S'(\bar{u})^* : \mathbf{Y}' \rightarrow \mathbf{H}^0(\Gamma)'$ shows up. It is connected to the solution of the so-called adjoint equation. In fact, it holds [20]:

Theorem 3.3. *The action of $S'(\bar{u})^*$ can be characterized as follows: for given $\theta \in \mathbf{Y}'$ it holds*

$$S'(\bar{u})^* \theta = - \left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right) \Big|_{\Gamma_c}, \quad (3.12)$$

where $\lambda \in \mathbf{H}_0^1(\Omega) \cap W^{2,r}(\Omega)^n$ is the unique solution of the weak formulation

$$\int_{\Omega} (\nu \nabla \lambda \cdot \nabla v - (\bar{y} \cdot \nabla) \lambda v - (v \cdot \nabla) \lambda \bar{y}) \, dx = \langle \theta, v \rangle_{H^{-1}, H^1} \quad \forall v \in \mathbf{H}_0^1(\Omega) \quad (3.13)$$

and $\pi \in W^{1,r}(\Omega)$ the associated pressure field for all $r \in [2, \infty)$.

Proof. The representation of $S'(\bar{u})^*$ is proven for instance in [20]. It remains to investigate the regularity of λ . The right-hand side $\langle \theta, v \rangle_{H^{-1}, H^1}$ is given according to the previous Theorem 3.3 by

$$\langle \theta, v \rangle_{H^{-1}, H^1} = \int_{\Omega} (-\nu \Delta \tilde{\varphi} v - (v \cdot \nabla) \tilde{\varphi} \bar{y} - (\bar{y} \cdot \nabla) \tilde{\varphi} v) \, dx,$$

where we used the notation $\tilde{\varphi} = -\varphi_l + \xi \varphi_d$. By assumption, U_{ad} is a subset of $\mathbf{L}^\infty(\Gamma)$, hence $\bar{u} \in \mathbf{L}^\infty(\Gamma)$ and $\bar{y} \in \mathbf{L}^q(\Omega)$, $2 \leq q < \infty$ for $n = 2$, $2 \leq q \leq \infty$ for $n = 3$, cf. (2.5). Due to the high regularity of $\tilde{\varphi}$, compare Lemma 2.8, we can estimate with $\tilde{p} > n$ such that $\mathbf{W}^{2,\tilde{p}}(\Omega) \hookrightarrow \mathbf{W}^{1,\infty}(\Omega)$

$$|\langle \theta, v \rangle_{H^{-1}, H^1}| \leq c(1 + \|\bar{y}\|_{\mathbf{L}^q}) \|\tilde{\varphi}\|_{\mathbf{W}^{2,\tilde{p}}} \|v\|_{\mathbf{L}^r}$$

with $1/q + 1/\tilde{p} + 1/r = 1$. Since q and \tilde{p} can be chosen arbitrary large (but not equal to ∞), we obtain $\theta \in \mathbf{L}^q(\Omega)^n$, for all $q \in (2, \infty)$. Let us now estimate the addend on the left-hand side of (3.13) that comes from the nonlinearity of the state equation:

$$\left| \int_{\Omega} ((\bar{y} \cdot \nabla) \lambda v + (v \cdot \nabla) \lambda \bar{y}) \, dx \right| \leq c \|\bar{y}\|_{\mathbf{L}^q} \|\nabla \lambda\|_{\mathbf{L}^p} \|v\|_{\mathbf{L}^r} \quad (3.14)$$

with $1/q + 1/p + 1/r = 1$, $2 \leq q < \infty$.

Since (\bar{y}, \bar{u}) is non-singular, the equation (3.13) is uniquely solvable with solution $\lambda \in \mathbf{H}_0^1(\Omega)$. That is, estimate (3.14) holds with $p = 2$. We can interpret the adjoint state as the weak solution of a Stokes equation, where the terms in (3.14) are put on the right-hand side. This allows to apply known regularity results for the Stokes equation.

With $p = 2$ the estimate (3.14) holds for all $r > 2$, hence the functional in (3.14) is in $\mathbf{L}^{r'}(\Omega)$ for all $r' < 2$. The regularity result by Galdi [11, Lemma IV.6.1] gives in a first step the regularities $\lambda \in \mathbf{W}^{2,r'}(\Omega)$ and $\pi \in W^{1,r'}(\Omega)$ for all $r' < 2$. By embedding arguments, we have then $\nabla \lambda \in L^p(\Omega)^{n,n}$, where p depends on n : $p \in (2, \infty)$ for $n = 2$; $p \in (2, 6)$ for $n = 3$.

In the 2d-case, (3.14) is valid for all $p < \infty$, which allows us to chose r arbitrary small with $r > 1$. That is, the functional involving \bar{y} and $\nabla\lambda$ is in $\mathbf{L}^q(\Omega)$ for all $q < \infty$. Again applying the regularity result for the Stokes equation, we find $\lambda \in \mathbf{W}^{2,q}(\Omega)$ and $\pi \in W^{1,q}(\Omega)$ for all $q < \infty$.

For the three-dimensional case, we obtain similarly $\lambda \in \mathbf{W}^{2,q}(\Omega)$ and $\pi \in W^{1,q}(\Omega)$ for all $q < 6$. By continuous imbeddings, $\nabla\lambda \in L^\infty(\Omega)^{n,n}$ follows, which gives after applying again Galdi's regularity result $\lambda \in \mathbf{W}^{2,q}(\Omega)$ and $\pi \in W^{1,q}(\Omega)$ for all $q < \infty$. \square

The adjoint pressure π is only determined up to constant. This fact is usually circumvented by requiring $\int_\Omega \pi dx = 0$. Here, it is not necessary to fix the constant, since the constant does not change the variational inequality due to the construction of $\mathbf{H}^0(\Gamma)$

$$(\pi \mathbf{n}, v)_{L^2(\Gamma_c)} = ((\pi + c)\mathbf{n}, v)_{L^2(\Gamma_c)} \quad \forall c \in \mathbb{R}, v \in \mathbf{H}^0(\Gamma).$$

Furthermore, the variational inequality (3.11b) can be written as a non-smooth equation.

Corollary 3.4. *Let the assumptions of the previous theorem be fulfilled. Then the variational inequality (3.11b) is equivalent the following condition. For each connected component $\Gamma_j \in \Gamma$ with $\Gamma_j \cap \Gamma_c \neq \emptyset$ there is $\eta_j \in \mathbb{R}$ such that the pointwise representation holds for a.a. $x \in \Gamma_c$*

$$\bar{u}(x) = \text{proj}_U \left(-\frac{1}{\alpha} \left(-\left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right) - f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u}) + \eta_j \mathbf{n} \right) \right), \quad x \in \Gamma_c \cap \Gamma_j, \quad (3.15)$$

and the zero net-mass conditions

$$\int_{\Gamma_c \cap \Gamma_j} \bar{u} \cdot \mathbf{n} d\gamma = 0 \quad \forall j : \Gamma_j \cap \Gamma_c \neq \emptyset$$

are satisfied. Here, $\text{proj}_U : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the Euclidean projection in \mathbb{R}^n onto the set U .

Proof. At first, the variational inequality (3.11b) is equivalent to

$$\bar{u} = \text{proj}_{U_{ad}} \left(-\frac{1}{\alpha} \left(-\left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right) - f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u}) \right) \right),$$

where $\text{proj}_{U_{ad}} : \mathbf{L}^2(\Gamma) \rightarrow \mathbf{H}^0(\Gamma)$ is the projection with respect to the $\mathbf{L}^2(\Gamma)$ -norm on U_{ad} . That is, \bar{u} solves the minimization problem

$$\min \frac{1}{2} \left\| u + \frac{1}{\alpha} \left(-\left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right) - f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u}) \right) \right\|_{\mathbf{L}^2(\Gamma)}^2$$

subject to

$$\int_{\Gamma_c} u \cdot \mathbf{n} d\gamma = 0, \quad \forall j : \Gamma_j \cap \Gamma_c \neq \emptyset, \\ u(x) \in U \text{ a.e. on } \Gamma_c.$$

Then there exists Lagrange multipliers η_j associated to the integral constraints in this auxiliary problem, and the variational inequality

$$\left(\alpha \bar{u} - \left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right) - f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u}) + \left(\sum_j \eta_j \chi_j \right) \mathbf{n}, u - \bar{u} \right) \geq 0$$

holds for all $u \in L^2(\Gamma)$ satisfying $u(x) \in U$ a.e. on Γ_c , $u(x) = 0$ on $\Gamma \setminus \Gamma_c$. By standard arguments [27], it can be proven that this variational inequality is equivalent to the projection representation as claimed. \square

Unfortunately, the argument $-f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u})$ of the projection depends on the control itself. This term involves no smoothing operation. Hence, we cannot conclude higher regularity of optimal controls from the projection representation, such that \bar{u} has the same regularity as $\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n}$. Moreover, the non-smooth formulation of the variational inequality is not suitable for semi-smooth Newton methods.

4 Finite-dimensional control set

We will now consider a regularized version of the optimal control problem (3.1)–(3.4). In particular, the controls will now be taken from $\mathbf{H}^{1/2}(\Gamma)$, which leads to higher regularity of the associated states. For $\mathbf{H}^{1/2}(\Gamma)$ -controls one has the following regularity result [11, Theorem VIII.4.1].

Lemma 4.1. *For every $u \in \mathbf{H}^{1/2}(\Gamma)$ the very weak solution y belongs to $\mathbf{H}^1(\Omega)$, and the trace of y coincides with the control u on the boundary Γ .*

Due to this regularity result, we can retract some of the reformulations of the boundary integrals in Section 2.4. For states $y \in \mathbf{H}^1(\Omega)$, we define the functionals

$$\hat{f}_i(y) := - \int_{\Omega} (\nu \nabla y \cdot \nabla \varphi_i + (y \cdot \nabla) y \varphi_i) dx, \quad i \in \{d, l\},$$

compare (2.8). Then \hat{f}_i is twice continuously Fréchet differentiable from $\mathbf{H}^1(\Omega)$ to \mathbb{R} . And it holds

$$\hat{f}_i(y) = \int_{\Gamma} \left(\nu \frac{\partial y}{\partial \mathbf{n}} - p \mathbf{n} \right) \varphi_i d\gamma = f_i(y, u), \quad i \in \{d, l\},$$

for smooth states y associated to controls u .

Additionally, let us introduce a finite-dimensional control space. Let e_i , $i = 1 \dots l$, be linearly independent functions from $\mathbf{H}^{1/2}(\Gamma)$ with support on Γ_c . Let $Q \in \mathbb{R}^l$ be the set of allowed coefficients, which is assumed to be non-empty, closed and convex. Then we define the set of admissible controls as

$$U_{ad,q} := \left\{ u \in \mathbf{H}^{1/2}(\Gamma) : u = \sum_{i=1}^l q_i e_i, \quad q \in Q \right\}.$$

Instead of this construction, we could have added a penalization term like $\beta \|u\|_{\mathbf{H}^{1/2}}$ to the cost functional. However, this additional term is not justified physically. Moreover, the optimality system would involve a variational

inequality with a non-local differential operator on Γ_c . Since the main focus of this section is to derive first-order necessary optimality conditions, we chose the finite-dimensional case for the sake of clarity.

Now, we are considering the following optimization problem, henceforth called (\mathbf{P}_l) : Minimize

$$J(y, u) := -\hat{f}_l(y) + \frac{\alpha}{2} \|u\|_{\mathbf{H}^0}^2 \quad (4.1)$$

subject to the very weak form of

$$\begin{aligned} -\nu \Delta y + (y \cdot \nabla) y + \nabla p &= 0 & \text{in } \Omega \\ \operatorname{div} y &= 0 & \text{in } \Omega \\ y &= u & \text{on } \Gamma_c, \\ y &= 0 & \text{on } \Gamma \setminus \Gamma_c, \end{aligned} \quad (4.2)$$

the control constraints

$$u \in U_{ad,q} \quad (4.3)$$

and the integral state constraint

$$\hat{f}_d(y) \leq D_0. \quad (4.4)$$

Due to the same reasons as above, existence of solutions cannot be proven directly. Here, we would have to work with similar modifications to (\mathbf{P}_l) as in Section 3.1 above. Rather, we would like to derive a first-order optimality system. To this end, let us assume that (\bar{y}, \bar{u}) is a non-singular and locally optimal solution of (\mathbf{P}_l) . Moreover, let us assume that a linearized Slater point for the state constraint exists, similarly defined as in (3.10)

Then one can argue as above to obtain:

Theorem 4.2. *Let (\bar{y}, \bar{u}) a non-singular local optimal solution for (\mathbf{P}_l) . Let us assume that there $\tilde{u} \in U_{ad,q} \cap \mathcal{O}(\bar{u})$ such that the linearized Slater condition (3.10) is satisfied. Then there exists a multiplier $\xi \geq 0$, an adjoint state $\lambda \in \mathbf{H}_0^1(\Omega) \cap W^{2,r}(\Omega)^n$, and an adjoint pressure $\pi \in W^{1,r}(\Omega)$, $r \in [2, \infty)$, such that (λ, π) is the weak solution of*

$$\begin{aligned} -\nu \Delta \lambda + (\nabla \bar{y})^T \lambda - (\bar{y} \cdot \nabla) \lambda + \nabla \pi &= -\nu \Delta(\varphi_l - \xi \varphi_d) + (\nabla \bar{y})^T(\varphi_l - \xi \varphi_d) \\ &\quad - (\bar{y} \cdot \nabla)(\varphi_l - \xi \varphi_d) & \text{in } \Omega \\ \operatorname{div} \lambda &= 0 & \text{in } \Omega \\ \lambda &= 0 & \text{on } \Gamma, \end{aligned}$$

and such that the variational inequality

$$\left(\alpha \bar{u} - \left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right), u - \bar{u} \right)_{L^2(\Gamma_c)} \geq 0 \quad \forall u \in U_{ad,q},$$

and the complementarity condition

$$\xi(\hat{f}_d(\bar{y}) - D_0) = 0, \quad \xi \geq 0, \quad \hat{f}_d(\bar{y}) \leq D_0$$

are satisfied.

Under some additional assumptions, we can simplify this system even more. Here, we will replace the functions φ_l and φ_d with differently defined functions. Let us assume that there exists functions $(\varphi_i, \pi_i) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$, $i \in \{d, l\}$, such that it holds

$$\begin{aligned} \operatorname{div} \varphi_i &= 0 && \text{on } \Omega \\ \varphi_i &= e_i && \text{on } \Gamma_b \\ \varphi_i &= 0 && \text{on } \Gamma \setminus \Gamma_b \\ \nu \frac{\partial \varphi_i}{\partial \mathbf{n}} - \pi_i \mathbf{n} &= 0 && \text{on } \Gamma_c. \end{aligned} \tag{4.6}$$

Of course, φ_i cannot be chosen as solutions of a Stokes system, since the above conditions represent over-determined boundary conditions. With this choice of auxiliary functions, all result remain valid, since we have never used that (φ_i, π_i) should be solutions of a Stokes equation. Introducing a new adjoint state as the difference of $\varphi_l - \xi \varphi_d$ and the adjoint state given by Theorem 4.2, we obtain

Theorem 4.3. *Let the assumptions of Theorem 4.2 be satisfied. Assume there exists $(\varphi_i, \pi_i) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$, $i \in \{d, l\}$, such that (4.6) is satisfied.*

Then there exists a multiplier $\xi \geq 0$, an adjoint state $\lambda \in \mathbf{H}_0^1(\Omega) \cap W^{2,r}(\Omega)^n$, and an adjoint pressure $\pi \in W^{1,r}(\Omega)$, $r \in [2, \infty)$, such that (λ, π) is the weak solution of

$$\begin{aligned} -\nu \Delta \lambda + (\nabla \bar{y})^T \lambda - (\bar{y} \cdot \nabla) \lambda + \nabla \pi &= 0 && \text{in } \Omega \\ \operatorname{div} \lambda &= 0 && \text{in } \Omega \\ \lambda &= e_l - \xi e_d && \text{on } \Gamma_b \\ \lambda &= 0 && \text{on } \Gamma \setminus \Gamma_b, \end{aligned}$$

and such that the variational inequality

$$\left(\alpha \bar{u} - \left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right), u - \bar{u} \right)_{L^2(\Gamma_c)} \geq 0 \quad \forall u \in U_{ad,q},$$

and the complementarity condition

$$\xi (\hat{f}_d(\bar{y}) - D_0) = 0, \quad \xi \geq 0, \quad \hat{f}_d(\bar{y}) \leq D_0$$

are satisfied.

Please note that this system does not involve the functions φ_i in the adjoint equation and in the variational inequality, which was the case for the optimality systems obtained in Theorems 3.2, 3.3, and 4.2. This makes this system favorable for computations, and it is used for the solution algorithm that we employed in our numerical experiments.

5 Second-order sufficient optimality condition

The presentation of second-order sufficient optimality condition follows [6, 29]. In order to define strongly active sets for the convex control constraint $u(x) \in U$, let us recall some definitions from the theory of convex sets. For a convex set

$C \in \mathbb{R}^n$ and an element $u \in C$, we denote by $\mathcal{N}_C(u)$ and $\mathcal{T}_C(u)$ the normal cone and polar cone of tangents of C at the point u , which are defined by

$$\begin{aligned}\mathcal{N}_C(u) &:= \{z \in \mathbb{R}^n : z^T(v - u) \leq 0 \forall v \in C\}, \\ \mathcal{T}_C(u) &:= \{z \in \mathbb{R}^n : z^T v \leq 0 \forall v \in \mathcal{N}_C(u)\}.\end{aligned}$$

Further, we need the linear subspaces

$$N_C(u) = \text{cl span } \mathcal{N}_C(u), \quad T_C(u) = N_C(u)^\perp.$$

Now, we want to use these notations with $C = U_{ad}$. Let be given an admissible control $u \in U_{ad}$. It is well-known, that the sets $\mathcal{N}_{U_{ad}}(u)$, $\mathcal{T}_{U_{ad}}(u)$, $N_{U_{ad}}(u)$ and $T_{U_{ad}}(u)$ admit a pointwise representation as U_{ad} itself. Let us introduce the projection operations on the sets $N_{U_{ad}}(u)$ and $T_{U_{ad}}(u)$ for an admissible function $u \in U_{ad}$. We define

$$w_N(x) = \text{Proj}_{[N_U(u(x))]}(w(x)),$$

which is the pointwise projection of $w(x)$ on the space $N_U(u(x))$ of normal directions of U at $u(x)$. Its orthogonal counterpart is denoted by

$$w_T(x) = \text{Proj}_{[T_U(u(x))]}(w(x)).$$

The relative interior of a convex set is defined by

$$\text{ri } C := \{x \in \text{aff } C : \exists \varepsilon > 0 \ B_\varepsilon(x) \cap \text{aff } C \subset C\},$$

its complement in C is called the relative boundary

$$\text{rb } C := C \setminus \text{ri } C.$$

The distance of a point $u \in \mathbb{R}^n$ to a set $C \subset \mathbb{R}^n$ is defined by

$$\text{dist}(u, C) := \inf_{x \in C} |u - x|.$$

Let (\bar{y}, \bar{u}) be a fixed admissible pair that fulfills the first-order necessary optimality condition of Theorem 3.2. For brevity, we define

$$j_u := \alpha \bar{u} - f'_{l,u}(\bar{y}, \bar{u}) + \xi f'_{d,u}(\bar{y}, \bar{u}) - \left(\nu \frac{\partial \lambda}{\partial \mathbf{n}} - \pi \mathbf{n} \right),$$

which represents the total derivative of the objective functional $J(y(u), u)$ with respect to u .

For $\varepsilon > 0$, we define by

$$\Gamma_\varepsilon := \{x \in \Gamma : \text{dist}(-j_u(x), \text{rb } \mathcal{N}_U(\bar{u}(x))) > \varepsilon\},$$

the set of strongly active control constraints for \bar{u} . Now, the ε -critical cone $C_\varepsilon(\bar{u})$ is made up of all $(z, h) \in \mathbf{Y} \times \mathbf{H}^0(\Gamma)$ satisfying the following conditions (5.1)-(5.4):

$$\begin{aligned}-\nu \Delta z + (\bar{y} \cdot \nabla)z + (z \cdot \nabla)\bar{y} + \nabla \mu &= 0 & \text{in } \Omega \\ \text{div } z &= 0 & \text{in } \Omega \\ z &= h & \text{on } \Gamma_c, \\ z &= 0 & \text{on } \Gamma \setminus \Gamma_c,\end{aligned} \tag{5.1}$$

$$\begin{cases} f'_d(\bar{y}, \bar{u})(z, h) = 0, & \text{if } f_d(\bar{y}, \bar{u}) = D_0 \text{ and } \xi > 0 \\ f'_d(\bar{y}, \bar{u})(z, h) \leq 0, & \text{if } f_d(\bar{y}, \bar{u}) = D_0 \text{ and } \xi = 0, \end{cases}, \quad (5.2)$$

$$h \in T_{U_{ad}}(\bar{u}), \quad (5.3)$$

$$h_N = 0 \text{ on } \Gamma_\varepsilon. \quad (5.4)$$

We will require coercivity of the following bilinear form, which is the second derivative of the Lagrangian of **(P)**:

$$\begin{aligned} \mathcal{L}''(\bar{y}, \bar{u}, \lambda, \xi)[(z_1, h_1), (z_2, h_2)] = & \\ & - \int_{\Omega} (z_1 \cdot \nabla)(\xi\varphi_d - \varphi_l - \lambda)z_2 + (z_2 \cdot \nabla)(\xi\varphi_d - \varphi_l - \lambda)z_1 dx \\ & + \int_{\Gamma_c} \alpha h_1 h_2 + (h_1 \cdot \mathbf{n})(\varphi_l - \xi\varphi_d) \cdot h_2 + (h_2 \cdot \mathbf{n})(\varphi_l - \xi\varphi_d) \cdot h_1 d\gamma \end{aligned}$$

Applying the methods of proof of e.g. [6, 29] one can show

Theorem 5.1. *Let (\bar{y}, \bar{u}) be an admissible non-singular point for the optimal control problem and fulfill the first-order necessary optimality condition of Theorem 3.2 with associated λ, ξ .*

Assume further that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\mathcal{L}''(\bar{y}, \bar{u}, \lambda, \xi)[(z, h)^2] \geq \delta \|h\|_{\mathbf{H}^0}^2$$

holds for all pairs $(z, h) \in C_\varepsilon(\bar{u})$.

Then there exist $\alpha > 0$ and $\tau > 0$ such that

$$J(v) \geq J(\bar{v}) + \alpha \|u - \bar{u}\|_{\mathbf{H}^0}^2$$

holds for all admissible pairs (y, u) with $\|u - \bar{u}\|_{\mathbf{L}^\infty} \leq \tau$.

6 Numerical experiments

In this section, we want to provide numerical results for the problem under consideration. The computational domain, depicted in Figure 1, is a 2D generic high-lift configuration consists of a NACA4412 main airfoil at 8° angle of attack and a NACA4415 flap with a deflection angle of 37° . The Reynolds number was given as $Re = 106.25$ based on the chord length L_{ref} and the free stream velocity $u_\infty = 1$. The inflow u_∞ acts as a inhomogeneous dirichlet boundary condition on the inflow boundary Γ_{in} . The control boundary Γ_c was modelled by a nonhomogeneous dirichlet condition, where the limited suction and/or blowing occurs on small slot on the flap. A no-slip boundary condition, i.e. homogeneous Dirichlet condition, was used for the remaining airfoil Γ_b and the wall boundaries Γ_{wall} . At the outflow Γ_{out} , we applied a so called "do-nothing"-condition:

$$\nu \frac{\partial y}{\partial n} - pn = 0.$$

For more details of the configuration see the technical report [4].

As already mentioned above, we solved the optimality system given by Theorem 4.3. Due to the presence of the do-nothing boundary condition, we can drop

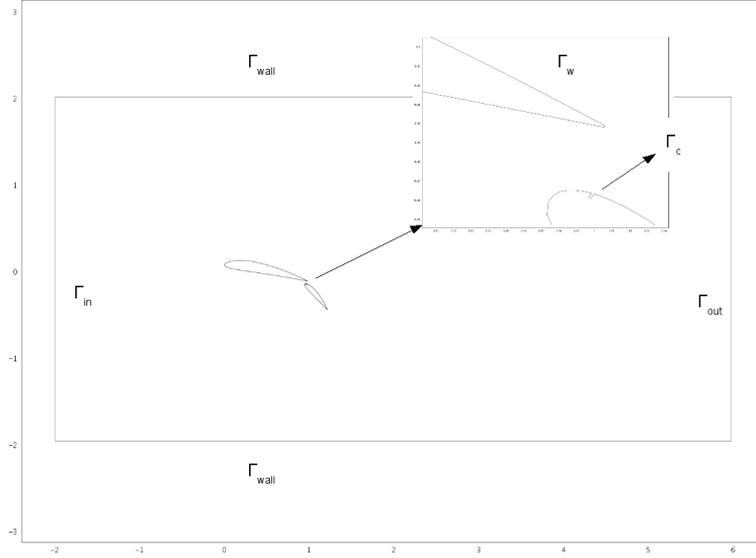


Figure 1: The domain

the constraint $\int_{\Gamma_j} u \cdot n d\gamma = 0$, which was incorporated to guarantee existence of divergence-free solutions. With this simplification, the variational inequality and the complementarity condition in the optimality system given by Theorem 4.3 are equivalent to

$$\bar{u} = \text{proj}_U \left(\frac{1}{\alpha} \left(\nu \frac{\partial \lambda}{\partial n} - \pi n \right) \right) \text{ a.e. in } \Gamma_c$$

and

$$\xi = \max(0, \xi + \hat{f}_d(\bar{y}) - D_0).$$

This enables us to eliminate the control variable by means of the projection. Then we want to solve the following system consisting of the state equation with control eliminated by the projection formula, the equation for the Lagrange

multiplier ξ and the associated adjoint equation, see also [4]:

$$\begin{aligned}
-\nu\Delta y + (y \cdot \nabla)y + \nabla p &= 0 && \text{in } \Omega \\
\operatorname{div} y &= 0 && \text{in } \Omega \\
y|_{\Gamma_c} &= \operatorname{proj}_U \left(\frac{1}{\alpha} \left(\nu \frac{\partial \lambda}{\partial n} - \pi n \right) \right) && \text{on } \Gamma_c \\
y &= 0 && \text{on } \Gamma_{wall} \cup \Gamma_b \setminus \Gamma_c \\
y &= u_\infty && \text{on } \Gamma_{in} \\
\nu \frac{\partial y}{\partial n} - pn &= 0 && \text{on } \Gamma_{out} \\
-\nu\Delta \lambda + (\nabla y)^T \lambda - (y \cdot \nabla)\lambda + \nabla \pi &= 0 && \text{in } \Omega \\
\operatorname{div} \lambda &= 0 && \text{in } \Omega \\
\lambda &= e_l - \xi e_d && \text{on } \Gamma_w \\
\lambda &= 0 && \text{on } \Gamma_{in} \cup \Gamma_{wall} \\
\nu \frac{\partial \lambda}{\partial n} - \pi n + (y \cdot n)\lambda &= 0 && \text{on } \Gamma_{out} \\
\xi &= \max(0, \xi + \hat{f}_d(y) - D_0).
\end{aligned}$$

We used the commercial FEM-solver COMSOL Multiphysics with a damped Newton method for the nonlinear system. The partial differential equations were discretized using Taylor-Hood finite elements, i.e. piecewise quadratic polynomials for the velocity and piecewise linear polynomials for the pressure.

Since the equation for ξ does not allow for a direct solution of the system, we choose the following algorithm. In the first step, we solved the optimal control problem without the state constraint. If for the computed solution the state constraint was fulfilled then this solution solves also the state-constrained problem. Otherwise, we solved the optimal control problem with the active state constraint $f_d(y, u) = D_0$.

We obtained for the uncontrolled problem a lift of $C_a = \frac{F_A}{0.5u_\infty^2 L_{ref}} = 1.299$ and a drag of $C_d = \frac{D_A}{0.5u_\infty^2 L_{ref}} = 0.701$, where F_A is the resulting lift force, D_A the drag force and $L_{ref} = 1.275$ is the reference length of the wing, see Figure 2 for a streamline plot of the velocity field, and Figure 3 for a plot of the absolute values of the uncontrolled velocity field and the pressure field.

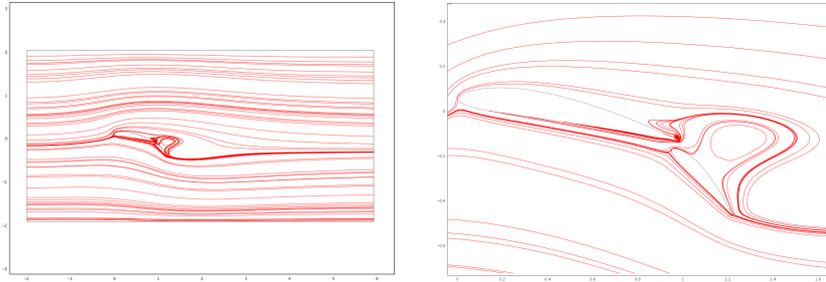


Figure 2: Uncontrolled case: velocity field with zoom on the wing (right).

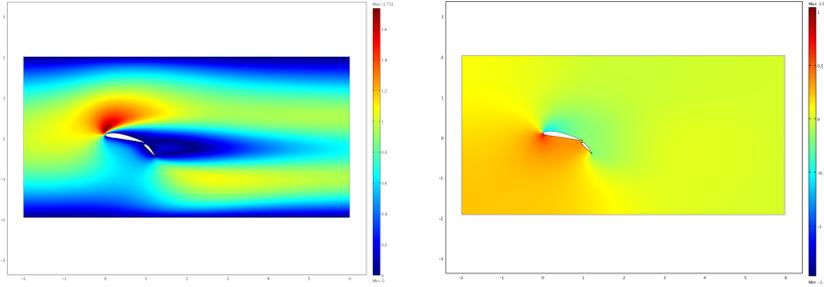


Figure 3: Uncontrolled case: absolute value of velocity field (left), pressure field (right).

Now let us report about the outcome of the optimization. Here, we chose the control cost parameter $\alpha = 0.001$, and the control constraints as box constraints $U = [-1, +1]$. At first, we computed solution for the case without any drag constraint. The optimal control is given by the maximal possible suction, which is natural from a physical point of view. The obtained optimized lift is $C_a = 1.313$ and the drag is $C_d = 0.72$, which is a lift gain of 1.1%. The controlled velocity field can be seen in Figure 4. The adjoint velocity field and pressure are plotted in Figures 5 and 6.

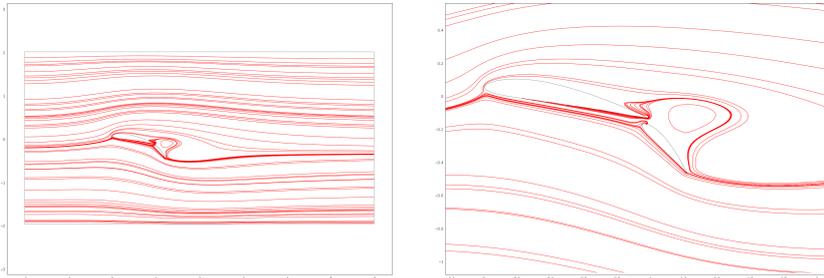


Figure 4: Controlled case: velocity field with zoom on the wing (right).

In the next step, we choose as upper bound for the drag $D_0 = 0.717$. Hence, we expect that this constraint will be active at the solution. In fact, for the computed solution we obtain $C_d = 0.717$. Moreover, due to this restriction the computed lift is $C_a = 1.3127$, which is less than for the case without state constraints, but which is still better than in the uncontrolled situation.

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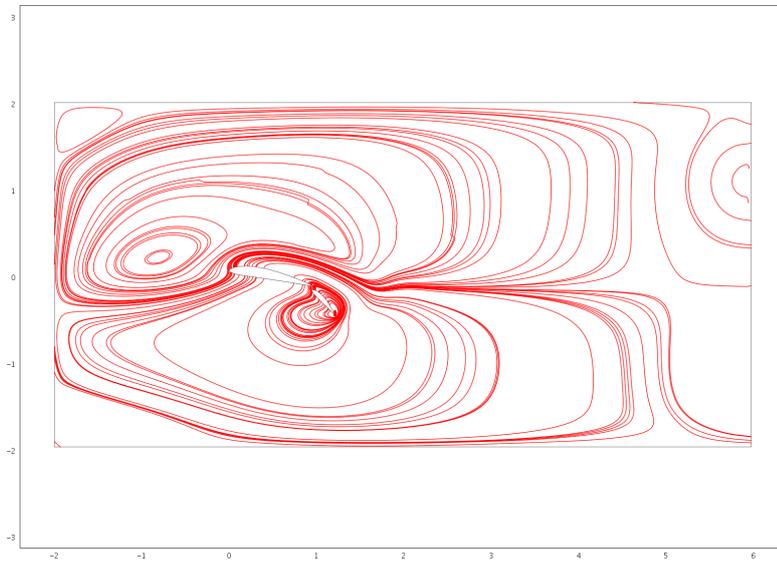


Figure 5: Controlled case: adjoint velocity field

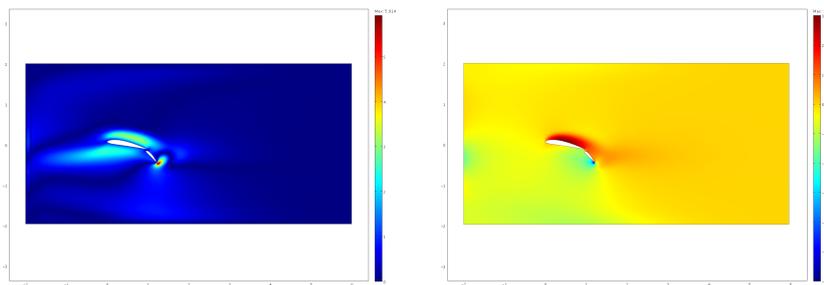


Figure 6: Controlled case: absolute value of adjoint velocity field (left), adjoint pressure (right).

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