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A Subspace Correction Method for Discontinuous Galerkin discretizations of linear elasticity equations

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A SUBSPACE CORRECTION METHOD FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS OF LINEAR ELASTICITY EQUATIONS

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1. Introduction

We study preconditioning techniques for discontinuous Galerkin discretizations of linear elasticity problems in primal (displacement) formulation. We propose a space splitting which gives rise to uniform preconditioners for the Interior Penalty (IP) Finite Element (FE) discretizations recently introduced in the works [12, 18, 19]. For the case when Dirichlet boundary conditions are imposed on the entire boundary, the action of the preconditioner is equivalent to solving several (2 or 3) Laplace equations and the condition number of the preconditioned system is uniformly bounded with respect to both the Poisson ratio and the mesh size. However, when natural (i.e. traction free) boundary conditions are prescribed on part of the boundary the situation is much more subtle, and we present here a preconditioning technique which reduces the solution of the linear algebraic system corresponding to the IP Galerkin method to a solution of a discretization with nonconforming Crouzeix-Raviart elements.

There are several works on preconditioning discretized linear elasticity equations for conforming or non-conforming finite element methods [5, 11, 14, 15, 16]. However, to our knowledge the works related to the preconditioning of the Discontinuous Galerkin (DG) discretizations of linear elasticity equations are very limited. Our work is focused on preconditioning a particular type of DG methods namely interior penalty methods, and we construct a uniform preconditioner for the Symmetric Interior Penalty Galerkin (SIPG) discretization. The main ingredient is a natural splitting of the DG space. Such a splitting was introduced in [4] in the context of designing subspace correction methods and also considered in [8] in a different context. In [4] it was shown that subspace correction methods for a discretization of scalar elliptic equations, based on such a natural splitting of the DG space lead to uniform preconditioners for the symmetrized DG schemes and to uniformly convergent iterative methods for the nonsymmetric DG schemes. Here we have also extended some of the results from [4] to vector field problems, including results for Nonsymmetric Interior Penalty (NIPG) and Incomplete Interior Penalty (IIPG) discretizations.

The rest of the paper is organized as follows. We introduce the problem and the basic notation in §2. Next, in §3 we introduce the corresponding DG discretizations and recall some of their stability and approximation properties. In the last section §4, we introduce the subspace correction methods, and we prove that they give rise to a uniform preconditioner of the symmetric IP method.

2. Problem formulation and notation

Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$ be a convex polygon or polyhedron and let $\boldsymbol{u}$ be a vector field in $\mathbb{R}^d$, defined on a domain $\Omega \subset \mathbb{R}^d$ such that $\boldsymbol{u} \in [H^2(\Omega)]^d$. The elasticity tensor, which we denote by $\mathcal{C}$, is a linear operator, i.e., $\mathcal{C} : \mathbb{R}^{d\times d}_{\rm sym} \mapsto \mathbb{R}^{d\times d}_{\rm sym}$, acting on a symmetric matrix $A \in \mathbb{R}^{d\times d}_{\rm sym}$, in the following way:

$$\mathcal{C}\, A = 2\mu A + \lambda\, \mathrm{trace}(A) I.$$

One can show that the linear operator $\mathcal{C}$ is selfadjoint and has two eigenvalues: (1) a simple eigenvalue equal to $(2\mu + d\lambda)$ corresponding to the identity matrix; (2) an eigenvalue equal to 2μ, corresponding to the $\frac{d(d-1)}{2} - 1$ dimensional space of traceless, symmetric, real matrices. Thus for $d = 2, 3$, $\lambda \geq 0$ and $\mu > 0$, we always have that

$$2\mu\langle A : A\rangle \leq \langle \mathcal{C}A : A\rangle \leq (2\mu + d\lambda)\langle A : A\rangle. \tag{2.1}$$

Here, and also later on we denote by $\langle\cdot,\cdot\rangle$ the Euclidean (resp. by $\langle\cdot : \cdot\rangle$ the Frobenius) scalar product for two vectors (resp. tensors) in $\mathbb{R}^d$ (resp. $\mathbb{R}^{d\times d}$), i.e.,

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{k=1}^{d} u_k v_k, \qquad \langle \boldsymbol{v} : \boldsymbol{w}\rangle = \sum_{j=1}^{d}\sum_{k=1}^{d} u_{jk} v_{jk}.$$

The corresponding products in $[L_2(\Omega)]^d$ and $[L_2(\Omega)]^{d\times d}$ are

$$(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \langle \boldsymbol{v}, \boldsymbol{w}\rangle, \qquad (\boldsymbol{v} : \boldsymbol{w}) = \int_\Omega \langle \boldsymbol{v} : \boldsymbol{w}\rangle.$$

We denote by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T)$ the symmetric part of the gradient of $\boldsymbol{u}$. Consider now the linear elasticity problem for finding a displacement field $\boldsymbol{u} \in [H^1_D(\Omega)]^d$, which for a given force $\boldsymbol{f} \in [L^2(\Omega)]^d$ is the unique minimizer of the energy functional $\mathcal{J}(\boldsymbol{u})$, where

$$\mathcal{J}(\boldsymbol{u}) := \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u})) - (\boldsymbol{f}, \boldsymbol{u}). \tag{2.2}$$

The Euler-Lagrange equations corresponding to the Minimization Problem (2.2) comprise the following well known system of linear PDEs for the unknown displacement $\boldsymbol{u}$:

$$\begin{aligned} -\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) &= f, && \text{on } \Omega,\\ \langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{n}\rangle &= 0, && \text{on } \Gamma_N,\\ \boldsymbol{u} &= \boldsymbol{g}, && \text{on } \Gamma_D. \end{aligned}$$

Here, following a standard convention, $\boldsymbol{u}$ takes prescribed values on a closed part of the boundary Γ_D (Dirichlet boundary) and satisfies natural (traction free) boundary conditions on the rest of the boundary Γ_N and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$.

3. Discontinuous Galerkin methods for linear elasticity equations

3.1. **Domain partitioning.** Let $\mathcal{T}_h$ be a shape-regular family of partitions of Ω into d-dimensional simplexes T (triangles if $d = 2$ and tetrahedrons if $d = 3$). We denote by h_T the diameter of T and we set $h = \max_{T\in\mathcal{T}_h} h_T$. We also assume that $\mathcal{T}_h$ is conforming in the sense that it does not contain hanging nodes. A face (shared by two neighboring elements or being part of the boundary) is denoted by E. Clearly, such a face is a $(d-1)$ dimensional simplex, that is, a line segment in two dimensions and a triangle in three dimensions. We denote the set of all faces by $\mathcal{E}_h$, and the collection of all interior faces and boundary faces by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$, respectively. Further, the set of Dirichlet faces is denoted by $\mathcal{E}_h^D$, and the set of Neumann faces by $\mathcal{E}_h^N$. We thus have,

$$\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial, \quad \mathcal{E}_h^D = \mathcal{E}_h^\partial \cap \Gamma_D, \quad \mathcal{E}_h^N = \mathcal{E}_h^\partial \cap \Gamma_N, \quad \mathcal{E}_h^\partial = \mathcal{E}_h^D \cup \mathcal{E}_h^N.$$

Some mesh dependent notation is related to different bilinear forms and inner products which we use in the definitions and the analysis that follows. For two vector (tensor) fields $\boldsymbol{v}$ and $\boldsymbol{w}$, which are sufficiently smooth so that the integrals below exist, we denote

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \int_T \langle \boldsymbol{v}, \boldsymbol{w}\rangle,$$

We also introduce an inner product on all (or part) of the faces of a given triangulation, which we define by the following relation

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle_{\mathcal{E}} = \sum_{E\in\mathcal{E}} \int_E \langle \boldsymbol{v}, \boldsymbol{w}\rangle,$$

where $\mathcal{E} \subset \mathcal{E}_h$.

Finally, let us introduce two function spaces that we will need in the derivation of the discretizations. The space of piecewise smooth functions is defined by

$$[H^2(\mathcal{T}_h)]^d = \left\{ \boldsymbol{u} \in [L^2(\Omega)]^d \text{ such that } \boldsymbol{u}\big|_T \in [H^2(T)]^d, \quad \forall\, T \in \mathcal{T}_h \right\}.$$

In addition the piecewise linear DG space is defined by

$$V^{\mathrm{DG}} := \{ u \in L^2(\Omega) \text{ such that } u\big|_T \in \mathbb{P}^1(T), \quad \forall\, T \in \mathcal{T}_h \},$$

where $\mathbb{P}^1(T)$ is the space of linear polynomials in T. The corresponding space of vector valued functions is then

$$\boldsymbol{V}^{\mathrm{DG}} := [V^{\mathrm{DG}}]^d.$$

3.2. **Trace operators (average and jump) on $E \in \mathcal{E}_h$.** To define the average and jump trace operators for an interior face $E \in \mathcal{E}_h^o$, and any $T \in \mathcal{T}_h$, such that $E \in \partial T$ we set $\boldsymbol{n}_{E,T}$ to be the unit outward (with respect to T) normal vector to E. With every face $E \in \mathcal{E}_h^o$ we also associate a unit vector n_E which is orthogonal to the $(d-1)$ dimensional affine variety (line in 2D and plane in 3D) containing the face. For the boundary faces, we always set $\boldsymbol{n}_E = \boldsymbol{n}_{E,T}$, where T is the *unique* element for which we have $E \subset \partial T$. In our setting, for the interior faces, the particular direction of $\boldsymbol{n}_E$ is not important, although it is important that this direction is fixed. For every face $E \in \mathcal{E}_h$, we define $T^+(E)$ and $T^-(E)$ as follows:

$$\begin{aligned} T^+(E) &:= \{ T \in \mathcal{T}_h \text{ such that } E \subset \partial T, \text{ and } \langle \boldsymbol{n}_E, \boldsymbol{n}_{E,T} \rangle > 0 \}, \\ T^-(E) &:= \{ T \in \mathcal{T}_h \text{ such that } E \subset \partial T, \text{ and } \langle \boldsymbol{n}_E, \boldsymbol{n}_{E,T} \rangle < 0 \}. \end{aligned} \tag{3.1}$$

It is immediate to see that both sets defined above contain *no more than* one element, that is: for every face we have exactly one $T^+(E)$ and for the interior faces we also have exactly one $T^-(E)$. For the boundary faces we only have $T^+(E)$. In the following, we will often write $T^\pm$ instead of $T^\pm(E)$, when this does not cause confusion and ambiguity.

For a given function $\boldsymbol{w} \in [L_2(\Omega)]^d$ and a fixed face E the average and jump trace operators for $E \in \mathcal{E}_h^o$ are as follows:

$$\{\!\{ \boldsymbol{w} \}\!\} := \left(\frac{\boldsymbol{w}^+ + \boldsymbol{w}^-}{2} \right), \qquad [\![\boldsymbol{w}]\!] := (\boldsymbol{w}^+ - \boldsymbol{w}^-), \tag{3.2}$$

where $\boldsymbol{w}^+$ and $\boldsymbol{w}^-$ denote respectively, the traces of $\boldsymbol{w}$ onto E taken from within the interior of T^+ and T^-. On boundary faces $E \in \mathcal{E}_h^\partial$, we set $\{\!\{ \boldsymbol{w} \}\!\} = \boldsymbol{w}$ and $[\![\boldsymbol{w}]\!] = \boldsymbol{w}$. This notation is a bit different from the classical one (see [2]). We employ it here, because it seems to lead to a shorter description of the subspace splitting and the preconditioner that we consider later.

We end this subsection by introducing the classical Crouzeix-Raviart finite element space

$$V^{\mathrm{CR}} = \left\{ v \in L^2(\Omega) \,:\, v_{|_T} \in \mathbb{P}^1(T), \ \forall T \in \mathcal{T}_h \text{ and } \mathcal{P}_E^0 [\![v]\!] = 0, \ \forall\, E \in \mathcal{E}_h^o \right\}. \tag{3.3}$$

The corresponding space of vector valued functions is

$$\boldsymbol{V}^{\mathrm{CR}} := [V^{\mathrm{CR}}]^d \tag{3.4}$$

Here for a given face E, $\mathcal{P}_E^0 : L^2(E) \mapsto \mathbb{P}^0(E)$ denotes the L^2-projection onto the constant function on E defined (for both scalar and vector valued functions) by

$$\mathcal{P}_E^0 \boldsymbol{w} = \frac{1}{|E|} \int_E \boldsymbol{w}, \quad \text{for all} \quad \boldsymbol{w} \in [L^2(E)]^d. \tag{3.5}$$

Finally, we introduce the energy norm on $\boldsymbol{V}^{\rm DG}$ which is the natural energy norm for the IP discretizations of linear elasticity. For $\boldsymbol{v} \in [H^2(\mathcal{T}_h)]^d$ we define

$$|||\boldsymbol{v}|||^2 = \sum_{T\in\mathcal{T}_h} \langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_T + \sum_{E\in\mathcal{E}_h} \langle h_E^{-1}[\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!]\rangle_E. \tag{3.6}$$

3.3. **Weighted residual approach for discretization.** We now derive the DG discretizations which we are interested in. The derivation uses the weighted residual approach described in [7] (see also [3]). If we now assume that we are to approximate a sufficiently smooth solution to (2.3) (e.g., $\boldsymbol{u} \in [H^2(\Omega)]^d$), then we may rewrite the continuous problem (2.3) as follows: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that

$$\begin{aligned} -\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) &= f, && \text{on } T \in \mathcal{T}_h,\\ [\![\langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{n}\rangle]\!]_E &= 0, && \text{on } E \in \mathcal{E}_h^o \cup \mathcal{E}_h^N,\\ [\![\boldsymbol{u}]\!]_E &= 0, && \text{on } E \in \mathcal{E}_h^o,\\ [\![\boldsymbol{u}-\boldsymbol{g}]\!]_E &= 0, && \text{on } E \in \mathcal{E}_h^D, \end{aligned} \tag{3.7}$$

where we recall that $\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\mathrm{trace}(\boldsymbol{\varepsilon}(\boldsymbol{u}))I$. Following [7], we next introduce a variational formulation of (3.7) by considering four (vectorial) operators

$$\begin{aligned} &\mathcal{B}_0 : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{T}_h)]^d, && \mathcal{B}_1 : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^o)]^d\\ &\mathcal{B}_2 : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^o \cup \mathcal{E}_h^N)]^d, && \mathcal{B}_1^\partial : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^D)]^d, \end{aligned}$$

and weighting each equation in (3.7) appropriately. This then amounts to considering the following problem: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that for all $v \in [H^2(\mathcal{T}_h)]^d$

$$\begin{aligned} (-\mathrm{div}\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - f, \mathcal{B}_0(\boldsymbol{v}))_{\mathcal{T}_h} \quad + \quad &\langle [\![\langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{n}\rangle]\!], \mathcal{B}_2(\boldsymbol{v})\rangle_{\mathcal{E}_h^o\cup\mathcal{E}_h^N}\\ &+\langle [\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v})\rangle_{\mathcal{E}_h^o}\\ &+\langle [\![\boldsymbol{u}-\boldsymbol{g}]\!], \mathcal{B}_1^\partial(\boldsymbol{v})\rangle_{\mathcal{E}_h^D} = 0. \end{aligned} \tag{3.8}$$

Obviously, different choices of the operators $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_1^\partial$ above will give rise to different variational formulations and, consequently to different DG methods. We refer to [7, Theorem 6] for sufficient conditions on the operators $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_1^\partial$ which guarantee[1] the uniqueness of the solution of (3.8).

To derive the IP methods of interest, we now set $\mathcal{B}_0(\boldsymbol{v}) = \boldsymbol{v}$ and $\mathcal{B}_2(\boldsymbol{v}) = \{\!\{\boldsymbol{v}\}\!\}$ in (3.8), to obtain that

$$\begin{aligned} &(-\mathrm{div}\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{v})_{\mathcal{T}_h} + \langle [\![\langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{n}\rangle]\!], \mathcal{B}_2(\boldsymbol{v})\rangle_{\mathcal{E}_h^o\cup\mathcal{E}_h^N} + \langle [\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v})\rangle_{\mathcal{E}_h^o\cup\mathcal{E}_h^D}\\ &\quad = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} + \langle [\![\boldsymbol{g}]\!], \mathcal{B}_1^\partial(\boldsymbol{v})\rangle_{\mathcal{E}_h^D}\ . \end{aligned} \tag{3.9}$$

Setting

$$\mathcal{F}(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} + \langle [\![\boldsymbol{g}]\!], \mathcal{B}_1^\partial(\boldsymbol{v})\rangle_{\mathcal{E}_h^D}, \tag{3.10}$$

and integrating by parts the first term on the left side of (3.9) then leads to

$$(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} - \langle \{\!\{\langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{n}\rangle\}\!\}, [\![\boldsymbol{v}]\!]\rangle_{\mathcal{E}_h^o\cup\mathcal{E}_h^N} + \langle [\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v})\rangle_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} = \mathcal{F}(\boldsymbol{v}). \tag{3.11}$$

Finally, we define the operator $\mathcal{B}_1(\boldsymbol{v})$ as

$$\mathcal{B}_1(\boldsymbol{v}) := \theta\{\!\{\langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{n}\rangle\}\!\} + \alpha_0\beta_0\mathcal{P}_E^0[\![\boldsymbol{v}]\!] + \alpha_1\beta_1[\![\boldsymbol{v}]\!]. \tag{3.12}$$

[1] We note that in [7] the Laplace model problem is analyzed, but the arguments for the present elasticity problem, are basically the same.

Recall that $\mathcal{P}_E^0$ is the projection on the constant functions on E (as seen from (3.5). There are five parameters involved in the definition of $\mathcal{B}_1$, namely, θ, β_0, β_1, α_0 and α_1. The parameter θ is just for convenience: the symmetric SIPG method corresponds to $\theta = -1$, and non-symmetric discretizations correspond to $\theta = 0$ (IIPG) and $\theta = 1$ (NIPG). The other four parameters are chosen following [12] so that the resulting DG discretization is consistent and stable. As proposed in [12] the parameters β_0 and β_1 in (3.12), depend on the Lamé constants λ and μ and are

$$\beta_0 := 2(\lambda + \mu), \qquad \beta_1 := 2\mu \; . \tag{3.13}$$

The remaining two parameters, α_0 and α_1 are at our disposal and they will serve to obtain different schemes.

3.4. **Weak formulations and bilinear forms.** The weak form of the equation (3.7) is obtained by substituting the expression for $\mathcal{B}_1(\cdot)$ from (3.12) in (3.11). We set

$$\begin{aligned} a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) &:= \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h} \int_E \langle h_E^{-1} [\![\boldsymbol{u}]\!], \mathcal{P}_E^0 [\![\boldsymbol{v}]\!] \rangle, \\ a_{j,1}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) &:= \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h} \int_E \langle h_E^{-1} [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!] \rangle. \end{aligned} \tag{3.14}$$

We now consider

$$a_j([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) = a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) + a_{j,1}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]).$$

Then we introduce the following weak formulation of Problem (3.7): Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\, \boldsymbol{w} \in [H^2(\mathcal{T}_h)]^d. \tag{3.15}$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ is given by

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{w}) = \mathcal{A}_0(\boldsymbol{u}, \boldsymbol{w}) + a_{j,1}([\![\boldsymbol{u}]\!], [\![\boldsymbol{w}]\!]), \tag{3.16}$$

where

$$\begin{aligned} \mathcal{A}_0(\boldsymbol{u}, \boldsymbol{w}) \;=\; & (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{T}_h} - \langle \{\!\{\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n}\}\!\} [\![\boldsymbol{w}]\!] \rangle \\ & + \theta \langle [\![\boldsymbol{u}]\!] \{\!\{\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w})\boldsymbol{n}\}\!\} \rangle + a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{w}]\!]). \end{aligned} \tag{3.17}$$

It is straightforward to see that

$$\begin{aligned} \mathcal{A}(\boldsymbol{u}, \boldsymbol{w}) \;=\; & (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{T}_h} - \langle \{\!\{\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n}\}\!\} [\![\boldsymbol{w}]\!] \rangle \\ & + \theta \langle [\![\boldsymbol{u}]\!] \{\!\{\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w})\boldsymbol{n}\}\!\} \rangle + a_j([\![\boldsymbol{u}]\!], [\![\boldsymbol{w}]\!]). \end{aligned} \tag{3.18}$$

Finally, to obtain the discrete formulation, we replace $[H^2(\mathcal{T}_h)]^d$ in (3.15) by $\boldsymbol{V}^{\mathrm{DG}}$, and we get the family of discrete problems: Find $\boldsymbol{u}_h \in \boldsymbol{V}^{\mathrm{DG}}$ such that:

$$\mathcal{A}(\boldsymbol{u}_h, \boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\, \boldsymbol{w} \in \boldsymbol{V}^{\mathrm{DG}}. \tag{3.19}$$

As we mentioned earlier, for $\theta = -1$, we recover the SIPG discretization for the elasticity system introduced in [12]. The values $\theta = 0$ and $\theta = 1$ correspond to the IIPG and NIPG discretizations, respectively.

3.5. **Stability and approximation.** We summarize below some of the results on a priori error estimation, stability and consistency proved in [12]. The a-priori error analysis done in [12] shows that by taking α_0 sufficiently large the method is stable and does not lock as $\lambda \to \infty$.

From the derivation that we presented, it is clear that if the exact solution to (2.3) is sufficiently regular, then the resulting method is consistent. Moreover, the following stability estimate has been shown in [12, Proposition 2.2]:

$$c\,|||\boldsymbol{v}|||^2 + (2\mu+\lambda)(\alpha_0 - c_0)\|h^{-1/2}P_0[\![\boldsymbol{v}]\!]\|^2_{\mathcal{E}_h} \le a(\boldsymbol{v},\boldsymbol{v}),$$

for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}$. The constant c in the estimate is independent of h, α_0, α_1, μ, and λ, provided that $\alpha_0 > c_0$, with c_0 sufficiently large and $\alpha_1 \ge c_1 > 0$.

In addition, the following a priori error estimate holds [12, Theorem 2.5]:

$$\begin{aligned}|||\boldsymbol{u}-\boldsymbol{u}_h||| + ((2\mu+\lambda)(\alpha_0-c_0))^{1/2}\|h^{-1/2}P_0[\![\boldsymbol{u}_h]\!]\|_{\mathcal{E}_h} \\ \le Ch\left((2\mu)^{1/2}\|\boldsymbol{u}\|_{H^2(\Omega)} + (\lambda)^{1/2}\|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^1(\Omega)}\right),\end{aligned}$$

where C is a constant independent of h, α_0, α_1, μ, and λ.

Combining the error estimate given above with the elliptic regularity estimate

$$\|\boldsymbol{u}\|_{H^2(\Omega)} + \lambda\|\boldsymbol{\nabla}\cdot\boldsymbol{u}\|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L_2(\Omega)},$$

which holds, for example, in the case of a convex polygonal domain Ω (see [6]) shows that the method does not lock as $\lambda \to \infty$. One then obtains the following a priori estimate in terms of the data $\boldsymbol{f}$ (see [12, Corollary 2.6]):

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| + ((2\mu+\lambda)(\alpha_0-c_0))^{1/2}\|h^{-1/2}P_0[\![\boldsymbol{u}_h]\!]\|_E \le Ch\|\boldsymbol{f}\|_{L^2(\Omega)}.$$

Remark 3.1. *The method (3.19), (respectively its restriction to $\boldsymbol{V}^{CR}$), remains stable if we replace* $a_{j,1}([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!])$ *in* (3.14) *with*

$$a_{j,1}([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]) = \alpha_1\beta_1\sum_{E\in\mathcal{E}_h}\int_E h_E^{-1}[\![\langle\boldsymbol{u},\boldsymbol{n}_E\rangle]\!][\![\langle\boldsymbol{v},\boldsymbol{n}_E\rangle]\!].$$

4. Preconditioning

4.1. **Space decomposition.** We next present a decomposition of the piecewise linear vectorial DG space that will play a key role in the construction of iterative solvers. This decomposition was introduced in [4] for scalar functions; its extension to vector-valued functions is more or less straightforward. We will therefore omit proofs which are just an easy modification of the corresponding proofs in the scalar case. However, we will review the main ingredients and ideas behind such proofs, since these ideas play an important role in the analysis of the preconditioner given later on.

Following [4] we introduce the space complementary to V^{CR} in V^{DG},

$$\mathcal{Z} = \left\{z \in L^2(\Omega)\ :\ z_{|_T}\ \in\ \mathbb{P}^1(T)\ \forall T\in\mathcal{T}_h\ \text{ and } \mathcal{P}^0_E\{\!\{z\}\!\} = 0\ \forall\, E\in\mathcal{E}^o_h\right\}. \tag{4.1}$$

The corresponding space of vector valued functions is

$$\boldsymbol{\mathcal{Z}} = [\mathcal{Z}]^d. \tag{4.2}$$

To describe the basis functions associated with the spaces (3.4), (4.2), let $\varphi_{E,T}$ denote the canonical scalar Crouzeix-Raviart (CR) basis function on T, dual to the degree of freedom at the mass center of the face E, and extended as zero outside T. For $E\in\partial T$, $E'\in\partial T$, the function $\varphi_{E,T}$ satisfies

$$\varphi_{E,T}(m_{E'}) = \begin{cases} 1 & \text{if}\quad E = E', \\ 0 & \text{otherwise,}\end{cases}$$

and also we have

$$\varphi_{E,T} \in \mathbb{P}^1(T), \quad \varphi_{E,T}(x) = 0, \forall\, x \notin T.$$

For all $\boldsymbol{u} \in \boldsymbol{V}^{\mathrm{DG}}$ we then have

$$\boldsymbol{u}(x) = \sum_{T\in\mathcal{T}_h}\sum_{E\in\partial T} \boldsymbol{u}_T(m_E)\varphi_{E,T}(x) = \sum_{E\in\mathcal{E}_h} \boldsymbol{u}^+(m_E)\varphi_E^+(x) + \sum_{E\in\mathcal{E}_h^o} \boldsymbol{u}^-(m_E)\varphi_E^-(x), \tag{4.3}$$

where in the last identity we have just changed the order of summation and used the short hand notation $\varphi_E^{\pm}(x) := \varphi_{E,T^{\pm}}(x)$ together with

$$\boldsymbol{u}^{\pm}(m_E) := \boldsymbol{u}_{T^{\pm}}(m_E) = \frac{1}{|E|}\int_E \boldsymbol{u}^{\pm} ds, \qquad \forall\, E \in \mathcal{E}_h^o, \;:\; E = \partial T^+ \cap \partial T^-,$$
$$\boldsymbol{u}(m_E) := \boldsymbol{u}_T(m_E) = \frac{1}{|E|}\int_E \boldsymbol{u}_T ds, \qquad \forall\, E \in \mathcal{E}_h^\partial, \text{ such that } E = \partial T \cap \partial\Omega.$$

We recall the definitions of $T^+(E)$ and $T^-(E)$ given in §3.2 (see equation (3.1)) and set

$$\begin{aligned} \varphi_E^{CR} &= \varphi_{E,T^+(E)} + \varphi_{E,T^-(E)}, && \forall\, E \in \mathcal{E}_h^o, \\ \varphi_E^{CR} &= \varphi_{E,T^+(E)}, && \forall\, E \in \mathcal{E}_h^N. \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \psi_E^z &= \frac{\varphi_{E,T^+(E)} - \varphi_{E,T^-(E)}}{2}, && \forall\, E \in \mathcal{E}_h^o, \\ \psi_E^z &= \varphi_{E,T^+(E)}, && \forall\, E \in \mathcal{E}_h^D. \end{aligned} \tag{4.5}$$

Some clarification is needed here. Note that from the definition of $\varphi_{E,T^+(E)}$ and $\varphi_{E,T^-(E)}$ for an interior edge $E \in \mathcal{E}_h^o$, it does not follow that their sum is even defined on the edge E, since it is just a sum of two functions from $L^2(\Omega)$. However, $(\varphi_{E,T^+(E)} + \varphi_{E,T^-(E)})$ has a representative which is continuous across E and this representative is denoted here with $\boldsymbol{\varphi}_E^{CR}$. It is also easy to check that $[\![\psi_E^z]\!] = 1$ on E, a property which we will use later. Clearly, $\{\varphi_E^{CR}\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}$ are linearly independent, and $\{\boldsymbol{\psi}_{E,T}^z\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}$ are linearly independent. A simple argument then shows that

$$\boldsymbol{V}^{\mathrm{CR}} = \operatorname{span}\left\{\{\varphi_E^{CR}\boldsymbol{e}_k\}_{k=1}^d\right\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}, \quad \boldsymbol{\mathcal{Z}} = \operatorname{span}\left\{\{\psi_E^z\boldsymbol{e}_k\}_{k=1}^d\right\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}.$$

Here $\boldsymbol{e}_k$, $k = 1, \dots, d$ is the k-th canonical basis vector in $\mathbb{R}^d$. Hence by performing a change of basis in (4.3), we have obtained a “natural” splitting of $\boldsymbol{V}^{\mathrm{DG}} = \boldsymbol{V}^{\mathrm{CR}} \oplus \boldsymbol{\mathcal{Z}}$. This is summarized in the next proposition.

Proposition 4.1. *For any $\boldsymbol{u} \in \mathbf{V}^{DG}$ there exist unique $\boldsymbol{v} \in \mathbf{V}^{CR}$ and a unique $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ such that*

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{z} \quad \text{and} \quad \begin{aligned} \boldsymbol{v} &= \textstyle\sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}\left(\frac{1}{|E|}\int_E \{\!\{\boldsymbol{u}\}\!\} ds\right)\varphi_E^{CR}(x) \in \mathbf{V}^{CR}, \\ \boldsymbol{z} &= \textstyle\sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}\left(\frac{1}{|E|}\int_E [\![\boldsymbol{u}]\!] ds\right)\boldsymbol{\psi}_{E,T^+}^z(x) \in \boldsymbol{\mathcal{Z}}. \end{aligned} \tag{4.6}$$

We refer to [4, Proposition 3.1] for the proof in the scalar case and further discussion. In case of vector-valued functions the proposition above is proved in the same way as in the scalar case, proceeding component-by-component. The next Lemma is straightforward to prove, by applying the proof for its scalar version componentwise. It however shows that the splitting we have proposed is orthogonal with respect to the inner product defined by $\mathcal{A}_0(\cdot,\cdot)$.

Lemma 4.2. *Let $\boldsymbol{u} \in \boldsymbol{V}^{DG}$ be such that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{z}$ with $\boldsymbol{v} \in \boldsymbol{V}^{CR}$ and $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$. Let $\mathcal{A}_0(\cdot,\cdot)$ be the bilinear form defined in* (3.17)*. Then,*

$$\mathcal{A}_0(\boldsymbol{v}, \boldsymbol{z}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}^{CR}, \quad \forall\, \boldsymbol{z} \in \boldsymbol{\mathcal{Z}}. \tag{4.7}$$

Furthermore if $\mathcal{A}_0(\cdot,\cdot)$ is symmetric ($\theta=-1$), then $\mathcal{A}_0(\boldsymbol{v},\boldsymbol{z})=\mathcal{A}_0(\boldsymbol{z},\boldsymbol{v})=0$ $\forall \boldsymbol{v}\in \boldsymbol{V}^{CR}$, $\forall \boldsymbol{z}\in\boldsymbol{\mathcal{Z}}$ and the decomposition (4.6) *is $\mathcal{A}_0$-orthogonal; $\boldsymbol{V}^{CR}\perp_{\mathcal{A}_0}\boldsymbol{\mathcal{Z}}$.*

Remark 4.3. *From the definition of the spaces $\boldsymbol{V}^{CR}$ and $\boldsymbol{\mathcal{Z}}$ it is easy to see that*

$$\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{z}\|_{0,T}^2=\langle[\![\boldsymbol{z}]\!],\{\!\{\nabla\boldsymbol{z}\}\!\}\rangle_{\mathcal{E}_h}.$$

Applying the Schwarz inequality, one then gets the following estimate

$$\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{z}\|_{0,T}^2\leq C\|h^{-1/2}P_0[\![\boldsymbol{z}]\!]\|_{0,\mathcal{E}_h}^2,$$

which is a straightforward way to see that the restriction of the SIPG scheme ($\theta=-1$) on the space $\boldsymbol{\mathcal{Z}}$ is positive definite. This inequality is used later to prove that the restriction of $\mathcal{A}(\cdot,\cdot)$ to $\boldsymbol{\mathcal{Z}}$ is well conditioned.

4.2. **Strengthened Cauchy-Schwarz inequality.** In this subsection we provide an estimate on the norm of the off-diagonal element in the 2×2 block splitting of the stiffness matrix, corresponding to the space splitting $\boldsymbol{V}^{\rm DG}=\boldsymbol{V}^{\rm CR}\oplus\boldsymbol{\mathcal{Z}}$. To prove such an estimate we need two auxiliary results (Lemma 4.4 and Lemma 4.5) which we prove in the appendix. We prove these results in two spatial dimensions and note that similar results hold in the three dimensional case, although the proofs are more elaborate.

Consider an interior edge $E\in\mathcal{E}_h^o$ and denote by x_m and x_n its end-points and by $x_{mn}=(x_n+x_m)/2$ its midpoint. Observe then that for any $\boldsymbol{v}\in\boldsymbol{V}^{\rm CR}$ it follows by definition of the space $\boldsymbol{V}^{\rm CR}$ that

$$[\![\boldsymbol{v}]\!](x_{mn})=0,\quad\text{and hence}\quad [\![\boldsymbol{v}]\!](x_m)=-[\![\boldsymbol{v}]\!](x_n). \tag{4.8}$$

Moreover, from the definition of the space $\boldsymbol{\mathcal{Z}}$ we have

$$\{\!\{\boldsymbol{z}\}\!\}(x_{mn})=0\quad\Longrightarrow\quad \boldsymbol{z}^+(x_{mn})=-\boldsymbol{z}^-(x_{mn})\quad\Longrightarrow\quad [\![\boldsymbol{z}]\!](x_{mn})=2\boldsymbol{z}^+(x_{mn}), \tag{4.9}$$

and since $\boldsymbol{z}\in\boldsymbol{\mathcal{Z}}$ is linear $[\![\boldsymbol{z}]\!](x_{mn})=\frac{1}{2}([\![\boldsymbol{z}]\!](x_m)+[\![\boldsymbol{z}]\!](x_n))$.

Lemma 4.4. *If $z\in\mathcal{Z}$ then the following inequality holds with a constant $C\leq 16$.*

$$\sum_{E\in\mathcal{E}_h}([\![z]\!]_{E,m}-[\![z]\!]_{E,n})^2\leq C\sum_{E\in\mathcal{E}_h}([\![z]\!]_{E,m}+[\![z]\!]_{E,n})^2 \tag{4.10}$$

Lemma 4.5. *If $z\in\mathcal{Z}$ then the following inequality holds with a constant $\gamma_1\leq\sqrt{\frac{C}{3+C}}$. Here C is the constant from Lemma 4.4.*

$$\sum_{E_{mn}\in\mathcal{E}_h}([\![z]\!]_{E,m}-[\![z]\!]_{E,n})^2\leq\gamma_1^2\sum_{E_{mn}\in\mathcal{E}_h}2\left(([\![z]\!]_{E,m}+[\![z]\!]_{E,n})^2+[\![z]\!]_{E,m}^2+[\![z]\!]_{E,n}^2\right) \tag{4.11}$$

From the equation (3.18), we have that for any $\boldsymbol{u},\boldsymbol{w}\in\boldsymbol{V}^{\rm DG}$, we can write $\boldsymbol{u}=\boldsymbol{z}+\boldsymbol{v}$, and $\boldsymbol{w}=\boldsymbol{\psi}+\boldsymbol{\varphi}$, where $\boldsymbol{z},\boldsymbol{\psi}\in\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{v},\boldsymbol{\varphi}\in\boldsymbol{V}^{\rm CR}$. Thus the bilinear form becomes $\mathcal{A}(\boldsymbol{u},\boldsymbol{w}):=\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{\psi},\boldsymbol{\phi}))$.

The preconditioner is readily suggested by the form of $\mathcal{A}_0(\cdot,\cdot)$. Note that for traction free boundary conditions, $\mathcal{A}_0(\cdot,\cdot)$ is not equivalent to $\mathcal{A}(\cdot,\cdot)$ (see [10]), and in fact, even for bounded values of the Lamé constant λ the restriction of $A_0(\cdot,\cdot)$ on $\boldsymbol{V}^{\rm CR}$ is singular and does not satisfy the discrete analogue of the Korn's inequality. However, it is easy to see that for Dirichlet conditions ($\Gamma_D=\Gamma$), the two bilinear forms $\mathcal{A}_0(\cdot,\cdot)$ and $\mathcal{A}(\cdot,\cdot)$ are equivalent. A simple calculation shows that

$$\mathcal{A}_0((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{\psi},\boldsymbol{\phi}))=\mathcal{A}_0(\boldsymbol{z},\boldsymbol{\psi})+\mathcal{A}_0(\boldsymbol{v},\boldsymbol{\phi}).$$

This in turn suggests that a reasonable choice for an approximation of $\mathcal{A}(\cdot,\cdot)$ is

$$\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{\psi},\boldsymbol{\phi}))=\mathcal{A}(\boldsymbol{z},\boldsymbol{\psi})+\mathcal{A}(\boldsymbol{v},\boldsymbol{\phi}). \tag{4.12}$$

The following algorithm describes the application of a preconditioner, which is based on the bilinear form in the equation (4.12).

Algorithm 4.6. *Let $\boldsymbol{r} \in [L_2(\Omega)]^d$ be given. Then the action of the preconditioner on $\boldsymbol{r}$ is the function $\boldsymbol{u} \in \boldsymbol{V}^{DG}$ which is obtained from the following three steps.*

1. *Find $z \in \boldsymbol{\mathcal{Z}}$ such that*
$$\mathcal{A}(\boldsymbol{z}, \boldsymbol{\psi}^z) = (\boldsymbol{r}, \boldsymbol{\psi}^z)_{\mathcal{T}_h} \quad \text{for all} \quad \boldsymbol{\psi}^z \in \boldsymbol{\mathcal{Z}}.$$
2. *Find $\boldsymbol{v} \in \boldsymbol{V}^{CR}$ such that*
$$\mathcal{A}(\boldsymbol{v}, \boldsymbol{\varphi}) = (\boldsymbol{r}, \boldsymbol{\varphi})_{\mathcal{T}_h} \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{V}^{CR}.$$
3. *Set $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{v}$.*

We will prove that this algorithm provides a uniform preconditioner for $\mathcal{A}(\cdot, \cdot)$ in Theorem 4.9. The following Lemma is crucial for this proof. We prove the Lemma in two spatial dimension only, however, a similar proof works also in the three dimensional case.

Lemma 4.7. *The following inequality holds for any $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and any $\boldsymbol{v} \in \boldsymbol{V}^{CR}$*
$$a(\boldsymbol{z}, \boldsymbol{v})^2 \le \gamma^2 a(\boldsymbol{z}, \boldsymbol{z}) a(\boldsymbol{v}, \boldsymbol{v})$$
where $\gamma < 1$ and γ depends only on α_0, α_1 and the constant from Lemma 4.4.

Proof. Choose α_0 large enough such that for all $\boldsymbol{u} \in \boldsymbol{V}^{\mathrm{DG}}$ we have

$$(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{T}_h} - \langle \{\!\{\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n}\}\!\}, [\![\boldsymbol{u}]\!] \rangle_{\mathcal{E}_h} - \langle [\![\boldsymbol{u}]\!], \{\!\{\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n}\}\!\} \rangle_{\mathcal{E}_h} + \alpha_0 a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{u}]\!]) \ge 0. \tag{4.13}$$

Then it is sufficient to prove that there exists $\gamma = \gamma(\alpha_1) < 1$ such that for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{CR}}$ the inequality holds.
$$[a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!])]^2 \le \gamma^2 a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{z}]\!]) a_{j,1}([\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!])$$
By Simpson formula, (4.8), Schwarz inequality, and Lemma 4.5 we get

$$\begin{aligned}
[a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!])]^2 &= \frac{\alpha_1^2\beta_1^2}{36} \sum_{i=1}^{d} \left(\sum_E [\![z_m^i]\!][\![v_m^i]\!] + [\![z_n^i]\!][\![v_n^i]\!] \right)^2 \\
&= \frac{\alpha_1^2\beta_1^2}{36} \sum_{i=1}^{d} \left(\sum_E \left([\![z_m^i]\!] - [\![z_n^i]\!]\right) [\![v_m^i]\!] \right)^2 \\
&\le \frac{\alpha_1^2\beta_1^2}{36} \sum_{i=1}^{d} \left(\sum_E \left([\![z_m^i]\!] - [\![z_n^i]\!]\right)^2 \sum_E \left([\![v_m^i]\!]\right)^2 \right) \\
&\le \frac{\gamma^2\alpha_1^2\beta_1^2}{36} \sum_{i=1}^{d} \left(2 \sum_E [\![v_m^i]\!]^2 \left(\sum_E \left([\![z_m^i]\!] + [\![z_n^i]\!]\right)^2 + [\![z_m^i]\!]^2 + [\![z_n^i]\!]^2 \right) \right) \\
&= \frac{\gamma^2\alpha_1^2\beta_1^2}{36} \sum_{i=1}^{d} \left(\sum_E \left([\![v_m^i]\!]^2 + [\![v_n^i]\!]^2\right) \left(\sum_E \left([\![z_m^i]\!] + [\![z_n^i]\!]\right)^2 + [\![z_m^i]\!]^2 + [\![z_n^i]\!]^2 \right) \right) \\
&= \gamma^2 a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{z}]\!]) a_{j,1}([\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!]).
\end{aligned}$$

Here $\gamma \le \gamma_1 = \sqrt{\frac{C}{C+3}}$, see Lemma 4.5. □

Remark 4.8. *If α_0 is large enough to satisfy (4.13) then Lemma 4.7 holds with $\gamma = \gamma_1$ from Lemma 4.5 for all $\alpha_1 > 0$.*

We are now in a position to prove that the preconditioner given by Algorithm 4.6 is uniform with respect to the mesh size and the problem parameters.

Theorem 4.9. *Let $\mathcal{A}(\cdot,\cdot)$ be the symmetric bilinear form defined by (3.18) where $\theta = -1$ and $\mathcal{B}(\cdot,\cdot)$ be the bilinear form defined by (4.12). Then the following estimates hold for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{CR}$*

$$\frac{1}{1+\gamma}\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \frac{1}{1-\gamma}\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})). \tag{4.14}$$

The constant $\gamma < 1$ is the constant from Lemma 4.7.

Proof. Using Lemma 4.7 we have

$$-2\gamma\sqrt{\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\,\mathcal{A}(\boldsymbol{v},\boldsymbol{v})} \le 2\mathcal{A}(\boldsymbol{z},\boldsymbol{v}) \le 2\gamma\sqrt{\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\,\mathcal{A}(\boldsymbol{v},\boldsymbol{v})}$$

and since $-a^2 - b^2 \le 2ab \le a^2 + b^2$ for any real numbers a and b we obtain

$$(1-\gamma)\left(\mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v})\right) \le \mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v}) + 2\mathcal{A}(\boldsymbol{z},\boldsymbol{v}) \le (1+\gamma)\left(\mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v})\right)$$

which is the same as

$$(1-\gamma)\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le (1+\gamma)\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v}))$$

and thus (4.14) holds with the same constant $\gamma < 1$ as used in the estimate of Lemma 4.7. □

Remark 4.10. *Note that γ is uniformly bounded that means the bound $\gamma \le q < 1$ holds independently of the mesh size h and of the Lamé parameters λ and μ for some constant $q < 1$.*

This result shows that it is possible to construct a uniform block-diagonal preconditioner for the matrix related to the bilinear form (3.18). The application of this preconditioner corresponds to solving a system with (4.12) and thus is equivalent to solving one subproblem on $\boldsymbol{\mathcal{Z}}$ and one subproblem on the space $\boldsymbol{V}^{\mathrm{CR}}$. As we will show the subproblem on $\boldsymbol{\mathcal{Z}}$ is well conditioned (see Lemma 4.13) and its solution can be done efficiently. Hence, the only remaining issue is to construct a uniform preconditioner for the subproblem on the space $\boldsymbol{V}^{\mathrm{CR}}$. For the case of Dirichlet boundary conditions on the entire boundary–the so-called pure displacement problem–it is known how to construct optimal order multilevel preconditioners that are robust with respect to the parameter λ, see e.g. [11] and the references therein. For mixed boundary conditions or pure Neumann boundary conditions (the traction free case), however, it is much more difficult to devise a robust optimal order method. This question is subject of current research work.

4.3. **On the conditioning of the subproblem in the space complementary to $\boldsymbol{V}^{\mathbf{CR}}$.** The next Lemma establishes the conditions required to ensure coercivity for $\mathcal{A}(\cdot,\cdot)$ restricted to the space $\boldsymbol{\mathcal{Z}}$ and it will be used in the proof of Lemma 4.13. The constants in the estimates given in this section depend on the constant in the trace inequality (see [1]) and the constant in the inverse inequality (see [9, Theorem 17.2, pp.135]). These two constants we denoted below by C_t and C_{inv} respectively.

Lemma 4.11. *Let $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ be the bilinear forms defined in* (3.18) *and* (3.17)*. Then, for α_0 sufficiently large both $\mathcal{A}_0(\cdot,\cdot)$ and $\mathcal{A}(\cdot,\cdot)$ restricted to $\boldsymbol{\mathcal{Z}}$ are coercive. Furthermore, for the non-symmetric methods NIPG and IIPG, both $\mathcal{A}^{DG}(\cdot,\cdot)$ and $\mathcal{A}_0^{DG}(\cdot,\cdot)$, restricted to $\boldsymbol{\mathcal{Z}}$ are symmetric.*

Proof. Using the fact that $\boldsymbol{z} \in (\mathbb{P}^1(\mathcal{T}_h))^2$,

$$0 = (-\mathrm{div}\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{\psi})_{\mathcal{T}_h} = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}))_{\mathcal{T}_h} - \langle \{\!\{ \langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{n}\rangle \}\!\}, [\![\boldsymbol{\psi}]\!]\rangle_{\mathcal{E}_h} - \langle [\![\langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{n}\rangle]\!], \{\!\{\boldsymbol{\psi}\}\!\}\rangle_{\mathcal{E}_h^o},$$

and so from the definition (4.1) of the space $\boldsymbol{\mathcal{Z}}$, it follows that for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{\psi} \in \boldsymbol{\mathcal{Z}}$

$$(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}))_{\mathcal{T}_h} = \langle \{\!\{ \langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{n}\rangle \}\!\}, [\![\boldsymbol{\psi}]\!]\rangle_{\mathcal{E}_h}. \tag{4.15}$$

Substituting the above identity in the definition of the methods (3.18) gives

$$\mathcal{A}(\boldsymbol{z},\boldsymbol{\psi}) = \theta(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\psi}) : \boldsymbol{\varepsilon}(\boldsymbol{z}))_{\mathcal{T}_h} + \alpha_0\beta_0\langle [\![\boldsymbol{z}]\!], \mathcal{P}_E^0([\![\boldsymbol{\psi}]\!])\rangle_{\mathcal{E}_h^o\cup\Gamma_D} + \alpha_1\beta_1\langle [\![\boldsymbol{z}]\!], [\![\boldsymbol{\psi}]\!]\rangle_{\mathcal{E}_h^o\cup\Gamma_D}, \tag{4.16}$$

where $\theta=-1$ for the symmetric SIPG and $\theta=0$ and $\theta=1$ for the non-symmetric IIPG and NIPG, respectively. Taking $\boldsymbol{\psi}=\boldsymbol{z}$ in the above equation we get

$$\mathcal{A}^{DG}(\boldsymbol{z},\boldsymbol{z})=\alpha_0\beta_0\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}+\alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}+\theta\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|^2_{0,\mathcal{T}_h}, \tag{4.17}$$

and so for the non-symmetric NIPG and IIPG we have

$$\begin{aligned}\mathcal{A}^{IIPG}(\boldsymbol{z},\boldsymbol{z}) &\geq 2\alpha_0(\nu+\lambda)\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}+2\alpha_1\nu\|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D},\\ \mathcal{A}^{NIPG}(\boldsymbol{z},\boldsymbol{z}) &\geq 2\nu\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|^2_{0,\mathcal{T}_h}+2\alpha_0(\nu+\lambda)\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}\\ &\quad +2\alpha_1\nu\|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}.\end{aligned} \tag{4.18}$$

For the SIPG, we use the formulation (3.18) rather than (4.17). Taking $\boldsymbol{u}=\boldsymbol{z}$ and $\boldsymbol{w}=\boldsymbol{z}$ in (3.18) with $\theta=-1$ we have

$$\begin{aligned}\mathcal{A}^{SIPG}(\boldsymbol{z},\boldsymbol{z})&=(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{\psi}):\boldsymbol{\varepsilon}(\boldsymbol{z}))_{\mathcal{T}_h}+\alpha_0\beta_0\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}+\alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}\\ &\quad-2\,\langle\{\!\{\langle\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{n}\rangle\}\!\},[\![\boldsymbol{\psi}]\!]\rangle_{\mathcal{E}_h}.\end{aligned}$$

Using Cauchy-Schwarz, trace and inverse inequalities together with the arithmetic-geometric inequality and the bound on the maximum eigenvalue of $\boldsymbol{\mathcal{C}}$ it follows that

$$\begin{aligned}\langle\{\!\{\langle\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{n}\rangle\}\!\},[\![\boldsymbol{\psi}]\!]\rangle_{\mathcal{E}_h} &= \langle\{\!\{\langle\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z}),\boldsymbol{n}\rangle\}\!\},\mathcal{P}_E^0([\![\boldsymbol{\psi}]\!])\rangle_{\mathcal{E}_h}\\ &\leq \frac{C_t(1+C_{inv})}{\alpha_0\beta_0}\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{z})\|^2_{0,\mathcal{T}_h}+\frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}\\ &\leq \frac{C_t(1+C_{inv})}{\alpha_0}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|^2_{0,\mathcal{T}_h}+\frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}.\end{aligned}$$

Hence, we finally have

$$\begin{aligned}\mathcal{A}^{SIPG}(\boldsymbol{z},\boldsymbol{z}) &\geq (1-\frac{2C_t(1+C_{inv})}{\alpha_0})\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|^2_{0,\mathcal{T}_h}\\ &\quad+\alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D}\\ &\quad+\frac{\alpha_0}{2}\beta_0\|h_E^{-1/2}\mathcal{P}_E^0([\![\boldsymbol{z}]\!])\|^2_{0,\mathcal{E}_h^o\cup\Gamma_D},\end{aligned}$$

and therefore by taking $\alpha_0=\max(1,4C_t(1+C_{inv}))$ we ensure the stability also for SIPG. Finally, the symmetry of $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ restricted to $\boldsymbol{\mathcal{Z}}$ follows easily from the expression (4.16), independently of the value of θ. ☐

Remark 4.12. *We note that α_0 being sufficiently large in the statement of the Lemma 4.11 means that $\alpha_0\geq\alpha_0^*\geq 0$ where $\alpha_0^*=0$ for NIPG, and $\alpha_0^*>0$ for for IIPG, and α_0^* sufficiently large for SIPG.*

We now prove bounds on the eigenvalues of $\mathcal{A}_0(\cdot,\cdot)$ and $\mathcal{A}(\cdot,\cdot)$, when restricted to $\boldsymbol{\mathcal{Z}}$. Since these restrictions are symmetric (regardless of the value of θ by virtue of Lemma 4.11) the Lemma below gives such bounds.

Lemma 4.13. *Let $\boldsymbol{\mathcal{Z}}$ be the space defined in* (4.2). *Then for all $\boldsymbol{z}\in\boldsymbol{\mathcal{Z}}$, the following estimates hold*

$$h^{-2}\|\boldsymbol{z}\|_0^2\lesssim\mathcal{A}_0(\boldsymbol{z},\boldsymbol{z})\lesssim h^{-2}\|\boldsymbol{z}\|_0^2, \tag{4.19}$$

and also,

$$[(\alpha_0+c)\beta_0+\alpha_1\beta_1]h^{-2}\|\boldsymbol{z}\|_0^2\lesssim\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\lesssim[\alpha_0\beta_0+\alpha_1\beta_1]h^{-2}\|\boldsymbol{z}\|_0^2, \tag{4.20}$$

where β_0 and β_1 are as defined in (3.13).

Proof. Arguing as in [4, Lemma 5.3] (but now componentwise for vector-valued functions) one can prove that[2]

$$h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \sum_{E\in\mathcal{E}_h} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \lesssim h^{-2}\|\boldsymbol{z}\|_0^2 \,. \tag{4.21}$$

Therefore, taking into account the coercivity of $\mathcal{A}_0$ (cf. Lemma 4.11) we easily see that

$$\alpha_0\beta_0 h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \le \mathcal{A}_0(\boldsymbol{z},\boldsymbol{z}) \,,$$

and, further, taking into account the $L^2(\mathcal{E}_h)$ stability of the projection $\mathcal{P}_E^0$ that

$$\begin{aligned}(\alpha_0\beta_0+\alpha_1\beta_1)h^{-2}\|\boldsymbol{z}\|_0^2 &\lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\ &\lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + C\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h} h_E^{-1}\|[\![\boldsymbol{z}]\!]\|_{0,E}^2 \le \mathcal{A}(\boldsymbol{z},\boldsymbol{z}) \,.\end{aligned}$$

Hence, the lower bounds in (4.19) and (4.20) follow. We next show the upper bounds, using the continuity of $\mathcal{A}_0^{DG}(\cdot,\cdot)$ and $\mathcal{A}^{DG}(\cdot,\cdot)$. Taking into account the expression (4.17) and using Cauchy-Schwarz inequality together with (2.1) we get

$$\begin{aligned}\mathcal{A}_0^{DG}(\boldsymbol{z},\boldsymbol{z}) &\le \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\Gamma_D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + |\theta|\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2 \\ &\le \beta_0\left(\alpha_0\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{\mathcal{E}_h^o\cup\Gamma_D}^2 + C\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2\right) \,.\end{aligned}$$

Hence, the upper bound in (4.19) follows in a straightforward fashion from the trace inequality

$$h_E^{-1}\|[\![\boldsymbol{z}]\!]\|_{0,E}^2 \lesssim h_E^{-1}\|\boldsymbol{z}\|_{0,E}^2 \lesssim h_E^{-2}\|\boldsymbol{z}\|_{0,T}^2 + \|\nabla\boldsymbol{z}\|_{0,T}^2 \quad \forall\, \boldsymbol{z}\in H^1(T),$$

together with the standard inverse inequality

$$\|\nabla w\|_{0,T}^2 \lesssim h_T^{-2}\|w\|_{0,T}^2 \quad \forall\, w\in\mathbb{P}^k(T),\ k\ge 1 \quad \forall\, T\in\mathcal{T}_h \,,$$

and the inequality $\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h} \le \|\boldsymbol{\nabla}\boldsymbol{z}\|_{0,\mathcal{T}_h}$.

To show the upper bound in (4.20), from (4.17), the Cauchy-Schwarz inequality and (2.1), we note that

$$\begin{aligned}\mathcal{A}^{DG}(\boldsymbol{z},\boldsymbol{z}) &\le \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\Gamma_D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + |\theta|\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2 + \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\Gamma_D} h_E^{-1}\|[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\ &\le \beta_0\left(\alpha_0\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{\mathcal{E}_h^o\cup\Gamma_D}^2 + C\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2\right) + \alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|_{\mathcal{E}_h^o\cup\Gamma_D}^2 \,,\end{aligned}$$

and so arguing as before, using the trace inequality together with the inverse inequality, the upper bound in (4.20) also follows and the proof of the Lemma is complete. □

This lemma guarantees that the restrictions of both $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ to $\boldsymbol{\mathcal{Z}}$ are well-conditioned with respect to both, the mesh size and the Lamé constants λ,μ. Therefore, Lemma 4.13 guarantees that linear systems corresponding to these subproblems can be efficiently solved by the method of Conjugate Gradients (CG). A simple consequence of the well known estimate on the convergence of CG (see, e.g., [17, 13]) shows that the number of CG iterations required to achieve a fixed error tolerance is uniformly bounded, independently of the size of the problem and the parameters.

[2] It is here where the special structure of the space $\boldsymbol{\mathcal{Z}}$ is used.

Appendix A. Auxiliary results

A.1. **Proofs of Lemma 4.4 and Lemma 4.5.** We prove here the two auxiliary results, which we used in §4.2. We need some additional notation. Let E_{mn} denote any edge (not necessarily interior) connecting the two endpoints x_m and x_n. As before, let $x_{mn}:=\frac{x_m+x_n}{2}$ denote the midpoint of the face $E=E_{mn}$, and

$$\begin{aligned} [\![z]\!]_{E,m} &:= [\![z]\!]_E(x_m), & [\![v]\!]_{E,m} &:= [\![v]\!]_E(x_m), \\ [\![z]\!]_{E,n} &:= [\![z]\!]_E(x_n), & [\![v]\!]_{E,n} &:= [\![v]\!]_E(x_n), \\ [\![z]\!]_{E,mn} &:= [\![z]\!]_E(x_{mn}), & [\![v]\!]_{E,mn} &:= [\![v]\!]_E(x_{mn}). \end{aligned}$$

By z_{mn} we denote the degrees of freedom of a function $z\in\mathcal{Z}$, i.e., $z=\sum_{E=E_{mn}} z_{mn}\psi^z_{E_{mn}}$.

Then any function $v\in V^{\mathrm{CR}}$ has the properties

$$[\![v]\!]_{E,mn} = 0, \tag{A.1}$$

$$[\![v]\!]_{E,m} = -[\![v]\!]_{E,n}, \tag{A.2}$$

where $E=E_{mn}$ Moreover, any function $z\in\mathcal{Z}$ satisfies

$$[\![z]\!]_{E,mn} = \frac{1}{2}([\![z]\!]_{E,m}+[\![z]\!]_{E,n}), \tag{A.3}$$

$$z^+_{mn}:=z^+(x_{mn}) = -z^-(x_{mn})=:-z^-_{mn}, \tag{A.4}$$

$$[\![z]\!]_{E,mn} = [\![z_{mn}\psi^z_{E_{mn}}]\!]_{E,mn} = z_{mn}[\![\psi^z_{E_{mn}}]\!]_{E,mn} = z_{mn}s_{mn}, \tag{A.5}$$

where s_{mn} is either plus or minus one, i.e., $s_{mn}=\pm 1$.

Next, for any edge $E=E_{mn}$ we define two sets, that is, first the set of interior neighbor edges

$$\mathcal{E}^o_{mn}:=\{E'\in\mathcal{E}^o_h \ :\ \exists T\in\mathcal{T}_h \text{ such that } E'\subset T \text{ and } E_{mn}\subset T\}, \tag{A.6}$$

and, second the set of (Dirichlet) boundary neighbor edges

$$\mathcal{E}^D_{pq}:=\{E'\in\mathcal{E}^D_h \ :\ \exists T\in\mathcal{T}_h \text{ such that } E'\subset T \text{ and } E_{pq}\subset T\}. \tag{A.7}$$

By N^o_{mn} and N^D_{pq} we denote the cardinality of these two sets, i.e., $N^o_{mn}:=|\mathcal{E}^o_{mn}|$ and $N^D_{pq}:=|\mathcal{E}^D_{pq}|$.

Lemma (Lemma 4.4)**.** *The following inequality*

$$\sum_{E\in\mathcal{E}_h}([\![z]\!]_{E,m}-[\![z]\!]_{E,n})^2 \le C\sum_{E\in\mathcal{E}_h}([\![z]\!]_{E,m}+[\![z]\!]_{E,n})^2, \tag{A.8}$$

holds for some constant $C\le 16$.

Proof. Since $z\in\mathcal{Z}$ is linear on each triangle, for an arbitrary interior edge $E=E_{mn}$, i.e., $E_{mn}\in\mathcal{E}^o_h$, by using (A.3)–(A.5), we find the following relations:

$$\begin{aligned} [\![z]\!]_{E,m}-[\![z]\!]_{E,n} &= (z^+_m - z^+_n) - (z^-_m - z^-_n) \\ &= (z^+_m + z^+_k) - (z^+_k + z^+_n) - (z^-_m + z^-_l) + (z^-_l + z^-_n) \\ &= 2z^+_{mk} - 2z^+_{kn} - 2z^-_{ml} + 2z^-_{ln} \\ &= t_{mk}(z^+_{mk}-z^-_{mk}) - t_{kn}(z^+_{kn}-z^-_{kn}) + t_{ml}(z^+_{ml}-z^-_{ml}) - t_{ln}(z^+_{ln}-z^-_{ln}) \\ &= t_{mk}[\![z]\!]_{E,mk} - t_{kn}[\![z]\!]_{E,kn} + t_{ml}[\![z]\!]_{E,ml} - t_{ln}[\![z]\!]_{E,ln} \\ &= z_{mk}s_{mk}t_{mk} - z_{kn}s_{kn}t_{kn} + z_{ml}s_{ml}t_{ml} - z_{ln}s_{ln}t_{ln} \end{aligned} \tag{A.9}$$

where t_{ij} is either one or two, and s_{ij} is either plus or minus one, $(i,j)\in\{(m,k),(k,n),(m,l),(l,n)\}$. Similarly, for an edge $E=E_{pq}$ on the (Dirichlet) boundary, i.e., $E_{pq}\in\mathcal{E}^D_h$, we have

$$\begin{aligned} [\![z]\!]_{E,p}-[\![z]\!]_{E,q} &= (z^+_p - z^+_q) = (z^+_p + z^+_r) - (z^+_r + z^+_q) \\ &= 2z^+_{pr} - 2z^+_{rq} = t_{pr}[\![z]\!]_{E,pr} - t_{rq}[\![z]\!]_{E,rq} \\ &= z_{pr}s_{pr}t_{pr} - z_{rq}s_{rq}t_{rq} \end{aligned} \tag{A.10}$$

where t_{ij} is again either one or two, and s_{ij} is either plus or minus one.

Next, by using (A.9) and (A.10) and the Schwarz inequality (taking into account the possible values of t_{ij} and s_{ij}) we find

$$
\begin{aligned}
\sum_{E_{mn}\in\mathcal{E}_h} \left([\![z]\!]_{E,m} - [\![z]\!]_{E,n}\right)^2 &= \sum_{E_{mn}\in\mathcal{E}_h^o} \left([\![z]\!]_{E,m} - [\![z]\!]_{E,n}\right)^2 + \sum_{E_{pq}\in\mathcal{E}_h^D} \left([\![z]\!]_{E,p} - [\![z]\!]_{E,q}\right)^2 \\
&= \sum_{E_{mn}\in\mathcal{E}_h^o} \left(z_{mk}s_{mk}t_{mk} - z_{kn}s_{kn}t_{kn} + z_{ml}s_{ml}t_{ml} - z_{ln}s_{ln}t_{ln}\right)^2 \\
&\quad + \sum_{E_{pq}\in\mathcal{E}_h^D} \left(z_{pr}s_{pr}t_{pr} - z_{rq}s_{rq}t_{rq}\right)^2 \\
&\le \sum_{E_{mn}\in\mathcal{E}_h^o} (4N_{mn}^D + N_{mn}^o)\left(z_{mk}^2 + z_{kn}^2 + z_{ml}^2 + z_{ln}^2\right) \\
&\quad + \sum_{E_{pq}\in\mathcal{E}_h^D} (4N_{pq}^D + N_{pq}^o)\left(z_{pr}^2 + z_{rq}^2\right), \qquad \text{(A.11)}
\end{aligned}
$$

which depends on the cardinality of the sets defined in (A.6) and (A.7). In view of (A.5) the right hand side of (A.11) can be rewritten as

$$
\begin{aligned}
&\sum_{E_{mn}\in\mathcal{E}_h^o} (4 + 3N_{mn}^D)\left([\![z]\!]_{E,mk}^2 + [\![z]\!]_{E,kn}^2 + [\![z]\!]_{E,ml}^2 + [\![z]\!]_{E,ln}^2\right) \\
&+ \sum_{E_{pq}\in\mathcal{E}_h^D} (2 + 3N_{pq}^D)\left([\![z]\!]_{E,pr}^2 + [\![z]\!]_{E,rq}^2\right) \qquad \text{(A.12)}
\end{aligned}
$$

where we also used the fact that for interior edges $N_{mn}^D + N_{mn}^o = 4$ and for boundary edges $N_{pq}^D + N_{pq}^o = 2$. Next we observe that any fixed boundary edge appears at most twice in (A.12) and any fixed interior edge at most four times; however, any edge (boundary or interior edge) can appear in either one of the two sums in (A.12) or in both of them. Thence an upper bound for (A.12) is given by

$$
\begin{aligned}
&4\sum_{E_{mn}\in\mathcal{E}_h^o} (4 + 3N_{mn}^D)[\![z]\!]_{E,mn}^2 + 4\sum_{E_{pq}\in\mathcal{E}_h^D} (2 + 3N_{pq}^D)[\![z]\!]_{E,pq}^2 \\
= &\sum_{E_{mn}\in\mathcal{E}_h^o} (4 + 3N_{mn}^D)\, 4\left[\frac{1}{2}\left([\![z]\!]_{E,m} + [\![z]\!]_{E,m}\right)\right]^2 \\
&+ \sum_{E_{pq}\in\mathcal{E}_h^D} (2 + 3N_{pq}^D)\, 4\left[\frac{1}{2}\left([\![z]\!]_{E,p} + [\![z]\!]_{E,q}\right)\right]^2 \\
= &\sum_{E_{mn}\in\mathcal{E}_h^o} (4 + 3N_{mn}^D)\left([\![z]\!]_{E,m} + [\![z]\!]_{E,m}\right)^2 \\
&+ \sum_{E_{pq}\in\mathcal{E}_h^D} (2 + 3N_{pq}^D)\left([\![z]\!]_{E,p} + [\![z]\!]_{E,q}\right)^2 \qquad \text{(A.13)}
\end{aligned}
$$

where we also used (A.3). Finally, by defining $C := \max_{E_{mn}\in\mathcal{E}_h}(4 + 3N_{mn}^D)$ we obtain the desired result

$$
\sum_{E_{mn}\in\mathcal{E}_h} \left([\![z]\!]_{E,m} - [\![z]\!]_{E,n}\right)^2 \le C \sum_{E_{mn}\in\mathcal{E}_h} \left([\![z]\!]_{E,m} + [\![z]\!]_{E,n}\right)^2, \qquad \text{(A.14)}
$$

which proves the inequality (A.8) for a constant $C \le 16$. □

The next result follows from the lemma above and was crucial in proving Lemma 4.7.

Lemma (Lemma 4.5). *The following inequality*

$$\sum_{E_{mn}\in\mathcal{E}_h} (\llbracket z\rrbracket_{E,m} - \llbracket z\rrbracket_{E,n})^2 \leq \gamma_1^2 \sum_{E_{mn}\in\mathcal{E}_h} 2\left((\llbracket z\rrbracket_{E,m} + \llbracket z\rrbracket_{E,n})^2 + \llbracket z\rrbracket_{E,m}^2 + \llbracket z\rrbracket_{E,n}^2\right), \tag{A.15}$$

holds with $\gamma_1 \leq \sqrt{\frac{C}{3+C}} < 1$ *where* C *is the constant from Lemma 4.4.*

Proof. Using inequality (A.8) we derive

$$\frac{3}{3+C}\sum_{E_{mn}\in\mathcal{E}_h} (\llbracket z\rrbracket_{E,m} - \llbracket z\rrbracket_{E,n})^2 \leq \frac{3C}{3+C}\sum_{E_{mn}\in\mathcal{E}_h} (\llbracket z\rrbracket_{E,m} + \llbracket z\rrbracket_{E,n})^2 .$$

Hence

$$\begin{aligned}\sum_{E_{mn}\in\mathcal{E}_h} (\llbracket z\rrbracket_{E,m} - \llbracket z\rrbracket_{E,n})^2 \quad &\leq \quad \frac{C}{3+C}\sum_{E_{mn}\in\mathcal{E}_h} \big(2(\llbracket z\rrbracket_{E,m} + \llbracket z\rrbracket_{E,n})^2 + \\ &\quad (\llbracket z\rrbracket_{E,m} + \llbracket z\rrbracket_{E,n})^2 + (\llbracket z\rrbracket_{E,m} - \llbracket z\rrbracket_{E,n})^2\big),\end{aligned}$$

or, equivalently

$$\sum_{E_{mn}\in\mathcal{E}_h} (\llbracket z\rrbracket_{E,m} - \llbracket z\rrbracket_{E,n})^2 \leq \frac{C}{3+C}\sum_{E_{mn}\in\mathcal{E}_h} 2\left((\llbracket z\rrbracket_{E,m} + \llbracket z\rrbracket_{E,n})^2 + \llbracket z\rrbracket_{E,m}^2 + \llbracket z\rrbracket_{E,n}^2\right),$$

which shows that (A.15) holds with $\gamma_1 \leq \sqrt{\frac{C}{3+C}} \leq \sqrt{\frac{16}{19}}$. ☐

A.2. **A "multiplicative" relation.** This is to prove a basic relation used to derive (4.3) as well as (3.9). Let $\odot$ be a map $V \times W \mapsto U$, where U, V, and W are linear vector spaces over the real numbers. One may think that $\odot$ is some sort of multiplication. We assume that $\odot$ satisfies the following distributive laws:

$$a\odot(b+c) = a\odot b + a\odot c, \qquad (a+b)\odot c = a\odot c + b\odot c,$$

and we assume that for all $\xi \in \mathbb{R}$ and all $\eta \in \mathbb{R}$, we have:

$$(\xi a)\odot(\eta b) = (\xi\eta)(a\odot b). \tag{A.16}$$

We have the following identities, based on the definitions (3.2):

$$a^+\odot b^+ - a^-\odot b^- = \llbracket a\rrbracket \odot \{\!\{b\}\!\} + \{\!\{a\}\!\}\odot\llbracket b\rrbracket. \tag{A.17}$$

Proving this relation is indeed trivial. Some examples for which the reader should verify these identities are: (1) For real numbers a and b one may take as $\odot$ the usual multiplication of real numbers; (2) a and b elements of a real Hilbert space and $\odot$ inner product; (3) a and b are linear operators, and $\odot$ is then the multiplication of linear operators (which not necessarily commutative); (4) a is a linear operator and b is an element of a Hilbert space (like, a is a matrix and b is a vector).

From (3.2), we have that the right side of the identity (A.17) is

$$\llbracket a\rrbracket \odot \{\!\{b\}\!\} + \{\!\{a\}\!\}\odot\llbracket b\rrbracket = (a^+ - a^-)\odot\left(\frac{b^+ + b^-}{2}\right) + \left(\frac{a^+ + a^-}{2}\right)\odot(b^+ - b^-)$$

Using the distributive law, and (A.16) (linearity of $\odot$ with respect to scalar multiplication), we have

$$
\begin{aligned}
&(a^+ - a^-) \odot \left(\frac{b^+ + b^-}{2}\right) + \left(\frac{a^+ + a^-}{2}\right) \odot (b^+ - b^-) \\
= \;& \frac{1}{2}(a^+ - a^-) \odot (b^+ + b^-) + \frac{1}{2}(a^+ + a^-) \odot (b^+ - b^-) \\
= \;& \frac{1}{2}a^+ \odot (b^+ + b^-) - \frac{1}{2}a^- \odot (b^+ + b^-) + \frac{1}{2}a^+ \odot (b^+ - b^-) + \frac{1}{2}a^- \odot (b^+ - b^-) \\
= \;& \frac{1}{2}a^+ \odot b^+ + \frac{1}{2}a^+ \odot b^- - \frac{1}{2}a^- \odot b^+ - \frac{1}{2}a^- \odot b^- \\
& + \frac{1}{2}a^+ \odot b^+ - \frac{1}{2}a^+ \odot b^- + \frac{1}{2}a^- \odot b^+ - \frac{1}{2}a^- \odot b^- \\
= \;& \frac{1}{2}a^+ \odot b^+ - \frac{1}{2}a^- \odot b^- + \frac{1}{2}a^+ \odot b^+ - \frac{1}{2}a^- \odot b^- = a^+ \odot b^+ - a^- \odot b^-
\end{aligned}
$$

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