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A (ROUGH) PATHWISE APPROACH TO FULLY NON-LINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS.

MICHAEL CARUANA, PETER FRIZ AND HARALD OBERHAUSER

Abstract. In a series of papers, starting with [Fully nonlinear stochastic partial differential equations. C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), no. 9] Lions and Souganidis proposed a (pathwise) theory for fully non-linear stochastic partial differential equations. We present here a (partial) extension towards certain spatial dependence in the noise term. The main novelty is the use of rough path theory in this context [Lyons, Terry J.; Differential equations driven by rough signals. Rev. Mat. Iberoamericana 14 (1998), no. 2, 215–310].

1. Introduction

We trust the reader is familiar with the rudiments of (second order) viscosity theory [7, 8] and rough path theory [27, 28]. Recall that geometric rough paths arise from the (abstract) completion of $\mathbb{R}^d$-valued smooth paths in a p-variation (or $1/p$-Hölder) type "rough path" metric which involves the iterated integrals up to order $[p]$. As is well known this abstract completion can be realized as genuine path space,

$$C^{0,p\text{-var}}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right) \quad \text{resp. } C^{0,1/p\text{-Höl}}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$$

where $G^{[p]}\left(\mathbb{R}^d\right)$ is the free step-$[p]$ nilpotent group over $\mathbb{R}^d$, equipped with Carnot–Caratheodory metric, and in this context usually realized as a subset of $1+\mathfrak{t}^{[p]}\left(\mathbb{R}^d\right)$ where

$$\mathfrak{t}^{[p]}\left(\mathbb{R}^d\right) = \mathbb{R}^d \oplus \left(\mathbb{R}^d\right)^{\otimes 2} \oplus \cdots \oplus \left(\mathbb{R}^d\right)^{\otimes [p]}$$

is the natural statespace for (up to $[p]$) iterated integrals of a smooth $\mathbb{R}^d$-valued path. For instance, almost every realization of d-dimensional Brownian motion *plus Lévy's area* gives rise to the a path in the step-2 free nilpotent group over $\mathbb{R}^d$.

Following [22, 23, 24] we consider a real-valued function of time and space $u = u(t,x) \in \mathrm{BUC}([0,T]\times\mathbb{R}^n)$ which solves the fully-nonlinear partial differential equation

$$\begin{aligned} du &= F\left(t,x,Du,D^2u\right)dt + \sum_{i=1}^{d} H_i(x,Du)\,dz^i \\ &\equiv F\left(t,x,Du,D^2u\right)dt + H(x,Du)\,dz \end{aligned} \tag{1.1}$$

in viscosity sense. When $z : [0,T] \to \mathbb{R}^d$ is smooth then, subject to suitable conditions on F and H, this is a rather standard setting in viscosity theory. However, the ultimate aim here is to allow for $z = z(t)$ with little regularity in time, such as to cover the case when z is a Brownian

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motion[1]; the class of such *stochastic partial differential equations* (SPDEs), possibly generalized to $H = H(x, u, Du)$, is considered to be an important one and the reader can find a variety of examples (drawing from fields as diverse as filtering and stochastic control theory, pathwise stochastic control, interest rate theory, front propagation and phase transition in random media, ...) in the articles [23, 21]. We mention explicitly

Example 1 (Pathwise stochastic control; [23, 3, 4])**.** *Consider*

$$dX = b(X;\alpha)\,dt + W(X;\alpha)\circ d\tilde{B} + V(X)\circ dB,\ \ X_0 = x$$

where b, W, V *are (collections of) sufficiently nice vector fields (with* b, W *dependent on some control process* α*) and* $\tilde{B}, B$ *multi-dimensional (independent) Brownian motions. Define*

$$u(x,t,B) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\left(g(X_t) + \int_0^t f(X_s,\alpha_s)\,ds\right)\middle| B\right]$$

and write $L_\alpha = \sum W_i^2$ *for the linear second order differential operator, here in Hörmander form. Then, at least by a formal computation,*

$$\begin{aligned} du &= \inf_{\alpha\in\mathcal{A}} \left[b(x,\alpha)\,Du + L_a u + f(x,\alpha)\right]dt + Du\cdot V(x)\circ dB_t \\ &\equiv F\left(x, Du, D^2u\right)dt + H(x,Du)\circ dB(t)\,. \end{aligned}$$

Observe that $H = (H_1, H_2)$ *with* $H_i(x,p) = p\cdot V_i(x)$ *is linear in* p*.*

As pointed out in [22], classical (deterministic) second order viscosity theory can deal at best with $z \in W^{1,1}\left([0,T],\mathbb{R}^d\right)$, i.e. measurable dependence in time. As such, (1.1) with "Brownian" regularity of z (i.e. just below 1/2-Hölder) falls dramatically outside the scope of the deterministic theory. The results [22, 23, 24] are in fact *pathwise* and apply to any continuous path $z \in C\left([0,T],\mathbb{R}^d\right)$, this includes Brownian and even rougher sources of noise; however, the assumption was made that $H = H(Du)$ is independent of x. The rôle of x-dependence is an important one (as it arises in applications such as the above example) and far from harmless: the results of Lions–Souganidis imply that the map

$$z \in C^1\left([0,T],\mathbb{R}^d\right) \mapsto u(\cdot,\cdot) \in C([0,T],\mathbb{R}^n)$$

depends continuously on z *in uniform topology*; thereby giving existence/uniqueness results to

$$du = F\left(t,x,Du,D^2u\right)dt + \sum_{i=1}^d H_i(Du)\,dz^i$$

for *every* continuous path $z:[0,T]\to\mathbb{R}^d$. When the Hamiltonian depends on x, this ceases to be true although (cf. remark 3 in [22]); indeed, as may seen by taking $F\equiv 0$, $d=2$ and

$$H_i(x,p) = \langle p, V(x)\rangle \equiv \sum_{j=1}^n p_j V_i^j(x)\,, \tag{1.2}$$

where $i = 1,2$ and V_1, V_2 are two C^∞-bounded vector fields with Lie bracket $[V_1,V_2]\neq 0$. In this case, solving the characteristic equations shows that u is expressed in terms of the (inverse) flow associated to $dy = V_1(y)\,dz^1 + V_2(y)\,dz^2$, the solution of which does *not* depend continuously on the driving signal $z = \left(z^1, z^2\right)$ in uniform topology[2].

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[1]... in which case (1.1) is understood in Stratonovich form to avoid super-parabolicity assumptions, needed in the Itô-formulation.

[2]We shall push this remark much further in theorem 2 below.

The Lyons-theory of rough paths does exhibit an entire cascade of (p-variation or $1/p$-Hölder type rough path) metrics (for each $p \geq 1$) on path-space under which such ODE solutions are continuous functions of their driving signal. This suggests to extend the Lions–Souganidis theory from a pathwise to a *rough* pathwise theory. At present, we are able to do so for a rich class of fully-nonlinear F and Hamiltonians of the form (1.2). The proof of the following theorem, detailed in the section below, can be viewed as a first demonstration of the natural and powerful interplay of rough path and viscosity ideas. The use of rough path theory in the context of fully non-linear SPDEs was verbally conjectured by P.L. Lions in his Courant lecture (2003); in a sense the present paper gives an affirmative, if partial, answer to this conjecture. We have the following result[3].

Theorem 1. *Let* $(z^\varepsilon) \subset C^\infty\left([0,T], \mathbb{R}^d\right)$ *be Cauchy in (p-variation) rough path topology with rough path limit* $\mathbf{z} \in C^{0,p\text{-}var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$. *Assume* $u_0^\varepsilon \in \mathrm{BUC}(\mathbb{R}^n) \to u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ *locally uniformly and let* $u^\varepsilon \in \mathrm{BUC}([0,T]\times\mathbb{R}^n)$ *be a viscosity solution to*

$$du^\varepsilon = F\left(t,x,Du^\varepsilon,D^2u^\varepsilon\right)dt + Du^\varepsilon(t,x)\cdot V(x)\,dz^\varepsilon(t) = 0, \tag{1.3}$$
$$u^\varepsilon(0,\cdot) = u_0^\varepsilon, \tag{1.4}$$

where $F = F(t,x,p,X)$ *is continuous, degenerate elliptic such that* $\partial_t = F$ *satisfies* $\Phi^{(3)}$*-invariant comparison (cf. definition 1 below, also for a list of examples which satisfy this condition) and*

$$V = (V_1,\dots,V_d) \subset \mathrm{Lip}^{\gamma+2}(\mathbb{R}^n;\mathbb{R}^n) \text{ with } \gamma > p.$$

Assume that any such family $(u^\varepsilon : \varepsilon > 0)$ *is locally uniformly bounded*[4]. *Then (i) there exists* u, *only dependent on* $\mathbf{z}$ *but not on the particular approximating sequence, such that* $u^\varepsilon \to u$ *locally uniformly. We write (formally)*

$$du = F\left(t,x,Du,D^2u\right)dt + Du(t,x)\cdot V(x)\,d\mathbf{z}(t) = 0, \tag{1.5}$$
$$u(0,\cdot) = u_0, \tag{1.6}$$

and also $u = u^{\mathbf{z}}$ *when we want to indicate the dependence on* $\mathbf{z}$. *(ii) comparison holds in the sense that*

$$|u^{\mathbf{z}} - \hat{u}^{\mathbf{z}}|_{\infty;\mathbb{R}^n\times[0,T]} \leq |u_0 - \hat{u}_0|_{\infty;\mathbb{R}^n}$$

where $\hat{u}^{\mathbf{z}}$ *is defined as limit of* $\hat{u}^\eta$, *defined as in (1.3).*
(iii) the map $(\mathbf{z},u_0) \mapsto u^{\mathbf{z}}$ *from*

$$C^{p\text{-}var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right) \times \mathrm{BUC}(\mathbb{R}^n) \to \mathrm{BUC}([0,T]\times\mathbb{R}^n)$$

is continuous.

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[3] Unless otherwise stated we shall always equip BUC-spaces with the topology of locally uniform convergence.

[4] A simple sufficient conditions is boundedness of $F(\cdot,\cdot,0,0)$ on $[0,T]\times\mathbb{R}^n$, and the assumption that $u_0^\varepsilon \to u_0$ uniformly, as can be seen by comparison.

2. Proof of theorem

We shall always assume that $F = F(t,x,p,X)$ is continuous and degenerate elliptic[5] and that comparison holds for solutions of $\partial_t = F$. On a bounded domain a well-known sufficient condition[6] is

Condition 1 ([7, (3.14)]). *There exists a function $\theta : [0,\infty] \to [0,\infty]$ with $\theta(0+) = 0$, such that for each fixed $t \in [0,T]$,*

$$F(t,x,\alpha(x-\tilde{x}),X) - F(t,\tilde{x},\alpha(x-\tilde{x}),Y) \le \theta\left(\alpha|x-\tilde{x}|^2 + |x-\tilde{x}|\right)$$

whenever $\alpha > 0$, $x,\tilde{x} \in \mathbb{R}^n$, and $X,Y \in S^n$ (the space of $n \times n$ symmetric matrices) satisfy

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

In fact, this condition also guarantees comparison on $[0,T] \times \mathbb{R}^n$ provided the solutions are assumed to have suitable growth restrictions. In particular one shows that comparison holds for BUC-solutions in $[0,T] \times \mathbb{R}^n$, cf. the remarks preceding Theorem 2.1 in [24] for instance.

Remark 1. *A free benefit, cf.* [7, p.20]*, of condition 1 is that if F_γ satisfies Condition 1 for $\gamma \in \Gamma$ (some index set) with a common modulus θ, then* $\inf_\gamma F_\gamma$ *again satisfies condition 1; similar remarks apply to* $\sup_\beta \inf_\gamma F_{\beta,\gamma}$.

To state our key assumption on F we need some preliminary remark on the transformation behaviour of

$$Du = (\partial_1 u, \ldots, \partial_n u),\ D^2 u = (\partial_{ij} u)_{i,j=1,\ldots,n}.$$

under change of coordinates on $\mathbb{R}^n$ where $u = u(t,\cdot)$, for fixed t. Let us allow the change of coordinates to depend on t, say $v(t,\cdot) := u(t,\phi_t(\cdot))$ where $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Differentiating $v\left(t,\phi_t^{-1}(\cdot)\right) = u(t,\cdot)$ twice, followed by evaluation at $\phi_t(y)$, we have, with summation over repeated indices,

$$\begin{aligned}
\partial_i u(t,\phi_t(x)) &= \partial_k v(t,x)\,\partial_i \phi_t^{-1;k}|_{\phi_t(x)} \\
\partial_{ij} u(t,\phi_t(x)) &= \partial_{kl} v(t,x)\,\partial_i \phi_t^{-1;k}|_{\phi_t(x)} \partial_j \phi_t^{-1;l}|_{\phi_t(x)} + \partial_k v(t,x)\,\partial_{ij}\phi_t^{-1;k}|_{\phi_t(x)}.
\end{aligned}$$

We shall write this, somewhat imprecisely[7] but convenient, as

$$\begin{aligned}
Du|_{\phi_t(x)} &= \left\langle Dv|_x, D\phi_t^{-1}|_{\phi_t(x)}\right\rangle, \\
D^2 u|_{\phi_t(x)} &= \left\langle D^2 v|_x, D\phi_t^{-1}|_{\phi_t(x)} \otimes D\phi_t^{-1}|_{\phi_t(x)}\right\rangle + \left\langle Dv|_x, D^2\phi_t^{-1}|_{\phi_t(x)}\right\rangle.
\end{aligned} \tag{2.1}$$

Let us now introduce $\Phi^{(k)}$ as the class of all flows of C^k-diffeomorphisms of $\mathbb{R}^n$, $\phi = (\phi_t : t \in [0,T])$, such that $\phi_0 = \mathrm{Id}\ \forall \phi \in \Phi^{(k)}$ and such that ϕ_t and ϕ_t^{-1} have k bounded derivatives, uniformly in $t \in [0,T]$. We say that $\phi(n) \to \phi$ in $\Phi^{(k)}$ iff for all multi-indices α with $|\alpha| \le k$

$$\partial_\alpha \phi(n) \to \partial_\alpha \phi_t,\ \partial_\alpha \phi(n)^{-1} \to \partial_\alpha \phi_t^{-1} \text{ locally uniformly in } [0,T] \times \mathbb{R}^n.$$

[5] $F(\ldots,X) \ge F(\ldots,Y)$ if $X \ge Y$ in the sense of symmetric matrices.

[6] ... which, en passant, implies degenerate ellipticity, cf. page 18 in [7, (3.14)].

[7] Strictly speaking, one should view $\left(Du, D^2u\right)|.$ as *second order* cotangent vector, the pull-back of $\left(Dv, D^2v\right)|_x$ under ϕ_t^{-1}.

Definition 1 ($\Phi^{(k)}$**-invariant comparison**)**.** *Let $k \geq 2$ and*

(2.2)
$$F^{\phi}\left((t,x,p,X)\right) := F\left(t, \phi_t(x), \left\langle p, D\phi_t^{-1}|_{\phi_t(x)}\right\rangle, \left\langle X, D\phi_t^{-1}|_{\phi_t(x)} \otimes D\phi_t^{-1}|_{\phi_t(x)}\right\rangle + \left\langle p, D^2\phi_t^{-1}|_{\phi_t(x)}\right\rangle\right)$$

We say that $\partial_t = F$ satisfies $\Phi^{(k)}$-invariant comparison if, for every $\phi \in \Phi^{(k)}$, comparison holds for BUC solutions of $\partial_t - F^{\phi} = 0$. More precisely, if u is a sub- and v a super-solution to this equation (in viscosity sense, both BUC)and $u(0,\cdot) \leq v(0,\cdot)$ then $u \leq v$ on $[0,T] \times \mathbb{R}^n$.

Example 2 (F linear)**.** *Suppose that $\sigma(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n'}$ and $b(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in t and Lipschitz continuous in x, uniformly in $t \in [0,T]$. If $F(t,x,p,X) = \mathrm{Tr}\left[\sigma(t,x)\sigma(t,x)^T X\right] + b(t,x)\cdot p$, then $\Phi^{(3)}$-invariant comparison holds. Although this is a special case of the following example, let us point out that F^{ϕ} is of the same form as F with σ, b replaced by*

$$\begin{aligned} \sigma^{\phi}(t,x)^k_m &= \sigma^i_m(t,\phi_t(x))\,\partial_i\phi_t^{-1;k}|_{\phi_t(x)}, \quad k=1,\dots,n; m=1,\dots,n' \\ b^{\phi}(t,x)^k &= \left[b^i(t,\phi_t(x))\,\partial_i\phi_t^{-1;k}|_{\phi_t(x)}\right] + \sum_{i,j}\left(\sigma^i_m\sigma^j_m\partial_{ij}\phi_t^{-1;k}|_{\phi_t(y)}\right), \quad k=1,\dots,n. \end{aligned}$$

By defining properties of flows of diffeomorphisms, $t \mapsto \partial_i\phi_t^{-1;k}|_{\phi_t(x)}, \partial_{ij}\phi_t^{-1;k}|_{\phi_t(y)}$ is continuous and the C^3-boundedness assumption inherent in our definition of $\Phi^{(3)}$ ensures that σ^{ϕ}, b^{ϕ} are Lipschitz in x, uniformly in $t \in [0,T]$. As is well known, this ensures that comparison holds for BUC solutions of $\partial_t - F^{\phi} = 0$. By applying theorem 1 to this class of linear F we recover the results of [5].

Example 3 (F quasi-linear)**.** *Let*

$$F(t,x,p,X) = \mathrm{Tr}\left[\sigma(t,x,p)\sigma(t,x,p)^T X\right] + b(t,x,p)\,. \tag{2.3}$$

We assume $b = b(t,x,p) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and Lipschitz continuous in x and p, uniformly in $t \in [0,T]$. We also assume that $\sigma = \sigma(t,x,p) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n'}$ is a continuous map such that

- *$\sigma(t,\cdot,p)$ is bounded and Lipschitz continuous, uniformly in $(t,p) \in [0,T] \times \mathbb{R}^n$;*
- *there exists a constant $c_1 > 0$, such that*[8]

$$\forall p,q \in \mathbb{R}^n : |\sigma(t,x,p) - \sigma(t,x,q)| \leq c_1 \frac{|p-q|}{1+|p|+|q|} \tag{2.4}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^n$.

Then $\Phi^{(3)}$-invariant comparison holds for $\partial_t = F$ with F given by (2.3). To see this we proceed as follows. For brevity denote

$$\begin{aligned} p &= \alpha(x-\tilde{x}), J_\cdot = D\phi_t^{-1}|_{\phi_t(\cdot)}, H_\cdot = D^2\phi_t^{-1}|_{\phi_t(\cdot)} \\ \sigma_\cdot &= \sigma(t,\phi_t(\cdot),\langle p, J_\cdot\rangle), a_\cdot = \sigma_\cdot\sigma_\cdot^T,\ b_\cdot = b(t,\phi_t(\cdot),\langle p,J_\cdot\rangle) \end{aligned}$$

so that

$$\begin{aligned} F^{\phi}(t,x,p,X) &= \mathrm{Tr}\left[a_x\left(\langle X, J_x \otimes J_x\rangle + \langle p, H_x\rangle\right)\right] + b_x \\ &= \mathrm{Tr}\left[J_x a_x J_x^T X\right] + b_x + \mathrm{Tr}\left[a_x\langle p, H_x\rangle\right]. \end{aligned}$$

[8]A condition of this type also appears also in [1].

Hence

$$F^{\phi}(t,\tilde{x},p,Y)-F^{\phi}(t,x,p,X)=\underbrace{\mathrm{Tr}\left[J_{\tilde{x}}a_{\tilde{x}}J_{\tilde{x}}^{T}Y-J_{x}a_{x}J_{x}^{T}X\right]}_{=:(i)}+\underbrace{b_{\tilde{x}}-b_{x}}_{=:(ii)}+\underbrace{\mathrm{Tr}\left[a_{\tilde{x}}\langle p,H_{\tilde{x}}\rangle-a_{x}\langle p,H_{x}\rangle\right]}_{=:(iii)}.$$

To estimate (i) note that $J_x a_x J_x^T = J_x\sigma_x (J_x\sigma_x)^T$. *The* $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ *matrix*

$$\begin{pmatrix} (J_x\sigma_x)(J_x\sigma_x)^T & J_x\sigma_x (J_{\tilde{x}}\sigma_{\tilde{x}})^T \\ (J_{\tilde{x}}\sigma_{\tilde{x}})(J_x\sigma_x)^T & J_{\tilde{x}}\sigma_{\tilde{x}}(J_{\tilde{x}}\sigma_{\tilde{x}})^T \end{pmatrix}$$

is positive semidefinite and thus we can multiply it to both sides of the inequality

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The resulting inequality is stable under evaluating the trace and so one gets

$$\begin{aligned}\mathrm{Tr}\left[J_{\tilde{x}}\sigma_{\tilde{x}}(J_{\tilde{x}}\sigma_{\tilde{x}})^T\cdot Y - J_x\sigma_x (J_x\sigma_x)^T\cdot X\right] &\le 3\alpha\mathrm{Tr}\Big[(J_x\sigma_x)(J_x\sigma_x)^T - J_x\sigma_x(J_{\tilde{x}}\sigma_{\tilde{x}})^T \\ &\quad -J_{\tilde{x}}\sigma_{\tilde{x}}(J_x\sigma_x)^T + J_{\tilde{x}}\sigma_{\tilde{x}}(J_{\tilde{x}}\sigma_{\tilde{x}})^T\Big] \\ &= 3\alpha\mathrm{Tr}\left[(J_x\sigma_x - J_{\tilde{x}}\sigma_{\tilde{x}})(J_x\sigma_x - J_{\tilde{x}}\sigma_{\tilde{x}})^T\right] \\ &= 3\alpha\left\|J_x\sigma_x - J_{\tilde{x}}\sigma_{\tilde{x}}\right\|^2\end{aligned}$$

(using that $\mathrm{Tr}\left[\cdot\,\cdot^T\right]$ *defines an inner product for matrices and gives rise to the Frobenius matrix norm* $\|.\|$*). Hence, by the triangle inequality and Lipschitzness of the Jacobian of the flow (which follows a fortiori from the boundedness of the second order derivatives of the flow),*

$$\begin{aligned}\|J_x\sigma_x - J_{\tilde{x}}\sigma_{\tilde{x}}\| &\le \|J_x\sigma_x - J_x\sigma_{\tilde{x}}\| + \|J_x\sigma_{\tilde{x}} - J_{\tilde{x}}\sigma_{\tilde{x}}\| \\ &\le \|J_x\|\,\|\sigma_x-\sigma_{\tilde{x}}\| + \|J_x - J_{\tilde{x}}\|\,\|\sigma_{\tilde{x}}\| \\ &\le \|J_x\|\,\|\sigma_x-\sigma_{\tilde{x}}\| + c_2(\sigma,\phi)\,|x-\tilde{x}|\end{aligned}$$

Since $\sigma(t,\cdot,q)$ *is Lipschitz continuous (uniformly in* $(t,q)\in[0,T]\times\mathbb{R}^n$*) and* $\phi_t(\cdot)$ *is Lipschitz continuous (uniformly in* $t\in[0,T]$*), we can use our assumption (2.4) on* σ*, to see*

$$\|\sigma_x - \sigma_{\tilde{x}}\| \le (const)\times|x-\tilde{x}|\,. \tag{2.5}$$

Indeed,

$$\begin{aligned}\|\sigma_x-\sigma_{\tilde{x}}\| &= \|\sigma(t,\phi_t(x),p\cdot J_x) - \sigma(t,\phi_t(\tilde{x}),p\cdot J_{\tilde{x}})\| \\ &\le \|\sigma(t,\phi_t(x),p\cdot J_x) - \sigma(t,\phi_t(\tilde{x}),p\cdot J_x)\| \\ &\quad + \|\sigma(t,\phi_t(\tilde{x}),p\cdot J_x) - \sigma(t,\phi_t(\tilde{x}),p\cdot J_{\tilde{x}})\| \\ &\le c_2(\sigma,\phi)\,|x-\tilde{x}| + c_1\frac{\alpha\,|x-\tilde{x}|\,|J_x - J_{\tilde{x}}|}{1+\alpha\,|x-\tilde{x}|\,(|J_x|+|J_{\tilde{x}}|)};\end{aligned}$$

and, noting that $\phi_t \circ \phi_t^{-1} = Id$ *and* $\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \|D\phi_t|_x\| \leq c_3$ *implies* $\|J_x\| = \left\|D\phi_t^{-1}|_{\phi_t(x)}\right\| \geq 1/c_3$*, we have*

$$\begin{aligned} c_1 \frac{\alpha |x-\tilde{x}| |J_x - J_{\tilde{x}}|}{1+\alpha |x-\tilde{x}| (|J_x| + |J_{\tilde{x}}|)} &\leq |x-\tilde{x}| \cdot \frac{c_1 \alpha |J_x - J_{\tilde{x}}|}{\alpha |x-\tilde{x}| (|J_x| + |J_{\tilde{x}}|)} \\ &\leq |x-\tilde{x}| \frac{c_4(\sigma,\phi) |x-\tilde{x}|}{|x-\tilde{x}| (|J_x| + |J_{\tilde{x}}|)} \\ &\leq c_5(\sigma,\phi) |x-\tilde{x}| . \end{aligned}$$

Putting things together we have

$$|(i)| \leq c_6(\sigma,\phi)\, \alpha |x-\tilde{x}|^2 .$$

As for (ii)*, we have that,*

$$\begin{aligned} |b_x - b_{\tilde{x}}| &\leq |b(t,\phi_t(x), \langle p, J_x\rangle) - b(t,\phi_t(\tilde{x}), \langle p, J_x\rangle)| \\ &\quad + |b(t,\phi_t(\tilde{x}), \langle p, J_x\rangle) - b(t,\phi_t(\tilde{x}), \langle p, J_{\tilde{x}}\rangle)| \\ &\leq c_7(b) \left(|\phi_t(x) - \phi_t(\tilde{x})| + |p| |J_{\tilde{x}} - J_x|\right) \end{aligned}$$

where $c_7(b)$ *is the (uniform in* $t \in [0,T]$*) Lipschitz bound for* $b(t,\cdot,\cdot)$*. To get the required estimate we again use the regularity of the flow. Finally, for* (iii)*,*

$$\begin{aligned} (iii) &= \operatorname{Tr}\left[a_{\tilde{x}} \langle p, H_{\tilde{x}}\rangle - a_{\tilde{x}} \langle p, H_x\rangle\right] + \operatorname{Tr}\left[a_{\tilde{x}} \langle p, H_x\rangle - a_x \langle p, H_x\rangle\right] \\ &= \operatorname{Tr}\left[a_{\tilde{x}} \langle p, H_{\tilde{x}} - H_x\rangle\right] + \operatorname{Tr}\left[(a_{\tilde{x}} - a_x) \langle p, H_x\rangle\right] . \end{aligned}$$

Using Cauchy-Schwartz (with inner product $\operatorname{Tr}\left[\cdot \cdot \cdot^T\right]$*) and* $p = \alpha(x-\tilde{x})$ *it is clear that boundedness of* H *and* a *(i.e.* $\sup_x |H_x| < \infty$ *uniformly in* $t \in [0,T]$ *and similarly for* a*) and Lipschitz continuity (i.e.*$|H_x - H_{\tilde{x}}| \leq (const) \times |x-\tilde{x}|$ *uniformly in* $t \in [0,T]$ *and similar for* a*) will suffice to obtain the (desired) estimate*

$$|(iii)| \leq c_8 \times \alpha |x-\tilde{x}|^2 .$$

Only Lipschitz continuity of $a_x = \sigma_x \sigma_x^T$ *requires a discsussion. But this follows, thanks to boundedness of* $\sup_x |\sigma_x|$*, from showing Lipschitzness of* $x \mapsto \sigma_x = \sigma(t,\phi_t(x), \langle p, J_x\rangle)$ *uniformly in* $t \in [0,T]$ *which was already seen in (2.5). This shows that* F^ϕ *satisfies condition 1 for any* $\phi \in \Phi^{(3)}$*.*

Example 4 (F of Hamilton-Jacobi-Bellman type)**.** *From the above examples and remark 1, we see that* $\Phi^{(3)}$*-invariant comparison holds when* F *is given by*

$$F(t,x,p,X) = \inf_{\gamma\in\Gamma} \left\{ \operatorname{Tr}\left[\sigma(t,x;\gamma)\,\sigma(t,x;\gamma)^T X\right] + b(t,x;\gamma)\cdot p \right\},$$

the usual non-linearity in the Hamilton-Jacobi-Bellman equation, and more generally

$$F(t,x,p,X) = \inf_{\gamma\in\Gamma} \left\{ \operatorname{Tr}\left[\sigma(t,x,p;\gamma)\,\sigma(t,x,p;\gamma)^T \cdot X\right] + b(t,x,p;\tau) \right\}$$

whenever the conditions in examples 2 and 3 are satisfied uniformly with respect to $\gamma \in \Gamma$*.*

Example 5 (F of Isaac type)**.** *Similarly,* $\Phi^{(3)}$*-invariant comparison holds for*

$$F(t,x,p,X) = \sup_\beta \inf_\gamma \left\{ \operatorname{Tr}\left[\sigma(t,x;\beta,\gamma)\,\sigma(t,x;\beta,\gamma)^T X\right] + b(t,x;\beta,\gamma)\cdot p \right\},$$

(such non-linearities arise in Isaac equation in the theory of differential games), and more generally

$$F(t,x,p,X) = \sup_\beta \inf_\gamma \left\{ \operatorname{Tr}\left[\sigma(t,x,p;\beta,\gamma)\,\sigma(t,x,p;\beta,\gamma)^T \cdot X\right] + b(t,x,p;\beta,\gamma) \right\}$$

whenever the conditions in examples 2 and 3 are satisfied uniformly with respect to $\beta \in \mathcal{B}$ *and* $\gamma \in \Gamma$*, where* $\mathcal{B}$ *and* Γ *are arbitrary index sets.*

Lemma 1. *Let* $z : [0,T] \to \mathbb{R}^d$ *be smooth and assume that we are given* C^3*-bounded vector fields*[9] $V = (V_1, \dots, V_d)$*. Then ODE*

$$dy_t = V(y_t)\, dz_t, \quad t \in [0,T]$$

has a unique solution flow (of C^3*-diffeomorphisms)* $\phi = \phi^z \in \Phi^{(3)}$*.*

Proof. Standard, e.g. chapter 4 in [14]. □

Proposition 1. *Let* z, V *and* ϕ *be as in lemma 1. Then* u *is a viscosity sub- (resp. super-) solution (always assumed* BUC*) of*

$$\dot{u}(t,x) = F\left(t, x, Du, D^2u\right) - Du(t,x) \cdot V(x)\, \dot{z}(t) \tag{2.6}$$

if and only if $v(t,x) := u(t, \phi_t(x))$ *is a viscosity sub- (resp. super-) solution of*

$$\dot{v}(t,x) = F^\phi\left(t, x, Dv, D^2v\right) \tag{2.7}$$

where F^ϕ *was defined in (2.2).*

Proof. Set $y = \phi_t(x)$. When u is a classical sub-solution, it suffices to use the the chain-rule and definition of F^ϕ to see that

$$\begin{aligned} \dot{v}(t,x) &= \dot{u}(t,y) + Du(t,y) \cdot \dot{\phi}_t(x) = \dot{u}(t,y) + Du(t,y) \cdot V(y)\, \dot{z}_t \\ &\leq F\left(t, y, Du(t,y), D^2u(t,y)\right) = F^\phi\left(t, x, Dv(t,x), D^2v(t,x)\right). \end{aligned}$$

The case when u is a viscosity sub-solution of (2.6) is not much harder: suppose that $(\bar{t}, \bar{x})$ is a maximum of $v - \xi$, where $\xi \in C^2([0,T] \times \mathbb{R}^n)$ and define $\psi \in C^2([0,T] \times \mathbb{R}^n)$ by $\psi(t,y) = \xi\left(t, \phi_t^{-1}(y)\right)$. Set $\bar{y} = \phi_{\bar{t}}(\bar{x})$ so that

$$F\left(\bar{t}, \bar{y}, D\psi(\bar{t},\bar{y}), D^2\psi(\bar{t},\bar{y})\right) = F^\phi\left(\bar{t}, \bar{x}, D\xi(\bar{t},\bar{x}), D^2\xi(\bar{t},\bar{x})\right).$$

Obviously, $(\bar{t}, \bar{y})$ is a maximum of $u - \psi$, and since u is a viscosity sub-solution of (2.6) we have

$$\dot{\psi}(\bar{t},\bar{y}) + D\psi(\bar{t},\bar{y})\, V(\bar{y})\, \dot{z}(\bar{t}) \leq F\left(\bar{t}, \bar{y}, D\psi(\bar{t},\bar{y}), D^2\psi(\bar{t},\bar{y})\right).$$

On the other hand, $\xi(t,x) = \psi(t, \phi_t(x))$ implies $\dot{\xi}(\bar{t},\bar{x}) = \dot{\psi}(\bar{t},\bar{y}) + D\psi(\bar{t},\bar{y})\, V(\bar{y})\, \dot{z}(\bar{t})$ and putting things together we see that

$$\dot{\xi}(\bar{t},\bar{x}) \leq F^\phi\left(\bar{t}, \bar{x}, D\xi(\bar{t},\bar{x}), D^2\xi(\bar{t},\bar{x})\right)$$

which says precisely that v is a viscosity sub-solution of (2.7). Replacing maximum by minimum and $\leq$ by $\geq$ in the preceding argument, we see that if u is a super-solution of (2.6), then v is a super-solution of (2.7).
Conversely, the same arguments show that if v is a viscosity sub- (resp. super-) solution for (2.7), then $u(t,y) = v\left(t, \phi^{-1}(y)\right)$ is a sub- (resp. super-) solution for (2.6). □

We can now give the proof of our main result.

[9] In particular, if the vector fields are Lip^γ, $\gamma > p + 2$, $p \geq 1$, then they are also C^3-bounded.

Proof. **(Theorem 1.)** Using Lemma 1, we see that $\phi^{\varepsilon} \equiv \phi^{z^{\varepsilon}}$, the solution flow to $dy = V(y)\,dz^{\varepsilon}$, is an element of $\Phi \equiv \Phi^{(3)}$. Set $F^{\varepsilon} := F^{\phi^{\varepsilon}}$. From Proposition 1, we know that u^{ε} is a solution to

$$du^{\varepsilon} = F\left(t, y, Du^{\varepsilon}, D^2 u^{\varepsilon}\right) dt - Du^{\varepsilon}(t,y) \cdot V(y)\,dz^{\varepsilon}(t)\,, \;\; u^{\varepsilon}(0,\cdot) = u_0^{\varepsilon}$$

if and only if v^{ε} is a solution to $\partial_t - F^{\varepsilon} = 0$. By assumption of Φ-invariant comparison,

$$|v^{\varepsilon} - \hat{v}^{\varepsilon}|_{\infty;\mathbb{R}^n \times [0,T]} \leq |v_0 - \hat{v}_0|_{\infty;\mathbb{R}^n}\,.$$

where $v^{\varepsilon}, \hat{v}^{\varepsilon}$ are viscosity solution to $\partial_t - F^{\varepsilon} = 0$. Let $\phi^{\mathbf{z}}$ denote the solution flow to the rough differential equation

$$dy = V(y)\,d\mathbf{z}.$$

Thanks to $\mathrm{Lip}^{\gamma+2}$-regularity of the vector fields $\phi^{\mathbf{z}} \in \Phi$, and in particular a flow of C^3-diffeomorphisms. Set $F^{\mathbf{z}} = F^{\phi^{\mathbf{z}}}$. The "universal" limit theorem [27] holds, in fact, on the level of flows of diffeomorphisms (see [26] and [14, Chapter 11] for more details) tells us that, since z^{ε} tends to $\mathbf{z}$ in rough path sense,

$$\phi^{\varepsilon} \to \phi^{\mathbf{z}} \text{ in } \Phi$$

so that, by continuity of F (more precisely: uniform continuity on compacts), we easily deduce that

$$F^{\varepsilon} \to F^{\mathbf{z}} \text{ locally uniformly.}$$

From the method of semi-relaxed limits (Lemma 6.1 and Remarks 6.2, 6.3 and 6.4 in [7], see also [8]) the pointwise (relaxed) limits

$$\begin{aligned} \bar{v} \;&:= \limsup{}^{*}\, v^{\varepsilon}, \\ \underline{v} \;&:= \liminf{}_{*}\, v^{\varepsilon}, \end{aligned}$$

are viscosity (sub resp. super) solutions to $\partial_t - F^{\mathbf{z}} = 0$, with identical initial data. As the latter equation satisfies comparison, one has trivially uniqueness and hence $v := \bar{v} = \underline{v}$ is the unique (and continuous, since $\bar{v}, \underline{v}$ are respectively upper resp. lower semi-continuous) solution to

$$\partial_t v = F^{\mathbf{z}} v\,, \;\; v(0,\cdot) = u_0(\cdot)\,.$$

Moreover, using a simple Dini-type argument (e.g. [7, p.35]) one sees that this limit must be uniform on compacts. It follows that v is the unique solution to

$$\partial_t v = F^{\mathbf{z}} v\,, \;\; v(0,\cdot) = u_0(\cdot)$$

(hence does not depend on the approximating sequence to $\mathbf{z}$) and the proof of (i) is finished by setting

$$u^{\mathbf{z}}(t,x) := v\left(t, (\phi_t^{\mathbf{z}})^{-1}(x)\right).$$

(ii) The comparison $|u^{\mathbf{z}} - \hat{u}^{\mathbf{z}}|_{\infty;[0,T]\times\mathbb{R}^n} \leq |u_0 - \hat{u}_0|_{\infty;\mathbb{R}^n}$ is a simple consequence of comparison for $v, \hat{v}$ (solutions to $\partial_t v = F^{\mathbf{z}} v$). At last, to see (iii), we argue in the very same way as in (i), starting with

$$F^{\mathbf{z}_n} \to F^{\mathbf{z}} \text{ locally uniformly}$$

to see that $v^n \to v$ locally uniformly, i.e. uniformly on compacts. □

3. Applications to stochastic partial differential equations

Applications to SPDEs are path-by-path, i.e. by taking $\mathbf{z}$ to be a typical realization of Brownian motion and Lévy's area, $\mathbf{B}(\omega) \equiv (B, A)$, also known as enhanced Brownian motion or Brownian rough path. The continuity property (iii) of our theorem 1 easily allows to identify (1.5) with $\mathbf{z} = \mathbf{B}(\omega)$ as Stratonovich solution to the non-linear SPDE

$$du = F\left(t, x, Du, D^2 u\right) dt + Du\,(t, x) \cdot V\,(x) \circ dB, \;\; u\,(0, \cdot) = u_0.$$

Let us mention some typical applications.

(Approximation results) *Any* approximation result to $\mathbf{B}$ in rough path topology implies a corresponding (weak or strong) limit theorem for such SPDEs: it suffices that an approximation to B converges in rough path topology; as is well known (e.g. [14, Chapter 13] and the references therein) examples include piecewise linear -, mollifier - and Karhunen-Loeve approximations, as well as (weak) Donsker type random walk approximations [2]. A slightly more interesting example is the following[10].

Theorem 2. *Let $V = (V_1, \ldots, V_d)$ be a collection of C^∞-bounded vector fields on $\mathbb{R}^n$ and B a d-dimensional standard Brownian motion. Then, for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \{1, \ldots, d\}^N$, $N \geq 2$, there exists (piecewise) smooth approximations $\left(z^k\right)$ to B, with each z^k only dependent on $\left\{B\,(t) : t \in D^k\right\}$ where $\left(D^k\right)$ is a sequence of dissections of $[0, T]$ with mesh tending to zero, such that almost surely*

$$z^k \to B \text{ uniformly on } [0, T],$$

but u^k, solutions to

$$du^k = F\left(t, x, Du^k, D^2 u^k\right) dt + Du^k\,(t, x) \cdot V\,(x)\, dz^k, \;\; u^k\,(0, \cdot) = u_0 \in \mathrm{BUC}\,(\mathbb{R}^n)\,,$$

(with assumptions on F as formulated in theorem 1) converge almost surely locally uniformly to the solution of the "wrong" differential equation

$$du = \left[F\left(t, x, Du, D^2 u\right) + Du\,(t, x) \cdot V_\alpha\,(x)\right] dt + Du\,(t, x) \cdot V\,(x) \circ dB$$

where V_α is the bracket-vector field given by $V_\alpha = \left[V_{\alpha_1}, \left[V_{a_2}, \ldots \left[V_{\alpha_{N-1}}, V_{\alpha_N}\right]\right]\right]$.

Proof. The rough path regularity of $\mathbf{B}(\omega)$ implies that higher iterated (Stratonvich) integrals are deterministically defined; see [25, First thm.]. Doing this up to level N yields a (rough path) $S_N(\mathbf{B})$ and we perturbe it in the highest level, linearly in the

$$\left[e_{\alpha_1}, \left[e_{a_2}, \ldots \left[e_{\alpha_{N-1}}, e_{\alpha_N}\right]\right]\right] \text{-direction}$$

of $S_N(\mathbf{B})$ viewed as element in the step-N free nilpotent Lie algebra. This yields a (level-N) rough path $\tilde{\mathbf{B}}$ and we can find approximations $\left(z^k\right)$ that converge almost surely to $\tilde{\mathbf{B}}$ in rough path topology (see [10]). One identifies standard RDEs driven by $\tilde{\mathbf{B}}$ as RDEs-with-drift (driven along the original vector fields by $d\mathbf{B}$, and along V_α by dt). The resulting identification obviously holds on the level of RDE flows and thus

$$u^{z^k}(t, x) = v\left(t, \left(\phi_t^{z^k}\right)^{-1}(x)\right) \to u^{\tilde{\mathbf{B}}}(t, x) = v\left(t, \left(\phi_t^{\tilde{\mathbf{B}}}\right)^{-1}(x)\right)$$

[10] The following theorem also holds when the Stratonovich differential $\circ dB$ is replaced by dz for some $z \in C^1\left([0, T], \mathbb{R}^d\right)$; it can then be viewed as a non-trivial assertion about the behaviour of non-linear parabolic equations with coefficients that exhibit highly oscillatory behaviour in time.

The flow identification then implies that

$$du = F\left(t, x, Du, D^2 u\right) dt + Du\,(t, x) \cdot V\,(x)\, d\tilde{\mathbf{B}}$$

is equivalent to the equation with $V\,(x)\, d\tilde{\mathbf{B}}$ replaced by $V\,(x)\, d\mathbf{B} + V_\alpha\,(x)\, dt$. □

(Support results) In conjunction with known support properties of $\mathbf{B}$ (e.g. [20] in p-variation rough path topology or [9] for a conditional statement in Hölder rough path topology) continuity of the SPDE solution as a function of $\mathbf{B}$ immediately implies Stroock–Varadhan type support descriptions for such SPDEs. Let us note that, to the best of our knowledge, results of this type are new for such non-linear SPDEs and may be hard to obtain without rough path technology; in the linear case, approximations and support of SPDEs have been studied in great detail [19, 18, 16, 15, 17].

(Large deviation results) Another application of our continuity result is the ability to obtain large deviation estimates when B is replaced by εB with $\varepsilon \to 0$; indeed, given the known large deiviation behaviour of $\left(\varepsilon B, \varepsilon^2 A\right)$ in rough path topology (e.g. [20] in p-variation and [11] in Hölder rough path topology) it suffices to recall that large deviation principles are stable under continuous maps. Again, large deviation estimates for non-linear SPDEs in the small noise limit appear to be new and may be hard to obtain without rough paths theory.

(SPDEs with non-Brownian noise) Yet another benefit of our approach is the ability to deal with SPDEs with non-Brownian and even non-semimartingale noise. For instance, one can take $\mathbf{z}$ as (the rough path lift of) fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$, cf. [6] or [13], a regime which is "rougher" than Brownian and notoriously difficult to handle; or a diffusion with uniformly elliptic generator in divergence form with measurable coefficients; see [12]. Much of the above (approximations, support, large deviation) results also extend, as is clear from the respective results in the above-cited literature.

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