Evolution by Non-Convex Functionals

P. Elbau, M. Grasmair F. Lenzen, O. Scherzer

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Peter Elbau * Markus Grasmair † Frank Lenzen ‡

Otmar Scherzer §

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Abstract

We establish a semi-group solution concept for morphological differential equations, such as the mean curvature flow equation. The proposed method consists in generating flows from generalized minimizers of non-convex energy functionals. We use relaxation and convexification to define generalized minimizers. The main part of this work consists in verification of the solution concept by comparing analytical, rotationally invariant solutions of the mean curvature flow equation and iterative minimizer of a non-convex energy functional.

1 Introduction

Semi-group theory is a powerful concept for proving existence of solutions of gradient flow equations

$$\begin{aligned} \partial_t u &\in -\partial\mathcal{R}(u)\,, \\ u(0) &= u_0\,, \end{aligned} \tag{1}$$

where $\mathcal{R}\colon X \to [0,+\infty]$ is a sequentially lower semi-continuous, convex energy functional on a Hilbert Space X. In (1) we use the notation that ∂ with a subscript denotes the derivative of a function with respect to the subscript and without subscript ∂ denotes a subdifferential.

Standard results on gradient flows state that equation (1) has a solution u whenever $u_0 \in \overline{\mathcal{D}(\mathcal{R})}$, and that $u(t)$, $t > 0$, can be computed as limit of the sequence $u_{t/n}^{(n)}$ defined by

$$u_{t/n}^{(n)} := \left(\mathrm{Id} + \tfrac{t}{n}\,\partial\mathcal{R}\right)^{-n} u_0$$

as $n \to \infty$ (cf. [3, 5, 6, 14, 16]). Equivalently, one can consider the iteration $u_{t/n}^{(0)} := u_0$ and

$$u_{t/n}^{(k)} := \arg\min\left\{\frac{1}{2}\|u - u_{t/n}^{(k-1)}\|^2 + \frac{t}{n}\mathcal{R}(u) : u \in U\right\}, \qquad k = 1,\ldots,n\,. \tag{2}$$

*Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria (peter.elbau@oeaw.ac.at)

†Inst. of Mathematics, University of Innsbruck, Austria (markus.grasmair@uibk.ac.at)

‡Inst. of Mathematics, University of Innsbruck, Austria (frank.lenzen@uibk.ac.at)

§Inst. of Mathematics, University of Innsbruck, Austria (otmar.scherzer@uibk.ac.at) & Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria

Then, again, the equality $u(t) = \lim_{n\to\infty} u_{t/n}^{(n)}$ holds.

In [2], this idea has been extended to gradient flows on metric spaces (X, d) by considering the iterative minimization of the functional

$$\mathcal{F}(\tau, \tilde{u}; u) := d(u, \tilde{u})^2 + \tau\mathcal{R}(u)$$

instead. Again, the limit of the iteration solves a gradient flow equation.

Let us now consider a more general form for both the energy functional and the distance measure. For $\tau > 0$ and $\tilde{u} \in X$ let $\mathcal{F}(\tau, \tilde{u}; \cdot)$ be a family of functionals defined by

$$\mathcal{F}(\tau, \tilde{u}; u) = \mathcal{S}(\tilde{u}; u) + \tau\mathcal{R}(u),$$

where $\mathcal{S}\colon X \times X \to [0, +\infty]$ and $\mathcal{R}\colon X \to [0, +\infty]$ are (not necessarily convex) functionals on X. Since we waive the convexity of $\mathcal{S}$ and $\mathcal{R}$, we are no longer able to characterize minimizers of $\mathcal{F}$ based on the subgradients of $\mathcal{S}(u)$ and $\mathcal{R}(u)$. Moreover, using *direct methods* (see [11, 7]), we can only assume existence of *generalized* minimizers. Nevertheless we can apply the semi-group concept and define an iterative method analogously to (2): We define the sequence $\{u_\tau^{(k)}\}_{k\in\mathbb{N}_0}$ by $u_\tau^{(0)} := u_0$ and

$$u_\tau^{(k)} \in \left\{u \in X : u \text{ is a generalized minimizer of } \mathcal{F}(\tau, u_\tau^{(k-1)}; \cdot)\right\} . \qquad (3)$$

Here, the generalized minimizers are defined via the relaxation of $\mathcal{F}$. Again, the sequence $\{u_{t/n}^{(n)}\}$ at least formally defines a flow on X.

In Section 3 we apply this concept to a particular functional for which the associated formal gradient flow equation resembles the mean curvature flow equation (see [10]). In the remaining parts of the paper we show that this relation is not just formal. We provide a rigorous proof that for rotationally symmetric, continuous, and initial data with monotonous rotational part the mean curvature flow is in fact generated by this non-convex functional. This example suggests that our non-convex semi-group concept is a reasonable solution concept for geometrical PDEs.

2 Evolution by Non-convex Functionals

Let X be a topological space, and let $\mathcal{R}\colon X \to [0, +\infty]$ and $\mathcal{S}\colon X\times X \to [0, +\infty]$. Assume that for every $u_0 \in X$ there exists $u \in X$ such that $\mathcal{S}(u_0; u) + \mathcal{R}(u) \neq +\infty$. For $\tau > 0$ and $u_0 \in X$ we define the functional $\mathcal{F}(u_0, \tau; \cdot)\colon X \to [0, +\infty]$ setting

$$\mathcal{F}(\tau, u_0; u) := \mathcal{S}(u_0; u) + \tau\mathcal{R}(u) .$$

We want to define a flow as limit for $n \to \infty$ of the sequence $\{u_{t/n}^{(n)}\}$ defined by the iteration $u_0^{(n)} := u_0$ and

$$u_{t/n}^{(k)} := \arg\min\left\{\mathcal{F}(t/n, u_{t/n}^{(k-1)}; u) : u \in U\right\} , \qquad k = 1, \ldots, n .$$

Since a minimizer need not exist for arbitrary non-convex functionals $\mathcal{R}$ and $\mathcal{S}$, it is necessary to work with generalized minimizers instead, which are defined as minimizers of relaxed functionals.

For the following definition of relaxation recall that a functional $\mathcal{F}$ on the topological space X is called *sequentially coercive*, if all of its sublevel sets are sequentially precompact. Moreover, it is called *proper*, if there exists some $u \in X$ with $\mathcal{F}(u) < \infty$. Finally, the functional $\mathcal{F}$ is *sequentially lower semi-continuous* if

$$\liminf_k \mathcal{F}(u^{(k)}) \geq \mathcal{F}(u) \qquad \text{whenever} \qquad u^{(k)} \to u \ .$$

Definition 1. Let $\mathcal{F}\colon X \to [0, +\infty]$ be sequentially coercive and proper on the topological space X. Assume moreover that $\mathcal{F}$ is bounded from below. We define the *relaxation* of $\mathcal{F}$ as

$$\mathcal{F}_{\mathcal{R}} := \sup\big\{\mathcal{J}\colon X \to [0, +\infty] : \mathcal{J} \leq \mathcal{F} \text{ is sequentially lower semi-continuous } \big\} \ .$$

■

The following facts easily follow from the definition of $\mathcal{F}_{\mathcal{R}}$:

- Since $\mathcal{F}$ is bounded from below, there exists a constant functional $\mathcal{J}_0 = c$ below $\mathcal{F}$, Since the constant functional is sequentially lower semi-continuous, $\mathcal{F}_{\mathcal{R}}$ is well defined.
- The functional $\mathcal{F}_{\mathcal{R}}\colon X \to [0, +\infty]$ is sequentially lower semi-continuous (cf. [12, Lemma 1.5]). Since the functional $\mathcal{F}_{\mathcal{R}}$ is sequentially coercive and proper, it therefore attains a minimum (cf. [12, Lemma 1.4]).

Moreover, every minimizer of the relaxed functional $\mathcal{F}_{\mathcal{R}}$ can be considered a generalized minimizer of the original functional $\mathcal{F}$.

Proposition 2. *Every minimizer u of $\mathcal{F}_{\mathcal{R}}$ is a generalized minimizer of $\mathcal{F}$. That is, either u is a minimizer of $\mathcal{F}$ or there exists a sequence $(u^{(k)})$ converging to u such that $\mathcal{F}(u^{(k)}) \to \inf_{v \in X} \mathcal{F}(v)$.*

More generally, we have the characterization of the relaxed functional

$$\mathcal{F}_{\mathcal{R}}(u) = \min\big\{\liminf_k \mathcal{F}(u^{(k)}) : u^{(k)} \to u\big\}$$

for every $u \in X$.

Proof. Cf. [4, Prop. 1.31]. □

We are now ready to introduce our concept of semi-groups generated by non-convex energy functionals and similarity terms.

Definition 3. Let $\mathcal{R}\colon X \to [0, +\infty]$ and $\mathcal{S}\colon X \times X \to [0, +\infty]$. For $\tau > 0$ and $\tilde{u} \in X$ define the functional $\mathcal{F}(\tau, \tilde{u}; \cdot)\colon X \to [0, +\infty]$ by

$$\mathcal{F}(\tau, \tilde{u}; u) := \mathcal{S}(\tilde{u}; u) + \tau \mathcal{R}(u) \ .$$

Assume that $\mathcal{F}(\tau, \tilde{u}; \cdot)$ is sequentially coercive for every $\tau > 0$ and $\tilde{u} \in X$.

For every $\tau > 0$ we define the piecewise constant approximation $\overline{u}_\tau\colon \mathbb{R}_{\geq 0} \to X$ by $\overline{u}_\tau(0) = u_0$ and

$$\overline{u}_\tau(t) \in \arg\min \mathcal{F}_{\mathcal{R}}\big(\tau, u((k-1)\tau); \cdot\big) \ , \qquad (k-1)\tau < t \leq k\tau \ , \ k \in \mathbb{N} \ .$$

We define the flow generated by $\mathcal{F}$ by

$$U(t) := \overline{\big\{u(t) : \text{there exist } \tau_\ell \to 0 \text{ with } u(t) = \lim_{\ell \to \infty} \overline{u}_{\tau_\ell}(t)\big\}} \ . \tag{4}$$

■

We note that for some time t the set $U(t)$ can be empty, in which case no solution of the evolution process exists at the time t.

3 MCM and Semi-Group Theory - Analytical Justification

The main motivation for the considerations above is the *mean curvature flow* equation

$$\begin{aligned} \partial_t u &= |\nabla u| \, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \qquad \text{on } \mathbb{R}^m \times (0, \infty)\,, \\ u(0) &= u_0 \qquad\qquad\qquad\qquad \text{on } \mathbb{R}^m\,, \end{aligned} \tag{5}$$

and its relation to the evolution defined by a non-convex similarity term and the total variation as energy functional.

This NCBV functional (*non-convex bounded variation*) on $X := L^1(\mathbb{R}^m)$ is defined for $u \in W^{1,1}(\mathbb{R}^m)$ as

$$\mathcal{F}(\tau, \tilde{u}; u) := \int_{\mathbb{R}^m} \left(\frac{(u - \tilde{u})^2}{2|\nabla u|} + \tau |\nabla u| \right) \mathrm{d}x \tag{6}$$

and $\mathcal{F}(\tau, \tilde{u}; u) = +\infty$ for $u \notin W^{1,1}(\mathbb{R}^m)$. Since in general, minimization of $\mathcal{F}$ with respect to the last variable u is considered, we will omit the dependence on τ and $\tilde{u}$ whenever possible and write $\mathcal{F}(u) := \mathcal{F}(\tau, \tilde{u}; u)$ instead.

The following computations, which are purely formal and by no means mathematically rigorous, provide the link between iterative minimization of $\mathcal{F}$ and the mean curvature flow.

Formally, the gradient of the functional $\mathcal{F}$ is

$$\partial \mathcal{F}(u) = \frac{u - \tilde{u}}{|\nabla u|} + \nabla \cdot \left(\left(\frac{(u - \tilde{u})^2}{2|\nabla u|^2} - \tau \right) \frac{\nabla u}{|\nabla u|} \right)\,. \tag{7}$$

Therefore, a minimizer u of $\mathcal{F}$ is expected to satisfy the optimality condition

$$\frac{u - \tilde{u}}{\tau} \in |\nabla u| \, \nabla \cdot \left(\left(1 - \frac{(u - \tilde{u})^2}{2\tau |\nabla u|^2} \right) \frac{\nabla u}{|\nabla u|} \right)\,. \tag{8}$$

Now denote $\Delta_\tau u := (u - \tilde{u})/\tau$. Then (8) reads as

$$\Delta_\tau u \in |\nabla u| \nabla \cdot \left(\left(1 - \tau \frac{(\Delta_\tau u)^2}{2|\nabla u|^2} \right) \frac{\nabla u}{|\nabla u|} \right)\,.$$

Interpreting $\Delta_\tau u$ as finite difference approximation of $\partial_t u$, a formal passage to the limit $\tau \to 0$ yields the mean curvature flow equation.

We will now show that the flow generated by the functional (6) in fact approximates the solution of the mean curvature equation, if the initial data u_0 is absolutely continuous, compactly supported, rotationally invariant, and its radial part is strictly monotonous on its support.

In order to prove this result, we start with the solution of the mean curvature equation (5) for rotationally invariant initial data $u_0 \in C^1(\mathbb{R}^m)$. In this case there exists a function $v_0 \in C^1(\mathbb{R}_{\geq 0})$ such that

$$u_0(x) = v_0(|x|) \qquad \text{for every } x \in \mathbb{R}^m\,. \tag{9}$$

In this setting, the solution u of the mean curvature equation is rotationally invariant. Therefore, there also exists a function $v\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$u(x,t) = v(|x|,t)\,, \quad (x,t) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}\,. \tag{10}$$

Thus, the mean curvature equation reduces to the linear partial differential equation

$$\partial_t v(r,t) = \frac{m-1}{r}\,\partial_r v(r,t)\,, \quad v(r,0) = v_0(r)\,. \tag{11}$$

Since the solution of this initial value problem is constant along the characteristic curves $t \mapsto (r(t),t)$ defined by $\partial_t r = -(m-1)/r$, we obtain the solution

$$v(r,t) = v_0\big(\sqrt{r^2 + 2(m-1)t}\,\big)\,, \quad (r,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\,.$$

We now calculate a solution of the flow generated by the NCBV functional. First, we will determine the relaxation $\mathcal{F}_{\mathcal{R}}$ of the functional $\mathcal{F}$ in $BV(\mathbb{R}^m)$, the space of functions $u \in L^1(\mathbb{R}^m)$ of bounded variation.

Recall to that end that the distributional gradient Du of a function $u \in BV(\mathbb{R}^m)$ can be decomposed into

$$Du = \nabla u\,\mathcal{L}^m + D^s u,$$

where $D^s u$ denotes the singular part of the signed measure Du. For further details see e.g. [1, 8].

In the following, we denote by $L^\infty_{\mathrm{c}}(\mathbb{R}^m)$ the space of essentially bounded functions $\tilde{u} \in L^\infty(\mathbb{R}^m)$ with compact support, and by $BV_{\mathrm{c}}(\mathbb{R}^m)$ the space of all compactly supported functions $\tilde{u} \in BV(\mathbb{R}^m)$.

Theorem 4. *Let $\tilde{u} \in L^\infty_{\mathrm{c}}(\mathbb{R}^m)$ and $\tau > 0$. Then the relaxation $\mathcal{F}_{\mathcal{R}}$ of $\mathcal{F}$ with respect to the L^1-norm on $BV(\mathbb{R}^m)$ coincides with its convexification $\mathcal{F}_c$ defined by*

$$\mathcal{F}_c(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^m} f\big(|u-\tilde{u}|, |\nabla u|; \tau\big)\mathrm{d}x + \tau |D^s u|(\mathbb{R}^m)\,, & \text{if } u \in BV(\mathbb{R}^m)\,, \\ +\infty\,, & \text{if } u \notin BV(\mathbb{R}^m)\,, \end{cases} \tag{12}$$

with

$$f(\xi,\eta;\tau) = \begin{cases} \dfrac{\xi^2}{2\eta} + \tau\eta, & \text{if } \sqrt{2\tau}\,\eta > \xi\,, \\ \sqrt{2\tau}\,\xi, & \text{if } \sqrt{2\tau}\,\eta \leq \xi\,. \end{cases} \tag{13}$$

Moreover, the functional $\mathcal{F}$ has a generalized minimizer $u_\tau \in BV(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ and

$$\operatorname{ess\,inf} \tilde{u} \leq u_\tau \leq \operatorname{ess\,sup} \tilde{u}\,.$$

In addition, $\operatorname{supp}(u_\tau)$ *is contained in the closed convex hull of* $\operatorname{supp}(\tilde{u})$ *and* $|Du_\tau|(\mathbb{R}^m) \leq |D\tilde{u}|(\mathbb{R}^m)$.

Proof. See Section 4. □

Corollary 5. *Let $u_0 \in BV(\mathbb{R}^m) \cap L^\infty_{\mathrm{c}}(\mathbb{R}^m)$ and $\tau > 0$. The the flow generated by the NCBV-functional (6) is well–defined.*

Proof. Theorem 4 shows that for every sequence (τ_ℓ) of time steps the corresponding piecewise constant approximations $\overline{u}_{\tau_\ell}$ form a bounded sequence in $BV(\mathbb{R}^m)$ and therefore have a weakly* convergent (with respect to the $BV(\mathbb{R}^m)$-topology) subsequence. From the same theorem it also follows that $\operatorname{supp}(\overline{u}_{\tau_\ell})$ is contained in the closed convex hull of $\Omega := \operatorname{supp}(u_0)$. Using that the embedding from $BV(\Omega)$ into $L^1(\Omega)$ is compact (see [1, Cor. 3.49]) it follows that $\overline{u}_{\tau_\ell}$ are convergent with respect to $L^1(\mathbb{R}^m)$. □

The next step is to restrict the problem of minimizing the functional $\mathcal{F}_\mathcal{R}$ over all functions $u \in BV_\mathrm{c}(\mathbb{R}^m)$ to a minimization problem over the rotationally invariant functions only. Before we can do this, we need to show that the minimization of the relaxed function $\mathcal{F}_\mathcal{R}$ preserves the rotational invariance of the initial data u_0.

Proposition 6. *Let $\tau \in \mathbb{R}_{>0}$ and $\tilde{u} \in L^\infty_\mathrm{c}(\mathbb{R}^m)$ an $\mathrm{SO}(m)$-invariant function. Then the functional $\mathcal{F}_\mathcal{R}(\tau, \tilde{u}; \cdot)$ has an $\mathrm{SO}(m)$-invariant minimizer u_τ.*

If, in addition, every $\mathrm{SO}(m)$-invariant minimizer u_τ of $\mathcal{F}_\mathcal{R}(\tau, \tilde{u}; \cdot)$ satisfies the condition

$$\mathcal{L}^m\big(\{x \in \operatorname{supp}(u_\tau) : \sqrt{2\tau}|\nabla u_\tau(x)| \le |u_\tau(x) - \tilde{u}(x)|\}\big) = 0\,, \tag{14}$$

then all the minimizers of $\mathcal{F}_\mathcal{R}(\tau, \tilde{u}; \cdot)$ are $\mathrm{SO}(m)$-invariant.

Proof. See Section 5. □

Proposition 6 shows that for a given rotationally invariant function $\tilde{u} : \mathbb{R}^m \to \mathbb{R}$ there exists a rotationally invariant minimizer of the relaxed NCBV functional $\mathcal{F}_\mathcal{R}(\tau, \tilde{u}; \cdot)$. In analogy to (9) and (10) we define the radial components of the rotationally invariant functions u and $\tilde{u}$ by $v(|x|) = u(x)$ and $\tilde{v}(|x|) = \tilde{u}(x)$. The relaxed NCBV functional then simplifies to

$$\mathcal{F}_\mathcal{R}(\tau, \tilde{u}; u) = m\omega_m \mathcal{G}(\tau, \tilde{v}; v)\,,$$

where $\mathcal{G}(\tau, \tilde{v}; \cdot) \colon \widehat{BV}_c \to [0, +\infty]$ is defined as

$$\mathcal{G}(\tau, \tilde{v}; v) := \int_0^\infty f\big(|v(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) r^{m-1} \mathrm{d}r + \tau \int_0^\infty r^{m-1} \mathrm{d}|D^s v|(r)\,,$$

and $\omega_m = \mathcal{L}^m(B_1)$ denotes the volume of the m-dimensional unit ball. Here $\widehat{BV}_c$ denotes the set of all functions $v \colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ which are the radial part of an $\mathrm{SO}(m)$-invariant function $u \in BV_\mathrm{c}(\mathbb{R}^m)$, i.e.,

$$\widehat{BV}_c = \big\{v \colon \mathbb{R}_{\ge 0} \to \mathbb{R} : \text{there exists } u \in BV_\mathrm{c}(\mathbb{R}^m) \text{ s.t. } u(x) = v(|x|)\,,\ x \in \mathbb{R}^m\big\}\,.$$

The following theorem states that the iterative minimization procedure introduced in Definition 3 for the functional $\mathcal{G}$ and with a strictly monotonous initial function $v_0 \in L^\infty_\mathrm{c}(\mathbb{R}_{\ge 0})$ converges to the solution of the mean curvature flow equation.

Theorem 7. *Let $v_0 \colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ be an absolutely continuous function with $\operatorname{supp}(v_0) = [0, R]$ for some $R > 0$, which is strictly monotonous on $[0, R]$. For $k \in \mathbb{N}$ let $v_\tau^{(0)} := v_0$ and*

$$v_\tau^{(k)} \in \arg\min\big\{\mathcal{G}(\tau, v_\tau^{(k-1)}; v) : v \in \widehat{BV}_c\big\}\,.$$

Then

$$\lim_{n\to\infty} v_{t/n}^{(n)}(r) = v_0\big(\sqrt{r^2+2(m-1)t}\,\big)$$

locally uniformly for $(r,t) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$.

Moreover, we have for all $k \in \mathbb{N}$ *and* $\tau \in \mathbb{R}_{>0}$ *the inequality*

$$|v_\tau^{(k)}(r) - v_\tau^{(k-1)}(r)| < \sqrt{2\tau}\,|\partial_r v_\tau^{(k)}(r)| \text{ for almost every } r \in (0,R). \tag{15}$$

Proof. See Section 8. □

In particular, the property (15) of Theorem 7 shows that every rotationally invariant minimizer of $\mathcal{F}_{\mathcal{R}}$ fulfills the condition (14) of Proposition 6. Therefore, all minimizers of $\mathcal{F}_{\mathcal{R}}$ are rotationally invariant and Theorem 7 completely describes the flow generated by the NCBV functional $\mathcal{F}$.

Corollary 8. *The mean curvature flow is equivalent to the flow generated by the NCBV functional in case the initial data is rotationally invariant, absolutely continuous, and its radial component is strictly monotonous on its support.*

4 Proof of Theorem 4

In this section we prove Theorem 4, which states that the relaxation of the NCBV functional coincides with its convexification and that minimization of $\mathcal{F}_{\mathcal{R}}$ does neither increase the norm nor the support of the input data u_0. This generalizes the results of [12, 13], where the relaxation has been computed on $BV(\Omega)$ for open and bounded subsets Ω of $\mathbb{R}^m$ with Lipschitz boundary.

Proof (of Thm. 4). For proving that the relaxed and convexified functional coincide, i.e., that the equality $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_c$ holds, we have to show that $\mathcal{F}_c$ is lower semi-continuous and that for every $u \in BV(\mathbb{R}^m)$ and $\varepsilon > 0$ there exists a function u_ε with $\|u_\varepsilon - u\|_2 < \varepsilon$ and $\mathcal{F}(u_\varepsilon) \le \mathcal{F}_c(u) + \varepsilon$. To that end it is convenient to define for every open and bounded set $\Omega \subset \mathbb{R}^m$ and $u \in L^1(\mathbb{R}^m)$ the localized functionals

$$\begin{aligned}\mathcal{F}(u;\Omega) &:= \int_\Omega \frac{(u-\tilde{u})^2}{2|\nabla u|} + \tau|\nabla u|\,\mathrm{d}x\,,\\ \mathcal{F}_c(u;\Omega) &:= \int_\Omega f\big(|u-\tilde{u}|, |\nabla u|; \tau\big)\,\mathrm{d}x + \tau|D^s u|(\Omega)\,.\end{aligned}$$

Here $|D^s u|(\Omega)$ denotes the singular part of the Radon measure $|Du|$ on Ω.

Using [13, Thm. 2], we obtain that $\mathcal{F}_c(\cdot,\Omega)$ is sequentially lower semi-continuous whenever $\Omega \subset \mathbb{R}^m$ is open and bounded with Lipschitz boundary. Since $\mathcal{F}_c(\cdot) = \mathcal{F}_c(\cdot,\mathbb{R}^m)$ is the supremum of all these functionals, it follows that also $\mathcal{F}_c$ is sequentially lower semi-continuous.

Let $r > 1$ such that $\operatorname{supp}(\tilde{u}) \subset B_r$. Assume that $u \in BV(\mathbb{R}^m)$ and that $\operatorname{supp}(u) \subset B_r$. Using [13, Thm. 2], it follows that for every $\varepsilon > 0$ there exists a function $v_\varepsilon \in W^{1,1}(B_{r+1})$ such that

$$\|v_\varepsilon - u\|_{L^1(B_{r+1})} \le \varepsilon \text{ and } \mathcal{F}(v_\varepsilon, B_{r+1}) \le \mathcal{F}_c(u, B_{r+1}) + \varepsilon\,. \tag{16}$$

A closer inspection of the proof of [13, Thm. 2] reveals that v_ε can be chosen to be an element of $W^{1,\infty}(B_{r+1})$.

Let R be a minimizer of the function $s \mapsto \int_{\partial B_s} |v_\varepsilon| \,\mathrm{d}\mathcal{H}^{m-1}$ in $[r, r+1]$ and let

$$C > \max\big\{\|v_\varepsilon\|_\infty, \|\nabla v_\varepsilon\|_\infty\big\} \,. \tag{17}$$

Define

$$\hat{u}_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } |x| \leq R\,, \\ v_\varepsilon\Big(\dfrac{Rx}{|x|}\Big) - C\,(|x| - R)\,\mathrm{sgn}\Big(v_\varepsilon\Big(\dfrac{Rx}{|x|}\Big)\Big) & \text{if } \Big|v_\varepsilon\Big(\dfrac{Rx}{|x|}\Big)\Big| \geq C(|x| - R) \\ & \text{and } |x| \geq R\,, \\ 0 & \text{else}\,, \end{cases} \tag{18}$$

and $\Omega_\varepsilon := B_R \cup \mathrm{supp}(\hat{u}_\varepsilon)$. Then

$$C \leq |\nabla \hat{u}_\varepsilon| \leq 2C \text{ on } \Omega_\varepsilon \setminus B_R. \tag{19}$$

Using the inequality $(R + r)^m \leq 2^m R^{m-1} r + R^m$ for $r \in [0, R]$ and (17), we obtain

$$\begin{aligned} \mathcal{L}^m(\Omega_\varepsilon \setminus B_R) &= \int_{S^{m-1}} \int_R^{R+|v_\varepsilon(Rx)|/C} t^{m-1}\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{m-1}(x) \\ &= \frac{1}{m} \int_{S^{m-1}} \Big(\Big(R + \frac{|v_\varepsilon(Rx)|}{C}\Big)^m - R^m\Big)\,\mathrm{d}\mathcal{H}^{m-1}(x) \\ &\leq \frac{2^m}{Cm} \int_{S^{m-1}} \big|v_\varepsilon(Rx)\big| R^{m-1}\,\mathrm{d}\mathcal{H}^{m-1}(x) \\ &\leq \frac{2^m}{Cm} \int_{\partial B_R} \big|v_\varepsilon(x)\big|\,\mathrm{d}\mathcal{H}^{m-1}(x) \,. \end{aligned}$$

As a consequence, the minimality of $\int_{\partial B_R} |v_\varepsilon| \,\mathrm{d}\mathcal{H}^{m-1}$ implies that

$$\begin{aligned} \mathcal{L}^m(\Omega_\varepsilon \setminus B_R) &\leq \frac{2^m}{Cm} \int_{\partial B_R} \big|v_\varepsilon(x)\big|\,\mathrm{d}\mathcal{H}^{m-1}(x) \\ &\leq \frac{2^m}{Cm} \int_r^{r+1} \int_{\partial B_t} \big|v_\varepsilon(x)\big|\,\mathrm{d}\mathcal{H}^{m-1}(x)\,\mathrm{d}t \\ &= \frac{2^m}{Cm} \|v_\varepsilon\|_{L^1(B_{r+1} \setminus B_r)} \\ &\leq \frac{2^m \varepsilon}{Cm} \,. \end{aligned}$$

Using (17) and (19) and that $\mathrm{supp}(\tilde{u}) \subset B_r \subset B_R$, we obtain that

$$\begin{aligned} \mathcal{F}(\hat{u}_\varepsilon; \Omega_\varepsilon \setminus B_R) &= \int_{\Omega_\varepsilon \setminus B_R} \frac{\hat{u}_\varepsilon^2}{2|\nabla \hat{u}_\varepsilon|} + \tau |\nabla \hat{u}_\varepsilon|\,\mathrm{d}x \\ &\leq \big(2\tau C + C/2\big) \mathcal{L}^m(\Omega_\varepsilon \setminus B_R) \\ &\leq \varepsilon(2\tau + 1/2) \frac{2^m}{m} \,. \end{aligned}$$

Since $\hat{u}_\varepsilon$ is absolutely continuous, it follows with (16) that

$$\begin{aligned} \mathcal{F}(\hat{u}_\varepsilon; \Omega_\varepsilon) &\leq \mathcal{F}(\hat{u}_\varepsilon; B_R) + \mathcal{F}(\hat{u}_\varepsilon; \Omega \setminus B_R) \\ &\leq \mathcal{F}(v_\varepsilon; B_R) + K\varepsilon \leq \mathcal{F}_c(u; \Omega_\varepsilon) + (K+1)\varepsilon \,, \end{aligned}$$

where $K := (2\tau + 1/2)\frac{2^m}{m}$.

Taking into account the definition (18) of $\hat{u}_\varepsilon$, we obtain that

$$\begin{aligned}\|\hat{u}_\varepsilon - u\|_{L^1(\Omega_\varepsilon)} &= \|v_\varepsilon - u\|_{L^1(B_R)} + \|\hat{u}_\varepsilon\|_{L^1(B_{r+1}\setminus B_R)} \\ &\le \varepsilon + \int_R^{r+1}\int_{\partial B_t} |\hat{u}_\varepsilon| \,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{m-1} \\ &\le \varepsilon + \int_r^{r+1}\int_{\partial B_t} |v_\varepsilon| \,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{m-1} \\ &= \varepsilon + \|v_\varepsilon\|_{L^1(B_{r+1}\setminus B_r)} \le 2\varepsilon\ .\end{aligned}$$

Now let $U_k \subset \mathbb{R}^m \setminus \Omega_\varepsilon$, $k \in \mathbb{N}$, be disjoint, open, bounded, and Lipschitz bounded sets such that $\mathcal{L}^m\big(\mathbb{R}^m \setminus \big(\Omega_\varepsilon \cup \bigcup_k U_k\big)\big) = 0$. Since $\operatorname{supp}(u)$ and $\operatorname{supp}(\tilde{u})$ are contained in B_r, which is a subset of Ω_ε, it follows that $\mathcal{F}_c(u; U_k) = 0$.

Therefore there exists for every $k \in \mathbb{N}$ a function $\hat{u}_k \in W^{1,\infty}(\mathbb{R}^m)$ with $\operatorname{supp}(u_k) \subset \overline{U}_k$ such that $\|\hat{u}_k\|_{L^1(U_k)} < \varepsilon/2^k$ and $\mathcal{F}(\hat{u}_k; U_k) < \varepsilon/2^k$. Setting

$$u_\varepsilon(x) := \begin{cases} \hat{u}_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon\ , \\ \hat{u}_k(x) & \text{if } x \in U_k,\ k \in \mathbb{N}\ , \end{cases}$$

it follows that $\|u_\varepsilon - u\|_{L^1(\mathbb{R}^m)} < 3\varepsilon$ and $\mathcal{F}(u_\varepsilon) \le \mathcal{F}_c(u) + (K+2)\varepsilon$. Since $\varepsilon > 0$ was arbitrary and K independent of ε, this shows that $\mathcal{F}_c(u) \ge \mathcal{F}_{\mathcal{R}}(u)$ whenever $u \in L^1(\mathbb{R}^m)$ is compactly supported.

Now let $u \in L^1(\mathbb{R}^m)$ be arbitrary. Then $\mathcal{F}_c(u) = \lim_{k\to\infty} \mathcal{F}_c(u\,\chi_{B_k})$, where χ_{B_k} denotes the characteristic function of B_k. Consequently,

$$\mathcal{F}_{\mathcal{R}}(u) \le \liminf_{k\to\infty} \mathcal{F}_{\mathcal{R}}(u\,\chi_{B_k}) \le \liminf_{k\to\infty} \mathcal{F}_c(u\,\chi_{B_k}) = \mathcal{F}_c(u)\,,$$

which shows that $\mathcal{F}_{\mathcal{R}}(u) = \mathcal{F}_c(u)$ for every $u \in L^1(\mathbb{R}^m)$.

Now let $u \in BV(\mathbb{R}^m)$. Denote by B the closed convex hull of $\operatorname{supp}(\tilde{u})$ and define $u_B := u\,\chi_B$. Then

$$|Du|\big((\mathbb{R}^m \setminus B) \cup \partial B\big) \ge \int_{\partial B} |u|\, d\mathcal{H}^{m-1} = |Du_B|\big((\mathbb{R}^m \setminus B) \cup \partial B\big)\ .$$

As a consequence, $\mathcal{F}_c(u_B) \le \mathcal{F}_c(u)$ with equality holding if and only if $u_B = u$. Similarly, defining $\hat{u}(x) := \max\big\{\min\{u(x), \sup\tilde{u}\}, \inf\tilde{u}\big\}$, one can show that $\mathcal{F}_c(\hat{u}) \le \mathcal{F}_c(u)$, and again, equality holds if and only if $\hat{u} = u$ (cf. [12, Lemma 4.12]). This shows that minimizing $\mathcal{F}_c$ on $L^1(\mathbb{R}^m)$ is equivalent to minimizing $\mathcal{F}_c$ on the compact set

$$S := \big\{u \in L^1(\mathbb{R}^m) : \inf\tilde{u} \le u \le \sup\tilde{u},\ \operatorname{supp}(u) \subset B\big\}\ .$$

Since $\mathcal{F}_c$ is lower semi-continuous, the existence of a minimizer u_τ, which is contained in S, follows.

From (12) and (13) it follows that

$$\tau|Du_\tau|(\mathbb{R}^m) \le \mathcal{F}_c(u_\tau) \le \mathcal{F}_c(\tilde{u}) = \tau|D\tilde{u}|(\mathbb{R}^m)\ . \qquad \square$$

5 Proof of Proposition 6

Proposition 6 states that, given an SO(m)-invariant function $\tilde{u}$, the functional $\mathcal{F}_{\mathcal{R}}(\tau, \tilde{u}; \cdot)$ has an SO(m)-invariant minimizer. In addition, it provides a criterion to determine whether every minimizer is SO(m)-invariant.

Proof (of Proposition 6). Theorem 4 implies the existence of a minimizer $u_\tau \in BV_{\mathrm{c}}(\mathbb{R}^m)$ of $\mathcal{F}_{\mathcal{R}}(\tau, \tilde{u}; \cdot)$. We then define the SO($m$)-invariant function $\bar{u}_\tau \in BV_{\mathrm{c}}(\mathbb{R}^m)$ by

$$\bar{u}_\tau(x) = \int_{\mathrm{O}(m)} u_\tau(Rx) \mathrm{d}\mu(R),$$

where μ denotes the Haar-measure on the orthogonal group O(m) of all real, orthogonal $m \times m$-matrices, normalized to fulfill $\mu(\mathrm{O}(m)) = 1$.

For the distributional gradient $D\bar{u}_\tau$ of $\bar{u}_\tau$, we find with $\psi_R(y) := R\phi(R^{-1}y)$

$$\begin{aligned}\int_{\mathbb{R}^m} \phi(x) \cdot \mathrm{d}D\bar{u}_\tau(x) &= -\int_{\mathbb{R}^m} \bar{u}_\tau(x) \nabla \cdot \phi(x) \mathrm{d}x \\ &= -\int_{\mathrm{O}(m)} \int_{\mathbb{R}^m} u_\tau(Rx) \nabla \cdot \phi(x) \mathrm{d}x \, \mathrm{d}\mu(R) \\ &= -\int_{\mathrm{O}(m)} \int_{\mathbb{R}^m} u_\tau(y) \nabla \cdot \psi_R(y) \mathrm{d}y \, \mathrm{d}\mu(R)\end{aligned}$$

for every function $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^m; \mathbb{R}^m)$. Therefore, using the definition of the distributional gradient of u_τ, we get

$$\begin{aligned}\int_{\mathbb{R}^m} \phi(x) \cdot \mathrm{d}D\bar{u}_\tau(x) &= \int_{\mathrm{O}(m)} \int_{\mathbb{R}^m} R\phi(R^{-1}y) \cdot \mathrm{d}Du_\tau(y) \mathrm{d}\mu(R) \qquad (20) \\ &= \int_{\mathrm{O}(m)} \int_{\mathbb{R}^m} \phi(x) \cdot R^{-1} \mathrm{d}Du_\tau(Rx) \mathrm{d}\mu(R).\end{aligned}$$

For the measure $|D\bar{u}_\tau|$, we thus have for all measurable sets $A \subset \mathbb{R}^m$ the inequality

$$|D\bar{u}_\tau|(A) \leq \int_{\mathrm{O}(m)} |Du_\tau|(RA) \mathrm{d}\mu(R). \qquad (21)$$

We next define the function

$$\begin{aligned}&F : BV(\mathbb{R}^m; [0, \infty)) \times \mathcal{M}(\mathbb{R}^m) \times \mathbb{R}_{>0} \to \mathbb{R} \cup \{\infty\}, \\ &F(\xi, \sigma; \tau) := \int_{\mathbb{R}^m} f(\xi(x), \sigma^a(x); \tau) \mathrm{d}x + \sigma^s(\mathbb{R}^m).\end{aligned}$$

Here, $BV(\mathbb{R}^m; [0, \infty))$ is the set of all functions $\xi : \mathbb{R}^m \to [0, \infty)$ with bounded variation and $\mathcal{M}(\mathbb{R}^m)$ denotes the space of finite Radon measures on $\mathbb{R}^m$. Moreover, $\sigma = \sigma^a \mathcal{L}^{(m)} + \sigma^s$ is the Lebesgue decomposition of the measure σ into its absolutely continuous and its singular part.

Defining the measure $\nu_\tau(A) := \int_{\mathrm{O}(m)} |Du_\tau|(RA) \mathrm{d}\mu(R)$, we get with the inequality (21) and the fact that the function f is monotonically increasing in its second argument that

$$F(\xi, |D\bar{u}_\tau|; \tau) \leq F(\xi, \nu_\tau; \tau), \qquad (22)$$

where the inequality is strict if $|D^s\bar{u}_\tau|(\mathbb{R}^m) < \nu_\tau^s(\mathbb{R}^m)$ or

$$\mathcal{L}^{(m)}(\{x \in \mathbb{R}^m : \sqrt{2\tau}|\nabla\bar{u}_\tau(x)| > \xi(x) \text{ and } |\nabla\bar{u}_\tau(x)| < \nu_\tau^a(x)\}) \neq 0. \tag{23}$$

With the estimate (22) and the monotonicity of f in its first argument, we thus find

$$\begin{aligned}\mathcal{F}_{\mathcal{R}}(\tau,\tilde{u};\bar{u}_\tau) &= F(|\bar{u}_\tau - \tilde{u}|, |D\bar{u}_\tau|;\tau) \\ &\leq F(|\bar{u}_\tau - \tilde{u}|, \nu_\tau;\tau) \\ &\leq F\Big(\int_{\mathrm{O}(m)} |u_\tau \circ R - \tilde{u}\circ R| \mathrm{d}\mu(R), \nu_\tau;\tau\Big).\end{aligned} \tag{24}$$

Now, since the function $f(\cdot,\cdot;\tau)$ is convex, we see that the functional $F(\cdot,\cdot;\tau)$ is convex as well. We may therefore apply Jensen's inequality to get

$$\mathcal{F}_{\mathcal{R}}(\tau,\tilde{u};\bar{u}_\tau) \leq \int_{\mathrm{O}(m)} F(|u_\tau\circ R - \tilde{u}\circ R|, |Du_\tau|\circ R;\tau)\mathrm{d}\mu(R),$$

and since $F(\xi\circ R, \sigma\circ R;\tau) = F(\xi,\sigma;\tau)$ for every $R \in \mathrm{O}(m)$, we finally have

$$\mathcal{F}_{\mathcal{R}}(\tau,\tilde{u};\bar{u}_\tau) \leq F(|u_\tau - \tilde{u}|, |Du_\tau|;\tau) = \mathcal{F}_{\mathcal{R}}(\tau,\tilde{u};u_\tau).$$

If we further know that

$$\mathcal{L}^m(\{x\in\mathbb{R}^m \mid \sqrt{2\tau}|\nabla\bar{u}_\tau(x)| \leq |\bar{u}_\tau(x) - \tilde{u}(x)|\}) = 0,$$

then, according to (23), the inequality (24) is strict if $|D\bar{u}_\tau|(\mathbb{R}^m) < \nu_\tau(\mathbb{R}^m)$. It therefore only remains to show that the equality $|D\bar{u}_\tau|(\mathbb{R}^m) = |Du_\tau|(\mathbb{R}^m)$ implies that u_τ is $\mathrm{SO}(m)$-invariant.

So, let $|D\bar{u}_\tau|(\mathbb{R}^m) = |Du_\tau|(\mathbb{R}^m)$. In a first step, we will show that the Radon–Nikodým derivative $\frac{\mathrm{d}Du_\tau}{\mathrm{d}|Du_\tau|}(x)$ is orthogonal to the sphere. Introducing the function

$$\bar{\phi}(x) = \int_{\mathrm{O}(m)} R\phi(R^{-1}x)\mathrm{d}\mu(R),$$

we see from equation (20) that $|D\bar{u}_\tau|(\mathbb{R}^m)$ can be written as

$$|D\bar{u}_\tau|(\mathbb{R}^m) = \sup_{\bar{\phi}\in\mathcal{C}} \int_{\mathbb{R}^m} \bar{\phi}(x)\cdot\mathrm{d}Du_\tau(x),$$

where $\mathcal{C}$ denotes the set of all functions $\bar{\phi} \in C_{\mathrm{c}}^\infty(\mathbb{R}^m;\mathbb{R}^m)$ with $\|\bar{\phi}\|_\infty \leq 1$ and $\bar{\phi}(Rx) = R\bar{\phi}(x)$ for every $x\in\mathbb{R}^m$ and every $R\in\mathrm{O}(m)$.

The symmetry property $\bar{\phi}(Rx) = R\bar{\phi}(x)$ implies that $\bar{\phi}(x) = \epsilon(x)|\phi(x)|\frac{x}{|x|}$, $\epsilon(x) \in \{-1,1\}$, and therefore

$$|D\bar{u}_\tau|(\mathbb{R}^m) \leq \int_{\mathbb{R}^m} \left|\frac{x}{|x|}\cdot\frac{\mathrm{d}Du_\tau}{\mathrm{d}|Du_\tau|}(x)\right| \mathrm{d}|Du_\tau|(x).$$

The condition $|D\bar{u}_\tau|(\mathbb{R}^m) = |Du_\tau|(\mathbb{R}^m)$ thus demands that

$$\left|\frac{x}{|x|}\cdot\frac{\mathrm{d}Du_\tau}{\mathrm{d}|Du_\tau|}(x)\right| = 1 \quad \text{for } |Du_\tau|\text{-almost every } x\in\mathbb{R}^m, \tag{25}$$

that is, $\frac{\mathrm{d}Du_\tau}{\mathrm{d}|Du_\tau|}(x)$ is orthogonal to the sphere.

Now note that for $|Du_\tau|$-almost every $x \in \mathbb{R}^m$ the vector $\frac{\mathrm{d}Du_\tau}{\mathrm{d}|Du_\tau|}(x)$ equals the inner normal to the boundary of the level sets $\Omega_t := \{y \in \mathbb{R}^m : u_\tau(y) \geq t\}$, $u_\tau^-(x) \leq t \leq u_\tau^+(x)$ (cf. [9, Thm. 4.5.9, (17), (25)]). Using the coarea formula, it follows that the same equality holds for $\mathcal{H}^{m-1}$-almost every $x \in \partial\Omega_t$ for almost every $t \in \mathbb{R}$. Consequently, (25) implies that almost all normals to level sets of u_τ point in direction orthogonal to spheres centered at zero. This, however, is only possible, if the boundary of each level set of u_τ is the union of spheres, in other words, if u_τ is constant on each sphere ∂B_r. $\square$

6 Properties of Rotationally Invariant Minimizers

In order to investigate the iterative minimization procedure of Definition 3 for the functional $\mathcal{G}$, we will collect some properties of the minimizers $v_\tau \in \widehat{BV}_c$ of the functional $\mathcal{G}(\tau, \tilde{v}; \cdot)$.

Assumption 9. *$\tilde{v} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and satisfies $\tilde{v} \in \widehat{BV}_c$,* $\mathrm{supp}(\tilde{v}) = [0, R]$*, and it is not locally constant on $(0, R)$, i.e.,*

$$\mathcal{L}^1\big(\{r \in (0,R) : \partial_r \tilde{v}(r) = 0\}\big) = 0 \, .$$

Before we come to the minimizers, let us first mention a few properties of the function f found in the integrand of $\mathcal{G}$. For its partial derivatives, we find directly from its definition (13) that

$$\partial_\xi f(\xi, \eta; \tau) = \begin{cases} \dfrac{\xi}{\eta} \, , & \text{if } \sqrt{2\tau}\,\eta > \xi \, , \\ \sqrt{2\tau} \, , & \text{if } \sqrt{2\tau}\,\eta \leq \xi \, , \end{cases} \tag{26}$$

$$\partial_\eta f(\xi, \eta; \tau) = \begin{cases} \tau - \dfrac{\xi^2}{2\eta^2} \, , & \text{if } \sqrt{2\tau}\,\eta > \xi \, , \\ 0 \, , & \text{if } \sqrt{2\tau}\,\eta \leq \xi \, . \end{cases} \tag{27}$$

Thus, $f(\cdot, \cdot; \tau) \in C^1\big(\mathbb{R}^2_{\geq 0} \setminus \{(0,0)\}\big)$ and we have

$$\partial_\eta f(\xi, \eta; \tau) \in [0, \tau] \quad \text{ for every } (\xi, \eta) \in \mathbb{R}^2_{\geq 0} \setminus \{(0,0)\} \tag{28}$$

and

$$\partial_\eta f(\xi, 0; \tau) = 0 \quad \text{ for every } \xi > 0 \, . \tag{29}$$

In particular, we have the estimates

$$\big\|\partial_\xi f(\cdot, \cdot; \tau)\big\|_\infty \leq \sqrt{2\tau} \quad \text{ and } \quad \big\|\partial_\eta f(\cdot, \cdot; \tau)\big\|_\infty \leq \tau \, .$$

Let now $v_\tau \in \widehat{BV}_c$ be a minimizer of the functional $\mathcal{G}(\tau, \tilde{v}; \cdot)$. We define the sets

$$\begin{aligned} \Omega_\tau &:= \big\{r \in (0,R) : |v_\tau(r) - \tilde{v}(r)| < \sqrt{2\tau}|\partial_r v_\tau(r)|\big\} \, , \\ \Sigma_\tau &:= \big\{r \in (0,R) : |v_\tau(r) - \tilde{v}(r)| \geq \sqrt{2\tau}|\partial_r v_\tau(r)|\big\} \, . \end{aligned} \tag{30}$$

In the sequel we will rewrite the Euler–Lagrange equations for the functional $\mathcal{G}(\tau,\tilde v;\cdot)$ in terms of the function

$$g_\tau\colon (0,R)\to\mathbb{R}\,,$$
$$g_\tau(r) := \begin{cases} \dfrac{v_\tau(r)-\tilde v(r)}{\partial_r v_\tau(r)}, & \text{if } r\in\Omega_\tau\,,\\ \sqrt{2\tau}, & \text{if } r\in\Sigma_\tau\,. \end{cases} \tag{31}$$

To this end, we remark that we can express the partial derivatives of the integrand of $\mathcal{G}$ with the function g_τ only. For $r\in(0,R)$ we namely have the relations

$$\begin{aligned} \partial_\xi f\big(|v_\tau(r)-\tilde v(r)|,|\partial_r v_\tau(r)|;\tau\big) &= |g_\tau(r)|\,,\\ \partial_\eta f\big(|v_\tau(r)-\tilde v(r)|,|\partial_r v_\tau(r)|;\tau\big) &= \tau-\frac{g_\tau^2(r)}{2}\,. \end{aligned} \tag{32}$$

Moreover, from the defintion (31), we see that the function g_τ is bounded. Indeed,

$$g_\tau(r)\in(-\sqrt{2\tau},\sqrt{2\tau}] \qquad \text{for every } r\in(0,R). \tag{33}$$

Lemma 10. *The functions g_τ and v_τ satisfy the following conditions:*

1. *The function g_τ^2 is locally absolutely continuous on $(0,R]$ and solves the differential equation*
$$r\,\partial_r(g_\tau^2)(r) = 2(m-1)\tau - 2r\,g_\tau(r) - (m-1)\,g_\tau^2(r) \tag{34}$$
almost everywhere on Ω_τ.
2. *The function $r\mapsto \operatorname{sgn}\big(\partial_r v_\tau(r)\big)$ is locally constant on Ω_τ.*
3. *The function $r\mapsto g_\tau(r)\operatorname{sgn}\big(v_\tau(r)-\tilde v(r)\big)$ is continuous on Ω_τ.*
4. *The function g_τ satisfies $g_\tau(r)\neq 0$ for every $r\in(0,R)$.*
5. *The function g_τ is locally absolutely continuous on every subset of $(0,R)$ where it is continuous.*

Here, all the properties of the functions are only valid with the right choice of representative. We especially want to stress that therefore Item 3 does not exclude points in Ω_τ where $v_\tau=\tilde v$.

Proof. Since, by assumption, the function $\tilde v$ is not locally constant on its support, we find that

$$\begin{aligned} &\mathcal{L}^1\big(\{r\in(0,R): \partial_r v_\tau(r)=0 \text{ and } v_\tau(r)=\tilde v(r)\}\big)\\ &\qquad = \mathcal{L}^1\big(\{r\in(0,R): \partial_r \tilde v(r)=0 \text{ and } v_\tau(r)=\tilde v(r)\}\big) = 0\,. \end{aligned}$$

Therefore, the condition that the variational derivative of $\mathcal{G}$ in direction of a function $\phi\in C_c^1([0,R))$ at the minimizer v_τ of $\mathcal{G}(\tau,\tilde v;\cdot)$ vanishes reads as

$$\begin{aligned} 0 &= \lim_{t\to 0}\frac{\mathcal{G}(\tau,\tilde v;v_\tau+t\phi)-\mathcal{G}(\tau,\tilde v;v_\tau)}{t}\\ &= \int_0^R \partial_\xi f\big(|v_\tau(r)-\tilde v(r)|,|\partial_r v_\tau(r)|;\tau\big)\operatorname{sgn}\big(v_\tau(r)-\tilde v(r)\big) r^{m-1}\phi(r)\,\mathrm{d}r\\ &\qquad + \int_0^R \partial_\eta f\big(|v_\tau(r)-\tilde v(r)|,|\partial_r v_\tau(r)|;\tau\big)\operatorname{sgn}\big(\partial_r v_\tau(r)\big) r^{m-1}\partial_r\phi(r)\,\mathrm{d}r\,. \end{aligned} \tag{35}$$

We now choose for $0 < r_0 < R$ the function $\phi = \phi_\varepsilon = \chi_{(0,r_0)} * \rho_\varepsilon$, where $\rho_\varepsilon(r) := \varepsilon^{-m}\rho(r/\varepsilon)$ and ρ is some mollifier. We refer to [8] for the basic definition of a mollifier and its properties.

Passing to the limit $\varepsilon \to 0$, it follows that

$$\begin{aligned}\int_0^{r_0} \partial_\xi f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big) r^{m-1}\,\mathrm{d}r \\ = \partial_\eta f\big(|v_\tau(r_0) - \tilde{v}(r_0)|, |\partial_r v_\tau(r_0)|; \tau\big) \operatorname{sgn}\big(\partial_r v_\tau(r_0)\big) r_0^{m-1} \end{aligned} \tag{36}$$

for almost every $r_0 \in (0, R)$.

Since the integrand of the left hand side of (36) is the integral of a bounded function, it follows that the function

$$r \mapsto \partial_\eta f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(\partial_r v_\tau(r)\big) \tag{37}$$

is locally absolutely continuous on $(0, R)$.

Because of the properties (28) and (29) of the function $\partial_\eta f$, the local absolute continuity of the function (37) also implies the local absolute continuity of the function

$$r \mapsto \partial_\eta f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) = \tau - \frac{g_\tau^2(r)}{2}$$

on $(0, R)$. Thus, g_τ^2 is locally absolutely continuous on $(0, R)$, and therefore, in particular, continuous and differentiable almost everywhere on $(0, R)$.

Moreover, since by definition of the set Ω_τ the function $\tau - g_\tau^2/2$ does not become zero on Ω_τ, the continuity of the map (37) implies that the function $r \mapsto \operatorname{sgn}\big(\partial_r v_\tau(r)\big)$ is locally constant on Ω_τ. Therefore, the function $\Omega_\tau \to \big(\sqrt{-2\tau}, \sqrt{2\tau}\big)$,

$$r \mapsto g_\tau(r) \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big) = |g_\tau(r)| \operatorname{sgn}\big(\partial_r v_\tau(r)\big)$$

is continuous on Ω_τ.

Differentiating equation (36), we find that

$$r^{m-1} g_\tau(r) = \partial_r\Big(\tau r^{m-1} - \frac{r^{m-1} g_\tau(r)^2}{2}\Big) \qquad \text{for almost every } r \in \Omega_\tau\,,$$

which evaluates to the differential equation (34). This finishes the proof of the first three items.

To prove Item 4, we note that the right hand side of (34) is strictly positive for

$$\begin{aligned} g_{\tau,-}(r) := -\frac{r}{m-1} - \sqrt{\frac{r^2}{(m-1)^2} + 2\tau} < g_\tau(r) \\ < -\frac{r}{m-1} + \sqrt{\frac{r^2}{(m-1)^2} + 2\tau} =: g_{\tau,+}(r)\,. \end{aligned} \tag{38}$$

Consequently, the derivative of the non-negative function g_τ^2 is strictly positive on a neighbourhood of every point $r_0 \in \Omega_\tau$ with $g_\tau(r_0) = 0$, which is impossible.

If therefore g_τ is continuous on some compact interval $I \subset (0, R)$, then either $g_\tau = \sqrt{g_\tau^2}$ or $g_\tau = -\sqrt{g_\tau^2}$ on I. But since the square root is a locally Lipschitz continuous function on $\mathbb{R}_{>0}$, this implies that g_τ is absolutely continuous on I, proving Item 5. $\square$

Since the function g_τ only captures the absolutely continuous part of the distributional derivative Dv_τ, we would still need to deal with the singular part $D^s v_\tau$. But it turns out that the minimizer v_τ is in fact locally absolutely continuous and so the singular part is zero.

Lemma 11. *The function v_τ is locally absolutely continuous, that is, the singular part of Dv_τ satisfies $|D^s v_\tau|(\mathbb{R}_{>0}) = 0$.*

Proof. Since v_τ is a minimizer of the convex functional $\mathcal{G}(\tau, \tilde{v}; \cdot)$, the one-sided variational derivative

$$\delta_\phi \mathcal{G}(\tau, \tilde{v}; v_\tau) := \lim_{t \to 0^+} \frac{\mathcal{G}(\tau, \tilde{v}; v_\tau + t\phi) - \mathcal{G}(\tau, \tilde{v}; v_\tau)}{t}$$

exists and is non-negative for all variations $\phi \in \widehat{BV}_c$.

Let us now assume that $|D^s v_\tau|\big((0, R)\big) \neq 0$. Then there exists $r_0 \in (0, R)$ such that

$$|D^s v_\tau|\big([r_0 - \varepsilon, r_0]\big) \neq 0 \qquad \text{for every } \varepsilon \in (0, r_0) \ .$$

We further choose for $\varepsilon \in \big(0, \min\{r_0, R - r_0\}\big)$ a function $\phi \in \widehat{BV}_c$ with $\phi|_{[r_0,\infty)} \in C^1\big([r_0, \infty)\big)$ and

$$\begin{aligned} &\phi(r) = 0 && \text{for } r \in \mathbb{R}_{\geq 0} \setminus (r_0 - \varepsilon, r_0 + \varepsilon), \\ &\phi(r) = -D^s v_\tau\big([r_0 - \varepsilon, r]\big) && \text{for } r \in (r_0 - \varepsilon, r_0] \ . \end{aligned}$$

Then in particular $\partial_r \phi(r) = 0$ for $r \notin [r_0, r_0 + \varepsilon]$ and

$$D\phi\big([r_0 - \varepsilon, r]\big) = D^s\phi\big([r_0 - \varepsilon, r]\big) = -D^s v_\tau\big([r_0 - \varepsilon, r]\big) \tag{39}$$

for $r \in [r_0 - \varepsilon, r_0]$. Thus we find that

$$\begin{aligned} &\delta_\phi \mathcal{G}(\tau, \tilde{v}; v_\tau) \\ &\quad = \int_{r_0 - \varepsilon}^{r_0 + \varepsilon} \partial_\xi f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big) r^{m-1} \phi(r) \, \mathrm{d}r \\ &\qquad + \int_{r_0}^{r_0 + \varepsilon} \partial_\eta f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(\partial_r v_\tau(r)\big) \, r^{m-1} \partial_r \phi(r) \, \mathrm{d}r \\ &\qquad\qquad - \tau \int_{[r_0 - \varepsilon, r_0]} r^{m-1} \, \mathrm{d}|D^s \phi|(r) \ . \end{aligned} \tag{40}$$

Now define the function $\psi \in C^1_c([0, R))$ setting $\psi(r) = \phi(r)$ for $r > r_0$ and $\psi(r) = \phi(r_0)$ for $r \leq r_0$. Since v_τ is a minimizer of $\mathcal{G}(\tau, \tilde{v}; \cdot)$, it follows that the Euler–Lagrange equation (35) holds for ψ and therefore

$$\begin{aligned} &\int_{r_0}^{r_0 + \varepsilon} \partial_\eta f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(\partial_r v_\tau(r)\big) \, r^{m-1} \partial_r \phi(r) \, \mathrm{d}r \\ &\quad = -\int_0^{r_0 + \varepsilon} \partial_\xi f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big) r^{m-1} \psi(r) \, \mathrm{d}r \\ &\quad = -\int_{r_0}^{r_0 + \varepsilon} \partial_\xi f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big) r^{m-1} \phi(r) \, \mathrm{d}r \\ &\qquad - \phi(r_0) \int_0^{r_0} \partial_\xi f\big(|v_\tau(r) - \tilde{v}(r)|, |\partial_r v_\tau(r)|; \tau\big) \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big) r^{m-1} \, \mathrm{d}r \ . \end{aligned}$$

Therefore (40) and (36) imply that

$$\begin{aligned}
&\delta_\phi \mathcal{G}(\tau,\tilde v;v_\tau)\\
&\quad= \int_{r_0-\varepsilon}^{r_0} \partial_\xi f\bigl(|v_\tau(r)-\tilde v(r)|,|\partial_r v_\tau(r)|;\tau\bigr)\operatorname{sgn}\bigl(v_\tau(r)-\tilde v(r)\bigr) r^{m-1}\phi(r)\,\mathrm{d}r\\
&\qquad\quad - \partial_\eta f\bigl(|v_\tau(r_0)-\tilde v(r_0)|,|\partial_r v_\tau(r_0)|;\tau\bigr)\operatorname{sgn}\bigl(\partial_r v_\tau(r_0)\bigr)\, r_0^{m-1}\phi(r_0)\\
&\qquad\qquad\qquad - \tau\int_{[r_0-\varepsilon,r_0]} r^{m-1}\,\mathrm{d}|D^{\mathrm{s}}\phi|(r)\,. \qquad (41)
\end{aligned}$$

From (39) we obtain that

$$|D^s\phi|\bigl([r_0-\varepsilon,r_0]\bigr) = |D\phi|\bigl([r_0-\varepsilon,r_0]\bigr) \ge C := \sup\bigl\{|\phi(r)| : r\in[r_0-\varepsilon,r_0]\bigr\}\,.$$

Using the relation (32) between g_τ and the partial derivatives of f, the inequality (41) therefore implies that

$$\begin{aligned}
\delta_\phi \mathcal{G}(\tau,\tilde v;v_\tau) &\le \int_{r_0-\varepsilon}^{r_0} |g_\tau(r)|\,|\phi(r)| r^{m-1}\,\mathrm{d}r + \Bigl(\tau - \frac{g_\tau^2(r_0)}{2}\Bigr) r_0^{m-1}|\phi(r_0)|\\
&\qquad - C\tau(r_0-\varepsilon)^{m-1}\\
&\le C\Bigl(\sqrt{2\tau}\, r_0^{m-1}\varepsilon + \tau\bigl(r_0^{m-1}-(r_0-\varepsilon)^{m-1}\bigr) - \frac{r_0^{m-1}}{2}\, g_\tau^2(r_0)\Bigr)\,,
\end{aligned}$$

which becomes negative for sufficiently small ε. Thus the function v_τ is absolutely continuous on the interval $(0,R)$.

Theorem 4 implies that $v_\tau(r)=0$ for $r>R$. It therefore only remains to show that v_τ is continuous at the point R. So assume that $\lim_{r\to R^-} v_\tau(r) = \Delta \ne 0$. Using the continuity of $\tilde v$, we would then find some $\varepsilon>0$ such that $\bigl(v_\tau(r)-\tilde v(r)\bigr)\operatorname{sgn}(\Delta)>0$ for all $r\in(R-\varepsilon,R)$. The variational derivative in the direction of $\phi = -\operatorname{sgn}(\Delta)\,\chi_{(R-\varepsilon,R)} \in \widehat{BV}_c$ would then be

$$\begin{aligned}
\delta_\phi \mathcal{G}(\tau,\tilde v;v_\tau) = -\int_{R-\varepsilon}^{R} &\partial_\xi f\bigl(|v_\tau(r)-\tilde v(r)|,|\partial_r v_\tau(r)|;\tau\bigr)\, r^{m-1}\,\mathrm{d}r\\
&\qquad - \tau\bigl(R^{m-1}-(R-\varepsilon)^{m-1}\bigr)\,,
\end{aligned}$$

which is negative and thus contradicts the minimality of v_τ. □

These results now allow us to calculate the minimizer v_τ by first solving the differential equation (34) for the function g_τ and then calculating v_τ from the differential equation $g_\tau(r)\partial_r v_\tau(r) = v_\tau(r)-\tilde v(r)$ on Ω_τ. One remaining problem in this approach is that the function g_τ does not need to be continuous, though. Therefore, the solution of the differential equation (34) depends on the position of the discontinuities of g_τ. The next section will thus be dedicated to the behavior of this function.

7 Behavior of a Rotational Invariant Minimizer

Let again $v_\tau \in \widehat{BV}_c$ be a minimizer of the functional $\mathcal{G}(\tau,\tilde v;\cdot)$, where we still make the Assumption 9 for $\tilde v$. To analyze the behavior of $v_\tau \in \widehat{BV}_c$, we will proceed as indicated at the end of the previous section, namely by investigating the function g_τ defined by (31).

We have seen in Lemma 10 that g_τ is locally absolutely continuous on every set where it is continuous. We therefore may solve the differential equation (34) on every interval where we know g_τ to be continuous.

As a first step, we have to derive estimates for the set of discontinuities of g_τ. Lemma 10 states that $g_\tau \operatorname{sgn}(v_\tau - \tilde{v})$ is continuous on the set where $g_\tau^2 \neq 2\tau$. Therefore, g_τ can only have discontinuities at points $r \in (0, R)$ where either $g_\tau^2(r) = 2\tau$ or $v_\tau(r) = \tilde{v}(r)$.

Lemma 12. *Assume that $I \subset [0, R]$ is an interval such that either $v_\tau(r) \geq \tilde{v}(r)$ for all $r \in I$ or $v_\tau(r) \leq \tilde{v}(r)$ for all $r \in I$. Then the set $(\Sigma_\tau \cup \{0\}) \cap I$ consists of at most one element.*

Proof. We perform the proof in the case that $\tilde{v}(r) \leq v_\tau(r)$ for all $r \in I$. The other case can be proven analogously.

Since the minimizer v_τ satisfies the Euler–Lagrange equation (36) and $\partial_\eta f\big(|v_\tau(r_0) - \tilde{v}(r_0)|, |\partial_r v_\tau(r_0)|; \tau\big) = 0$ for $r_0 \in \Sigma_\tau$, we see that the equation

$$\int_0^{r_0} |g_\tau(r)| r^{m-1} \operatorname{sgn}\big(v_\tau(r) - \tilde{v}(r)\big)\, \mathrm{d}r = 0 \tag{42}$$

is a necessary condition for r_0 to be contained in $\Sigma_\tau \cup \{0\}$. By Item 4 in Lemma 10, the function $g_\tau(r)$ is not equal to zero and since $v_\tau(r) = \tilde{v}(r)$ is for $\partial_r v_\tau(r) \neq 0$ only possible if $g_\tau(r) = 0$, we know that $\tilde{v} < v_\tau$ almost everywhere.

Thus, the integrand in (42) is positive almost everywhere on I. Therefore, there exists at most one element $r_0 \in I$ for which (42) holds. □

Lemma 13. *Assume that $I \subset [0, R]$ is an interval such that $\tilde{v}$ is strictly monotonous and absolutely continuous on I. Then there exists at most one point $r_0 \in I$ with $v_\tau(r_0) = \tilde{v}(r_0)$.*

Proof. Assume that $r_0 \in I$ satisfies $v_\tau(r_0) = \tilde{v}(r_0)$. We know from Lemma 12 that the set Σ_τ is countable. Therefore, by the definition of the function g_τ, the minimizer v_τ solves the initial value problem

$$g_\tau(r)\, \partial_r v_\tau(r) = v_\tau(r) - \tilde{v}(r) \text{ for almost every } r \in I\,, \text{ and } v_\tau(r_0) = \tilde{v}(r_0)\,.$$

This differential equation can be solved explicitly, and we find

$$v_\tau(r) = \tilde{v}(r) - \int_{r_0}^{r} \exp\Big(\int_s^r \frac{1}{g_\tau(y)} \mathrm{d}y\Big)\, \partial_r \tilde{v}(s)\, \mathrm{d}s\,.$$

Since $\tilde{v}$ is strictly monotonous on I, this shows that $v_\tau(r) \neq \tilde{v}(r)$ for every $r \in I \setminus \{r_0\}$. □

Before we now analyze the behavior of the absolutely continuous parts of the function g_τ, we formulate a statement concerning the estimation of the solution of an ordinary differential equation.

Lemma 14. *Let $\varepsilon > 0$ and $a < c < b \in \mathbb{R}$. Assume that $F\colon (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous and that y, $z\colon (a, b) \to \mathbb{R}$ are absolutely continuous functions with*

$$\partial_r y(r) \leq F\big(r, y(r)\big) - \varepsilon \text{ and } \partial_r z(r) \geq F\big(r, z(r)\big)\,, \text{ for almost every } r \in (a, b)\,.$$

If $y(c) \leq z(c)$, then $y(r) \leq z(r)$ for all $r \in (c,b)$. Conversely, if $y(c) \geq z(c)$, then $y(r) \geq z(r)$ for all $r \in (a,c)$.

Proof. We first consider the case where $y(c) \leq z(c)$. Let us assume the existence of $r_1 \in (c,b)$ such that $y(r_1) > z(r_1)$ and define

$$r_0 := \max\bigl\{r \in [c, r_1] : y(r) \leq z(r)\bigr\} .$$

Then, in particular, $y(r_0) = z(r_0)$ and $y(r) > z(r)$ for every $r \in (r_0, r_1]$.

Because of the continuity of the functions F, y, and z, there exists $r_2 > r_0$ such that

$$\partial_r y(r) \leq F\bigl(r, y(r_0)\bigr) - \varepsilon/2 \leq \partial_r z(r) \text{ for every } r \in (r_0, r_2) .$$

Consequently, it follows that for every $r \in (r_0, r_2)$

$$y(r) = y(r_0) + \int_{r_0}^{r} \partial_r y(s)\,\mathrm{d}s \leq z(r_0) + \int_{r_0}^{r} \partial_r z(r)\,\mathrm{d}s = z(r) ,$$

which contradicts our choice of the point r_0. Therefore, $y(r) \leq z(r)$ for every $r \in (c,b)$.

In the case where $y(c) \geq z(c)$, we apply the first part of the assertion to the functions $\tilde{y}(r) = -y(-r)$, $\tilde{z}(r) = -z(-r)$, and $\tilde{F}(r,y) = F(-r,-y)$. □

With the help of Lemma 14, we will in the following construct upper and lower bounds for the function g_τ.

Lemma 15. *For every $r \in (0,R)$, we have $\bigl|g_\tau(r)\bigr| \geq g_{\tau,+}(r)$, where $g_{\tau,+}(r)$ is defined as in* (38).

In particular,

$$\bigl|g_\tau(r)\bigr| \geq \frac{(m-1)\,\tau}{r + (m-1)\sqrt{\tau}/\sqrt{2}} \text{ for every } r \in (0,R) . \tag{43}$$

Proof. Since $\mathcal{L}^1(\Sigma_\tau) = 0$ and $(0,R) = \Omega_\tau \cup \Sigma_\tau$, it follows from Lemma 10 that g_τ satisfies for almost all $r \in (0,R)$ the differential equation

$$\partial_r\bigl(g_\tau^2\bigr)(r) = F_\tau\bigl(r, g_\tau(r)\bigr) \text{ where } F_\tau(r,g) = \frac{m-1}{r}\bigl(2\tau - g^2\bigr) - 2g . \tag{44}$$

We remark that (38) implies that $F_\tau(r,g)$ is negative for $g \in \bigl(g_{\tau,+}(r), \sqrt{2\tau}\bigr]$ and positive, in particular, for $g \in \bigl(-\sqrt{2\tau}, g_{\tau,+}(r)\bigr)$.

Let us now assume that $g_\tau(r_0) < g_{\tau,+}(r_0)$ for some $r_0 \in (0,R)$. Since

$$\partial_r\bigl(g_{\tau,+}^2\bigr)(r) < 0 = F_\tau\bigl(r, |g_{\tau,+}(r)|\bigr) \text{ and } \partial_r\bigl(g_\tau^2\bigr)(r) \geq F_\tau\bigl(r, |g_\tau(r)|\bigr) , \tag{45}$$

we may apply Lemma 14 to find that $|g_\tau(r)| \leq g_{\tau,+}(r)$ for all $r \in (0, r_0)$. In particular, the function $|g_\tau|$ is monotonically increasing on $(0, r_0)$. This, together with (45) and the monotonicity of F_τ with respect to the second component gives

$$\partial_r\bigl(g_\tau^2\bigr)(r) \geq \frac{m-1}{r}\bigl(2\tau - g_\tau^2(r_0)\bigr) - 2|g_\tau(r_0)| > 0 \text{ for almost every } r \in (0, r_0) .$$

This inequality, however, would imply that $\lim_{r\to 0} g_\tau^2(r) = -\infty$, which is impossible. □

Lemma 16. *Assume that g_τ is continuous on the interval $[a,b]$ for some a, $b \in (0,R)$, and assume that $g_\tau(a) < 0$. Then $b - a \le \sqrt{2\tau} - |g_\tau(a)|$.*

Proof. Since, by assumption, g_τ is continuous on $[a,b]$, and therefore absolutely continuous by Lemma 10, the differential equation (34) simplifies to

$$\partial_r g_\tau(r) = \frac{m-1}{2r\, g_\tau(r)}\big(2\tau - g_\tau^2(r)\big) - 1 \quad \text{for every } r \in (a,b)\ . \tag{46}$$

Since $g_\tau(a) < 0$ and, by Lemma 10, $g_\tau(r) \neq 0$ for all $r \in (0,R)$, the continuity of g_τ on $[a,b]$ implies that $g_\tau(r) < 0$ for all $r \in [a,b]$. Using that the right hand side of (46) is smaller or equal -1 for $g_\tau(r) < 0$, it follows that

$$\partial_r g_\tau(r) \le -1 \text{ for every } r \in (a,b)\,,$$

and therefore

$$g_\tau(r) \le g_\tau(a) - r + a \text{ for every } r \in (a,b)\ . \tag{47}$$

Since $g_\tau(r) \in \big(-\sqrt{2\tau}, 0\big)$ for every $r \in [a,b]$ (which follows from (33) and the fact that $g_\tau(r)$ is negative), the inequality (47) gives $b - a \le \sqrt{2\tau} + g_\tau(a) = \sqrt{2\tau} - |g_\tau(a)|$. □

Lemma 17. *Assume that g_τ is continuous on the interval $[a,b]$ for some $a, b \in (0,R)$, and satisfies $g_\tau(a) > 0$. Moreover, for $\varepsilon \in (0,1)$ let*

$$g_{\tau,\varepsilon}(r) := \frac{\tau(m-1)}{r(1-\varepsilon)} \text{ for every } r \in (0,R)\ . \tag{48}$$

Then,

$$g_\tau(r) \le g_{\tau,\varepsilon}(r) \text{ for every } r \in \big(a + \sqrt{2\tau}/\varepsilon, b\big)\ . \tag{49}$$

Proof. On

$$\Gamma_\tau := \Big\{ r \in (a,b) : g_\tau(r) \ge g_{\tau,\varepsilon}(r) \Big\}\,,$$

the solution g_τ of the differential equation (34) satisfies

$$\partial_r g_\tau(r) \le -\varepsilon - \frac{\tau\,(m-1)^2}{2r^2\,(1-\varepsilon)} \le -\varepsilon \text{ for every } r \in \Gamma_\tau\ . \tag{50}$$

Let $(c,d) \subset \Gamma_\tau$, then by integration of (50) and using (33), which implies that $g_\tau(r) \in \big(0, \sqrt{2\tau}\big]$, it follows that

$$0 < g_\tau(d) \le g_\tau(c) - \varepsilon(d-c) \le 2\tau - \varepsilon(d-c)\ .$$

Therefore, an interval in Γ_τ can have at most length $\sqrt{2\tau}/\varepsilon$.

We further remark that the function $g_{\tau,\varepsilon}(r)$ satisfies

$$\partial_r (g_{\tau,\varepsilon}^2)(r) > F_\tau\big(r, g_{\tau,\varepsilon}(r)\big) \text{ for every } r \in (0,R)\,,$$

where F_τ is defined as in (44). From (34) we know that $\partial_r(g_\tau^2)(r) = F_\tau\big(r, g_\tau(r)\big)$ for almost every $r \in (a,b)$. Lemma 14 therefore implies that $g_\tau(r) \le g_{\tau,\varepsilon}(r)$ for every $r \in (r_0, b)$ in case $g_\tau(r_0) \le g_{\tau,\varepsilon}(r_0)$ holds for some $r_0 \in (a,b)$. □

8 Proof of Theorem 7

If the initial function $v_0 \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and strictly monotonous on its support $[0, R]$, the results of Section 6 and 7 are sufficient to prove Theorem 7.

We define for $\tau > 0$ the sequence by $v_\tau^{(0)} = v_0$ and

$$v_\tau^{(k)} \in \arg\min\bigl\{\mathcal{G}(\tau, v_\tau^{(k-1)}; v) : v \in \widehat{BV_c}\bigr\}\,, \quad k \in \mathbb{N}\,.$$

The goal is to show that $\lim_{n\to\infty} v_{t/n}^{(n)}(r) = v_0\bigl(\sqrt{r^2 + 2(m-1)t}\bigr)$ for every $(r,t) \in \mathbb{R}^2_{>0}$. That is, it approximates the solution of the rotationally invariant mean curvature equation (11).

Proof (of Theorem 7). Because of Lemma 11, the functions $v_\tau^{(k)}$, $k \in \mathbb{N}$, $\tau > 0$, are absolutely continuous.

We now show by induction that for every $\tau > 0$ all the functions $v_\tau^{(k)}$ are strictly monotonous on $[0, R]$. Thus, let us assume that $v_\tau^{(k-1)}$ is strictly monotonous on $[0, R]$.

Since $v_\tau^{(k-1)}(R) = v_\tau^{(k)}(R) = 0$, Lemma 13 applied to the interval $[0, R]$ implies that $v_\tau^{(k)}(r) \neq v_\tau^{(k-1)}(r)$ for every $r \in [0, R)$. Since the functions $v_\tau^{(k)}$ and $v_\tau^{(k-1)}$ are continuous and pointwise different in $(0, R)$, one of them has to strictly lie above the other, i.e.,

$$\operatorname{sgn}(v_\tau^{(k)} - v_\tau^{(k-1)}) = \text{constant}\,. \tag{51}$$

From Lemma 12 applied to the interval $[0, R]$, it then follows that

$$\Sigma_{k,\tau} := \{r \in (0, R) : \bigl|v_\tau^{(k)}(r) - v_\tau^{(k-1)}(r)\bigr| \geq \sqrt{2\tau}\,\bigl|\partial_r v_\tau^{(k)}(r)\bigr|\} = \emptyset\,.$$

Thus from Lemma 10 (where g_τ is replaced by $g_{k,\tau}$), we get that the function

$$g_{k,\tau}(r) := \frac{v_\tau^{(k)}(r) - v_\tau^{(k-1)}(r)}{\partial_r v_\tau^{(k)}(r)}\,, \quad r \in (0, R)\,, \tag{52}$$

is continuous and different from zero. In particular, the last two properties imply that $g_{k,\tau}$ does not change sign in $(0, R)$. Now, assume that $g_{k,\tau} < 0$ on $(0, R)$, then from Lemma 16, it follows that $R \leq \sqrt{2\tau} - \limsup_{r\to 0}|g_{k,\tau}(r)| = 0$ by Lemma 15. Thus, we have $g_{k,\tau} > 0$ on $(0, R)$.

Therefore, the function

$$r \mapsto \operatorname{sgn}\bigl(\partial_r v_\tau^{(k)}(r)\bigr) = \operatorname{sgn}\bigl(v_\tau^{(k)}(r) - v_\tau^{(k-1)}(r)\bigr)$$

is constant and different from zero on $(0, R)$ (cf. (51)). This shows that $v_\tau^{(k)}$ is strictly monotonous.

Moreover, the functions $g_{\tau,k}$, $k \in \mathbb{N}$, are absolutely continuous on $(0, R)$ and solve the differential equation (34) which is independent of the functions $v_{k,\tau}$. Therefore $g_{\tau,k} = g_\tau$ where g_τ denotes the absolutely continuous solution of (34).

For $\varepsilon > 0$ let $r > r_{\varepsilon,\tau} := \frac{m-1}{2\varepsilon}\sqrt{2\tau}$. Using the positivity of g_τ and (43) and (49) in Lemma 15 and Lemma 17, respectively, it follows that

$$\left|\frac{\tau}{g_\tau(r)} - \frac{r}{m-1}\right| < \frac{r}{m-1}\,\varepsilon\,. \tag{53}$$

We define the strictly increasing function

$$G_{\varepsilon,\tau}\colon (0,R) \to \mathbb{R}\,, \qquad G_{\varepsilon,\tau}(r) = \int_{r_{\varepsilon,\tau}}^{r} \frac{\tau}{g_\tau(y)}\,\mathrm{d}y\ .$$

From the defintion (52) of $g_{k,\tau}$, we find the initial value problem

$$\partial_r v_\tau^{(k)}(r) = \frac{1}{g_\tau(r)}\big(v_\tau^{(k)}(r) - v_\tau^{(k-1)}(r)\big)\,, \qquad v_\tau^{(k)}(R) = 0\,,$$

for the minimizer $v_\tau^{(k)}$. It follows by variation of constants that

$$v_\tau^{(k)}(r) = \int_r^\infty \frac{1}{g_\tau(s)} \exp\Big(-\int_r^s \frac{1}{g_\tau(y)}\,\mathrm{d}y\Big)\, v_\tau^{(k-1)}(s)\,\mathrm{d}s\ .$$

Substituting therein $s = G_{\varepsilon,\tau}^{-1}\big(w_k + G_{\varepsilon,\tau}(r)\big)$, we find

$$v_\tau^{(k)}(r) = \int_0^\infty \frac{1}{\tau} \exp\Big(-\frac{w_k}{\tau}\Big)\, v_\tau^{(k-1)}\big(G_{\varepsilon,\tau}^{-1}\big(w_k + G_{\varepsilon,\tau}(r)\big)\big)\,\mathrm{d}w_k\ .$$

Using this formula iteratively, we finally get

$$v_\tau^{(k)}(r) = \int_{\mathbb{R}^k_{>0}} \frac{1}{\tau^k} \exp\Big(-\frac{1}{\tau}\sum_{j=1}^k w_j\Big)\, v_0\big(G_{\varepsilon,\tau}^{-1}\big(\textstyle\sum_{j=1}^k w_j + G_{\varepsilon,\tau}(r)\big)\big) \displaystyle\prod_{j=1}^k \mathrm{d}w_j\ .$$

Substituting now $z_\ell = \sum_{j=1}^\ell w_j$, $1 \le \ell \le k$, and integrating out the variables z_ℓ with $\ell < k$, we are left with (setting $z = z_k$)

$$v_\tau^{(k)}(r) = \int_0^\infty \frac{z^{k-1}}{\tau^k (k-1)!} \exp\Big(-\frac{z}{\tau}\Big)\, v_0\big(G_{\varepsilon,\tau}^{-1}\big(z + G_{\varepsilon,\tau}(r)\big)\big)\,\mathrm{d}z\ .$$

Choosing $\tau = t/n$ and $\varepsilon = \sqrt[4]{\tau}$, it follows from (53) that

$$\lim_{n\to\infty} G_{\varepsilon,\tau}^{-1}\big(z + G_{\varepsilon,\tau}(r)\big) = \sqrt{2(m-1)z + r^2}$$

locally uniformly for $(z,r) \in \mathbb{R}_{>0} \times \big(0,\infty\big)$. Now we find with Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{(n/\mathrm{e})^n \sqrt{2\pi n}} = 1$$

that for every $r \in \big(r_{\varepsilon,\tau}, \infty\big)$

$$\lim_{n\to\infty} v_{t/n}^{(n)}(r) = \lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} \int_0^\infty \mathrm{e}^{-(n-1)(z-1-\log z)} \mathrm{e}^{1-z} v_0\big(\sqrt{2(m-1)tz + r^2}\big)\,\mathrm{d}z\ .$$

The function $z - 1 - \log z$ in the exponent herein is now positive on $\mathbb{R}_{>0} \setminus \{1\}$ and has the Taylor expansion

$$z - 1 - \log z = \frac{(z-1)^2}{2} + \sum_{\ell=3}^\infty \frac{(1-z)^\ell}{\ell}$$

around its minimum point 1. In the limit $n \to \infty$, only the lowest order term will contribute and so we can write

$$\lim_{n\to\infty} v_{t/n}^{(n)}(r) = \lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} \int_0^\infty \mathrm{e}^{-(n-1)\frac{(z-1)^2}{2}} \mathrm{e}^{1-z} v_0\big(\sqrt{2(m-1)tz + r^2}\big)\,\mathrm{d}z\ .$$

Because the functions $\sqrt{\frac{n}{2\pi}}\mathrm{e}^{-(n-1)\frac{(z-1)^2}{2}}$ converge for $n \to \infty$ to the δ-distribution at the point 1, it follows that

$$\lim_{n\to\infty} v_{t/n}^{(n)}(r) = v_0\big(\sqrt{2(m-1)t + r^2}\big) .$$

This completes the proof. □

9 Conclusion

We have introduced a concept of gradient flows generated by non-convex energy terms and distance measures. As in standard semi-group theory, the flow is defined by iterative minimization of a functional composed by the distance measure and the scaled energy. Since the existence of minimizers, however, is strongly related to convexity, it is necessary to use generalized minimizers instead, which are defined by relaxation.

The main motivation for considering *non-convex flows* are geometric PDEs, which cannot be treated by a standard semi-group theory. We have shown that our theory applies to mean curvature motion with special initial data. From the calculations in this paper it seems that the identity also holds for more complex rotationally invariant data; however, the identity could not be established rigorously. We have demonstrated that our notion of solution can serve as a solution concept of the mean curvature motion for arbitrary integrable initial data, for which, up to now, no solution concept exists.

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