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OPTIMAL CONTROL FOR MULTI-PHASE FLUID STOKES PROBLEMS

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Abstract. Optimal control for a system consistent of the viscosity dependent Stokes equations coupled with a transport equation for the viscosity is studied. Motivated by lack of sufficient regularity of the adjoint equations artificial diffusion is introduced to the transport equation. The asymptotic behavior of the regularized system is investigated. Optimality conditions for the regularized optimal control problems are obtained and again the asymptotic behavior is analyzed. Lack of uniqueness of solutions to the underlying system is another source of difficulties for the problem under investigation.

1. Introduction

The focus of this work is to establish an approach for optimal control multi-phases fluid flow. More specifically we consider the problem

$$\min J(\eta, \mathbf{u}) = \frac{1}{2}\|\eta - \widetilde{\eta}\|^2_{L^2(Q)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{L^2(Q)}, \tag{1.1}$$

subject to

$$\begin{cases} \mathbf{y}_t - \operatorname{div}(\eta(\nabla \mathbf{y})) + \nabla p = \mathbf{u}, \\ \operatorname{div} \mathbf{y} = 0, \quad \mathbf{y}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{y}|_{t=0} = \mathbf{y}_0, \\ \eta_t + \mathbf{y} \cdot \nabla \eta = 0, \\ \eta|_{t=0} = \eta_0. \end{cases} \tag{1.2}$$

Let us describe the various terms in this problem formulation. Here Ω is a bounded domain in $\mathbb{R}^2$, $T > 0$ and $Q = (0, T) \times \Omega$. The spatio-temperally dependent vector field $\mathbf{y}$ presents the velocity of the fluid, p its pressure, and η is the nonconstant viscosity of the fluid. Further $\mathbf{y}_0$ and η_0 are the initial velocity and viscosity respectively. The control variable is denoted by $\mathbf{u}$. The control problem consists in finding $\mathbf{u}$ such that the corresponding state-control vector $(\mathbf{y}, \eta, p, \mathbf{u})$ minimizes $J(\eta, \mathbf{u})$, where $\tilde{\eta}$ is given and fixed.

If $\tilde{\eta}$ is chosen as $\eta_1 + (\eta_2 - \eta_1)\chi_{\hat{\Omega}}$ where $\chi_{\hat{\Omega}}$ is the characteristic function of a set $\hat{\Omega} \subset \Omega$, then (1.1), (1.2) represents the problem of determining a control $\mathbf{u}$ such that the interface between the two fluids with viscosities η_1 and η_2 coincides with the boundary $\partial\hat{\Omega}$ of $\hat{\Omega}$.

One of the key issues in optimal control with partial equations as constraints consists in establishing existence and first order necessary optimality conditions,

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which are expressed in the form of optimality systems. Here we shall establish existence by means of a compensated compactness argument. To obtain an optimality system, one can rely on a Lagrangian formalism, for example. Proceeding formally we introduce adjoint variables for the velocity, the pressure and the viscosity and denote them by $(\mathbf{z}, q, \xi)$, and denote by $e_i(\mathbf{y}, p, \eta) = 0$, $i = 1, 2, 3$ the momentum, the mass, and the transport equations respectively. Defining the formal Lagrangian

$$\mathcal{L}(\mathbf{y}, p, \eta, \mathbf{u}; \mathbf{z}, q, \xi) = J(\eta, \mathbf{u}) + \langle \mathbf{z}, e_1(\mathbf{y}, p, \eta) \rangle + \langle q, e_2(\mathbf{y}, p, \eta) \rangle + \langle \xi, e_3(\mathbf{y}, p, \eta) \rangle,$$

and setting the first derivatives with respect to $(\mathbf{y}, p, \eta, \mathbf{u})$ equal to zero, we obtain formal adjoint equations

$$\begin{cases} -\mathbf{z}_t - \operatorname{div}(\eta \nabla \mathbf{z}) - \eta \nabla \xi + \nabla q = \mathbf{0}, \\ \operatorname{div} \mathbf{z} = 0, \quad \mathbf{z}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{z}|_{t=T} = \mathbf{0}, \\ -\xi_t + \mathbf{y} \cdot \nabla \xi + \nabla \mathbf{y} : \nabla \mathbf{z} = -(\eta - \widetilde{\eta}), \\ \xi|_{t=T} = 0, \end{cases} \tag{1.3}$$

where $\nabla \mathbf{y} : \nabla \mathbf{z}$ is the matrix inner product of Frobenius type, and the optimality condition:

$$\alpha \mathbf{u} = \mathbf{z}. \tag{1.4}$$

Combining the primal equations (1.2), the adjoint equations (1.3) and the optimality condition (1.4) provides the formal optimality system. These equations are indeed only formal since the transport equations in (1.2) and (1.4) have no smoothing properties. Hence $\mathbf{z}$ is strictly less regular in space than H^1 and ξ is strictly less regular in space than L^1. The bilinear coupling in (1.3) is the source of significant difficulties in analyzing this equation.

This lack of regularity of the adjoint equations motivates the introduction of a smoothing step. In the present work, we introduce artificial diffusion to the transport equation, which results in

$$\eta_t^\epsilon - \epsilon \triangle \eta^\epsilon + \mathbf{y}^\epsilon \cdot \nabla \eta^\epsilon = 0, \tag{1.5}$$

and investigate the optimal control problems for the regularized system.

Let us briefly outline the contents of the paper. In Section 2 we gather technical result which will be used throughout the remainder of the paper. The experienced reader can proceed directly to Section 3 where the regularized primal problems are investigated. The existence of solutions for each $\epsilon > 0$ is shown by means of Schauder's fixed point theorem. It is further shown that as $\epsilon \to 0^+$ limit points of regularized solutions satisfy (1.2), where the solution concept for the transport equations is that of regularized solutions in the sense of DiPerna-Lions. In Sections 4 and 5 two optimal control formulations, with controls in $L^2(Q)$ the other with controls $L^2(0, T; V^*)$ are studied. In each case optimality systems are rigorously derived and convergence of the optimal control problems as $\epsilon \to 0^+$ is investigated. Lack of uniqueness of solutions to (1.1) significantly complicates this analysis.

The invesigations of this paper can certainly be extended in several aspects. Similar results as presented here should also hold true if the Stokes equations are replaced by the Navier Stokes equations with the nonconstant density function. More involved cost-functionals, and cost functionals involving the velocity $\mathbf{y}$ can be treated by the same techniques as used in this paper.

Finally let us give only a few comments on the multi-phase fluid model that is used in this paper. If $\eta_0 \in \{\eta_1, \dots \eta_L\}$, with η_i constants strictly larger than zero, then the renormalized solution $\eta(t,x) \in \{\eta_1, \dots \eta_L\}$ as well, see e.g. [11, 12]. The transport equation in (1.2) is a variational formulation, posed on all of the domain Ω, of the immiscibility condition along the interfaces occupied by fluids with different viscosity, as proved in [11], Lemma 2.3. Of course, once the regularization is introduced the solution η_ϵ will not satisfy $\eta(t,x) \in \{\eta_1, \dots \eta_L\}$, but rather mushy regions will arise. In [1] an improved model is investigated, which allows for shear rate dependent viscosities and which takes into consideration surface tension along the interfaces of different fluids. A different analytic framework for (1.2) is based on viscosity solutions. Global existence is shown in [8] under the assumption that the difference of the viscosities of two different fluids is sufficiently small. Finally global existence to multiphase viscous flow is also investigated in [5], again under the condition that the viscosities of the fluids in different phases do not differ too much.

2. Preliminaries and Notations

Let Ω be an open bounded domain in R^2 with Lipschitz boundary. We use standard notation $W^{m,p}$ and H^m for the Sobolev space, and we simplify the norm of H^m as $\|f\|_m = \|f\|_{H^m}$.

We will repeatedly use the following inequalities. The generic constant C does not depends on the choice of u.

- Poincaré inequality: For any $u \in H_0^1$ or $u \in H^1 \cap L_0^2$, we have
$$\|u\| \leq C\|\nabla u\|.$$
- Hölder inequality:
$$\int_\Omega |fgh| \leq \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}$$
where p, q, $r > 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.
- Sobolev inequality:
$$\|u\|_{L^4} \leq C\|u\|^{1/2}\|u\|_1^{1/2},$$
$$\|u\|_{L^p} \leq C\|u\|_1, \quad \text{for any } 1 \leq p < \infty,$$
where $\Omega \in R^2$;
- Gronwall's inequality:
Let $y(t)$ be a nonnegative, absolutely continuous function in $[0,t]$ and satisfy for almost every t, the differential inequality:
$$y'(t) \leq a(t)y(t) + b(t), \tag{2.1}$$
where $a(t)$ and $b(t)$ are nonnegative, summable functions in $[0,t]$. Then we have:
$$y(t) \leq e^{\int_0^t a(s)ds}\left[y(0) + \int_0^t b(s)ds\right]. \tag{2.2}$$
- Aubin's Lemma: (c.f. [2])
Let X_0, X_1, X_2 be Banach spaces such that
$$X_0 \subset X_1 \subset X_2, \quad X_i \text{ is reflexive for } i = 0, 1,$$

and the injection of X_0 into X_1 is compact. Let $1 < p_i < \infty$, $i = 0, 1$. Then the space
$$W = L^{p_0}(X_0) \cap W^{1,p_1}(X_2)$$
is compactly imbedded in $L^{p_0}(X_1)$.

For the Stokes equation, the following divergence free spaces are useful.
$$\begin{aligned}
&\mathcal{V} = \{\mathbf{u} \in C_0^\infty(\Omega), \operatorname{div} \mathbf{u} = 0\},\\
&\mathbf{H} = \overline{\mathcal{V}}^{\mathbf{L}^2} = \{\mathbf{u} \in \mathbf{L}^2, \operatorname{div} \mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n} = 0\},\\
&\mathbf{V} = \overline{\mathcal{V}}^{\mathbf{H}_0^1} = \{\mathbf{u} \in \mathbf{H}_0^1, \operatorname{div} \mathbf{u} = 0\},
\end{aligned}$$
where $\mathbf{H}$ and $\mathbf{V}$ are equipped with the norm induced by $\mathbf{L}^2$ and $\mathbf{H}_0^1$. We identify the dual space of $\mathbf{H}$ as itself, and define the dual space of $\mathbf{V}$ as $\mathbf{V}^*$. We also introduce the projection operator P from $\mathbf{L}^2$ to its divergence free subspace $\mathbf{H}$. By the Helmholtz-Hodge decomposition theorem (c.f. [14]), we have:
$$\mathbf{L}^2 = \mathbf{H} \oplus \nabla H^1.$$

Now we introduce time dependent function spaces. For any function $f : [0, T] \to B$, where B is a given Banach space, we denote $f \in L^p(B)$ if
$$\begin{cases} \int_0^T \|f(t)\|_B^p dt < \infty, & 1 \le p < \infty,\\ \operatorname{ess\,sup}_{0\le t\le T} \|f(t)\|_B < \infty, & p = \infty. \end{cases}$$
The norm is defined by $\|f\|_{L^p(B)} = (\int_0^T \|f(t)\|_B^p)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_{L^\infty(B)} = \operatorname{ess\,sup}_{0\le t\le T} \|f(t)\|_B$. If $f(0) \in B$ and $\frac{\partial f}{\partial t} \in L^2(B)$, we denote it by $f \in H^1(B)$. We will use Q to represent the time-space domain, i.e.
$$Q = (0, T) \times \Omega.$$
For time dependent test functions, we denote:
$$\mathcal{V}_T = C^1([0, T], \mathcal{V}), \quad C_T = C^1([0, T], C_0^\infty(\Omega)).$$
For any function $f \in C_T$ or $\mathbf{f} \in \mathcal{V}_T$, we can define a operator γ_0 by taking the initial data: $\gamma_0(f) = f(0)$ or $\gamma_0(\mathbf{f}) = \mathbf{f}(0)$. We will use this operator in section 4 and 5.

The Stokes operator is defined as $\Lambda : \mathbf{V} \to \mathbf{V}^*$, $\Lambda\mathbf{y} = -\triangle\mathbf{y}$. If the domain is $\mathbf{H}^2 \cap \mathbf{V}$ then the Stokes operator Λ is defined from $\mathbf{H}^2 \cap \mathbf{V}$ to $\mathbf{H}$ by $\Lambda\mathbf{y} = -P\triangle\mathbf{y}$. We will not distinct the above two operators and denote them both by Λ. One can verify the Stokes operator has the following properties (see [14]):

- Λ is positive, self-adjoint operator,
- Λ is bijective from $\mathbf{V}$ to $\mathbf{V}^*$, hence it is an isomorphism from $\mathbf{V}$ to $\mathbf{V}^*$,
- $\|\mathbf{y}\|_1 \approx \|\Lambda^{1/2}\mathbf{y}\| \approx \|\Lambda\mathbf{y}\|_{\mathbf{V}^*}$, for all $\mathbf{y} \in \mathbf{V}$,
- $\|\mathbf{u}\|_{\mathbf{V}^*} \approx \|\Lambda^{-1/2}\mathbf{u}\| \approx \|\Lambda^{-1}\mathbf{u}\|_1$, for all $\mathbf{u} \in \mathbf{V}^*$,
- Λ is bijective from $\mathbf{H}^2 \cap \mathbf{V}$ to $\mathbf{H}$, hence it is an isomorphism from $\mathbf{H}^2 \cap \mathbf{V}$ to $\mathbf{H}$,
- $\|\mathbf{y}\|_2 \approx \|\Lambda^{1/2}\mathbf{y}\|_1 \approx \|\Lambda\mathbf{y}\|$, for all $\mathbf{y} \in \mathbf{H}^2 \cap \mathbf{V}$,
- $\|\mathbf{u}\| \approx \|\Lambda^{-1/2}\mathbf{u}\|_1 \approx \|\Lambda^{-1}\mathbf{u}\|_2$, for all $\mathbf{u} \in \mathbf{H}$,

where $\approx$ means that the norms are equivalent.

We denote two viscosities as m and M, and initial viscosity η_0 satisfies $0 < m \le \eta_0(\mathbf{x}) \le M < \infty$. The generic constants C and C_i only depend on Ω, m, M, T, the initial velocity $\mathbf{y}_0$, the initial viscosity η_0 and extenal force. C may be different in different cases, and C_i can be fixed in advance. C_ϵ is also constant but may depend on the choice of ϵ.

3. Existence and Convergence for the Approximated System

As we mentioned in the introduction part, the governing equation for the multi-phases immiscible incompressible fluid reads:

$$\mathbf{y}_t - \mathrm{div}(\eta \nabla \mathbf{y}) + \nabla p = \mathbf{u}, \tag{3.1}$$

$$\mathrm{div}\, \mathbf{y} = 0, \tag{3.2}$$

$$\mathbf{y}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{y}|_{t=0} = \mathbf{y}_0, \tag{3.3}$$

$$\eta_t + \mathbf{y} \cdot \nabla \eta = 0, \tag{3.4}$$

$$\eta|_{t=0} = \eta_0. \tag{3.5}$$

Where $\mathbf{y}$ presents the velocity of the fluid, η is the viscosity of the fluid, $\mathbf{y}_0$ and η_0 are the initial velocity and viscosity respectively. To avoid the pressure term, we can put the system into the weak form: given $\mathbf{u} \in L^2(\mathbf{V}^*)$ and $\mathbf{y}_0 \in \mathbf{H}$, find $(\eta, \mathbf{y}) \in L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*)$, such that

$$\mathbf{y}_t - \mathrm{div}(\eta \nabla \mathbf{y}) = \mathbf{u}, \quad \text{in } L^2(\mathbf{V}^*),$$
$$\eta_t + \mathbf{y} \cdot \nabla \eta = 0, \quad \text{in } L^2(H^{-1}),$$

with the initial conditions (3.3) and (3.5). The existence of a solution can be found in [11, 12].

We take a singular perturbation to the system (3.1) - (3.5) and arrive at the following approximating system:

$$\mathbf{y}^\epsilon_t - \mathrm{div}(\eta^\epsilon (\nabla \mathbf{y}^\epsilon)) + \nabla p^\epsilon = \mathbf{u}, \tag{3.6}$$

$$\mathrm{div}\, \mathbf{y}^\epsilon = 0, \tag{3.7}$$

$$\mathbf{y}^\epsilon|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{y}^\epsilon|_{t=0} = \mathbf{y}_0, \tag{3.8}$$

$$\eta^\epsilon_t - \epsilon \triangle \eta^\epsilon + \mathbf{y}^\epsilon \cdot \nabla \eta^\epsilon = 0, \tag{3.9}$$

$$\eta^\epsilon|_{t=0} = \eta^\epsilon_0, \quad \eta^\epsilon|_{\partial\Omega} = m, \tag{3.10}$$

where ϵ is a positive constant and η^ϵ_0 is an approximation of η_0 which satisfies

$$m \le \eta^\epsilon_0 \le M, a.e. \quad \eta^\epsilon_0 \to \eta_0 \text{ in } L^2(Q) \text{ as } \epsilon \to 0^+. \tag{3.11}$$

To avoid the pressure term, we can consider the following equivalent approximating system. Given $\mathbf{u} \in L^2(\mathbf{V}^*)$ and $\mathbf{y}_0 \in \mathbf{H}$, find $(\eta^\epsilon, \mathbf{y}^\epsilon) \in L^2(\mathbf{H}^1) \cap L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap L^\infty(\mathbf{H}) \cap L^2(\mathbf{V}^*)$, such that

$$\mathbf{y}^\epsilon_t - \mathrm{div}(\eta^\epsilon (\nabla \mathbf{y}^\epsilon)) = \mathbf{u}, \quad \text{in } L^2(\mathbf{V}^*), \tag{3.12}$$

$$\eta^\epsilon_t - \epsilon \triangle \eta^\epsilon + \mathbf{y}^\epsilon \cdot \nabla \eta^\epsilon = 0, \quad \text{in } L^2(H^{-1}), \tag{3.13}$$

with the initial conditions and boundary conditions (3.8) and (3.10). Moreover, if $\mathbf{u} \in L^2(\mathbf{H})$, $\mathbf{y}_0 \in \mathbf{V}$ and $\eta^\epsilon_0 - m \in H^1_0$, we find $(\eta^\epsilon, \mathbf{y}^\epsilon) \in L^2(\mathbf{H}^2) \cap L^\infty(Q) \cap L^\infty(H^1) \cap H^1(L^2) \times L^2(\mathbf{H}^2) \cap L^\infty(\mathbf{V}) \cap L^2(\mathbf{H})$, such that,

$$\mathbf{y}^\epsilon_t - P(\mathrm{div}(\eta^\epsilon \nabla \mathbf{y}^\epsilon)) = \mathbf{u}, \quad \text{in } L^2(\mathbf{H}), \tag{3.14}$$

$$\eta^\epsilon_t - \epsilon \triangle \eta^\epsilon + \mathbf{y}^\epsilon \cdot \nabla \eta^\epsilon = 0, \quad \text{in } L^2(Q), \tag{3.15}$$

with the initial conditions and boundary conditions (3.8) and (3.10). The equivalence of (3.6) - (3.10) to (3.12) - (3.13) and (3.14) - (3.15) is based on Helmholtz-Hodge decomposition theorem (c.f. [14]). For simplicity, we will not distinct the

PDE form with its variational form in the paper, i.e. the pressure is omitted in the statement and the proof.

During this section, $\mathbf{u}$ is fixed as a given function in $L^2(\mathbf{V}^*)$. We have the following existence and convergence results.

Theorem 3.1 (Existence for fixed ϵ)**.** *For any given positive constant ϵ, $\mathbf{u} \in L^2(\mathbf{V}^*)$, $\mathbf{y}_0 \in \mathbf{H}$, $\eta_0^\epsilon \in L^2$ with $m \le \eta_0^\epsilon \le M$ a.e., system (3.6) - (3.10) has at least one solution which satisfies $(\eta^\epsilon, \mathbf{y}^\epsilon) \in L^2(\mathbf{H}^1) \cap L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap L^\infty(\mathbf{H}) \cap L^2(\mathbf{V}^*)$.*

The proof for existence is based on fixed point argument, and is given in subsection 3.1 - 3.3.

Theorem 3.2 (Convergence for $\epsilon \to 0^+$)**.** *Assume that $\mathbf{u} \in L^2(\mathbf{V}^*)$, $\mathbf{y}_0 \in \mathbf{H}$, and η_0^ϵ satisfies (3.11) as $\epsilon \to 0^+$. Let $(\eta^\epsilon, \mathbf{y}^\epsilon) \in L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*)$ be any solution of (3.6) - (3.10). Then there exists a sequence $\epsilon^n \to 0^+$ such that $\mathbf{y}^n \to \mathbf{y}$ in $\mathbf{L}^2(Q)$, $\eta^n \to \eta$ in $L^2(Q)$ and $(\eta, \mathbf{y}) \in L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*)$ satisfies the equations (3.1) - (3.5), and η is renormalized solution.*

The definition and property of renormalized solution can be found in [11]. The proof of convergence is given in subsection 3.4.

3.1. **A-priori Estimate for the Stokes Equation.** We fix η as a measurable function satisfying $m \le \eta \le M a.e.$ For the Stokes equation

$$\mathbf{y}_t - \operatorname{div}(\eta \nabla \mathbf{y}) + \nabla p = \mathbf{u}, \tag{3.16}$$

$$\operatorname{div} \mathbf{y} = 0, \tag{3.17}$$

$$\mathbf{y}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{y}|_{t=0} = \mathbf{y}_0, \tag{3.18}$$

existence and uniqueness of a weak solution can be obtained by standard argument similar to the constant viscosity case, and we have the a-priori estimate

$$\sup_{0 \le t \le T} \|\mathbf{y}(t,\cdot)\|^2 + \int_0^T \|\mathbf{y}\|_1^2 \, dt \le \|\mathbf{y}_0\| + C \int_0^T \|\mathbf{u}\|_{\mathbf{V}^*}^2 \, dt. \tag{3.19}$$

Then $\mathbf{y}_t$ can be estimated as:

$$\|\mathbf{y}_t\|_{\mathbf{V}^*} = \sup_{\mathbf{v} \in \mathbf{V}} \frac{\langle \mathbf{y}_t, \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|} = \sup_{\mathbf{v} \in \mathbf{V}} \frac{\langle \mathbf{u}, \mathbf{v} \rangle - (\eta \nabla \mathbf{y}, \nabla \mathbf{v})}{\|\nabla \mathbf{v}\|} \le \|\mathbf{u}\|_{\mathbf{V}^*} + M \|\nabla \mathbf{y}\|, \tag{3.20}$$

and by virtue of (3.19), we have

$$\|\mathbf{y}_t\|_{L^2(\mathbf{V}^*)} \le C \|\mathbf{f}\|_{L^2(\mathbf{V}^*)}. \tag{3.21}$$

We notice that the estimates (3.19) and (3.21) only depend on $\mathbf{u}$, $\mathbf{y}_0$, Ω, m and M. Hence there exists a constant C_1 such that:

$$\|\mathbf{y}\|_{L^\infty(\mathbf{H})} + \|\mathbf{y}\|_{L^2(\mathbf{V})} + \|\mathbf{y}_t\|_{L^2(\mathbf{V}^*)} \le C_1. \tag{3.22}$$

To avoid the pressure term in the Stokes equation (3.16) - (3.18) and make the proof simple, we can rewrite the Stokes equation in the following equivalent form: find $\mathbf{y} \in L^2(\mathbf{V}) \cap L^\infty(\mathbf{H}) \cap H^1(\mathbf{V}^*)$ which satisfies

$$\mathbf{y}_t - \operatorname{div}(\eta(\nabla \mathbf{y})) = \mathbf{f}, \text{ in } \mathcal{V}_T^*$$
$$\mathbf{y}|_{t=0} = \mathbf{y}_0.$$

3.2. **A-priori Estimate for the Convection-diffusion Equation.** For $\epsilon > 0$ as a fixed positive constant, and given $\mathbf{y}$ which satisfies (3.22), we consider the following convection-diffusion equation:

$$\eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta = 0, \tag{3.23}$$

$$\eta|_{t=0} = \eta_0^\epsilon, \quad \eta|_{\partial\Omega} = m. \tag{3.24}$$

Existence and uniqueness of the weak solution can be obtained by standard arguments, and we have the following a priori estimate:

$$\|\eta(t,\cdot) - m\|^2 + 2\epsilon\int_0^t \|\nabla\eta\|^2 = \|\eta_0^\epsilon - m\|^2. \tag{3.25}$$

In fact, shifting η by a constant function m, we find that $\bar{\eta} = \eta - m$ satisfies the same parabolic equation (3.23) with initial condition $\eta_0^\epsilon - m$ and homogenous Dirichlet boundary condition. Multiplying the resulting equation (3.23) by $\bar{\eta}$ gives (3.25).

By virtue the maximum principle for parabolic equations, we have

$$m \le \eta(t,\mathbf{x}) \le M. \tag{3.26}$$

The maximum principle we used here is Theorem 7.2 in [9] pp. 188. We need to check that the coefficient $\mathbf{y} = (y_1, y_2)$ satisfies $y_i^2 \in L^2(Q)$, i.e. $r = q = 2$ in that theorem. From Sobolev's inequality, we have

$$\int_0^T\int_\Omega y_i^4 dxdt \le \int_0^T \|\mathbf{y}\|_{L^4}^4 dt \le C\int_0^T \|\mathbf{y}\|^2\|\mathbf{y}\|_1^2 dt,$$

and since $\mathbf{y} \in L^2(\mathbf{V}) \cap L^\infty(\mathbf{H})$, we obtain $y_i^2 \in L^2(L^2)$.

We proceed to estimate the time derivatives:

$$\begin{aligned}\|\eta_t\|_{H^{-1}} &= \sup_{v\in H_0^1}\frac{\langle\eta_t, v\rangle}{\|\nabla v\|} = \sup_{v\in H_0^1}\frac{\langle\epsilon\triangle\eta - \mathbf{y}\cdot\nabla\eta, v\rangle}{\|\nabla v\|}\\ &= \sup_{v\in H_0^1}\frac{-\epsilon(\nabla\eta,\nabla v) + (\mathbf{y}\eta,\nabla v)}{\|\nabla v\|} = \|\mathbf{y}\eta - \epsilon\nabla\eta\|.\end{aligned}$$

With the help of (3.25), (3.26) and since $\mathbf{y} \in L^2(\mathbf{V})$, we obtain

$$\|\eta\|_{L^2(H^{-1})} \le C. \tag{3.27}$$

Since the constants in (3.25) and (3.27) only depend on m, M and C_1, we can define two constants C_2 and $C_{2,\epsilon}$ such that

$$\|\eta\|_{L^2(H^1)} + \|\eta_t\|_{L^2(H^{-1})} \le C_2, \tag{3.28}$$

$$\|\eta\|_{L^2(H^1)} + \|\eta_t\|_{L^2(H^{-1})} \le C_{2,\epsilon}, \tag{3.29}$$

Similar to the Stokes equation, we can also rewrite equation (3.23) - (3.24) in the following equivalent way: find $\eta \in L^2(H^1) \cap H^1(H^{-1})$, such that

$$\begin{aligned}&\eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta = 0, \text{ in } C_T^*,\\ &\eta|_{t=0} = \eta_0, \quad \eta|_{\partial\Omega} = m.\end{aligned}$$

3.3. **Proof to Theorem 3.1.** We will prove existence for the approximating system (3.6) - (3.10) by Schauder's fixed point theorem (c.f. [3]). Since ϵ is fixed in Theorem 3.1, the notation η and $\mathbf{y}$ without subscript ϵ are used in the proof for simplicity of notation. We define two Banach spaces as

$$E_1 = L^2(H^1) \cap H^1(H^{-1}), \quad \mathbf{E}_2 = L^\infty(\mathbf{H}) \cap L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*). \tag{3.30}$$

Let $K_1 \subset E_1$ and $\mathbf{K}_2 \subset \mathbf{E}_2$ be given by:

$$K_1 = \{\eta : \|\eta\|_{E_1} \le C_{2,\epsilon}\ , m \le \eta \le M\ a.e.\}$$
$$\mathbf{K}_2 = \{\mathbf{y} : \|\mathbf{y}\|_{\mathbf{E}_2} \le C_1\},$$

where C_1 and $C_{2,\epsilon}$ are defined in estimates (3.22) and (3.29). For any $\eta \in K_1$, the estimate (3.22) guarantees the existence of a velocity $\mathbf{y} \in \mathbf{K}_2$ to the Stokes equation (3.16) - (3.18). Similarly for any $\mathbf{y} \in \mathbf{K}_2$, the estimates (3.26) and (3.29) imply that the convection-diffusion equation (3.23) - (3.24) can be solved for a new viscosity $\xi \in K_1$. This combined mapping is defined as $\tau(\eta) = \xi$ and we note that τ maps K_1 into itself. Theorem 3.1 follows directly from the following Lemma.

Lemma 3.3. *Let the same assumption hold in Theorem 3.1. Then the map τ defined as above is well-defined and has at least one fixed point in K_1.*

Proof: Clearly τ is well-defined. We will prove existence for fixed point. First we notice that K_1 is a closed, bounded and convex set in E_1 and $\mathbf{K}_2$ is also a closed bounded set in $\mathbf{E}_2$. To apply Schauder's fixed point theorem, we need to prove that τ is continuous and compact. While proving these facts we always use the notation $\eta \to \mathbf{y} \to \xi$ (with or without subscript n). The proof is based on the following claims:

1. If $\{\eta^n\} \subset K_1$ is a weak convergent sequence in E_1 with limit η, then there exists a subsequence $\{\mathbf{y}^{n_j}\} \subset \mathbf{K}_2$ and $\mathbf{y} \in \mathbf{K}_2$ such that $\mathbf{y}^{n_j} \rightharpoonup \mathbf{y}$ weak-star in $\mathbf{E}_2$.
2. If $\{\mathbf{y}^n\} \subset \mathbf{K}_2$ is a weak star convergent sequence in $\mathbf{E}_2$ with limit $\mathbf{y}$, then there exists a subsequence $\{\xi^{n_j}\} \subset K_1$ and $\xi \in K_1$ such that $\xi^{n_j} \to \xi$ strongly in E_1.
3. Consider a sequence $\{a^n\}$ in a Banach space B. If for any subsequence of $\{a^n\}$ (denoted by $\{a^{n_i}\}$), we can pick up a sub-subsequence $\{a^{n_{i_j}}\}$ such that $\{a^{n_{i_j}}\}$ converges to $a \in B$ in the strong or the weak or the weak-star topology, then $\{a^n\}$ converges to a in the same topology.

Proof to claim 1. Since $m \le \eta^n \le M, a.e.$, estimate (3.22) implies that $\{\mathbf{y}^n\} \subset \mathbf{K}_2$. Hence we can choose a subsequence $\{\mathbf{y}^{n_i}\}$ (still denoted by $\{\mathbf{y}^n\}$ for simplicity) and $\mathbf{z} \in \mathbf{K}_2$, such that

$$\mathbf{y}^n \rightharpoonup \mathbf{z} \quad \text{in } L^2(\mathbf{V}),$$
$$\mathbf{y}^n \rightharpoonup \mathbf{z} \quad \text{in } L^\infty(\mathbf{H}) \text{ in the weak-star topology},$$
$$\mathbf{y}^n_t \rightharpoonup \mathbf{z}_t \quad \text{in } L^2(\mathbf{V}^*).$$

Since $\{\eta^n\} \subset K_1$, Aubin's Lemma implies that there exists a subsequence (still denoted by η^n) which converges to η strongly in $L^2(Q)$. Then strong convergence of η^n in $L^2(Q)$ and weak convergence of $\nabla\mathbf{y}^n$ in $L^2(Q)$ imply convergence of $\eta^n\nabla\mathbf{y}^n$ in the distribution sense. By definition, $\mathbf{y}^n$ solves the Stokes equation

$$\mathbf{y}^n_t - \mathrm{div}(\eta^n(\nabla\mathbf{y}^n)) = \mathbf{f} \quad \text{in } \mathcal{V}^*_T.$$

After passage to the limit, for any test function $\mathbf{v} \in \mathcal{V}_T$, we have

$$\begin{aligned}
&\langle \mathbf{y}_t^n, \mathbf{v}\rangle_Q \to \langle \mathbf{x}_t, \mathbf{v}\rangle_Q,\\
&\langle \mathrm{div}(\eta^n(\nabla \mathbf{y}^n)), \mathbf{v}\rangle_Q = -\langle \eta^n(\nabla \mathbf{y}^n), \nabla \mathbf{v}\rangle_Q\\
&\to -\langle \eta(\nabla \mathbf{x}), \nabla \mathbf{v}\rangle_Q = \langle \mathrm{div}(\eta(\nabla \mathbf{x})), \mathbf{v}\rangle_Q,
\end{aligned}$$

and hence $\mathbf{z}$ satisfies the following equation

$$\mathbf{z}_t - \mathrm{div}(\eta(\nabla \mathbf{z})) = \mathbf{f} \quad \text{in } \mathcal{V}_T^*.$$

For the initial condition, we notice that $\mathbf{E}_2$ is compactly embedded into $C([0,T],\mathbf{H})$ (see [6]). Since $\mathbf{y}^n|_{t=0} = \mathbf{y}_0$ for all n, we have $\mathbf{z}|_{t=0} = \mathbf{y}_0$. Then uniqueness of the Stokes equation implies that $\mathbf{z} = \mathbf{y}(\eta)$.

Proof to the claim 2. Since $\{\mathbf{y}^n\} \subset \mathbf{K}_2$, inequality (3.29) implies that $\{\xi^n\} \subset K_1$. We can choose a subsequence (still denoted by $\{\xi^n\}$) and $\psi \in K_1$ such that

$$\begin{aligned}
&\xi^n \rightharpoonup \psi \quad \text{in } L^2(H^1),\\
&\xi_t^n \rightharpoonup \psi_t \quad \text{in } L^2(H^{-1}).
\end{aligned}$$

By definition, ξ^n satisfies the equation

$$\xi_t^n - \epsilon \triangle \xi^n + \mathbf{y}^n \cdot \nabla \xi^n = 0 \quad \text{in } \mathcal{C}_T^*.$$

Since $\mathbf{y}^n \in \mathbf{K}_2$, Aubin's Lemma implies that there exists a subsequence (still denoted by $\mathbf{y}^n$) converging to $\mathbf{y}$ strongly in $L^2(\mathbf{H})$. Then strong convergence of $\mathbf{y}^n$ in $L^2(Q)$ and weak convergence of $\nabla \xi^n$ in $L^2(Q)$ imply convergence of $\mathbf{y}^n \cdot \nabla \xi^n$ in the distribution sense. After passage to the limit, for any test function $\phi \in C_T$, we have

$$\begin{aligned}
&\langle \xi_t^n, \phi\rangle_Q \to \langle \psi_t, \phi\rangle_Q,\\
&\langle \triangle \xi^n, \phi\rangle_Q \to \langle \triangle \psi, \phi\rangle_Q,\\
&\langle \mathbf{y}^n \cdot \nabla \xi^n, \phi\rangle_Q \to \langle \mathbf{y} \cdot \nabla \psi, \phi\rangle_Q.
\end{aligned}$$

This implies that ψ satisfies the equation

$$\psi_t - \epsilon \triangle \psi + \mathbf{y} \cdot \nabla \psi = 0 \quad \text{in } \mathcal{C}_T^*.$$

The initial condition can be treated similarly as in the proof to claim 1. Since E_1 is compactly embedded into $C([0,T],L^2)$, we have $\psi|_{t=0} = \eta_0^\epsilon$. The boundary condition can also be treated by shifting every function by the constant function m and replacing H^1 by H_0^1 in the proof. Then by virtue of the uniqueness of the convection-diffusion equation, we have $\psi = \xi$.

Now we need to prove the strong convergence of $\{\xi^n\}$ in E_1. Defining $\varphi^n = \xi^n - \xi$, $\mathbf{z}^n = \mathbf{y}^n - \mathbf{y}$, we find that φ^n satisfies

$$\varphi_t^n - \epsilon \triangle \varphi^n + \mathbf{z}^n \cdot \nabla \xi^n + \mathbf{y} \cdot \nabla \varphi^n = 0,$$

with zero boundary and initial condition. Multiplying φ^n on both sides of the above equation, we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi^n\|^2 + \epsilon\|\nabla \varphi^n\|^2 + \langle \mathbf{z}^n \cdot \nabla \xi^n, \varphi^n\rangle = 0. \tag{3.31}$$

Since $\mathbf{z}^n$ is divergence free and $m \leq \xi^n \leq M$ *a.e.*, we have

$$\langle \mathbf{z}^n \cdot \nabla \xi^n, \varphi^n\rangle = -\langle \mathbf{z}^n \xi^n, \nabla \varphi^n\rangle \leq \frac{\epsilon}{2}\|\nabla \varphi^n\|^2 + C_\epsilon \|\mathbf{z}^n\|^2.$$

Substituting into (3.31) and integrating in time gives

$$\sup_{0\le t\le T}\|\varphi^n\|^2+\int_0^T\|\varphi^n\|_1^2dt\le C_\epsilon\int_0^T\|\mathbf{z}^n\|^2dt. \tag{3.32}$$

The time derivative can be evaluated as

$$\|\varphi_t^n\|_{H^{-1}}=\sup_{v\in H_0^1}\frac{\langle\varphi_t^n,v\rangle}{\|\nabla v\|}=\sup_{v\in H_0^1}\frac{\langle-\epsilon\nabla\varphi^n+\xi^n\mathbf{z}^n+\varphi^n\mathbf{y}^n,\nabla v\rangle}{\|\nabla v\|}$$

$$\le\epsilon\|\varphi^n\|_1+M\|\mathbf{z}^n\|+\|\mathbf{y}^n\|_{L^4}^{1/2}\|\varphi^n\|_{L^4}^{1/2}\le C_\epsilon(\|\varphi^n\|_1+\|\mathbf{z}^n\|+\|\mathbf{y}^n\|\|\varphi^n\|_1+\|\mathbf{y}^n\|_1\|\varphi^n\|).$$

Since $\mathbf{y}^n\in\mathbf{K}_2$, we have

$$\|\varphi_t^n\|_{L^2(H^{-1})}\le C_\epsilon(\|\varphi^n\|_{L^2(H^1)}+\|\mathbf{z}^n\|_{L^2(Q)}+\|\varphi^n\|_{L^\infty(L^2)}).$$

By virtue of (3.19) and $\|\mathbf{z}^n\|_{L^2(Q)}\to0$, we conclude that $\varphi^n\to0$ strongly in X.

Proof to the claim 3. See [13].

Proof to the Lemma. Continuity of τ. Consider a sequence $\{\eta^n\}\subset K_1$ and $\eta\in K_1$, such that $\eta^n\to\eta$ in E_1. By virtue of claim 1 and claim 3, we obtain $\{\mathbf{y}^n\}\subset\mathbf{K}_2$, $\mathbf{y}\in\mathbf{K}_2$ and $\mathbf{y}^n\rightharpoonup\mathbf{y}$ in $\mathbf{E}_2$ in the weak star topology. Then by claim 2 and claim 3, we have $\xi^n\to\xi$ in E_1 as desired. Compactness of τ follows from claim 1 and claim 2. □

3.4. **Proof to Theorem 3.2.** From Theorem 3.1, there exists at least one solution $(\eta^\epsilon,\mathbf{y}^\epsilon)\in K_1\times\mathbf{K}_2$ for the system (3.6) - (3.10). From estimates (3.22), (3.25), (3.26) and (3.28), we have the a-priori estimates

$$m\le\eta^\epsilon\le M\ a.e., \tag{3.33}$$

$$\|\mathbf{y}^\epsilon\|_{L^\infty(\mathbf{H})}+\|\mathbf{y}^\epsilon\|_{L^2(\mathbf{V})}+\|\mathbf{y}_t^\epsilon\|_{L^2(\mathbf{V}^*)}\le C_1, \tag{3.34}$$

$$\|\eta^\epsilon(t,\cdot)-m\|\le\|\eta_0^\epsilon-m\|,\quad\|\eta_t^\epsilon\|_{L^2(H^{-1})}\le C_2. \tag{3.35}$$

Since $\{(\eta^\epsilon,\mathbf{y}^\epsilon)\}$ is a bounded set in $L^\infty(Q)\cap H^1(H^{-1})\times L^2(\mathbf{V})\cap H^1(\mathbf{V}^*)$, we can find a subsequence which is denoted by $(\eta^n,\mathbf{y}^n)$ and $(\xi,\mathbf{z})$ in $L^\infty(Q)\cap H^1(H^{-1})\times L^2(\mathbf{V})\cap H^1(\mathbf{V}^*)$, such that

$$\begin{aligned}&\mathbf{y}^n\rightharpoonup\mathbf{z}\text{ in }L^2(\mathbf{V}),\\&\mathbf{y}_t^n\rightharpoonup\mathbf{z}_t\text{ in }L^2(\mathbf{V}^*),\\&\eta^n\rightharpoonup\xi\text{ in }L^\infty(Q)\text{ weak star},\\&\eta_t^n\rightharpoonup\xi_t\text{ in }L^2(H^{-1}).\end{aligned}$$

We recall that $(\eta^n,\mathbf{y}^n)$ satisfy the equations:

$$\begin{aligned}&\mathbf{y}_t^n-\operatorname{div}(\eta^n(\nabla\mathbf{y}^n))=\mathbf{f}\text{ in }\mathcal{V}_T^*,\\&\eta_t^n-\epsilon^n\triangle\eta^n+\mathbf{y}^n\cdot\nabla\eta^n=0,\text{ in }C_T^*,\end{aligned}$$

with the same initial and boundary condition as (3.8) and (3.10). Since $\mathbf{y}^n\to\mathbf{z}$ strongly in $L^2(Q)$ by Aubin's Lemma and $\eta^n\rightharpoonup\xi$ in $L^2(Q)$, we have $\mathbf{y}^n\eta^n$ converges to $\mathbf{z}\,\xi$ in the distribution sense. Choosing a test function $\phi\in C_T$ in the convection-diffusion equation, we find for $\epsilon^n\to0^+$,

$$\begin{aligned}&\langle\eta_t^n,\phi\rangle_Q\to\langle\xi_t,\phi\rangle_Q,\quad\epsilon^n\langle\triangle\eta^n,\phi\rangle_Q=\epsilon^n\langle\eta^n,\triangle\phi\rangle_Q\to0,\\&\langle\mathbf{y}^n\cdot\nabla\eta^n,\phi\rangle_Q=-\langle\mathbf{y}^n\eta^n,\nabla\phi\rangle_Q\to-\langle\mathbf{z}\xi,\nabla\phi\rangle_Q=\langle\mathbf{z}\cdot\nabla\xi,\phi\rangle_Q.\end{aligned}$$

Hence $(\mathbf{z}, \xi)$ satisfies the transport equation

$$\xi_t + \mathbf{z} \cdot \nabla \xi = 0, \text{ in } \mathcal{C}_T^*.$$

Since $L^2(L^2) \cap H^1(H^{-1})$ is compactly embedded into $C([0,T], H^{-1})$, and $\eta_0^{\epsilon^n} \to \eta_0$ in L^2, the initial condition for ξ is η_0. If we restrict our test function to be zero at time T, i.e. $\{\phi : \phi \in C_T, \phi(T) = 0\}$, one can check that

$$\int_0^T \int_\Omega \xi(\phi_t + \mathbf{z} \cdot \nabla \phi) dxdt + \int_\Omega \eta_0 \phi(0) dx = 0.$$

According to Theorem 4.1 in [11], the weak solution ξ is also a renormalized solution and satisfies $\|\xi(t,\cdot)\| = \|\eta_0\|$. By the property of renormalized solutions (choosing $\beta(s) = (s-m)^2$ in [11]), we have $\|\xi(t,\cdot) - m\| = \|\eta_0 - m\|$ for a.e. $t \in [0,T]$. Therefore by (3.35), we have

$$\limsup \|\eta^n - m\|_{L^2(Q)} \leq \sqrt{T} \lim \|\eta_0^{\epsilon^n} - m\| = \sqrt{T}\|\eta_0 - m\| = \|\xi - m\|_{L^2(Q)}. \tag{3.36}$$

Weak lower semi-continuous of norm implies that

$$\liminf \|\eta^n - m\|_{L^2(Q)} \geq \|\xi - m\|_{L^2(Q)}. \tag{3.37}$$

Combining (3.36) and (3.37) leads to $\|\eta^n - m\|_{L^2(Q)} \to \|\xi - m|_{L^2(Q)}$ and hence $\eta^n \to \xi$ in $L^2(Q)$. Strong convergence of η^n in $L^2(Q)$ and weak convergence of $\nabla \mathbf{y}^n$ in $L^2(Q)$ imply convergence of $\eta^n \nabla \mathbf{y}^n$ in the distribution sense. For test functions $\mathbf{v} \in \mathcal{V}_T$, we have

$$\begin{aligned}
&\langle \mathbf{y}_t^n, \mathbf{v}\rangle_Q \to \langle \mathbf{z}_t, \mathbf{v}\rangle_Q, \\
&\langle \mathrm{div}(\eta^n \nabla \mathbf{y}^n), \mathbf{v}\rangle_Q = -\langle \eta^n \nabla \mathbf{y}^n, \nabla \mathbf{v}\rangle_Q \\
&\to -\langle \xi \nabla \mathbf{z}, \nabla \mathbf{v}\rangle_Q = \langle \mathrm{div}(\xi \nabla \mathbf{z}), \mathbf{v}\rangle_Q.
\end{aligned}$$

Hence $(\mathbf{z}, \xi)$ satisfies the Stokes equation

$$\mathbf{z}_t - \mathrm{div}(\xi(\nabla \mathbf{z})) = \mathbf{f} \quad \text{in } \mathcal{V}_T^*.$$

Since $L^2(\mathbf{V}) \cap L^2(\mathbf{V}^*)$ is compactly embedded into $C([0,T], H)$, we have $\mathbf{z}|_{t=0} = \mathbf{y}_0$. Hence $(\mathbf{z}, \xi)$ solves the system (3.1) - (3.5). This implies Theorem 3.2.

Corollary 3.4. Consider the same assumption as in Theorem 3.2. If η_0^ϵ, in addition to (3.11), also satisfies

$$\eta_0^\epsilon \to \eta_0, \text{ in } L^p(\Omega), \ 2 \leq p < \infty,$$

then we have

$$\eta^n \to \eta, \text{ in } L^p(Q).$$

Proof: After shifting by a constant function m, we denote $\bar{\eta}^\epsilon = \eta^\epsilon - m$. Hence $\bar{\eta}^\epsilon$ satisfies equation (3.9) with initial condition $\bar{\eta}^\epsilon|_{t=0} = \eta_0^\epsilon - m$ and zero Dirichlet boundary condition. It is known that $\bar{\eta}^\epsilon \in L^\infty(Q) \cap L^2(H^1)$. We can multiply $|\bar{\eta}^\epsilon|^{p-2}\bar{\eta}^\epsilon$ on both sides of equation (3.9) (where η^ϵ is replaced by $\bar{\eta}^\epsilon$). Since $\nabla(|\bar{\eta}^\epsilon|^{p-2}\bar{\eta}^\epsilon) = (p-1)|\bar{\eta}^\epsilon|^{p-2}\nabla\bar{\eta}^\epsilon$, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |\bar{\eta}^\epsilon|^p dx + \epsilon(p-1)\int_\Omega |\bar{\eta}^\epsilon|^{p-2}|\nabla\bar{\eta}^\epsilon|^2 dx = 0.$$

This implies that $\|\bar{\eta}^\epsilon(t)\|_{L^p(\Omega)} \leq \|\eta_0^\epsilon - m|_{L^p(\Omega)}$. After passage to the limit and by same argument as in the proof of Theorem 3.2, we have $\eta^n \rightharpoonup \eta$ in $L^p(Q)$ and η is the renormalized solution. From the property of renormalized solutions (choosing

$\beta(s) = |s-m|^p$ in [11]), we have $\|\eta(t,\cdot) - m\|_{L^p(\Omega)} = \|\eta_0 - m\|_{L^p(\Omega)}, a.e. t \in [0,T]$. Therefore

$\limsup \|\bar{\eta}^n - m\|_{L^p(Q)} \le T^{1/p} \lim \|\eta_0^{\epsilon^n} - m\|_{L^p(\Omega)} = T^{1/p}\|\eta_0 - m\|_{L^p(\Omega)} = \|\eta - m\|_{L^p(Q)}$.

Together with weak lower semi-continuous of norm, this implies that $\|\eta^n - m\|_{L^p(Q)} \to \|\eta - m|_{L^p(Q)}$ and hence $\eta^n \to \eta$ in $L^p(Q)$. □

4. Optimal Control in $\mathbf{L}^2(Q)$ Space

The abstract formulation for the optimal control is:

$$\min J(\mathbf{x},\mathbf{u}), \quad \text{such that } e(\mathbf{x},\mathbf{u}) = 0, \tag{4.1}$$

where $J(\mathbf{x},\mathbf{u})$ is the cost functional, $\mathbf{u}$ and $\mathbf{x}$ are the control and state variable respectively, and $e(\mathbf{x},\mathbf{u}) = 0$ denotes the underlying equation. For our control problem, the state variable $\mathbf{x} = (\eta, \mathbf{y})$ and the control variable is $\mathbf{u}$ itself. The underlying equation is either system (3.1) - (3.5) or (3.6) - (3.10).

In the L^2 control case, we choose the cost functional as

$$J(\eta,\mathbf{u}) = \frac{1}{2}\|\eta - \widetilde{\eta}\|^2_{L^2(Q)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(Q)}, \tag{4.2}$$

where $\widetilde{\eta}$ is a given function in $L^2(Q)$. The optimal control problem associated with original equation is

Problem 4.1.

$$\min J(\eta,\mathbf{u}), \quad \text{such that equation (3.1) - (3.5) hold.}$$

We have existence for the above problem.

Theorem 4.2. *Given $\mathbf{y}_0 \in \mathbf{H}$, $m \le \eta_0 \le M$ a.e., there exists at least one optimal solution $(\eta, \mathbf{y}, \mathbf{u})$ for Problem 4.1, and $(\eta, \mathbf{y}, \mathbf{u}) \in L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*) \times L^2(Q)$.*

Proof: Clearly Problem 4.1 is feasible, hence we can find a minimal sequence $(\mathbf{y}^n, \eta^n, \mathbf{u}^n)$, i.e. $\lim_{n\to\infty} J(\eta^n, \mathbf{u}^n) = \inf J(\eta, \mathbf{u})$ such that $(\eta^n, \mathbf{y}^n, \mathbf{u}^n)$ solve equations (3.1) - (3.5) (we use the equivalent weak form to avoid the pressure term). By the definition of J in (4.2), we know that $\{\mathbf{u}^n\}$ is bounded in $\mathbf{L}^2(Q)$. Hence $\{\mathbf{y}^n\}$, $\{\eta^n\}$ are also bounded in $L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*)$ and $L^\infty(Q) \cap H^1(H^{-1})$ respectively. After passing to the subsequence (still denote the subscript by n), we have

$$\begin{aligned} &\mathbf{u}^n \rightharpoonup \mathbf{u} \quad \text{in } \mathbf{L}^2(Q), \\ &\mathbf{y}^n \rightharpoonup \mathbf{y} \quad \text{in } L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*), \\ &\eta^n \rightharpoonup \eta \quad \text{weak star in } L^\infty(Q) \cap H^1(H^{-1}). \end{aligned}$$

We need to check that $(\mathbf{y}, \eta, \mathbf{u})$ satisfy equations (3.1) - (3.5). Since $\mathbf{y}^n \to \mathbf{y}$ in $L^2(Q)$ by Aubin's lemma and $\eta^n \rightharpoonup \eta$ in $L^2(Q)$, we obtain that $\mathbf{y}^n\eta^n$ converges to $\mathbf{y}\eta$ in the distribution sense, hence the transport equation (3.4) is satisfied. The initial condition can be obtained in a similar way as in Thoerem 3.2. Hence η is a renormalized solution. By the property of renormalized solution (c.f. [11]), $\|\eta^n\|_{L^2(Q)} = \sqrt{T}\|\eta_0\| = \|\eta\|_{L^2(Q)}$. This implies that $\eta^n \to \eta$ in $L^2(Q)$. Together with $\nabla\mathbf{y}^n \rightharpoonup \nabla\mathbf{y}$ in $L^2(Q)$, we have convergence of $\eta^n\nabla\mathbf{y}^n$ in the distribution sense. Therefore the Stokes equations (3.1) - (3.2) are also satisfied. The initial condition can be treated similarly as before. Hence $(\eta, \mathbf{y}, \mathbf{u})$ satisfies the underlying equation.

Lastly, since the norm $\|\cdot\|_{L^2(Q)}$ is a weakly lower semi-continuous functional, we obtain that $(\eta, \mathbf{u})$ provides a minimum for Problem 4.1. □

Now we move to the optimal control problem associated with the approximated system. For the $L^2(Q)$ control case, we assume that $\mathbf{y}_0 \in \mathbf{V}$, $\eta_0^\epsilon - m \in H_0^1$. Then equations (3.6) - (3.10) are equivalent to equations (3.14) - (3.15) with the same initial condition. To avoid the pressure term, we will use equations (3.14) - (3.15). Denote

$$e^\epsilon(\eta, \mathbf{y}, \mathbf{u}) = \begin{pmatrix} \mathbf{e}_{1,1}^\epsilon, & \mathbf{e}_{1,2}^\epsilon \\ e_{2,1}^\epsilon, & e_{2,2}^\epsilon \end{pmatrix} = \begin{pmatrix} \mathbf{y}_t - P\mathrm{div}(\eta\nabla\mathbf{y}) - \mathbf{u}, & \mathbf{y}(0) - \mathbf{y}_0 \\ \eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta, & \eta(0) - \eta_0^\epsilon \end{pmatrix}. \tag{4.3}$$

Then the optimal control problems for the approximated system are given by:

Problem 4.3.

$$\min J(\eta, \mathbf{u}), \quad \text{such that } e^\epsilon(\eta, \mathbf{y}, \mathbf{u}) = 0.$$

Similar to Theorem 4.2, we have existence for optimal solution for Problem 4.3.

Theorem 4.4. *Given $\mathbf{y}_0 \in \mathbf{V}$, $\eta_0^\epsilon - m \in H_0^1$ and $m \le \eta_0 \le M$ a.e., there exists at least one optimal solution $(\eta, \mathbf{y}, \mathbf{u})$ for Problem 4.3, and $(\eta - m, \mathbf{y}, \mathbf{u}) \in L^2(H^2 \cap H_0^1) \cap L^\infty(Q) \cap H^1(L^2) \times L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*) \times L^2(Q)$.*

In the remaining of this section, we discuss the optimality system for Problem 4.3 and how Problem 4.3 approximates Problem 4.1 when $\epsilon \to 0^+$. Subsection 4.1 contains the regularity estimates for equation (3.6) - (3.10). The theorems are state in subsection 4.2, i.e. Theorem 4.9 for the optimality system and Lemma 4.10 for the limit property.

4.1. **Regularity Estimates.** We will repeatedly use the following estimates.

Lemma 4.5.

$$\mathbf{y} \in L^2(\mathbf{H}^1) \cap L^\infty(\mathbf{L}^2), \eta \in L^2(H^2) \cap L^\infty(H^1) \quad \Rightarrow \mathbf{y}\cdot\nabla\eta \in L^2(Q), \tag{4.4}$$

$$\mathbf{y} \in L^\infty(\mathbf{H}^1), \eta \in L^2(H^2) \cap L^\infty(H^1) \quad \Rightarrow \mathbf{y}\cdot\nabla\eta \in L^3(Q), \tag{4.5}$$

$$\mathbf{y} \in L^2(\mathbf{H}^2) \cap L^\infty(\mathbf{H}^1), \eta \in L^2(H^2) \cap L^\infty(H^1) \quad \Rightarrow \nabla\eta\cdot\nabla\mathbf{y} \in L^2(Q). \tag{4.6}$$

Proof: The conclusions are based on the Hölder and Sobolev inequalities.

$$\int_0^T \int_\Omega |\mathbf{y}\cdot\nabla\eta|^2 dxdt = \int_0^T \|\mathbf{y}\cdot\nabla\eta\|_{L^2}^2 dt \le \int_0^T \|\mathbf{y}\|_{L^4}^2 \|\nabla\eta\|_{L^4}^2 dt$$
$$\le C\int_0^T \|\mathbf{y}\|\|\mathbf{y}\|_1\|\eta\|_1\|\eta\|_2 dt \le C(\|\mathbf{y}\|_{L^\infty(\mathbf{L}^2)}^2\|\mathbf{y}\|_{L^2(\mathbf{H}^1)}^2 + \|\eta\|_{L^\infty(H^1)}^2\|\eta\|_{L^2(H^2)}^2),$$
$$\int_0^T \int_\Omega |\mathbf{y}\cdot\nabla\eta|^3 dxdt = \int_0^T \|\mathbf{y}\cdot\nabla\eta\|_{L^3}^3 dt \le \int_0^T \|\mathbf{y}\|_{L^6}^3 \|\nabla\eta\|_{L^6}^3 dt$$
$$\le C\int_0^T \|\mathbf{y}\|_1^3\|\eta\|_1\|\eta\|_2^2 dt,$$
$$\int_0^T \int_\Omega |\nabla\eta\cdot\nabla\mathbf{y}|^2 dxdt = \int_0^T \|\nabla\eta\cdot\nabla\mathbf{y}\|^2 dt \le C\int_0^T \|\nabla\mathbf{y}\|_{L^4}^2 \|\nabla\eta\|_{L^4}^2 dt$$
$$\le C\int_0^T \|\mathbf{y}\|_1\|\mathbf{y}\|_2\|\eta\|_1\|\eta\|_2 dt \le C(\|\mathbf{y}\|_{L^\infty(\mathbf{H}^1)}^2\|\mathbf{y}\|_{L^2(\mathbf{H}^2)}^2 + \|\eta\|_{L^\infty(H^1)}^2\|\eta\|_{L^2(H^2)}^2).$$

□

To obtain a higher order regularity result for the solution to the system (3.6) - (3.10), we need to assume that the initial data $\mathbf{y}_0$ and η_0^ϵ satisfying higher regularity properties. Besides (3.11), let

$$\mathbf{y}_0 \in \mathbf{V}, \quad \eta_0^\epsilon - m \in H^2(\Omega) \cap H_0^1(\Omega) \cap W^{1,3}(\Omega). \tag{4.7}$$

The existence result (Theorem 3.1) implies that for any $\mathbf{u} \in L^2(\mathbf{H})$, there exists at least one $(\eta, \mathbf{y})$ satisfies (3.6) - (3.10) such that

$$\mathbf{y} \in L^2(\mathbf{V}) \cap L^\infty(\mathbf{H}) \cap H^1(\mathbf{V}^*), \quad \eta \in L^2(H^1) \cap L^\infty(Q) \cap H^1(H^{-1}). \tag{4.8}$$

Taking the inner product of (3.9) with $-\triangle\eta$, using

$$(\mathbf{y} \cdot \nabla\eta, \triangle\eta) \le \|\mathbf{y}\|_{L^4} \|\nabla\eta\|_{L^4} \|\eta\|_2 \le C_\epsilon \|\mathbf{y}\|^2 \|\mathbf{y}\|_1^2 \|\eta\|_1^2 + \frac{1}{2\epsilon} \|\triangle\eta\|^2,$$

and estimate (4.8) and $\eta_0^\epsilon \in H^1$, we have $\eta \in L^2(H^2) \cap L^\infty(H^1)$. Hence $\eta_t \in L^2(Q)$, and

$$\eta \in L^2(H^2) \cap L^\infty(H^1) \cap H^1(L^2). \tag{4.9}$$

After taking the time derivative in equation (3.9)

$$\eta_{tt} - \epsilon\triangle\eta_t + \mathbf{y}_t \cdot \nabla\eta + \mathbf{y} \cdot \nabla\eta_t = 0,$$

and taking the inner product with η_t in the above equation,

$$\frac{1}{2}\frac{d}{dt}\|\eta_t\| + \epsilon\|\nabla\eta_t\|^2 + \langle \mathbf{y}_t \cdot \nabla\eta, \eta_t\rangle = 0.$$

Since

$$\langle \mathbf{y}_t \cdot \nabla\eta, \eta_t\rangle = -\langle \mathbf{y}_t\eta, \nabla\eta_t\rangle \le \frac{\epsilon}{2}\|\nabla\eta_t\|^2 + C_\epsilon\|\mathbf{y}_t\|^2,$$

we find

$$\|\eta_t\|^2 \le C_\epsilon \int_0^t \|\mathbf{y}_t\|^2 + \|\eta_t(0)\|^2, \tag{4.10}$$

where $\|\eta_t(0)\| \le \epsilon\|\eta_0^\epsilon\|_2 + \|\mathbf{y}_0 \cdot \nabla\eta_0^\epsilon\| \le \epsilon\|\eta_0^\epsilon\|_2 + C\|\mathbf{y}_0\|_1\|\eta_0^\epsilon\|_2$. Moving η_t in equation (3.9) to the right hand side, the elliptic estimation gives

$$\|\triangle\eta\| \le C_\epsilon(\|\eta_t\| + \|\mathbf{y}\|_1). \tag{4.11}$$

Now we move to higher order regularity estimates for the Stokes equation. First we consider the time independent Stokes equation with nonconstant viscosity η.

Lemma 4.6. *Suppose that $\eta \in H^2$ and $m \le \eta \le M$ a.e., and that $\mathbf{y}$ solves following Stokes equation*

$$-\mathrm{div}(\eta(\nabla\mathbf{y})) + \nabla p = \mathbf{u}, \tag{4.12}$$

$$\mathrm{div}\mathbf{y} = 0, \quad \mathbf{y}|_\Omega = \mathbf{0}. \tag{4.13}$$

Then we have

$$\|\mathbf{y}\|_2 \le C(\|\mathbf{u}\| + \|\eta\|_1\|\eta\|_2\|\mathbf{u}\|_{\mathbf{V}^*}). \tag{4.14}$$

Proof: First we have

$$\|\mathbf{y}\|_1 + \|p\| \le C\|\mathbf{u}\|_{\mathbf{V}^*}. \tag{4.15}$$

Equation (4.12) can be rewritten as

$$-\triangle\mathbf{y} - \frac{\nabla\eta}{\eta} \cdot \nabla\mathbf{y} + \nabla(\frac{p}{\eta}) + \frac{p\nabla\eta}{\eta^2} = \frac{\mathbf{u}}{\eta}.$$

We multiply $\Lambda\mathbf{y} = -P\triangle\mathbf{y}$ on both sides of the above equation. Since

$$\langle\frac{\nabla\eta}{\eta}\cdot\nabla\mathbf{y},\Lambda\mathbf{y}\rangle \le C\|\eta\|_1^{1/2}\|\eta\|_2^{1/2}\|\mathbf{y}\|_1^{1/2}\|\mathbf{y}\|_2^{1/2}\|\Lambda\mathbf{y}\|,$$

$$\langle\frac{p\nabla\eta}{\eta^2},\Lambda\mathbf{y}\rangle \le C\|p\|^{1/2}\|p\|_1^{1/2}\|\eta\|_1^{1/2}\|\eta\|_2^{1/2}\|\Lambda\mathbf{y}\|,$$

$$\|p\|_1 \le \|\mathbf{u}\| + M\|\mathbf{y}\|_2 + \|\nabla\eta\|_{L^4}\|\nabla\mathbf{y}\|_{L^4} \le \|\mathbf{u}\| + C\|\mathbf{y}\|_2 + C\|\eta\|_1^{1/2}\|\eta\|_2^{1/2}\|\mathbf{y}\|_1^{1/2}\|\mathbf{y}\|_2^{1/2},$$

together with estimate (4.15) and the property of Stokes operator $\|\mathbf{y}\|_2 \le C\|\Lambda\mathbf{y}\|$, we have

$$\langle\frac{\nabla\eta}{\eta}\cdot\nabla\mathbf{y},\Lambda\mathbf{y}\rangle \le \frac{1}{4}\|\Lambda\mathbf{y}\|^2 + C\|\eta\|_1^2\|\eta\|_2^2\|\mathbf{f}\|_{\mathbf{V}^*}^2,$$

$$\langle\frac{p\nabla\eta}{\eta^2},\Lambda\mathbf{y}\rangle \le \frac{1}{4}\|\Lambda\mathbf{y}\|^2 + C\|\eta\|_1^2\|\eta\|_2^2\|\mathbf{f}\|_{\mathbf{V}^*}^2,$$

$$\langle\frac{\mathbf{u}}{\eta},\Lambda\mathbf{y}\rangle \le \frac{1}{4}\|\Lambda\mathbf{y}\|^2 + C\|\mathbf{u}\|^2.$$

Hence $\|\mathbf{y}\|_2 \le C(\|\mathbf{u}\| + \|\eta\|_1\|\eta\|_2\|\mathbf{u}\|_{\mathbf{V}^*})$.□

We next consider the time dependent Stokes equation (3.6) - (3.8). Multiplying $\mathbf{y}_t$ on both sides of equation (3.6) and noticing that $(\mathbf{y}_t, \nabla p) = 0$, we have

$$\|\mathbf{y}_t\|^2 + \frac{1}{2}\frac{d}{dt}(\eta\nabla\mathbf{y},\nabla\mathbf{y}) - \frac{1}{2}\langle\eta_t\nabla\mathbf{y},\nabla\mathbf{y}\rangle = (\mathbf{u},\mathbf{y}_t).$$

After moving $\mathbf{y}_t$ to the right hand side, using Lemma 4.6 and

$$\|\mathbf{y}_t\|_{\mathbf{V}^*} \le \|\mathbf{u}\|_{\mathbf{V}^*} + C\|\mathbf{y}\|_1,$$

we find

$$\|\mathbf{y}\|_2 \le C(\|\mathbf{u}\| + \|\mathbf{y}_t\|) + C\|\eta\|_1\|\eta\|_2(\|\mathbf{u}\|_{\mathbf{V}^*} + \|\mathbf{y}\|_1).$$

We also have the estimates

$$\begin{aligned}\langle\eta_t\nabla\mathbf{y},\nabla\mathbf{y}\rangle &\le C\|\eta_t\|\|\mathbf{y}\|_1\|\mathbf{y}\|_2 \le C\|\eta_t\|\|\mathbf{y}\|_1(\|\mathbf{u}\| + \|\mathbf{y}_t\|) + C\|\eta_t\|\|\mathbf{y}\|_1\|\eta\|_1\|\eta\|_2(\|\mathbf{u}\|_{\mathbf{V}^*} + \|\mathbf{y}\|_1)\\ &\le C\|\eta_t\|^2\|\mathbf{y}\|_1^2 + \|\mathbf{u}\|^2 + \frac{1}{2}\|\mathbf{y}_t\|^2 + C(\|\mathbf{y}\|_1^2\|\eta\|_1^2\|\eta\|_2^2 + \|\eta_t\|^2\|\mathbf{u}\|^2 + \|\eta_t\|\|\eta\|_1\|\eta\|_2\|\mathbf{y}\|_1^2).\end{aligned}$$

Defining $\beta(t) = \int_0^t \|\mathbf{y}_t\|^2 + \|\eta_t(0)\|^2$, $\gamma(t) = (\eta\nabla\mathbf{y},\nabla\mathbf{y})$, inequality (4.10) implies that $\|\eta_t\| \le C_\epsilon\beta(t)$. Hence

$$\frac{d}{dt}(\beta+\gamma) \le C_\epsilon(\|\eta_t\|^2\beta + \|\eta\|_1^2\|\eta\|_2^2\beta + \|\mathbf{u}\|^2\gamma) + \|\mathbf{u}\|^2 \le C_\epsilon(\|\eta_t\|^2\beta + \|\eta\|_1^2\|\eta\|_2^2 + \|\mathbf{u}\|^2)(\beta+\gamma) + \|\mathbf{u}\|^2.$$

Since we already have (4.9) and $\mathbf{u} \in L^2(Q)$, then by Gronwall inequality, we have $\mathbf{y}_t \in L^2(Q)$ and $\mathbf{y} \in L^\infty(\mathbf{V})$. Hence $\eta_t \in L^\infty(L^2)$ and $\eta \in L^\infty(H^2)$ (from (4.10) and (4.11)), which immediately gives $\mathbf{y} \in L^2(\mathbf{H}^2)$. Summing up we have

$$\mathbf{y} \in L^2(\mathbf{H}^2) \cap L^\infty(\mathbf{V}) \cap H^1(\mathbf{H}). \tag{4.16}$$

To guarantee the existence for a Lagrange multiplier, we need a slight better estimation for η. Due to (4.9) and (4.16), Lemma 4.5 implies $\mathbf{y}\cdot\nabla\eta \in L^3(Q)$. Then moving $\mathbf{y}\cdot\nabla\eta$ to right hand side for equation (3.9), and using Theorem 1.14 in [7], we obtain

$$\eta \in L^3(W^{2,3}), \quad \eta_t \in L^3(Q). \tag{4.17}$$

Combining the results, we find

Theorem 4.7. *Assume that (3.11) and (4.7) are satisfied. If* $\mathbf{u} \in L^2(\mathbf{H})$*, then every solution of systems (3.6) - (3.10) satisfies:*

$$\mathbf{y} \in L^2(\mathbf{H}^2) \cap L^\infty(\mathbf{V}) \cap H^1(\mathbf{H}), \quad \eta \in L^3(W^{2,3}), \quad \eta_t \in L^3(Q),$$

and

$$\|\mathbf{y}\|_{L^2(\mathbf{H}^2)\cap L^\infty(\mathbf{V})\cap H^1(\mathbf{H})} + \|\eta\|_{L^3(W^{2,3})} + \|\eta_t\|_{L^3(Q)} \leq C(\|\mathbf{u}\|_{L^2(Q)}),$$

where $C(\cdot)$ *maps bounded set to bounded set.*

4.2. **Optimal Control Problem Associated with Approximated Equations.** This section is devoted to deriving the first order optimality condition for Problem 4.3. Define the spaces

$$X_1 = \{\phi : \phi \in L^3(W^{2,3}), \phi_t \in L^3(Q)\}, \tag{4.18}$$

$$\mathbf{Y}_1 = \{\mathbf{v} : \mathbf{v} \in L^2(\mathbf{H}^2) \cap L^\infty(\mathbf{V}) \cap H^1(\mathbf{H})\}. \tag{4.19}$$

By a standard embedding result (c.f. [7]), $X_1 \subset L^\infty$. Recalling (4.3) for the definition of the nonlinear map e^ϵ, we have

Lemma 4.8. *For any fixed positive constant* ϵ*, the map* e^ϵ *acts from* $X_1 \times \mathbf{Y}_1 \times L^2(\mathbf{H})$ *to* $\begin{pmatrix} L^2(\mathbf{H}), & \mathbf{V} \\ L^3(Q), & W^{1,3} \end{pmatrix}$*. Moreover, it is Frechet differentiable.*

Proof: We first verify that e is well-defined. Recalling that $\gamma_0(\eta) = \eta(0)$ and $\gamma_0(\mathbf{y}) = \mathbf{y}(0)$. Here γ_0 is continuous from X_1 to $W^{1,3}$ (see e.g. Theorem 1.13 in [7]) and also continuous from $\mathbf{Y}_2$ to $\mathbf{V}$ (see e.g. [14]), then $e^\epsilon_{1,2}$ and $e^\epsilon_{2,2}$ are well defined. For any given $(\eta, \mathbf{y}, \mathbf{u}) \in X_1 \times \mathbf{Y}_1 \times L^2(\mathbf{H})$, by virtue of $X_1 \hookrightarrow L^\infty(Q)$ and Lemma 4.5, we have

$$\mathrm{div}(\eta\nabla\mathbf{y})) = \eta\triangle\mathbf{y} + \nabla\eta\cdot\nabla\mathbf{y} \in \mathbf{L}^2(Q), \quad \mathbf{y}\cdot\nabla\eta \in L^3(Q).$$

Hence $e^\epsilon_{1,1}$ and $e^\epsilon_{2,1}$ lie in $L^2(\mathbf{H})$ and $L^3(Q)$ respectively. Since $e^\epsilon_{1,2}$ and $e^\epsilon_{2,2}$ are linear operators, the differentiability is clear. For $e^\epsilon_{1,1}$ and $e^\epsilon_{2,1}$, consider the linearized equation at point $(\eta, \mathbf{y}, \mathbf{u})$ as

$$\frac{d}{d\mathbf{x}} \begin{pmatrix} e^\epsilon_{1,1}(\eta, \mathbf{y}, \mathbf{u}) \\ e^\epsilon_{2,1}(\eta, \mathbf{y}, \mathbf{u}) \end{pmatrix} (\delta\eta, \delta\mathbf{y}, \delta\mathbf{u}) = \begin{pmatrix} \delta\mathbf{y}_t - P(\mathrm{div}(\delta\eta\nabla\mathbf{y})) - P(\mathrm{div}(\eta\nabla\delta\mathbf{y})) - \delta\mathbf{u} \\ \delta\eta_t - \epsilon\triangle\delta\eta + \delta\mathbf{y}\cdot\nabla\eta + \mathbf{y}\cdot\nabla\delta\eta \end{pmatrix}. \tag{4.20}$$

We will check that the linearized equation is indeed the Frechet derivative. By calculation,

$$\begin{pmatrix} e^\epsilon_{1,1}(\eta+\delta\eta, \mathbf{y}+\delta\mathbf{y}, \mathbf{u}+\delta\mathbf{u}) \\ e^\epsilon_{2,1}(\eta+\delta\eta, \mathbf{y}+\delta\mathbf{y}, \mathbf{u}+\delta\mathbf{u}) \end{pmatrix} - \begin{pmatrix} e^\epsilon_{1,1}(\eta, \mathbf{y}, \mathbf{u}) \\ e^\epsilon_{2,1}(\eta, \mathbf{y}, \mathbf{u}) \end{pmatrix}$$
$$- \frac{d}{d\mathbf{x}} \begin{pmatrix} e^\epsilon_{1,1}(\eta, \mathbf{y}, \mathbf{u}) \\ e^\epsilon_{2,1}(\eta, \mathbf{y}, \mathbf{u}) \end{pmatrix} (\delta\eta, \delta\mathbf{y}, \delta\mathbf{u}) = \begin{pmatrix} P(\mathrm{div}(\delta\eta\nabla\delta\mathbf{y})) \\ \delta\mathbf{y}\cdot\nabla\delta\eta \end{pmatrix}.$$

Since

$$\|P(\mathrm{div}(\delta\eta\nabla\delta\mathbf{y})\|_{L^2(Q)} \leq \|\delta\eta\|_{L^\infty(Q)}\|\triangle\delta\mathbf{y}\|_{L^2(Q)} + \|\nabla\delta\eta\cdot\nabla\delta\mathbf{y}\|_{\mathbf{L}^2(Q)},$$

by $X_1 \hookrightarrow L^\infty(Q)$ and Lemma 4.5, we have

$$\|P(\mathrm{div}(\delta\eta\nabla\delta\mathbf{y})\|_{L^2(Q)} + \|\delta\mathbf{y}\cdot\nabla\delta\eta\|_{L^3(Q)} \leq C\|\delta\eta\|_{X_1}\|\delta\mathbf{y}\|_{\mathbf{Y}_1}.$$

Recalling the definition of Frechet derivative, we conclude that e^ϵ is differentiable with derivative e^ϵ_x. □

The existence of an optimal solution for problem 4.3 was already obtained in Theorem 4.4. We let $(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$ be one optimal solution. From Lemma 4.8, e^ϵ is differentiable, and hence $e_x^\epsilon(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$ also maps $X_1 \times \mathbf{Y}_1 \times L^2(\mathbf{H})$ to $\begin{pmatrix} L^2(\mathbf{H}), & \mathbf{V} \\ L^3(Q), & W^{1,3} \end{pmatrix}$. Moreover, this map is also surjective. In fact, for any $\begin{pmatrix} \mathbf{g}_1, & \mathbf{q}_1 \\ g_2, & q_2 \end{pmatrix} \in \begin{pmatrix} L^2(\mathbf{H}), & \mathbf{V} \\ L^3(Q), & W^{1,3} \end{pmatrix}$, we verify that there exists $(\delta\eta, \delta\mathbf{y}, \delta\mathbf{u})$ which satisfies:

$$\begin{pmatrix} \delta\mathbf{y}_t - P(\operatorname{div}(\delta\eta\nabla\mathbf{y}_\epsilon^*)) - P(\operatorname{div}(\eta_\epsilon^*\nabla\delta\mathbf{y})) - \delta\mathbf{u} & = \mathbf{g}_1 \\ \delta\eta_t - \epsilon\triangle\delta\eta + \delta\mathbf{y}\cdot\nabla\eta_\epsilon^* + \mathbf{y}_\epsilon^*\cdot\nabla\delta\eta & = g_2 \end{pmatrix},$$

with initial condition

$$\begin{pmatrix} \delta\mathbf{y}(0) & = \mathbf{q}_1 \\ \delta\eta(0) & = q_2 \end{pmatrix}.$$

One can choose $\delta\mathbf{y}$ solving the Stokes equation $\delta\mathbf{y}_t - P\triangle\delta\mathbf{y} = \mathbf{0}$ with initial condition $\delta\mathbf{y}(0) = \mathbf{q}_1$. Hence $\delta\mathbf{y} \in \mathbf{Y}_1$. By Lemma 4.5, $\delta\mathbf{y}\cdot\nabla\eta_\epsilon^* \in L^3(Q)$. Let $\delta\eta$ solves the equation

$$\delta\eta_t - \epsilon\triangle\delta\eta + \mathbf{y}_\epsilon^*\cdot\nabla\delta\eta = g_2 - \delta\mathbf{y}\cdot\nabla\eta_\epsilon^*$$

with initial condition q_2 and zero boundary condition. A similar argument as in Theorem 4.7 implies $\delta\eta \in X_1$. Then we choose $\delta\mathbf{u} = \delta\mathbf{y}_t - P(\operatorname{div}(\eta_\epsilon^*\nabla\delta\mathbf{y})) - P(\operatorname{div}(\delta\eta\nabla\mathbf{y}_\epsilon^*)) - \mathbf{g}_1$. By $X_1 \hookrightarrow L^\infty(Q)$ and Lemma 4.5, we have

$$\operatorname{div}(\eta_\epsilon^*\nabla\delta\mathbf{y}) = \eta_\epsilon^*\triangle\delta\mathbf{y} + \nabla\eta_\epsilon^*\cdot\nabla\delta\mathbf{y} \in L^2(Q), \quad \operatorname{div}(\delta\eta\nabla\mathbf{y}_\epsilon^*) \in L^2(Q).$$

Hence $\delta\mathbf{u} \in L^2(\mathbf{H})$, and surjectivity follows. The surjectivity of $e_x^\epsilon(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$ implies there exists a Lagrange multiplier $(\mathbf{z}, \xi) \in L^2(\mathbf{H}) \times L^{4/3}(Q)$, such that the following Lagrangian

$$\begin{aligned} \mathcal{L}(\eta, \mathbf{y}, \mathbf{u}, \xi, \mathbf{z}) = {} & \frac{1}{2}\|\eta - \widetilde{\eta}\|_{L^2(Q)}^2 + \frac{\alpha}{2}\|\mathbf{u}\|_{L^2(Q)}^2 \\ & + (\mathbf{z}, \mathbf{y}_t - P(\operatorname{div}(\eta\nabla\mathbf{y})) - \mathbf{u}) + \langle \xi, \eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta \rangle_{L^{4/3}(Q), L^3(Q)} \end{aligned} \tag{4.21}$$

has a stationary point $(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*, \xi, \mathbf{z})$, see e.g. [10]. In particular, we have

$$\begin{aligned} \mathcal{L}_\xi = 0, \mathcal{L}_\mathbf{z} = 0 \quad & \Rightarrow \text{primal equation}, \\ \mathcal{L}_\eta = 0, \mathcal{L}_\mathbf{y} = 0 \quad & \Rightarrow \text{adjoint equation}, \\ \mathcal{L}_\mathbf{u} = 0 \quad & \Rightarrow \text{optimal condition}. \end{aligned}$$

Expressing these facts in PDE form we obtain the following result.

Theorem 4.9. *Let $(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$ denote an optimal solution of Problem 4.3 and let $(\xi, \mathbf{z})$ be an associated Lagrange multiplier. Then they satisfy the following equations.*

Primal Equation:

$$\begin{aligned} & \mathbf{y}_t - P(\operatorname{div}(\eta\nabla\mathbf{y})) = \mathbf{u}, \\ & \mathbf{y}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{y}|_{t=0} = \mathbf{y}_0, \\ & \eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta = 0, \\ & \eta|_{t=0} = \eta_0^\epsilon, \quad \eta|_{\partial\Omega} = m. \end{aligned}$$

Adjoint Equation (in the weak sense):

$$\begin{aligned} & -\mathbf{z}_t - \operatorname{div}(\eta\nabla\mathbf{z}) - \eta\nabla\xi = \mathbf{0}, \\ & -\xi_t - \epsilon\triangle\xi + \mathbf{y}\cdot\nabla\xi + \nabla\mathbf{y} : \nabla\mathbf{z} = \eta - \widetilde{\eta}, \\ & \mathbf{z}|_{t=T} = \mathbf{0}, \quad \eta|_{t=T} = 0, \end{aligned}$$

where $\nabla \mathbf{y} : \nabla \mathbf{z}$ *is the matrix inner product of Frobenius type.*
Optimality Condition:

$$\alpha \mathbf{u} = \mathbf{z}.$$

Next we consider the relation between the minimum of Problem 4.1 and Problem 4.3. Let $(\eta^*, \mathbf{y}^*, \mathbf{u}^*)$ and $(\eta^*_\epsilon, \mathbf{y}^*_\epsilon, \mathbf{u}^*_\epsilon)$ be a minimizer for Problem 4.1 and 4.3 respectively, and define

$$\jmath^* = J(\eta^*, \mathbf{u}^*), \quad \jmath^\epsilon = J(\eta^*_\epsilon, \mathbf{u}^*_\epsilon).$$

Then we have

Lemma 4.10.

$$\jmath^* \leq \varliminf_{\epsilon \to 0^+} \jmath^\epsilon.$$

Proof: Consider any convergence subsequence $(\eta^*_n, \mathbf{y}^*_n, \mathbf{u}^*_n)$ for $\epsilon^n \to 0^+$ (we use n to replace ϵ^n for simplicity), and suppose it converges to pair $(\eta, \mathbf{y}, \mathbf{u})$ weak star in the space $L^\infty(Q) \cap H^1(H^{-1}) \times L^2(\mathbf{V}) \cap H^1(\mathbf{V}^*) \times \mathbf{L}^2(Q)$. By similar argument as in Theorem 3.2, one can find that $(\eta, \mathbf{y}, \mathbf{u})$ satisfies (3.1) - (3.5). By definition, we have

$$\jmath^* \leq J(\eta, \mathbf{u}) \leq \varliminf_{\epsilon^n \to 0^+} \jmath^n.$$

Since the above inequality holds for any convergence subsequence with $\epsilon^n \to 0^+$, we have the conclusion. □

From above lemma, after solving the approximated optimal control problem, we find an upper bound for $\jmath^*$, but it is no guarantee on lower bound. To obtain the lower bound, we will discuss the case of $L^2(\mathbf{V}^*)$ control in the next section.

5. Optimal Control in $L^2(\mathbf{V}^*)$ Space

Define the cost function J as

$$J(\eta, \mathbf{u}) = \frac{1}{2}\|\eta - \widetilde{\eta}\|^2_{L^2(Q)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{L^2(\mathbf{V}^*)} + \frac{\beta}{3}\|\nabla \mathbf{y}\|^3_{L^3(Q)}, \tag{5.1}$$

where β is a given small positive constant. The optimal control problem associated with equation (3.1) - (3.5) is:

Problem 5.1.

$$\min J(\eta, \mathbf{u}), \quad \text{such that equations (3.1) - (3.5) hold.}$$

The regularity term $\frac{\beta}{3}\|\nabla \mathbf{y}\|^3_{L^3(Q)}$ is utilized to pass to the limit with respect to $\epsilon \to 0^+$ in the bilinear form $\eta^\epsilon \nabla \mathbf{y}^\epsilon$. This is further discussed in Remark 5.1.

Theorem 5.2. *If* $\mathbf{y}_0 \in \mathbf{H} \cap \mathbf{W}^{1,3}(\Omega)$, *then there exists at least one optimal solution which satisfies* $(\eta, \mathbf{y}, \mathbf{u}) \in L^\infty(Q) \times L^2(\mathbf{V}) \cap L^3(\mathbf{W}^{1,3}) \cap H^1(\mathbf{V}^*) \times L^2(\mathbf{V}^*)$ *for the Problem 5.1.*

Proof: We only need to check the feasibility of the Problem 5.1. Choosing $\mathbf{y}(t) = \mathbf{y}_0$, we have $\mathbf{y} \in L^2(\mathbf{V}) \cap L^3(\mathbf{W}^{1,3})$ and $\mathbf{y}_t = \mathbf{0}$. Let $\eta(t)$ satisfies the equation

$$\eta_t + \mathbf{y} \cdot \nabla \eta = 0, \quad \eta(0) = \eta_0.$$

Due to Theorem 4.1 in [11], $\eta \in L^\infty(Q)$. Then choose $\mathbf{u} \in L^2(\mathbf{V}^*)$ such that

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(\mathbf{V}^*), L^2(\mathbf{V})} = (\eta \nabla \mathbf{y}, \nabla \mathbf{v}), \quad \forall \mathbf{v} \in L^2(\mathbf{V}).$$

One can verify that $(\eta, \mathbf{y}, \mathbf{u})$ satisfies equation (3.1) - (3.5) and the desired regularity assumption. Hence the Problem 5.1 is feasible. Then following the argument as in Theorem 4.2, we have the existence of an optimal solution. □

To consider the approximating problems, we define the function spaces for viscosity and velocity as

$$X_2 = L^2(H^2) \cap L^\infty(H^1) \cap H^1(L^2), \tag{5.2}$$

$$\mathbf{Y}_2 = \{\mathbf{v} : \mathbf{v} \in L^2(\mathbf{V}) \cap L^\infty(\mathbf{H}) \cap H^1(\mathbf{V}^*), \nabla \mathbf{v} \in \mathbf{L}^3(Q)\}. \tag{5.3}$$

Recalling the definition of $\mathcal{V}_T$, we clearly have that $\mathbf{Y}_2 = \overline{\mathcal{V}_T}^{\mathbf{Y}_2}$. Recalling the operator γ_0 on $\mathcal{V}$ as $\gamma_0(\mathbf{y}) = \mathbf{y}(0)$. We denote the kernel of $\imath_y$ by $\mathcal{V}_{T,0}$. Let the closure of $\mathcal{V}_{T,0}$ in $\mathbf{Y}_2$ be $\mathbf{Y}_{2,0}$ and $\mathbf{I}_y$ be the quotient space

$$\mathbf{I}_y = \mathbf{Y}_2 / \mathbf{Y}_{2,0}.$$

One can verify the linear map γ_0 can be continuously extended to $\mathbf{Y}_2$, such that $\imath_y : \mathbf{Y}_2 \to \mathbf{I}_y$ is surjective. The Banach space $\mathbf{I}_y$ is the initial data of the function in $\mathbf{Y}_2$, and we have

$$\mathbf{W}^{1,3} \cap \mathbf{H} \subset \mathbf{I}_y \subset \mathbf{H}.$$

If the nonlinear map e^ϵ is defined as

$$e^\epsilon(\eta, \mathbf{y}, \mathbf{u}) = \begin{pmatrix} \mathbf{e}^\epsilon_{1,1}, & \mathbf{e}^\epsilon_{1,2} \\ e^\epsilon_{2,1}, & e^\epsilon_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_t - \operatorname{div}(\eta \nabla \mathbf{y}) - \mathbf{u}, & \mathbf{y}(0) - \mathbf{y}_0 \\ \eta_t - \epsilon \triangle \eta + \mathbf{y} \cdot \nabla \eta, & \eta(0) - \eta_0^\epsilon \end{pmatrix}, \tag{5.4}$$

then we have

Lemma 5.3. *The map e^ϵ acts from $X_2 \times \mathbf{Y}_2 \times L^2(\mathbf{V}^*)$ to $\begin{pmatrix} L^2(\mathbf{V}^*), & \mathbf{I}_y \\ L^2(Q), & H^1 \end{pmatrix}$. Moreover, it is Frechet differentiable.*

Proof: The proof is quite similar to the proof of Lemma 4.8. First we check that e^ϵ is well defined. Notice that $X_2 \hookrightarrow L^6(Q)$. Then for any given $(\eta, \mathbf{y}, \mathbf{u}) \in X_2 \times \mathbf{Y}_2 \times L^2(\mathbf{V}^*)$, by virtue of Lemma 4.5,

$$\|\operatorname{div}(\eta \nabla \mathbf{y})\|_{L^2(\mathbf{V}^*)} \le \|\eta \nabla \mathbf{y}\|_{L^2(Q)} \le C \|\eta\|_{L^6(Q)} \|\nabla \mathbf{y}\|_{L^3(Q)},$$
$$\eta \in X_2, \mathbf{y} \in \mathbf{Y}_2 \Rightarrow \mathbf{y} \cdot \nabla \eta \in L^2(Q).$$

And by the definition of $\mathbf{I}_y$, the initial condition is also well defined. □

The optimal control problem associated with approximated system is:

Problem 5.4.

$$\min J(\eta, \mathbf{u}), \quad \text{such that } e^\epsilon(\eta, \mathbf{y}, \mathbf{u}) = 0.$$

Similar to previous case, we have the existence of an optimal solution this problem.

Theorem 5.5. *Assume $\mathbf{y}_0 \in \mathbf{I}_y$, then for Problem 5.4, there exists at least one optimal solution which satisfies $(\eta, \mathbf{y}, \mathbf{u}) \in X_2 \times \mathbf{Y}_2 \times L^2(\mathbf{V}^*)$.*

Then let $(\eta^*_\epsilon, \mathbf{y}^*_\epsilon, \mathbf{u}^*_\epsilon)$ be one optimal solution, the existence of the Lagrange multiplier $(\xi, \mathbf{z}) \in L^2(Q) \times L^2(\mathbf{V})$ can be obtained by the similar technique as in $L^2(Q)$ control case. The key step in the proof is to show surjectivity of the map

$e_x^\epsilon(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$, i.e. for any $\begin{pmatrix} \mathbf{g}_1, & \mathbf{q}_1 \\ g_2, & q_2 \end{pmatrix} \in \begin{pmatrix} L^2(\mathbf{V}^*), & \mathbf{I}_y \\ L^2(Q), & H^1 \end{pmatrix}$, we need to verify that there exists $(\delta\eta, \delta\mathbf{y}, \delta\mathbf{u})$ which satisfies:

$$\begin{pmatrix} \delta\mathbf{y}_t - \operatorname{div}(\delta\eta\nabla\mathbf{y}_\epsilon^*) - \operatorname{div}(\eta_\epsilon^*\nabla\delta\mathbf{y}) - \delta\mathbf{u} & = \mathbf{g}_1 \\ \delta\eta_t - \epsilon\triangle\delta\eta + \delta\mathbf{y}\cdot\nabla\eta_\epsilon^* + \mathbf{y}_\epsilon^*\cdot\nabla\delta\eta & = g_2 \end{pmatrix},$$

with initial condition

$$\begin{pmatrix} \delta\mathbf{y}(0) & = \mathbf{q}_1 \\ \delta\eta(0) & = q_2 \end{pmatrix}.$$

By definition of $\mathbf{I}_y$, we can find $\delta\mathbf{y} \in \mathbf{Y}_2$, such that $\delta\mathbf{y}(0) = \mathbf{q}_1$. Then let $\delta\eta$ satisfy the equation

$$\delta\eta_t - \epsilon\triangle\delta\eta + \mathbf{y}_\epsilon^*\cdot\nabla\delta\eta = g_2 - \delta\mathbf{y}\cdot\nabla\eta_\epsilon^*, \quad \delta\eta(0) = q_2,$$

with zero Dirichlet boundary condition. Lemma 4.5 implies that $\delta\mathbf{y}\cdot\nabla\eta_\epsilon^* \in L^2(Q)$, and hence $\delta\eta \in X_2$. Let

$$\mathbf{u} = \delta\mathbf{y}_t - \operatorname{div}(\delta\eta\nabla\mathbf{y}_\epsilon^*) - \operatorname{div}(\eta_\epsilon^*\nabla\delta\mathbf{y}) - \mathbf{g}_1.$$

We have

$$\langle\delta\mathbf{u}, \mathbf{v}\rangle = \langle\delta\mathbf{y}_t, \mathbf{v}\rangle + (\delta\eta\nabla\mathbf{y}_\epsilon^*, \nabla\mathbf{v}) + (\eta_\epsilon^*\nabla\delta\mathbf{y}, \nabla\mathbf{v}) - \langle\mathbf{g}_1, \mathbf{v}\rangle, \quad \forall\mathbf{v} \in L^2(\mathbf{V}).$$

Then $X_2 \hookrightarrow L^6(Q)$ implies $\delta\mathbf{u} \in L^2(\mathbf{V}^*)$, which verifies the surjectivity of the map $e_x^\epsilon(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$. Hence there exists a Lagrange multiplier $(\mathbf{z}, \xi) \in L^2(\mathbf{V}) \times L^2(Q)$, such that the following Lagrangian is stationary at the point $(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*, \xi, \mathbf{z})$:

$$\begin{aligned} \mathcal{L}(\eta, \mathbf{y}, \mathbf{u}, \xi, \mathbf{z}) &= \frac{1}{2}\|\eta - \widetilde{\eta}\|_{L^2(Q)}^2 + \frac{\alpha}{2}\|\mathbf{u}\|_{L^2(\mathbf{V}^*)}^2 + \frac{\beta}{3}\|\nabla\mathbf{y}\|_{L^3(Q)}^3 \\ &+\langle\mathbf{z}, \mathbf{y}_t - \operatorname{div}(\eta\nabla\mathbf{y}) - \mathbf{u}\rangle_{L^2(\mathbf{V}), L^2(\mathbf{V}^*)} + (\xi, \eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta). \end{aligned} \tag{5.5}$$

Similar to Theorem 4.9, the optimal system can be obtained by taking the derivative of Lagrangian:

$$\begin{aligned} &\mathcal{L}_\xi = 0, \mathcal{L}_\mathbf{z} = 0 && \Rightarrow \text{primal equation}, \\ &\mathcal{L}_\eta = 0, \mathcal{L}_\mathbf{y} = 0 && \Rightarrow \text{adjoint equation}, \\ &\mathcal{L}_\mathbf{u} = 0 && \Rightarrow \text{optimal condition}. \end{aligned}$$

The optimal solution $(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$ and the associated Lagrange multiplier $(\xi, \mathbf{z})$ satisfy the optimality system. We can also interpret the optimality system in PDE form:

Primal Equation:

$$\begin{aligned} &\mathbf{y}_t - \operatorname{div}(\eta\nabla\mathbf{y}) = \mathbf{u}, \\ &\mathbf{y}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{y}|_{t=0} = \mathbf{y}_0, \\ &\eta_t - \epsilon\triangle\eta + \mathbf{y}\cdot\nabla\eta = 0, \\ &\eta|_{t=0} = \eta_0^\epsilon, \quad \eta|_{\partial\Omega} = m. \end{aligned}$$

Adjoint Equation:

$$\begin{aligned} &-\mathbf{z}_t - \operatorname{div}(\eta\nabla\mathbf{z}) - \eta\nabla\xi - \operatorname{div}(|\nabla\mathbf{y}|\nabla\mathbf{y}) = \mathbf{0}, \\ &-\xi_t - \epsilon\triangle\xi + \mathbf{y}\cdot\nabla\xi + \nabla\mathbf{y} : \nabla\mathbf{z} = \eta - \widetilde{\eta}, \\ &\mathbf{z}|_{t=T} = \mathbf{0}, \quad \eta|_{t=T} = 0. \end{aligned}$$

Optimality Condition:

$$\alpha\mathbf{u} = -\triangle\mathbf{z},$$

where the first equation in the primal equation and the optimality condition are in space $L^2(\mathbf{V}^*)$.

Let $(\eta^*, \mathbf{y}^*, \mathbf{u}^*)$ and $(\eta_\epsilon^*, \mathbf{y}_\epsilon^*, \mathbf{u}_\epsilon^*)$ be one minimizer for problem 5.1 and problem 5.4 respectively, and define

$$\jmath^* = J(\eta^*, \mathbf{u}^*), \quad \jmath^\epsilon = J(\eta_\epsilon^*, \mathbf{u}_\epsilon^*).$$

Then Lemma 4.10 can be improved to the following theorem.

Theorem 5.6. *If we assume that $\eta_0^\epsilon \to \eta_0$ in $L^6(Q)$, then*

$$\jmath^* = \underline{\lim}_{\epsilon\to 0^+} \jmath^\epsilon.$$

Proof: Follow the argument in Lemma 4.10, we have

$$\jmath^* \leq \underline{\lim}_{\epsilon\to 0^+} \jmath^\epsilon.$$

To check the equality, we note that for any given $\mathbf{y} \in \mathbf{Y}_2$, we can find a unique $(\eta(\mathbf{y}, \epsilon), \mathbf{u}(\mathbf{y}, \epsilon)) \in X_2 \times L^2(\mathbf{V}^*)$ which satisfies equations (3.6) - (3.10). By definition

$$\jmath^\epsilon \leq J(\eta(\mathbf{y}, \epsilon), \mathbf{y}, \mathbf{u}(\mathbf{y}, \epsilon)), \quad \forall \mathbf{y} \in \mathbf{Y}_2.$$

Particularly, we can choose $\mathbf{y} = \mathbf{y}^*$ since $\mathbf{y}^* \in \mathbf{Y}_2$. If we denote $\widehat{\eta^\epsilon} = \eta(\mathbf{y}^*, \epsilon)$ and $\widehat{\mathbf{u}^\epsilon} = \mathbf{u}(\mathbf{y}^*, \epsilon)$, then

$$\mathbf{u}^* = \mathbf{y}_t^* - \operatorname{div}(\eta^* \nabla \mathbf{y}^*), \tag{5.6}$$

$$\eta_t^* + \mathbf{y}^* \cdot \nabla \eta^* = 0, \tag{5.7}$$

$$\widehat{\mathbf{u}^\epsilon} = \mathbf{y}_t^* - \operatorname{div}(\widehat{\eta^\epsilon} \nabla \mathbf{y}^*), \tag{5.8}$$

$$\widehat{\eta^\epsilon}_t - \epsilon \triangle \widehat{\eta^\epsilon} + \mathbf{y}^* \cdot \nabla \widehat{\eta^\epsilon} = 0, \tag{5.9}$$

where equation (5.7) is completed by the initial condition (3.5) and equation (5.9) is understood with the initial and boundary condition (3.10). By a similar argument as in Corollary 3.4, we can prove that after passing to a subsequence, $\widehat{\eta^n} \to \eta^*$ in $L^6(Q)$. Subtracting (5.6) from (5.8), we have

$$\|\mathbf{u}^* - \widehat{\mathbf{u}^n}\|_{L^2(\mathbf{V}^*)} = \sup_{\mathbf{v}\in L^2(\mathbf{V})} \frac{((\eta^* - \widehat{\eta^n})\nabla \mathbf{y}^*, \nabla \mathbf{v})_{L^2(Q)}}{\|\nabla \mathbf{v}\|_{L^2(Q)}} \leq \|\eta^* - \widehat{\eta^n}\|_{L^6(Q)} \|\nabla \mathbf{y}^*\|_{L^3(Q)}.$$

Since $\widehat{\eta^n} \to \eta^*$ in $L^6(Q)$ and $\nabla \mathbf{y}^* \in L^3(Q)$, we obtain $\widehat{\mathbf{u}^n} \to \mathbf{u}^*$ in $L^2(\mathbf{V}^*)$. Then

$$\jmath^n \leq J(\widehat{\eta^n}, \mathbf{y}^*, \widehat{\mathbf{u}^n}) \to J(\eta^*, \mathbf{y}^*, \mathbf{u}^*) = \jmath^*,$$

which gives

$$\jmath^* \geq \underline{\lim}_{\epsilon\to 0^+} \jmath^\epsilon.$$

This leads to the conclusion.□

Remark 5.1. *The above treatment in this section so far is to add an extra regularization term in the cost function, for both the original problem and approximated problem. Alternatively, we can keep the cost function in the original problem as in (5.10), and modify the cost function in the approximated problem to $J^\epsilon(\eta, \mathbf{y}, \mathbf{u}) = \frac{1}{2}\|\eta - \widetilde{\eta}\|_{L^2(Q)}^2 + \frac{\alpha}{2}\|\mathbf{u}\|_{L^2(\mathbf{V}^*)}^2 + \frac{\epsilon}{3}\|\nabla \mathbf{y}\|_{L^3(Q)}^3$. Then with the additional regularity assumption $\mathbf{y}^* \in \mathbf{Y}_2$ for the optimal solution $\mathbf{y}^*$ (of the original problem), we still have Theorem 5.6.*

Open Issue.
We close our paper by describing some open issues. For this purpose, we consider the quadratic cost function

$$J(\eta, \mathbf{u}) = \frac{1}{2}\|\eta - \widetilde{\eta}\|^2_{L^2(Q)} + \frac{\alpha}{2}\|\mathbf{u}\|_{\mathbf{L}^2(Q)}, \tag{5.10}$$

and the associated optimal control problems

Problem 5.7.

$$\min J(\eta, \mathbf{u}), \quad \text{such that equations (3.1) - (3.5) hold.}$$

Problem 5.8.

$$\min J(\eta, \mathbf{u}), \quad \text{such that equations (3.6) - (3.10) hold.}$$

Let $(\eta^*, \mathbf{y}^*, \mathbf{u}^*)$ and $(\eta^*_\epsilon, \mathbf{y}^*_\epsilon, \mathbf{u}^*_\epsilon)$ be optimal solutions for Problem 5.7 and Problem 5.8 respectively, and define

$$\jmath^* = J(\eta^*, \mathbf{u}^*), \quad \jmath^\epsilon = J(\eta^*_\epsilon, \mathbf{u}^*_\epsilon).$$

Then by similar arguments as in Lemma 4.10, we have the inequality

$$\jmath^* \le \underline{\lim}_{\epsilon\to 0^+} \jmath^\epsilon. \tag{5.11}$$

In general we can not expect that equality holds in (5.11). The gap between $\jmath^\epsilon$ and $\jmath^*$ is due to the non-uniqueness of the solution to the equations (3.1) - (3.5). If assumption 5.9 holds, then the equality also holds in (5.11).

Assumption 5.9. There exists an optimal solution $(\eta^*, \mathbf{y}^*, \mathbf{u}^*)$ for Problem 5.7 which can be approximated by triples $(\eta^n, \mathbf{y}^n, \mathbf{u}^n)$ which satisfy the system (3.6) - (3.10) (with $\epsilon = \epsilon^n$), where $\epsilon^n \to 0^+$. The approximation for η^* and $\mathbf{u}^*$ are in the strong topology of $L^2(Q)$.

To verify that equality holds in (5.11), we only need to prove

$$\jmath^* \ge \underline{\lim}_{\epsilon\to 0^+} \jmath^\epsilon.$$

By assumption 5.9, we can find a sequence of ϵ^n, such that $(\eta^n, \mathbf{y}^n, \mathbf{u}^n)$ solves system (3.6) - (3.10) with $\epsilon = \epsilon^n$, and $(\eta^n, \mathbf{u}^n) \to (\eta^*, \mathbf{u}^*)$ in the strong topology of $L^2(Q) \times \mathbf{L}^2(Q)$. Hence

$$\jmath^n \le J(\eta^n, \mathbf{u}^n) = \frac{1}{2}\|\eta^n - \widetilde{\eta}\|^2_{L^2(Q)} + \frac{\alpha}{2}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(Q)} \to J(\eta^*, \mathbf{u}^*) = \jmath^*.$$

This implies $\jmath^* \ge \underline{\lim}_{\epsilon\to 0^+} \jmath^\epsilon$.

The investigation of assumption 5.9 is still an open issue.

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