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We show that for the originally ill-posed inverse problem of calibrating a localized jump-diffusion process to given option price data, Tikhonov regularization can be used to get a well-posed optimization problem. Furthermore we prove stability as well as convergence of the regularized parameters using the forward partial integro-differential equation associated to the European call price. By providing a precise link between these parameters and the corresponding market models we are able to extend the results to the associated market models and hence to the model prices of exotic derivatives. Finally we indicate some possible applications.

1 Introduction

Since the work of Dupire [16] and Derman & Kani [14] the so called local volatility model has been a popular model among practitioners for pricing and in particular for mark to model purposes. The popularity on the one side stems from the fact that for the local volatility model simple and efficient option pricing schemes are available in the literature. However, the main reason for the popularity is its extreme flexibility with respect to the implied volatility surface, i.e. nearly any arbitrage-free European option prices can be obtained. In fact the marginals of any Itô process, in particular also processes depending on a multi-dimensional Brownian motion, can be mimicked by the solution to a stochastic differential equation with local volatility (see Gyöngy [25]).

The local volatility model also has severe drawbacks, though. For example the future implied volatility structure is implausible, which results from the typically observed flattening of the calibrated local volatility over time. This is especially problematic, because thereby also the skewness of the log-returns vanishes. Furthermore the asset price is a.s. continuous and hence the empirically observed price jumps cannot be adequately modeled leading to a sometimes serious underestimation of the downside-risk.

To overcome some of the above mentioned problems Carr et al. [8] introduced the local Lévy model. It is also an inhomogeneous model, but a more general stochastic process than the geometric Brownian motion, i.e. an exponential Lévy process, serves as base model. Hence also the homogeneous model is capable of describing skewed log-returns. Another advantage is that

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exponential Lévy models are known to fit the European option price surface much better than the Black-Scholes model is able to. Thus the necessary corrections of the model by means of the local functions may be expected to be less incisive than in the local volatility mode. Moreover the local Lévy model obviously also allows for jumps, i.e. the asset price lives on the space of càdlàg functions.

Having specified a market model for the asset-price a reliable calibration method is of course a prerequisite for using it to evaluate a portfolio. In order to guarantee a consistent calibration one should use available market data, which in equity markets typically are the quoted prices of standard European options. Among the most prominent calibration procedures are least-squares approaches, minimizing in some way the quadratic error between model and market prices of options (see e.g. Andersen & Andreasen [2] or Schoutens [38]), or equivalent martingale measure approaches, that minimize the distance of the equivalent risk-neutral measure to the objective one (see e.g. Avellaneda [3], Frittelli [23] or Goll & Rüschendorf [24]).

The above-mentioned approaches mainly deal with parametric models, i.e. models that depend on a finite set of parameters. In the Dupire-model and in the local Lévy model case the parameter is an infinite-dimensional object. Hence it is not straightforward to find a good and computationally tractable calibration procedure. Nevertheless, in [8] a formula to back out the local speed function, to be defined later, from observed option prices was found. Unfortunately this formula is valid only under special restrictions and a differentiation of the input data, i.e. the European option prices, is required. Hence the discrete data has to be interpolated and the calibrated parameter will, of course, depend heavily on the chosen interpolation method. The same problem also arises in the Dupire model, because also the famous Dupire formula, that relates the local volatility to the price of butterfly spreads of infinitesimal length, requires a differentiation of the data. Moreover the parameter identification problems in these models are ill-posed in general. That means that it is not clear, whether a solution exists and even if existence can be shown, the solution, i.e. the calibrated parameters, will not depend continuously on the data, which is especially problematic in the case of noisy data (see e.g. Engl [20]). At a first sight this seems to be a rather academic issue. However, suppose that an investor uses an inhomogeneous Markov model for mark-to-market purposes. She will expect that the model value of her portfolio roughly remains unchanged, if the market she is invested in only admits small changes. Nevertheless, this is not guaranteed, if the calibration procedure is unstable: Due to the absence of a continuous inverse the difference in the parameters may be large, which may dramatically affect the prices of exotic and path-dependent options and thus the portfolio value.

The technique of regularization of the original problem is well known to be able to overcome these stability problems and was successfully applied in various types of problems arising in physics, mechanics or medicine. Also in the finance literature, recently, regularization techniques were applied successfully to calibration problems; especially the local volatility model has attracted much attention (see e.g. Lagnado & Osher [31], Crepey [13]. Egger & Engl [17], Hein [26], Jiang et al. [29], Achdou & Pironneau [1], Egger et al. [18] and references therein). Another stream of research was followed by Cont & Tankov [10, 11], who calibrated the Lévy measure of an exponential Lévy process to market data and used the relative entropy with respect to the objective probability measure for the regularization. Belomestny & Reiss [4, 5] used a different kind of regularization to calibrate exponential Lévy market models using the characteristic function of the marginal distributions.

In this paper we discuss the calibration problem of generalized market models, that include, amongst others, the above-mentioned local volatility and jump-diffusion type models as special cases. Therefore we rely on the approach of Kindermann et al. [30] and use a parabolic partial integro-differential equation (PIDE) as definition of the European call price. In [30] it was shown that the local speed function can be robustly identified by applying Tikhonov-regularization. Here we generalize this result and consider the problem of identifying all model parameters, especially also the Lévy measure. Furthermore, in contrast to [30], we also discuss the market models associated to the identified parameters: In particular we prove existence of the calibrated asset price process, i.e. solvability of the corresponding stochastic integral equation, and provide stability and convergence results not only for the parameters, but also for the market models themselves.

The rest of the paper is structured as follows: In Section 2 we formally introduce the inverse problem and show that the call price, defined as expectation of a function of the asset price model, solves a certain PIDE (cf. [30]). In Section 3 we will show that the forward operator can equivalently to the probabilistic definition be defined via this PIDE and in Section 4 we explore some properties of this forward operator. Section 5 introduces the regularization technique and proves stability and convergence results for the regularized problem. Finally, in Section 6 we indicate some possible applications and conclude.

2 Model setup and the calibration problem

Let us now formally introduce the local Lévy model. Since we are interested in a pricing measure on the space of the càdlàg functions, we model the asset price S_t directly in the risk-neutral setting. Following Carr et al. [8] S_t is modeled as exponential of an inhomogeneous Markov process X_t of Lévy type, i.e.

$$S_T = S_0 \mathrm{e}^{(r-\eta)T} \mathrm{e}^{X_T}. \tag{1}$$

where r is the riskless interest rate and η is the dividend yield. The driving process X_t is defined as the solution of the stochastic integral equation

$$\begin{aligned} X_T = &-\int_0^T \frac{1}{2}\sigma^2(X_{t-}+(r-\eta)t,t)\ dt + \int_0^T \sigma(X_{t-}+(r-\eta)t,t)\ dW_t \\ &- \int_{(0,T]\times\mathbb{R}} \left(\mathrm{e}^x - 1 - x1_{\{x\geq -1\}}\right) \mu_{(X_{t-}+(r-\eta)t,t)}(dx,dt) \\ &+ \int_{(0,T]\times\mathbb{R}} x1_{\{x\geq -1\}} \left(m_{(X_{t-}+(r-\eta)t,t)}(dx,dt) - \mu_{(X_{t-}+(r-\eta)t,t)}(dx,dt)\right) \\ &+ \int_{(0,T]\times\mathbb{R}} x1_{\{x<-1\}} m_{(X_{t-}+(r-\eta)t,t)}(dx,dt), \end{aligned} \tag{2}$$

where W is a standard Brownian motion, m is the integer valued-random measure associated to the jumps of X independent of W and μ is its compensator, which is assumed to fulfill:

$$\mu_{(k,t)}(dt,dx) = a(k,t)\nu(dx)dt. \tag{3}$$

By this last assumption the distribution of the jumps remains unchanged over time, while the arrival rate does vary with time and asset price. This can be interpreted as modifying the speed of the jump term and thus in the sequel a will be called local speed function.
The existence of a solution to the stochastic integral equation (2) is not straightforward and we will discuss this issue in Section 3.

Note that the "truncation function" in (2) was chosen to be $x1_{\{x\geq -1\}}$. This is motivated by the observation that the expectation of the stock price is necessarily finite and hence $\int_{\mathbb{R}} \mathrm{e}^x \mu(dx) < \infty$ (cf. [36, Theorem 25.3]). However, since $\mathbb{E}[\log(S_t)]$ does not need to exist in general we cannot simply take x as truncation function above.
For the stock price process S_t an application of Itô's lemma shows that it can be described by

$$\begin{aligned} S_T &= S_0 + \int_0^T (r-\eta) S_{t-}\, dt + \int_0^T \sigma(\log(S_{t-}), t) S_{t-}\, dW_t \\ &\quad + \int_{(0,T]\times\mathbb{R}} S_{t-}\left(\mathrm{e}^x - 1\right)\left(m_{(\log(S_{t-}),t)}(dt, dx) - \mu_{(\lg(S_{t-}),t)}(dt, dx)\right). \end{aligned} \tag{4}$$

Hence $\mathrm{e}^{-(r-\eta)t} S_t$ is in fact a martingale, as necessary in the risk-neutral setting.

Remark 1. *For notational convenience we will always assume that r and η are constants. The analysis can, however, easily be extended to the case where r and η are deterministic functions of time.*

The market model (4) is very general and includes amongst others, the classic Black-Scholes market model, local volatility and local jump-diffusion models with possibly state- and time-dependent parameters. However, also due to the generality of the above model, it is rather tricky to calibrate. In order to calibrate the model in a consistent way, one should use market data, i.e. prices of traded options. The calibration procedures described in Dupire [16] or Carr et al. [8] only apply in special cases and require multiple derivatives of the data. Here we want to introduce and discuss a calibration method, which is more stable and does not rely on the differentiation of the data.
We assume that the given market data are prices of European call and put options with various strikes and maturities. The call-put parity allows us to concentrate solely on call options with model prices

$$\mathrm{C}(K,T) = \mathbb{E}_Q\left[\mathrm{e}^{-rt}(S_T - K)^+\right], \tag{5}$$

where K is the strike, T the maturity of the call and S_T is given by (1).
The free parameters in the asset price model, that have to be identified, are

- σ^2: the local volatility function, which we assume to fulfill $\sigma^2 \geq c > 0$.
- $a \geq 0$: the local speed function governing the speed at which the pure jump-component is running.
- ν: a Lévy measure, i.e. a Radon measure satisfying $\nu(\{0\}) = 0$ and

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\nu(dx) < \infty.$$

Our aim is to calibrate these parameters in a consistent way.
The calibration problem hence can be seen as follows: find parameters

$$\theta = (\sigma^2, a, \nu)$$

in equation (2), such that the model prices $\mathrm{C}(K,T)$ match the observed market prices for the liquid strikes and maturities.
To define the problem in a more formal and abstract way, we define the forward operator G:

$$G : \theta = (\sigma^2, a, \nu) \rightarrow \mathrm{C}^{(\theta)}(K,T), \tag{6}$$

where $\mathrm{C}^{(\theta)}$ is the model price of a European call defined by (5).
Then denoting the market price of a European call option with strike K and maturity T by $\mathrm{C}^{\mathrm{M}}(K,T)$ the calibration problem is to find a triple $\bar{\theta}$, such that

$$G^{(\bar{\theta})}(K,T) = \mathrm{C}^{\mathrm{M}}(K,T) \qquad (K,T) \in \mathcal{D}, \tag{7}$$

where $\mathcal{D}$ denotes the set of all pairs of maturities and strikes, for which data are available.
Furthermore $\bar{\theta}$ has to be such that the stochastic integral equation (2) admits a solution to ensure the existence of a market model associated to the identified parameter.
There are several obvious questions raised by formulation (7):

- Does there exist a solution $\bar{\theta}$?
- Is there a computationally tractable way to find $\bar{\theta}$?
- Does the solution depend continuously on the given data?

To answer these questions we reformulate the calibration problem and in particular modify the forward operator to obtain an analytically more tractable problem, for which parameter identification problems are better understood. In our setting this will result in using a certain PIDE as definition of the forward operator, which, of course, has to be shown to be in some sense equivalent to (6). In the context of the local volatility model this procedure was for example used by Crepey [13] and Egger & Engl [17].
As it turns out that the call price itself is not integrable, which leads to problems in defining a suitable solution space for the PIDE, we consider the following transformation of the call price (cf. Matache et al. [32, 33]).

$$c^{(\theta)}(k,T) = \mathrm{e}^{\eta T}\,\mathrm{C}^{(\theta)}(\mathrm{e}^k,T) - \left(S_0 - \mathrm{e}^k\right)^+, \tag{8}$$

where S_0 is the current asset price.
A PIDE for $c^{(\theta)}(k,T)$ can be derived in a similar manner as for $\mathrm{C}^{(\theta)}(K,T)$ itself, as was done in Carr et al. [8] and Carr & Javaheri [9]. For the convenience of the reader we replicate the calculations adopted to our setting here. First the following special form of the Meyer-Itô formula (cf. Protter [34, Theorem IV.68]) is used

$$\begin{aligned}(S_T - K)^+ = (S_0 - K)^+ &+ \int_{0+}^T 1_{\{S_{t-}>K\}}\, dS_t + \frac{1}{2}\int_{0+}^T \delta(S_{t-} - K)\sigma_0^2(S_{t-},t)S_{t-}^2\, dt \\ &+ \sum_{0<t\le T} \left(1_{\{S_{t-}>K\}}(K - S_t)^+ + 1_{\{S_{t-}\le K\}}(S_t - K)^+\right),\end{aligned}$$

where the second integral is the continuous part of the local time of S_t, which is formally expressed through Dirac's delta function δ here.
Taking expectations (with the respect to the risk-neutral measure) on both sides, using that e^{X_t}

is a martingale and plugging in (3) yields:

$$
\begin{aligned}
\mathrm{e}^{rT}\,\mathrm{C}(K,T) = (S_0-K)^+ &+ \int_0^T (r-\eta)\mathbb{E}\left[S_{t-}1_{\{S_{t-}>K\}}\right]\,dt \\
&+ \frac{1}{2}\int_0^T \mathbb{E}\left[\delta_K(S_{t-})\sigma_0^2(S_{t-},t)S_{t-}^2\right]\,dt \\
&+ \int_0^T \mathbb{E}\left[1_{\{S_{t-}>K\}}a_0(S_t,t)\int_{-\infty}^{\log(K/S_{t-})}(K-S_t e^x)\,\nu(dx)\right]\,dt \\
&+ \int_0^T \mathbb{E}\left[1_{\{S_{t-}\le K\}}a_0(S_{t-},t)\int_{\log(K/S_{t-})}^{\infty}(S_{t-}e^x-K)\,\nu(dx)\right]\,dt.
\end{aligned} \tag{9}
$$

Note that the interchange of the integrals is justified by Tonelli's theorem and the that last two terms are implied by the definition of the compensator.
As every integrand in (9) is $\mathcal{F}_0$-measurable (due to the expectations) and the time integrals are taken with respect to the Lebesgue measure, we can replace S_{t-} by S_t without changing the value of the integrals.
Next we exploit the following property of the call price $\mathrm{C}(K,T)$ (cf. [6]):

$$
\frac{\partial}{\partial K}\,\mathrm{C}(K,T) = \mathrm{e}^{-rT}(F_{S_T}(K)-1), \quad \text{i.e.} \quad F_{S_T}(K) = \mathrm{e}^{rT}\frac{\partial}{\partial K}\,\mathrm{C}(K,T)+1, \tag{10}
$$

where $\frac{\partial}{\partial K}\,\mathrm{C}(K,T)$ denotes the right derivative of the callprice with respect to the strike and $F_{S_T}(K) = Q[S_T \le K]$ denotes the cumulative distribution function of the asset price S_T at time T given S_0.
Since F_{S_T} might not be differentiable (see [12] for details on smoothness in the homogeneous case), in general there is no density function. However, we can make use of distribution theory (see e.g. [40]) and differentiate equation (10) formally to obtain

$$
\frac{d}{dK}F_{S_T}(K) = \mathrm{e}^{rT}\frac{\partial^2}{\partial K^2}\,\mathrm{C}(K,T).
$$

Note that the above equation generally holds in the sense of distributions and both sides may be generalized functions (or measures) instead of proper ones. This is one reason why we will deal with weak solution of PIDE's later on.
From equation (9) it is easily derived, that for $c^{(\theta)}$ similar relations hold, i.e.:

$$
\begin{aligned}
F_{S_T}(K) &= \frac{\mathrm{e}^{(r-\eta)T}}{K}\left(c_k^{(\theta)}(\log(K),T)+g_k(\log(K))\right)+1, \\
\frac{d}{dK}F_{S_T}(K) &= \frac{\mathrm{e}^{(r-\eta)T}}{K^2}\Big(c_{kk}^{(\theta)}(\log(K),T)+g_{kk}(\log(K)) \\
&\qquad - c_k^{(\theta)}(\log(K),T)-g_k(\log(K))\Big) =: f(\log(K),T),
\end{aligned} \tag{11}
$$

where $g(k) := \left(S_0-\mathrm{e}^k\right)^+$, $c_k^{(\theta)}$, g_k and $c_{kk}^{(\theta)}$, g_{kk} are the right and the second derivatives in sense of distributions with respect to k, respectively.
The derivatives of g with respect to k in sense of distributions are given by:

$$
\begin{aligned}
g_k(k) &:= \frac{d}{dk}(S-e^k)^+ = \begin{cases} -e^k & \text{for } k<\log(S) \\ 0 & \text{else} \end{cases} \\
g_{kk}(k) &:= \frac{d^2}{dk^2}(S-e^k)^+ = \begin{cases} -e^k & \text{for } k<\log(S) \\ S_0\delta(\log(S_0)-k) & \text{else.} \end{cases}
\end{aligned}
$$

Before proceeding we introduce the double-exponential tail ψ of the Lévy measure ν (cf. Carr et al. [8])

$$\psi(z) = \begin{cases} \int_{-\infty}^{z} (e^z - e^x)\, \nu(dx) & \text{for } z < 0 \\ \int_{z}^{\infty} (e^x - e^z)\, \nu(dx) & \text{for } z > 0. \end{cases} \tag{12}$$

One may recognize this function in the last two terms of equation (9), coming from the pure jump part of the process.
Plugging equations (11) and (8), as well as the definition of ψ, into (9) and differentiating with respect to T results in

$$\begin{aligned}(r-\eta)\mathrm{e}^{(r-\eta)T}\left(c^{(\theta)}(\log(K),T) + g(\log(K))\right) + \mathrm{e}^{(r-\eta)T} c_T^{(\theta)}(\log(K),T) = \\ (r-\eta)\int_K^\infty Y f(\log(Y),T)\ dY + \frac{1}{2}\sigma_0^2(K,T)K^2 f(\log(K),T) \\ + \int_K^\infty Y\psi(\log(K/Y))a_0(Y,T)f(\log(Y),T)\ dY \\ + \int_0^K Y\psi(\log(K/Y))a_0(Y,T)f(\log(Y),T)\ dY.\end{aligned}$$

Hence we find, writing k instead of $\log(K)$ and substituting $y = \log(Y)$ in the integrals

$$\begin{aligned} c_T^{(\theta)}(k,T) = & -(r-\eta)\left(c_k^{(\theta)}(k,T) + g_k(k)\right) \\ & + \frac{1}{2}\sigma^2(k,T)\left(c_{kk}^{(\theta)}(k,T) + g_{kk}(k) - c_k^{(\theta)}(k,T) - g_k(k)\right) \\ & + \int_{-\infty}^{\infty} \psi(k-y)a(y,T)\left(c_{kk}^{(\theta)}(y,T) + g_{kk}(y) - c_k^{(\theta)}(y,T) - g_k(y)\right)\ dy. \end{aligned} \tag{13}$$

Now introducing, for notational convenience, the following operators:

$$\begin{aligned} \mathcal{I}_\psi u &:= \psi * u = \int_{-\infty}^{\infty} \psi(k-y)u(y)\ dy \\ \mathcal{L}_\theta u &:= \left(r - \eta + \frac{\sigma^2(k,T)}{2}\right)u_k - \frac{\sigma^2(k,T)}{2}u_{kk} \\ &\quad -\mathcal{I}_\psi\left(a(.,T)(u_{kk} - u_k)\right), \end{aligned}$$

equation (13) can be written as

$$c_T^{(\theta)} + \mathcal{L}_\theta c^{(\theta)}(.,T) = -\mathcal{L}_\theta g, \tag{14}$$

which is a PIDE in the weak sense, as we have seen that $c^{(\theta)}$ may be and g is twice differentiable only in sense of distributions.
The no arbitrage assumption implies the following initial and boundary conditions, with which we supplement the above PIDE:

$$c^{(\theta)}(k,0) \equiv 0 \tag{15}$$

$$c^{(\theta)}(-\infty,T) = 0 \tag{16}$$

$$c^{(\theta)}(\infty,T) = 0. \tag{17}$$

In Carr et al. [8] a PIDE for the call price itself was derived, which reads

$$\begin{aligned} C_T &= -\eta C - (r-\eta)KC_K + \frac{\sigma_0^2(K,T)}{2}K^2 C_{KK} && (18)\\ &+ \int_0^\infty Y C_{KK}(Y,T)a_0(Y,T)\psi\left(\log\left(\frac{K}{Y}\right)\right)dY \qquad \text{on } \mathbb{R}^+\times[0,T^*]. && (19)\end{aligned}$$

The corresponding initial and boundary conditions are

$$C(K,0) = (S-K)^+ \tag{20}$$

and

$$C(0,T) = e^{-\eta T}S, \quad C(\infty,T)=0. \tag{21}$$

Turning to the calibration problem we now want to express the calibration problem in terms of the PIDE (14), since this formulation is analytically more tractable. Thus we define the forward operator H by

$$H:\theta=(\sigma^2,a,\nu) \quad\rightarrow\quad c^{(\theta)}(k,T) \quad k\in\mathbb{R},\ T\in[0,T^*] \tag{22}$$

with $c^{(\theta)}$ being the weak solution of the developed PIDE (14)–(17).
Of course H has to be shown to be equivalent in some sense to G. Assuming this equivalence for a moment, the calibration problem is then to find $\bar{\theta}$, such that

$$H(\bar{\theta})(k,T) = \mathrm{e}^{T\eta}\,\mathrm{C}^{\mathrm{M}}(\mathrm{e}^k,T) - (S_0-\mathrm{e}^k)^+ \qquad (\mathrm{e}^k,T)\in\mathcal{D} \tag{23}$$

holds with $\mathcal{D}$ being again the set of strikes and maturities, for which plain vanilla options are liquid.
In the following, unless explicitly stated otherwise, we refer to (22), when speaking about the forward operator.

3 Well-posedness of the forward operator

In the preceding section we assumed the existence of a market model with given parameters θ and although the forward operator is defined by a solution to a PIDE and (23) can be understood as the identification problem in the PIDE, we have to keep in mind, that ultimately we are interested in finding a market model, i.e. a weak solution to the stochastic integral equation (4), that explains the market option prices. Fortunately we can find constraints on the parameters, that guarantee that (4) is in fact really solveable.

Proposition 1. *If $\sigma^2 \geq c > 0$ and a are bounded and continuous, then there exists a unique (in the sense of Jacod and Shiryaev [28, III.2.37]) stochastic process admitting the representation (2).*

Proof. Using (3) the characteristics of X (in the sense of [28]) are given by

$$\begin{aligned} B_t(\omega) &= \int_0^t b(X_s(\omega),s)\,ds\\ \Gamma_t(\omega) &= \int_0^t \gamma(X_s(\omega),s)\,ds\\ \mu(\omega;dt\times dx) &= dt\times K_t(X_t(\omega),dx),\end{aligned}$$

where

$$\left.\begin{array}{l} b(X_t,t) = -\frac{1}{2}\sigma^2(\tilde{X}_{t-},t) - a(\tilde{X}_t,t)\int_{\mathbb{R}}\left(\mathrm{e}^z - 1 - z1_{\{z\geq -1\}}\right)\nu(dz) \\ \gamma(X_t,t) = \sigma(\tilde{X}_t,t) \\ K_t(X_t(\omega),dx) = a(\tilde{X}_t,t)\nu(dx) \end{array}\right\} \tag{24}$$

and $\tilde{X}_t = X_t + (r-\eta)t$.
Note that (24) describes a jump-diffusion in the sense of Jacod & Shiryaev [28, Definition III.2.18]. Furthermore b_t is bounded, because $(\mathrm{e}^z - 1 - z1_{|z|\leq 1}) \sim z^2$ for $z \to 0$ and $\int_1^\infty \mathrm{e}^x \nu(dx) < \infty$. Moreover by assumption the mapping $(y,t) \mapsto \sigma^2(y+(r-\eta)t,t)$ is bounded, continuous and also everywhere strictly positive, while the functions

$$a: \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R} \qquad (y,t) \mapsto a(y+(r-\eta)t,t)\int_A (|z^2 \wedge 1|)\nu(dx)$$

are continuous and bounded for all $A \in \mathfrak{B}(\mathbb{R})$.
Thus Theorem III.2.34 [28] (see Stroock [41] for a proof of that theorem using the infinitesimal generator of the process) can be applied and yields the existence of a unique solution to the martingale problem associated to the characteristics (24) and thus also a unique weak solution to the stochastic integral equation (2). Using [28, Corollary III.2.41] we also have the uniqueness in the claimed sense. □

Section 2 showed, that the $c^{(\theta)}$ defined by (8) fulfills (14) – (17) in the weak sense. However, in order to define the calibration problem as a parameter identification problem in a PIDE, we have to show that the solution to the PIDE is unique and that the call price is in the same space as this solution. Now we will prove that under certain assumptions on the parameters this is fulfilled and hence a one-to-one correspondence of the probabilistic definition (8) and the solution of the PIDE can be established in a certain space to be defined later.
Denoting the space of continuous and bounded functions by C_B, we pose the following assumptions.

Assumptions:

$$\sigma^2(k,T) \in C_B(\mathbb{R}\times[0,T^*]), \quad \sigma_k^2(k,T) \in L^\infty([0,T^*],L^2(\mathbb{R})). \tag{25}$$

$$a(k,T) \in C_B(\mathbb{R}\times[0,T^*]), \quad a_k(k,T) \in L^\infty([0,T^*],L^2(\mathbb{R})). \tag{26}$$

$$\nu \text{ is a Lévy measure fulfilling } \int_{x\geq 1} x\mathrm{e}^x\nu(dx) < \infty. \tag{27}$$

$$a(k,T) \geq 0 \quad \text{for } (k,T) \in \mathbb{R}\times[0,T^*]. \tag{28}$$

$$\sigma^2(k,T) \geq c > 0 \quad \text{for } (k,T) \in \mathbb{R}\times[0,T^*]. \tag{29}$$

Some remarks are in order to discuss the above-made assumptions.

Remark 2. *The assumptions on a and σ^2 by Proposition 1 in particular imply the existence of a market model associated to the given parameters. That is an important point, since in this way the calibrated parameters, which will be obtained by a parameter identification in a PIDE, in fact correspond to a model for the asset price.*

Remark 3. *Assumption* (26) *is equivalent ([36, Theorem 25.3]) to an assumption on the asset price defined in* (4)*, namely that*

$$\mathbb{E}\left[S_t \ln S_t\right] < \infty \quad \textit{for} \quad 0 \leq t \leq T^*.$$

Remark 4. *Neither of the assumptions* (25)–(28) *is very restrictive. Assumptions* (25) *and* (26) *are standard for inverse problems and imply some regularity of the parameter functions governing the stochastic process described by* (4)*. Condition* (28) *is posed in order to get a reasonable market model – in this way a can be interpreted as deterministic time change of the jump part of the asset price process. Since by the no-arbitrage assumption* $\mathbb{E}[S_t] = e^{(r-\eta)t}S_0 < \infty$ *also* (26) *is a rather mild extra-assumption. The only restraining assumption on the asset price process is* (29)*, since it rules out pure jump processes.*

As mentioned before, we want to show, that the PIDE (14) with homogeneous boundary conditions has a unique solution and that this solution is given by (5). In this way we can define the forward operator H via the PIDE. In the theory of PDE's and PIDE's, unique solvability can be defined in various ways. In the context of PIDE's arising from Lévy market models in finance, Cont & Voltchkova [12] considered viscosity solutions, while Matache et al. [32, 33] concentrated on weak solutions. Here we follow the latter approach and prove that the variational formulation of the PIDE (14) has a unique solution in the space $W_2^1(0,T^*)$, where, denoting the dual of $H^1(\mathbb{R})$ by $\left(H^1(\mathbb{R})\right)'$,

$$Y := W_2^1(0,T^*) = \Big\{ f | f \in L^2\left([0,T^*], H^1(\mathbb{R})\right); \frac{df}{dt} \in L^2\left([0,T^*], \left(H^1(\mathbb{R})\right)'\right); \max_T \|f\|_{L^2} < \infty \Big\}$$

and

$$H^1(\mathbb{R}) := \left\{ f \in L_2(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} |f(x)|^2 dx + \int_{\mathbb{R}} |f'(x)|^2 dx < \infty \right\}$$

is the usual Sobolev space.
For notational convenience we also introduce the space

$$Y' = L^2((0,T^*), \left(H^1(\mathbb{R})\right)').$$

In [30] the unique solvability of the PIDE was proved under a slightly stronger assumption than (25), namely

$$\sigma^2(k,T) \in L^\infty([0,T^*], W^{1,\infty}(\mathbb{R})).$$

However this does not affect the crucial part of the proof and the only extra ingredient needed is given by the following proposition.

Proposition 2. *For any* $\epsilon > 0$ *and* $u \in H^1(\mathbb{R})$

$$\left| \int_{\mathbb{R}} \sigma_k^2(k,T) u_k(k) u(k)\, dk \right| \le \|\sigma_k^2(.,T)\|_{L^2} \left(R(\epsilon)\|u\|_{L^2}\|u\|_{H^1(\mathbb{R})} + \epsilon \|u\|^2_{H^1(\mathbb{R})} \right)$$

with a positive $R(\epsilon)$ *depending on* ϵ.

Proof. By the Hoelder inequality

$$\left| \int_{\mathbb{R}} \sigma_k^2(k,T) u_k(k) u(k)\, dk \right| \le \|\sigma_k^2(.,T)\|_{L^2} \|u_k\|_{L^2} \|u\|_{L^\infty}. \tag{30}$$

Using the Sobolev embedding theorem, the interpolation inequality and Young's inequality we have for $1 > s > 1/2$ the estimate:

$$\|u\|_{L^\infty} \le R_1 \|u\|_{H^s} \le R_2 \|u\|_{H^1}^s \|u\|_{L^2}^{1-s} \le \left(R(\epsilon)\|u\|_{L^2} + \epsilon\|u\|_{H^1}\right).$$

Plugging the above in inequality (30) leads to the desired result. □

With the above proposition and following [30] we have that the operator $\mathcal{L}_\theta$ is coercive and continuous:

Proposition 3. *Let assumptions* (25)-(29) *hold, then there exist some constants* $b > 0$, γ *and* B *such that*

$$(-\mathcal{L}_\theta u, u)_{((H^1(\mathbb{R}))', H^1(\mathbb{R}))} \geq b\|u\|^2_{H^1(\mathbb{R})} - \gamma\|u\|^2_{L^2} \tag{31}$$

and

$$\left|(\mathcal{L}_a u, v)_{((H^1(\mathbb{R}))', H^1(\mathbb{R}))}\right| \leq B\|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})}. \tag{32}$$

Hence, standard parabolic theory gives the following

Theorem 1. *Under the assumptions of Proposition 3 there exists a unique weak solution* $c^{(\theta)} \in Y$ *of* (14)*, i.e. a solution satisfying*

$$(c^{(\theta)}_T, \phi)_{L^2} + (\mathcal{L}_\theta c^{(\theta)}, \phi)_{L^2} = (-\mathcal{L}_\theta g, \phi)_{L^2} \quad \forall \phi \in H^1(\mathbb{R}),\ T > 0, \tag{33}$$

$$c^{(\theta)}(.,0) \equiv 0, \quad c^{(\theta)}(-\infty,.) = 0, \quad c^{(\theta)}(\infty,.) = 0 \tag{34}$$

and there are constants R, R' *such that*

$$\|c^{(\theta)}\|_Y \leq R\|\mathcal{L}_\theta g\|_{Y'} \leq R'. \tag{35}$$

Note that Matache et al. [32, 33] showed a similar result for the backward equation under different parameter assumptions.
For the probabilistic definition of $c^{(\theta)}$ one can show

Proposition 4. *Let the assumptions* (25) – (28) *hold and* (σ^2, a, ν) *be such that there exists a semimartingale with representation* (4)*, then* $c^{(\theta)}$ *defined by* (8) *fullfills:*

$$c^{(\theta)}(.,T) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \tag{36}$$

$$\|c^{(\theta)}_k(.,T)\|_{L^\infty(\mathbb{R})} \leq S_0 \quad \text{and} \quad \|c^{(\theta)}_k(.,T)\|_{L^1(\mathbb{R})} \leq 2S_0, \tag{37}$$

where c_k *(as in Section 2) corresponds to the right derivative of the transformed call price.*

Proof. We start by proving (36). Note that

$$\begin{aligned}\left\|c^{(\theta)}(.,T)\right\|_{L^\infty} &= \sup_k \left(\mathrm{e}^{\eta T}\, \mathrm{C}(\mathrm{e}^k, T) - \left(S_0 - \mathrm{e}^k\right)^+\right) \\ &\leq \sup_k \left(\max\left\{\mathrm{e}^{\eta T}\, \mathrm{C}(\mathrm{e}^k, T), \left(S_0 - \mathrm{e}^k\right)^+\right\}\right) = S_0,\end{aligned}$$

where we used the no-arbitrage bound $0 \leq \mathrm{C}(\mathrm{e}^k, T) \leq \mathrm{e}^{-\eta T} S_0$.
Concerning the L^1-bound of $c^{(\theta)}$ splitting up the integral yields

$$\begin{aligned}\|c^{(\theta)}(.,T)\|_{L^1} &= \int_{\mathbb{R}} \left|\mathrm{e}^{-(r-\eta)T}\mathbb{E}\left[\left(S_T - \mathrm{e}^k\right)^+\right] - \left(S_0 - \mathrm{e}^k\right)^+\right| \, dk \\ &= \int_{-\infty}^{\log(S_0)} \left|\mathrm{e}^{-(r-\eta)T}\mathbb{E}\left[\left(S_T - \mathrm{e}^k\right)^+\right] - \left(S_0 - \mathrm{e}^k\right)\right| \, dk \\ &\quad + \int_{\log(S_0)}^{\infty} \mathrm{e}^{-(r-\eta)T}\mathbb{E}\left[\left(S_T - \mathrm{e}^k\right)^+\right] \, dk.\end{aligned} \tag{38}$$

Since $\mathbb{E}[\mathrm{e}^{-(r-\eta)T}S_T] = S_0$, for the first term we find

$$\int_{-\infty}^{\log(S_0)} \left| \mathrm{e}^{-(r-\eta)T}\mathbb{E}\left[\left(S_T - \mathrm{e}^k\right)^+\right] - \left(S_0 - \mathrm{e}^k\right)\right| \, dk =$$

$$\begin{aligned}
&\int_{-\infty}^{\log(S_0)} \left| \int_{\mathrm{e}^k}^{\infty} \mathrm{e}^{-(r-\eta)T} \left(s - \mathrm{e}^k\right) \, dF_{S_T}(s) - \left(S_0 - \mathrm{e}^k\right)\right| \, dk \\
=\ & \int_{-\infty}^{\log(S_0)} \left| \mathrm{e}^k \left(1 - \mathrm{e}^{-(r-\eta)T}\left(1 - F_{S_T}(\mathrm{e}^k)\right)\right) - \int_0^{\mathrm{e}^k} \mathrm{e}^{-(r-\eta)T} s \, dF_{S_T}(s) \right| \, dk, \\
\leq\ & \int_{-\infty}^{\log(S_0)} \mathrm{e}^k \left|1 - \mathrm{e}^{-(r-\eta)T}\left(1 - F_{S_T}(\mathrm{e}^k)\right)\right| \, dk + \int_{-\infty}^{\log(S_0)} \int_0^{\mathrm{e}^k} \mathrm{e}^{-(r-\eta)T} s \, dF_{S_T}(s) \, dk \\
\leq\ & BS_0 + \int_{-\infty}^{\log(S_0)} \mathrm{e}^{-(r-\eta)T}\mathrm{e}^k(1 - F_{S_T}(\mathrm{e}^k)) \, dk \quad \leq (B+1)S_0,
\end{aligned}$$

where for the last inequality we used that $\left|1 - \mathrm{e}^{-(r-\eta)T}\left(1 - F_{S_T}(\mathrm{e}^k)\right)\right|$ can be bounded uniformly in k and

$$\int_0^{\mathrm{e}^k} \mathrm{e}^{-(r-\eta)T} s \, dF_{S_T}(s) \leq \mathrm{e}^k \mathrm{e}^{-(r-\eta)T} \int_0^{\mathrm{e}^k} dF_{S_T}(s) = \mathrm{e}^{-(r-\eta)T}\mathrm{e}^k(1 - F_{S_T}(\mathrm{e}^k)).$$

Concerning the second term of (38), note that due to assumption (26)

$$\lim_{k\to\infty} k \int_{\mathrm{e}^k}^{\infty} \mathrm{e}^{-(r-\eta)T} \left(s - \mathrm{e}^k\right) \, dF_{S_T}(s) \leq \lim_{k\to\infty} \int_{\mathrm{e}^k}^{\infty} \log(s) s \, dF_{S_T}(s) = 0$$

Hence applying integration by parts yields

$$\begin{aligned}
\int_{\log(S_0)}^{\infty} \mathrm{e}^{-(r-\eta)T}\mathbb{E}\left[\left(S_T - \mathrm{e}^k\right)^+\right] \, dk &= \int_{\log(S_0)}^{\infty} \int_{\mathrm{e}^k}^{\infty} \mathrm{e}^{-(r-\eta)T} \left(s - \mathrm{e}^k\right) \, dF_{S_T}(s) \, dk \\
&\leq \int_{\log(S_0)}^{\infty} \mathrm{e}^{-(r-\eta)T} k\mathrm{e}^k \left(1 - F_{S_T}(\mathrm{e}^k)\right) \, dk - \log(S_0)c^{(\theta)}(\log(S_0).T) \\
&\leq \int_{S_0}^{\infty} \mathrm{e}^{-(r-\eta)T} \log(K) \left(1 - F_{S_T}(K)\right) \, dK - \log(S_0)c^{(\theta)}(\log(S_0).T) \quad < \infty,
\end{aligned}$$

where the last line again holds because of (26).
Now consider (37). We have

$$c_k^{(\theta)}(k,T) = \begin{cases} \mathrm{e}^k \left(1 - \mathrm{e}^{-(r-\eta)T}\mathbb{P}\left[S_T \geq \mathrm{e}^k\right]\right) & \text{for } k < \log(S_0) \\ -\mathrm{e}^{-(r-\eta)T}\mathrm{e}^k \, \mathbb{P}\left[S_T \geq \mathrm{e}^k\right] & \text{for } k \geq \log(S_0). \end{cases} \tag{39}$$

Thus

$$\begin{aligned}
\left|c_k^{(\theta)}(k,T)\right| &\leq \max\left\{1_{\{\mathrm{e}^k < S_0\}}\mathrm{e}^k, \mathrm{e}^{-(r-\eta)T}\mathrm{e}^k \, \mathbb{P}\left[S_T \geq \mathrm{e}^k\right]\right\} \\
&\leq \max\left\{S_0, \mathrm{e}^{-(r-\eta)T}\mathrm{e}^k \, \mathbb{P}\left[S_T \geq \mathrm{e}^k\right]\right\}.
\end{aligned}$$

Hence using that $\mathrm{e}^{-(r-\eta)T}\mathbb{E}[S_T] = S_0$ we find

$$\mathrm{e}^{-(r-\eta)T}\mathrm{e}^k \, \mathbb{P}\left[S_T \geq \mathrm{e}^k\right] = \mathrm{e}^{-(r-\eta)T}\mathrm{e}^k \int_{\mathrm{e}^k}^{\infty} 1 \, dF_{S_T}(s) \leq \mathrm{e}^{-(r-\eta)T} \int_{\mathrm{e}^k}^{\infty} s \, dF_{S_T}(s) \leq S_0,$$

which proves the first part of (37).
Turning to the second inequality we find

$$\|c_k^{(\theta)}(.,T)\|_{L^1} = \int_{-\infty}^{\infty} |c_k^{(\theta)}(k,T)|\; dk = \int_{-\infty}^{\infty} \mathrm{e}^k \left|1_{\{\mathrm{e}^k<S_0\}} - \mathrm{e}^{-(r-\eta)T}\mathbb{P}\left[S_T \geq \mathrm{e}^k\right]\right|\; dk$$
$$\leq S_0 + \int_{-\infty}^{\infty} \mathrm{e}^{-(r-\eta)T}\mathrm{e}^k\mathbb{P}\left[S_T \geq \mathrm{e}^k\right]\; dk = 2S_0.$$

□

Note that we did not use any assumption on the parameter-triple $\theta = (a, \psi, \sigma^2)$ other than the parameters to be such that (4) has a solution. Therefore Proposition 4 also holds in the case $\sigma^2 \equiv 0$ for suitably chosen a and ν.

Theorem 2. *The transformed call price $c^{(\theta)}$ defined by* (8) *is a weak solution in Y to the PIDE* (14) *with boundary conditions* (15) *–* (17)*.*

Proof. In Section 2 we showed that $c^{(\theta)}$ solves the PIDE in sense of distributions and we can conclude that $c^{(\theta)}$ solves

$$(c_T, \psi) + (\mathcal{L}c^{(\theta)}, \psi) = -(\mathcal{L}g, \psi) \tag{40}$$

for all $T > 0$ and smooth test-functions $\psi \in C_0^\infty(\mathbb{R})$.
The boundary conditions (15)–(17) are implied by the no-arbitrage assumption. Furthermore Proposition 4 and the Hoelder inequality yield that $c^{(\theta)}(.,T)$ as defined in (8) lies in $H^1(\mathbb{R})$ for all T. Due to the continuity of $\mathcal{L}(c^{(\theta)}, u)$, the estimate

$$(\mathcal{L}g, \psi) \leq \|\mathcal{L}g\|_{(H^1(\mathbb{R}))'}\|\psi\|_{H^1(\mathbb{R})} \leq \text{const } \|\psi\|_{H^1(\mathbb{R})}$$

and the fact that $C_0^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, we have that (40) is fulfilled for any $\psi \in H^1(\mathbb{R})$. Finally, this last fact also yields $c_T^{(\theta)} \in L^\infty\big((0,T^*), \big(H^1(\mathbb{R})\big)'\big)$, hence $c_T^{(\theta)} \in L^2\big((0,T^*), \big(H^1(\mathbb{R})\big)'\big)$ and especially $c^{(\theta)} \in Y$. □

4 Properties of the parameter to solution map

To investigate properties such as continuity of the parameter to solution map H, we have to define its domain, i.e. the space in which the parameter functions in the stochastic integral equation (2) are located in. Motivated by the assumptions on the parameters we consider the space:

$$\begin{aligned}\tilde{X} \;:=\; & \{\sigma^2(k,T) \in C_B(\mathbb{R}\times[0,T^*]) \,|\, \sigma_k^2(k,T) \in L^\infty([0,T^*], L^2(\mathbb{R})) \wedge\ \sigma^2(k,T) \geq c_0\} \\ & \times\{a(k,T) \in C_B(\mathbb{R}\times[0,T^*]) \,|\, a_k(k,T) \in L^\infty([0,T^*], L^2(\mathbb{R})) \wedge\ a(k,T) \geq 0\} \\ & \times\{\psi \,|\, \psi \text{ is a double exponential tail of a Lévy-measure satisfying (27)}\}\end{aligned} \tag{41}$$

with the norm

$$\begin{aligned}\|(\sigma^2, a, \psi)\|_{\tilde{X}} \;=\; & \|\sigma^2\|_{L^\infty(\mathbb{R}\times[0,T^*])} + \sup_{T\in[0,T^*]} \|\sigma_k^2(\cdot,T)\|_{L^2(\mathbb{R})} \\ & + \|a\|_{L^\infty(\mathbb{R}\times[0,T^*])} + \sup_{T\in[0,T^*]} \|a_k(\cdot,T)\|_{L^2(\mathbb{R})} + \|\psi\|_{L^1(\mathbb{R})}.\end{aligned}$$

Remark 5. *In the above, as well as in the rest of this section we view H as an operator $(\sigma^2, a, \psi(\nu)) \to c^{(\theta)}$, since only the double exponential tail ψ of ν, i.e. the integral transform defined in* (12)*, appears in the PIDE.*

As in the sequel we will repeatedly be confronted with the $\left(H^1(\mathbb{R})\right)'$-norms of certain functions, a technical proposition is given first. Consider the integral operators

$$I_1(u,v;a(.,T),\psi) := \int_{\mathbb{R}}\int_{\mathbb{R}} a(y,T)\,(u_{kk}(y)-u_k(y))\,\psi(k-y)\;dy\;v(k)\;dk \tag{42}$$

$$I_2(u,v;\sigma^2(.,T)) := \int_{\mathbb{R}} \sigma^2(k,T)\,(u_{kk}(k)-u_k(k))\;\;v(k)\;dk. \tag{43}$$

Proposition 5. *The $\left(H^1(\mathbb{R})\right)'$-norms of the functions*

$$\mathcal{I}_\psi(a(.,T)(u_{kk}-u_k)) \quad \text{and} \quad \sigma^2(.,T)(u_{kk}-u_k)$$

for $u \in H^1(\mathbb{R})$ can be bounded by

$$\begin{aligned}\|\mathcal{I}_\psi(a(.,T)(u_{kk}-u_k))\|_{(H^1(\mathbb{R}))'} &= \sup_{v\in H^1}\frac{|I_1(u,v;a(.,T),\psi)|}{\|v\|_{H^1}} \\ &\le B\Big(\|a(.,T)\|_{L^\infty}+\|a_k(.,T)\|_{L^2}\Big)\|\psi\|_{L^1}\|u\|_{H^1(\mathbb{R})}\end{aligned} \tag{44}$$

$$\begin{aligned}\left\|\sigma^2(.,T)(u_{kk}-u_k)\right\|_{(H^1(\mathbb{R}))'} &= \sup_{v\in H^1}\frac{\left|I_2(u,v;\sigma^2(.,T))\right|}{\|v\|_{H^1}} \\ &\le \tilde{B}\Big(\|\sigma^2(.,T)\|_{L^\infty}+\|\sigma_k^2(.,T)\|_{L^2}\Big)\|u\|_{H^1(\mathbb{R})}\end{aligned} \tag{45}$$

with positive constants B and $\tilde{B}$.

Proof. Applying a change of variables $\tilde{k} = k-y$, Fubini's theorem and integration by parts yields

$$\begin{aligned}|I_1(u,v;a(.,T),\psi)| = \Big| &\int_{\mathbb{R}}\int_{\mathbb{R}} a(y,T)u_k(y)v_k(\tilde{k}+y)\;dy\;\psi(\tilde{k})\;d\tilde{k} \\ &+\int_{\mathbb{R}}\int_{\mathbb{R}} a_k(y,T)u_k(y)v(\tilde{k}+y)\;dy\;\psi(\tilde{k})\;d\tilde{k} \\ &+\int_{\mathbb{R}}\int_{\mathbb{R}} a(y,T)u_k(y)v(\tilde{k}+y)\;dy\;\psi(\tilde{k})\;d\tilde{k}\Big|\end{aligned}$$

and hence

$$\begin{aligned}|I_1(u,v;a(.,T),\psi)| \;\le\; & \|a\|_{L^\infty}\|u_k\|_{L^2}\|v_k\|_{L^2}\|\psi\|_{L^1}+\|a_k\|_{L^2}\|u_k\|_{L^2}\|v\|_{L^\infty}\|\psi\|_{L^1} \\ &+\|a\|_{L^\infty}\|u_k\|_{L^2}\|v\|_{L^2}\|\psi\|_{L^1}.\end{aligned} \tag{46}$$

Using the Sobolev embedding theorem

$$\|v\|_{L^\infty} \le B_4\|v\|_{H^1(\mathbb{R})}$$

and plugging the above in inequality (46) proves (44).
Considering (45) with the help of applying integration by parts we immediately have:

$$\left|I_2(u,v;\sigma^2(.,T))\right| \;\le\; \|\sigma_k^2\|_{L^2}\|u_k\|_{L^2}\|v\|_{L^\infty}+\|\sigma^2\|_{L^\infty}\|u_k\|_{L^2}\|v_k\|_{L^2},$$

which together with the sobolev inequality proves the proposition. ◻

Proposition 6. *H is locally Lipschitz continuous from $\tilde{X} \to Y$.*

Proof. Let

$$v := H(\theta_1) - H(\theta_2) = c^{(\theta_1)} - c^{(\theta_2)}.$$

Then v solves the following PIDE

$$\begin{aligned} v_T + \mathcal{L}_{\theta_1} v &= \mathcal{L}_{\theta_2} g - \mathcal{L}_{\theta_1} g + \mathcal{L}_{\theta_2} c^{(\theta_2)} - \mathcal{L}_{\theta_1} c^{(\theta_2)} \\ &= \left(\mathcal{I}_{\psi_1} a^{(1)} - \mathcal{I}_{\psi_2} a^{(2)}\right)(g_{kk} - g_k) + \left(\mathcal{I}_{\psi_1} a^{(1)} - \mathcal{I}_{\psi_2} a^{(2)}\right)\left(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)}\right) \\ &\quad + \left(\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}\right)(g_{kk} - g_k) + \left(\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}\right)\left(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)}\right). \end{aligned} \tag{47}$$

As the left-hand side of the above PIDE is the same as in Theorem 1, the Gårding inequality holds. Thus, for a constant B, we get the estimate

$$\begin{aligned} \|v\|_Y \leq B \Bigg\| &\left(\mathcal{I}_{\psi_1} a^{(1)} - \mathcal{I}_{\psi_2} a^{(2)}\right)(g_{kk} - g_k) + \left(\mathcal{I}_{\psi_1} a^{(1)} - \mathcal{I}_{\psi_2} a^{(2)}\right)\left(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)}\right) \\ &+ \left(\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}\right)(g_{kk} - g_k) + \left(\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}\right)\left(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)}\right)\Bigg\|_{Y'}. \end{aligned}$$

With the triangle inequality, to estimate the right-hand side of the above equation, we have to consider four terms. As the procedure for the terms with $(g_{kk} - g_k)$ in place of $(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)})$ is similar, we will just treat the latter ones.
First we concentrate on the norm in the state space. Note that with the definitions (42) and (43) we can write

$$\begin{aligned} &\left(\left(\mathcal{I}_{\psi_1} a^{(1)}(.,T) - \mathcal{I}_{\psi_2} a^{(2)}(.,T)\right)\left(c_{kk}^{(\theta_2)}(.,T) - c_k^{(\theta_2)}(.,T)\right), u\right) = \\ &\qquad I_1(c^{(\theta_2)}(.,T), u; a^{(1)}(.,T) - a^{(2)}(.,T), \psi_2) + I_1(c^{(\theta_2)}(.,T), u; a^{(1)}(.,T), \psi_1 - \psi_2) \end{aligned}$$

and

$$\left(\left(\frac{\sigma_1^2(.,T)}{2} - \frac{\sigma_2^2(.,T)}{2}\right)\left(c_{kk}^{(\theta_2)}(.,T) - c_k^{(\theta_2)}(.,T)\right), u\right) = \frac{1}{2} I_2(c^{(\theta_2)}(.,T), u; \sigma_1^2(.,T) - \sigma_2^2(.,T)).$$

Hence with the estimates of Proposition 5 we find

$$\begin{aligned} &\left\|\left(\mathcal{I}_{\psi_1} a^{(1)}(.,T) - \mathcal{I}_{\psi_2} a^{(2)}(.,T)\right)\left(c_{kk}^{(\theta_2)}(.,T) - c_k^{(\theta_2)}(.,T)\right)\right\|_{(H^1(\mathbb{R}))'} \leq \\ &B_1\left(\|a^{(1)}(.,T) - a^{(2)}(.,T)\|_{L^\infty(\mathbb{R})} + \|a^{(1)}{}_k(.,T) - a^{(2)}{}_k(.,T)\|_{L^2(\mathbb{R})} + \|\psi_1 - \psi_2\|_{L^1}\right)\|c_k^{(\theta_2)}(.,T)\|_{L^2(\mathbb{R})} \end{aligned}$$

and

$$\begin{aligned} &\left\|\left(\frac{\sigma_1^2(.,T)}{2} - \frac{\sigma_2^2(.,T)}{2}\right)\left(c_{kk}^{(\theta_2)}(.,T) - c_k^{(\theta_2)}(.,T)\right)\right\|_{(H^1(\mathbb{R}))'} \leq \\ &\qquad B_2\left(\|\sigma_1^2(.,T) - \sigma_2^2(.,T)\|_{L^\infty(\mathbb{R})} + \|(\sigma_1^2(.,T) - \sigma_2^2(.,T)_k\|_{L^2(\mathbb{R})}\right)\|c_k^{(\theta_2)}(.,T)\|_{L^2(\mathbb{R})}. \end{aligned}$$

Thus, taking also the time-dependence into account and using that $\|c^{(\theta_2)}(.,T)\|_Y$ is bounded by Theorem 1, we can give the following estimates

$$\begin{aligned} \left\|\left(\mathcal{I}_{\psi_1} a^{(1)} - \mathcal{I}_{\psi_2} a^{(2)}\right)\left(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)}\right)\right\|_{Y'} &\leq B_3 \left\|(0, a^{(1)} - a^{(2)}, \psi_1 - \psi_2)\right\|_{\tilde{X}} \|c^{(\theta_2)}\|_Y \\ &\leq B_4 \|\theta_1 - \theta_2\|_{\tilde{X}} \end{aligned}$$

and

$$\left\| \left(\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right) \left(c_{kk}^{(\theta_2)} - c_k^{(\theta_2)} \right) \right\|_{Y'} \leq B_5 \left\| (\sigma_2^2 - \sigma_2^2, 0, 0) \right\|_{\tilde{X}} \| c^{(\theta_2)} \|_Y B_6 \left\| \theta_1 - \theta_2 \right\|_{\tilde{X}} .$$

As $g_k \in L^2(\mathbb{R})$, following the same steps for g instead of $c^{(\theta_2)}$ completes the proof. □

The Frechet derivative of the forward operator can be calculated by similar means.

Proposition 7. *$H : \tilde{X} \to Y$ is Frechet differentiable and the derivative $H'(\theta)$ is given by*

$$H'(\theta) : \; h = (h_1, h_2, h_3) \to v^{(\theta)},$$

where v solves

$$v_T^{(\theta)} + \mathcal{L}_\theta v^{(\theta)} = \frac{h_1}{2} \Big(\hat{c}_{kk}^{(\theta)} - \hat{c}_k^{(\theta)} \Big) + \mathcal{I}_\psi \Big(h_2 \left(\hat{c}_{kk}^{(\theta)} - \hat{c}_k^{(\theta)} \right) \Big) + \mathcal{I}_{h_3} \Big(a \left(\hat{c}_{kk}^{(\theta)} - \hat{c}_k^{(\theta)} \right) \Big) \tag{48}$$

with homogeneous boundary conditions and $\hat{c}^{(\theta)} = c^{(\theta)} + g$.
Furthermore $H'(\theta)$ is locally Lipschitz continuous.

Proof. The first step of the proof consists in showing that $H'(\theta)$ is well-defined. Since the left-hand side of (48) is the same as in Theorem 1, we just have to show, that the right-hand side is bounded in the Y'-norm. However, since it is similar to the one in the proof of Proposition 6, the estimates of Proposition 5 can be used to arrive at

$$\left\| \frac{h_1}{2} \Big(\hat{c}_{kk}^{(\theta)} - \hat{c}_k^{(\theta)} \Big) + \mathcal{I}_\psi \Big(h_2 \left(\hat{c}_{kk}^{(\theta)} - \hat{c}_k^{(\theta)} \right) \Big) + \mathcal{I}_{h_3} \Big(a \left(\hat{c}_{kk}^{(\theta)} - \hat{c}_k^{(\theta)} \right) \Big) \right\|_{Y'} \leq R_1 \|h\|_{\tilde{X}},$$

which proves the existence of a unique solution of the PIDE (48) and hence yielding well-definedness of $H'(\theta)$.
The next step is to show, that the defined operator $H'(\theta)$ is in fact the Frechet-derivative, thus we have to show

$$\lim_{\|h\|_{\tilde{X}} \to 0} \frac{\|H(\theta + h) - H(\theta) - H'(\theta)h\|_Y}{\|h\|_{\tilde{X}}} = 0. \tag{49}$$

Defining

$$u := H(\theta + h) - H(\theta) - H'(\theta)h,$$

straightforward calculations show that u solves

$$\begin{aligned} u_T + \mathcal{L}_\theta u \;\; = \;\; & \frac{h_1}{2} \Big(\big(\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} \big)_{kk} - \big(\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} \big)_k \Big) \\ & + \mathcal{I}_\psi \Big(h_2 \Big(\big(\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} \big)_{kk} - \big(\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} \big)_k \Big) \Big) \\ & + \mathcal{I}_{h_3} \Big(a \Big(\big(\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} \big)_{kk} - \big(\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} \big)_k \Big) \Big). \end{aligned}$$

Since

$$\hat{c}^{(\theta+h)} - \hat{c}^{(\theta)} = c^{(\theta+h)} - c^{(\theta)},$$

employing again Proposition 5 and the local Lipschitz-continuity of $c^{(\theta)}$ yields

$$\|u\|_Y \leq R_2 \|h\|_{\tilde{X}} \|c^{(\theta+h)} - c^{(\theta)}\|_Y \leq R_3 \|h\|_{\tilde{X}}^2$$

and therewith (49).
For the local Lipschitz-continuity of H' consider $w = H'(\theta_1) - H'(\theta_2)$ and observe that w solves

$$\left.\begin{aligned} w_T + \mathcal{L}_{\theta_1} w \;=\; & (\mathcal{L}_{\theta_2} - \mathcal{L}_{\theta_1})\, v^{(\theta_2)} \\ & + \tfrac{h_1}{2}\Big(\big(\hat{c}^{(\theta_1)} - \hat{c}^{(\theta_2)}\big)_{kk} - \big(\hat{c}^{(\theta_1)} - \hat{c}^{(\theta_2)}\big)_k\Big) \\ & + \mathcal{I}_\psi\Big(h_2\Big(\big(\hat{c}^{(\theta_1)} - \hat{c}^{(\theta_2)}\big)_{kk} - \big(\hat{c}^{(\theta_1)} - \hat{c}^{(\theta_2)}\big)_k\Big)\Big) \\ & + \mathcal{I}_{h_3}\Big(a\Big(\big(\hat{c}^{(\theta_1)} - \hat{c}^{(\theta_2)}\big)_{kk} - \big(\hat{c}^{(\theta_1)} - \hat{c}^{(\theta_2)}\big)_k\Big)\Big). \end{aligned}\right\} (\star)$$

As before the Y'-norm of $(\star)$ can be bounded by $R_4 \|h\|_{\tilde{X}} \|\theta_1 - \theta_2\|_{\tilde{X}}$.
Furthermore following the proof of Proposition 6,

$$\Big\| (\mathcal{L}_{\theta_2} - \mathcal{L}_{\theta_1})\, v^{(\theta_2)} \Big\|_{Y'} \leq R_5 \|h\|_{\tilde{X}} \|\theta_1 - \theta_2\|_{\tilde{X}},$$

which finally also yields the local Lipschitz-continuity of $H'(\theta)$. □

5 Regularizing the calibration problem

5.1 The Tikhonov functional

Turning to the calibration problem we have to reconstruct the parameters $\theta = (\sigma^2, a, \nu)$ from given observed data, which we now denote as y^δ, indicating that these might be noisy observations. The noise level is defined as the distance

$$\delta = \|y^\delta - y\|, \tag{50}$$

where $y = H(\bar{\theta})$ as in (22) with the true parameters $\bar{\theta} = (\bar{\sigma}^2, \bar{a}, \bar{\nu})$. In practice the true parameters are of course not known (in particular it is not even clear that there exists a parameter triple producing the data) and the noise level has to be specified in a different way. One example would be the norm of the difference between bid- and ask-prices.
The identification of the true parameters from observed data could, in principle, be done by inverting the forward operator, but due to the ill-posedness of the problem this does not lead to a stable calibration method. Hence, in order to identify the parameters from noisy data in a stable way, one has to use a regularization method. We attempt to apply non-linear Tikhonov regularization, since for this method many results on convergence are available in the literature. Furthermore it seems to be the standard method in engineering problems similar to the one we are facing here.
The first thing we therefore define is the Tikhonov functional:

$$J_\alpha(\theta) := \|H(\theta) - y^\delta\|^2 + \alpha \mathcal{R}(\theta), \tag{51}$$

where $\mathcal{R}$ is a regularization functional and α is the regularization parameter, which in some sense represents the importance of $\mathcal{R}$ in the calibration procedure.
The Tikhonov-regularized solution is then given by

$$\hat{\theta}_\alpha := \operatorname{argmin}_{\theta \in U} J(\theta), \tag{52}$$

where U is the search space, which we will come back to later.
For the data space norm we specify two possible choices here. Although we proved that the

forward operator is well-defined as a mapping into the Sobolev space $W_2^1(0,T)$, both are weighted L^2-norms, which is a natural choice, since in this way the data space is a Hilbert space and no differentiation of the noisy data is required.
The first norm we consider is a discrete one, i.e. with δ-like weights (cf. Cont & Tankov [10, 11]).

$$\|H(\theta) - y^\delta\|_a^2 = \sum_{(\mathrm{e}^k,T)\in\mathcal{D}} w_{(k,T)} \left(H(\theta)(k,T) - y^\delta(k,T)\right)^2, \tag{53}$$

where $\sum_{(\mathrm{e}^k,T)\in\mathcal{D}} w_{(k,T)} = 1$,

$$y^\delta(k,T) = \mathrm{e}^{\eta T}\,\mathrm{C}^{\mathrm{M}}(\mathrm{e}^k,T;\theta) - \left(S - \mathrm{e}^k\right)^+$$

and $\mathrm{C}^{\mathrm{M}}(K,T)$ is the arithmetic average of the bid and ask prices of a traded European call option with strike K and maturity T.
The second one is given by:

$$\|H(\theta) - y^\delta\|_b^2 = \sum_{T\in\mathcal{D}_T} \int_{-\infty}^{\infty} w(k,T) \left(H(\theta)(k,T) - y_I^\delta(k,T)\right)^2\, dk, \tag{54}$$

where $\mathcal{D}_T$ denotes the set of maturities, for which call prices are liquid, $w(k,T) \in L^\infty(\mathbb{R})$ $\forall T \in \mathcal{D}_T$, $\sum_{T\in\mathcal{D}_T}\int_{-\infty}^{\infty} w(k,T) = 1$ and y_I^δ is an interpolated version of y^δ.
The latter norm affords an interpolation in the state-variable at least on small intervals around the given data-points. However, this can be done quite efficiently, since the call price has to be decreasing and convex in the strike. That means, that the additional noise introduced by the interpolation can be expected to be small. Furthermore this second norm has some advantageous theoretical properties, as will be discussed later on. Note that in a numerical implementation both specifications are equivalent, since the call price has to be discretized anyway.
In both cases minimizing the Tikhonov functional defines a modified weighted least squares solution, i.e. setting $\alpha = 0$, $\hat\theta_0$ is exactly the least squares solution. Unfortunately also the problem of finding the least-squares solution is ill-posed (see e.g. Cont & Tankov [11]) implying that α should be chosen positive in the case of noisy data.
For the applicability of the Tikhonov regularization theory, we need the forward operator to be continuous and weakly continuous as a mapping to the weighted L^2-spaces. Proposition 6 yields continuity with respect to the Sobolev space $W_2^1(0,T)$, which immediately implies continuity in the norm specified in (54). However, since the norm in (53) can not be bounded by the $W_2^1(0,T)$-norm, in this case we have to show continuity by an extra proposition.

Proposition 8. *The PIDE* (14) *has a unique solution $c^{(\theta)}(k,T)$, that is continuous in k for all T. Furthermore defining the forward operator $H(\theta)$ to be this solution, it is Hoelder-continuous as mapping from $\tilde X$ to the space of continuous functions $C(\mathbb{R}\times[0,T^*])$ equipped with the supremum norm.*

Proof. First note that, if there were two solutions $c^{(\theta)}$ and u continuous in k for all T, then $\max_{0\le T\le T^*} \|c^{(\theta)}(.,T) - u(.,T)\|_{L^2} = 0$ and therefore $c^{(\theta)}(.,T) = u(.,T)$ for almost all k and all T. Thus due to the continuity in the state variable both are equal on $\mathbb{R}\times[0,T^*]$ and $\max_{\mathbb{R}\times[0,T^*]} |c^{(\theta)} - u| = 0$. Hence due to Theorem 2, $c^{(\theta)}$ defined by (8) is the unique continuous solution of the PIDE (14).
For the continuous dependence on the parameters we observe that

$$\begin{aligned}\|c^{(\theta_1)} - c^{(\theta_2)}\|_Y^2 &\ge \max_{0\le T\le T^*} \|c^{(\theta_1)}(.,T) - c^{(\theta_2)}(.,T)\|_{L^2}^2\\ &= \max_{0\le T\le T^*} \int_{-\infty}^{\infty} \left(c^{(\theta_1)}(k,T) - c^{(\theta_2)}(k,T)\right)^2\, dk.\end{aligned}$$

Assuming w.l.o.g. $S_0 = 1$, we have $-1 \leq c_k^{(\theta)}(k,T) \leq 0$ uniformly in T (see Proposition 4). Therefore it is clear that for all $(k,T) \in \mathbb{R} \times [0,T]$

$$|c^{(\theta_1)}(k,T) - c^{(\theta_2)}(k,T)| \geq (m - (\kappa - k))^+,$$

where $m = \max |c^{(\theta_1)} - c^{(\theta_2)}|$ and $(\kappa, \tau) = \operatorname{argmax} |c^{(\theta_1)} - c^{(\theta_2)}|$.
Thus

$$\begin{aligned} \max_{0 \leq T \leq T^*} \int_{-\infty}^{\infty} \left(c^{(\theta_1)}(k,T) - c^{(\theta_2)}(k,T) \right)^2 \, dk &\geq \int_{-s}^{s} (s - \tilde{k})^2 \, d\tilde{k} \\ &= \max_{0 \leq T \leq T^*} \frac{8s^3}{3} = \frac{8}{3} \max_{\mathbb{R} \times [0,T^*]} |c^{(\theta_1)} - c^{(\theta_2)}|^3, \end{aligned}$$

which proves the Hoelder continuity of H with respect to the topology induced by the supremum norm. □

The above in particular implies, that the continuous solution of (14), which is the one that a numeric scheme for the PIDE finds, depends in the weighted L^2-norm of (53) continuously on the parameters.

5.2 The regularization functional

We assume that the regularization functional $\mathcal{R}$ can be decomposed in the following way $\mathcal{R}(\theta) = \mathcal{R}_{\sigma^2}(\sigma^2) + \mathcal{R}_a(a) + \mathcal{R}_\nu$. For $\mathcal{R}_{\sigma^2}$, as well as for $\mathcal{R}_a$, we conveniently use a norm Sobolev space norm $\|\|_s$, that induces a finer topology than the respective strong topology in $\tilde{X}$. That means we choose $\|\|_s$ to fulfill

$$\begin{aligned} \|a - a_*\|_Z &\leq C_1 \|a - a_*\|_s \qquad \forall (a - a_*) \in H^s && \text{(55a)} \\ \|a_n - a_*\|_s &\leq C_2 \quad \Rightarrow \quad \exists_{a_{n_k}} a_{n_k} \text{ convergent in } Z, && \text{(55b)} \end{aligned}$$

where Z is the sub-space of $\tilde{X}$ containing the local speed functions a, i.e.

$$Z := \{a(k,T) \in C_B(\mathbb{R} \times [0,T^*]) \,|\, a_k(k,T) \in L^\infty([0,T^*], L^2(\mathbb{R}))\}.$$

Remember that the norm in Z is given by

$$\|a\|_Z := \|a\|_{L^\infty(\mathbb{R} \times [0,T^*])} + \sup_{t \in [0,T^*]} \|a_k(\cdot,T)\|_{L^2(\mathbb{R})}.$$

The same shall also hold for σ^2 instead of a.
Examples of norms, which satisfy this are the $H^2(\mathbb{R} \times [0,T^*])$ norm or the tensor product space $H^1(\mathbb{R}) \otimes H^1[0,T^*]$ norm. Note that by (55) the Lipschitz-continuity of the forward operator H implies weak lower semi-continuity and especially the weak closedness of H with respect to σ^2 and a.

For $\mathcal{R}_\nu$ the case is more involved. First measures are numerically harder to handle and second the weak topology of measures does not guarantee the weak closedness of level-sets. Thus a first idea might be to regularize the double exponential tail ψ instead of ν. However, noting that in view of the results in the preceding sections the natural regularization functional would be the L^1-norm, also for ψ the problem with the level sets arises. Furthermore we would have to guarantee that ψ is a double exponentail tail of some Lévy measure, which does not seem to be a trivial problem. Thus we concentrate on the regularization of the Lévy measure itself. However, since in general ν is not a finite measure, we first give the following proposition.

Proposition 9. *Every Lévy measure ν can be decomposed in two finite measures μ^+ and μ^- living on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, by*

$$\nu(dx) = \begin{cases} \frac{1+x^2}{x^2(1+xe^x)}\mu^+(dx) & \text{for } x > 0 \\ \frac{1+x^2}{x^2}\mu^-(dy) & \text{for } x < 0. \end{cases} \tag{56}$$

Proof. This follows directly from the properties of a Lévy measure. □

With this decomposition it is possible to consider the usual weak convergence on the space of measures and we will prove now that the thereby induce topology is finer than the topology of strong convergence in $\tilde{X}$. Define therefore ψ_n and ψ to be the double exponential tails associated to the Lévy measures parameterized via $\mu_n^{+,-}$ and $\mu^{+,-}$, respectively, and denote weak convergence by $\rightharpoonup$. Then we have the following proposition.

Proposition 10. *If $\mu_n^{+,-} \rightharpoonup \mu^{+,-}$, then $\psi_n \to \psi$ strongly in L^1.*

Proof. We will use the following assertion (for a proof we refer to Dunford & Schwartz [15, Theorem IV.8.12]).
Let $\zeta_n, \zeta \in L^1$. Then ζ_n converges strongly (in the L^1-norm) to ζ, if and only if

(i) $\zeta_n \rightharpoonup \zeta$ in L^1

(ii) $\forall \epsilon > 0$ and $\lambda(A) < \infty$: $\lambda\left(\{|\zeta_n - \zeta| > \epsilon\} \cap A\right) \to 0$,

where λ denotes the Lebesgue-measure. The second statement is often called *local convergence in measure.*
For (i) observe, that for $h \in L^\infty$

$$\begin{aligned} \int_{\mathbb{R}} h(z)\psi_n(z)\, dz &= \int_0^\infty h(z) \int_z^\infty (\mathrm{e}^x - \mathrm{e}^z)\, \frac{1+x^2}{x^2(1+x\mathrm{e}^x)}\, \mu_n^+(dx)\, dz \\ &\quad + \int_{-\infty}^0 h(z) \int_{-\infty}^z (\mathrm{e}^z - \mathrm{e}^x)\, \frac{1+x^2}{x^2}\, \mu_n^-(dx)\, dz. \end{aligned}$$

We only consider the first term in the last equation here, since for the second one the reasoning is similar.
The integrand is absolutely integrable. So Fubini's theorem can be applied and we can interchange the order of integration to get

$$\begin{aligned} &\int_0^\infty h(z) \int_z^\infty (\mathrm{e}^x - \mathrm{e}^z) \frac{1+x^2}{x^2(1+x\mathrm{e}^x)} \mu_n^+(dx)\, dz = \\ &\qquad \int_0^\infty \frac{1+x^2}{x^2(1+x\mathrm{e}^x)} \mu_n^+(dx) \int_0^x h(z)(\mathrm{e}^x - \mathrm{e}^z)\, dz. \end{aligned}$$

Note that for all $x \in \mathbb{R}^+$

$$\int_0^x (\mathrm{e}^x - \mathrm{e}^z)\, dz = x\mathrm{e}^x - \mathrm{e}^x + 1$$

and therefore

$$\left| \int_0^x h(z)(\mathrm{e}^x - \mathrm{e}^z)\, dz \right| \le \|h\|_{L^\infty([0,x])}(x\mathrm{e}^x - \mathrm{e}^x + 1). \tag{57}$$

Now define for $x > 0$:

$$\tilde{h}(x) = \frac{1+x^2}{x^2(1+x\mathrm{e}^x)} \int_0^x h(z)(\mathrm{e}^x - \mathrm{e}^z)\, dz.$$

Since the right-hand side of the above is continuous for $x > 0$, $\tilde{h} : \mathbb{R}^+ \to \mathbb{R}$ is measureable and due to equation (57) it is bounded also for $x \to 0$ and $x \to \infty$, respectively. Hence it follows that $\tilde{h} \in L^\infty(\mathbb{R}^+)$ and we have

$$\int_0^\infty \frac{1+x^2}{x^2(1+x\mathrm{e}^x)}\mu_n^+(dx) \int_0^x h(z)(\mathrm{e}^x - \mathrm{e}^z)\ dz = \int_0^\infty \tilde{h}(x)\ \mu_n^+(dx).$$

Therefore

$$\lim_{n\to\infty} \int_0^\infty \psi_n g(z)\ dz = \lim_{n\to\infty} \int_0^\infty \tilde{h}(x)\ \mu_n^+(dx) = \int_0^\infty \tilde{h}(x)\ \mu^+(dx) = \int_0^\infty \psi h(z)\ dz,$$

where the last equation can be arrived going backwards through the previous steps.
Now we are left to prove (ii). We may assume $z < 0$ in the following, since for $z > 0$ the reasoning is similar. For arbitrary, but fixed $z < 0$ we find

$$|\psi_n(z) - \psi(z)| = \left| \int_{-\infty}^z (\mathrm{e}^z - \mathrm{e}^x) \frac{1+x^2}{x^2} \left(\mu_n^+(dx) - \mu_n^-(dx)\right) \right| \to 0,$$

since $\mu_n^{+,-} \rightharpoonup \mu^{+,-}$ and $(\mathrm{e}^z - \mathrm{e}^x)\frac{1+x^2}{x^2} 1_{\{x\le z\}} \in L^\infty(\mathbb{R}^-)$. Hence ψ_n converges λ-a.e. to ψ, which implies (ii) (see e.g. [19, Satz VI.4.5]). □

Remark 6. *One could also directly regularize the Lévy measure without representing it by finite measures as in* (56) *(cf. Cont & Tankov [10, 11]). However, by regularizing the Lévy measure directly, e.g. by using relative entropy, typically its behavior around 0 has to be known in advance. Namely, if the prior Lévy measure ν_* is of infinite activity, i.e. $\int_{-\epsilon}^{\epsilon} \nu_*(dx) = \infty$, then also the calibrated measure will be of infinite activity and, if we assume for simplicity that ν_* admits a density f_*, then also the calibrated measure admits a density f satisfying*

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} f_*(x) \quad \textit{and} \quad \lim_{x\to 0-} f(x) = \lim_{x\to 0-} f_*(x).$$

This can be seen as advantage, or as disadvantage, depending on the application of the calibration procedure. Anyway, the topology induced by such a direct regularization functional is of course finer than the topology induced by the same funcitonal for the finite measures and hence the results in the following section apply also in these cases.

5.3 Convergence and stability

The classical Tikhonov-regularization theorems (see e.g. Engl et al. [21] or Seidmann & Vogel [39]) were stated for operators between Hilbert or reflexive Banach spaces. However, in recent years most of those results were extended to general Banach space operators. This theory allows to find conditions such that existence, stability and convergence of the regularized solution are ensured. In case the calibration problem is not uniquely solvable the true solution $\theta^\dagger$ will be defined as $\mathcal{R}$-minimizing solution.

$$\theta^\dagger = \operatorname{argmin}\{\mathcal{R}(\theta) \,|\, H(\theta)(k,T) = \mathrm{e}^{T\eta}\,\mathrm{C}^{(\mathrm{M})}(\mathrm{e}^k, T) - (S_0 - \mathrm{e}^k)^+\}. \tag{58}$$

To establish existence, stability and convergence results, we aim to use Theorems 3.1, 3.2 and 3.5 in Hofmann et al. [27] here. Therefore we have to show that assumptions ([27, 2.1]) are fulfilled in our setting.
Let H be the Forward operator as defined by (22). Then the assumptions to be proved are: There are Banach spaces U and V associated to topologies τ_U and τ_V, respectively, which are weaker than the respective norm-topologies, such that the following assertions hold:

(A) The norm in the observation space V is weakly lower semi-continuous with respect to τ_V.

(B) The forward operator is continuous with respect to the topologies τ_U and τ_V.

(C) The regularization functional $\mathcal{R}$ is convex, and τ_U lower semi-continuous

(D) The domain $D(H) \subset U$ of the forward operator is closed with respect to τ_U and the intersection D of it with the domain $D(\mathcal{R})$ of the regularization functional is not empty, i.e. $D := D(H) \cap D(\mathcal{R}) \neq \emptyset$.

(E) For all $\alpha > 0$ and $M > 0$ the level sets

$$\mathcal{M}_\alpha(M) := \{\theta \in D : J_\alpha(\theta) \leq M\}$$

are sequentially compact in the τ_U-topology.

For U we choose the following Banach space:

$$U = H_s \times H_s \times \mathrm{ca}(\mathbb{R}^+) \times \mathrm{ca}(\mathbb{R}^-), \tag{59}$$

where H_s is a Sobolev space such that (55) holds and $\mathrm{ca}(\mathbb{R}^+)$ denotes the Banach space of all countably additive measures on $\mathbb{R}^+$ equipped with the total variation norm.
Furthermore we define the data space V in Proposition 11 to be one of the weighted L^2-spaces as defined by (53) and (54), respectively, and τ_U and τ_V to be the topologies induced by the weak convergence.
Note that we did not specify any regularization functional $\mathcal{R}_\nu$ on the space of measures yet. However, for the proof of the above proposition no concrete choice is necessary. We first rely on abstract assumptions and will later give concrete examples, where these are satisfied. The first assumption we pose is

$$\mathcal{R}_\nu \text{ is convex and weakly lower semi-continuous.} \tag{60}$$

The second assumption is twofold: In view of condition (E) above, either the domain of the forward operator with respect to μ^+ and μ^- has to be assumed to be restricted to a tight set or R has to be supposed to be strong enough to assure the tightness of the level sets. Defining an initial guess $\theta_* = (\sigma_*^2, a_*, \nu_*)$ the domain $D(H)$ in the case of a strong enough $\mathcal{R}_\nu$ is hence given by

$$\begin{aligned} D(H) \quad = \quad & \Big\{\theta = (\sigma^2, a, \nu) \;\Big|\; \sigma^2 \geq c > 0 \wedge \left(\sigma^2 - \sigma_*^2\right) \in H_s \wedge a \geq 0 \wedge (a - a_*) \in H_s \\ & \wedge\, \mu^+, \mu^- \text{ are non-negative sigma-additive measures, i.e. } \mu^{+,-}(A) \geq 0\Big\}. \end{aligned} \tag{61}$$

and by $D(H) \cap H^s \times H^s \times S_1 \times S_2$, where S_1 and S_2 are tight subsets of $\mathrm{ca}(\mathbb{R}^+)$ and $\mathrm{ca}(\mathbb{R}^-)$, respectively.

Proposition 11. *Under the assumptions above, conditions (A)–(E) are met.*

With the above definitions we now give the proof of Proposition 11

Proof. (A) is obvious: Simply take the weak-topologies induced by the norms $\|\|_a$ and $\|\|_b$, respectively.
(B): Note that by Proposition 10 and the embedding condition (55) a weakly convergent sequence in U implies strong convergence in $\tilde{X}$. Thus the Lipschitz continuity of H (Proposition 6) implies

even strong convergence in both data spaces (for (53) also Propostion 8 is needed).
(C): is an explicit assumption stated before.
(D): For the parameter a, σ^2 this follows from the compact embedding assumption (55). Concerning the measures $\mu_n^{+,-} \rightharpoonup \mu^{+,-}$ implies that for any set of positive measure E

$$\int_E \left(\mathbb{I}_{\{\mathbb{R}+\}}(x)\mu^+(dx) + \mathbb{I}_{\{\mathbb{R}-\}}(x)\mu^-(dx)\right) \geq 0.$$

Thus the measure ν induced by μ^+ and μ^- is a Lévy measure and the domain of H is thus closed w.r.t. τ_U.
Furthermore, since for both observation norm-specifications $\|H(\theta) - y^\delta\| \leq S_0 \ \forall\theta \in \tilde{X}$ (see Proposition 4), $\theta_* = (\sigma_*^2, a_*, \nu_*)$ certainly satisfies $\|H(\theta_*) - y^\delta\|^2 + \mathcal{R}(\theta_*) < \infty$ and hence $D \neq \emptyset$.
(F) follows again from (55) for σ^2 and a, respectively, and from the assumptions and Prokhorov's theorem for μ^+ and μ^-. □

Now, finally, we are ready to state the main theorems about existence, stability and convergence of the Tikhonov-regularized solutions. Therefore we assume to be given an a-priori guess $\theta_* = (\sigma_*^2, a_*, \psi(\nu_*)) \in \tilde{X}$ and set

$$\mathcal{R}(\theta) = \|\sigma^2 - \sigma_*^2\|_s + \|a - a_*\|_s + \mathcal{R}_\nu(\nu - \nu_*), \tag{62}$$

where $\|\|_s$ meets (55) and $\mathcal{R}_\nu$ satisfies (60). Furthermore the level sets with respect to $\mathcal{R}_\nu$ are assumed to be weakly closed (either because of $\mathcal{R}$ or by using a tight domain).

Theorem 3. *Let $\mathcal{R}$ be as in* (62) *and $D(H)$ as in* (61). *Then the following statements hold true:*

- Existence*:*
  The Tikhonov functional (51) *admits a global minimum over U.*
- Stability:
  For fixed α this regularized solution depends stably on the data y_δ in the sense that, if y_δ is replaced by a sequence y_k converging to y_δ as $k \to \infty$, then the corresponding sequence of minima $\theta_{\alpha,k} = (\sigma_{\alpha,k}^2, a_{\alpha,k}, \nu_{\alpha,k})$ in (52) *has a subsequence k_l, such that σ_{α,k_l}^2 and a_{α,k_l} converge strongly in H^s and $\mu_{\alpha,k_l}^{+,-}$, converge weakly. Furthermore the limit of every convergent subsequence is a minimizer of* (52).

Proof. The first assertion readily follows from [27, Theorem 3.1], while Theorem 3.2 therein gives weak convergence of the parameters and for any weakly convergent subsequence $\theta_{\alpha,k_i} \rightharpoonup \bar{\theta}_\alpha$ and $\mathcal{R}(\theta_{\alpha,k_i}) \to \mathcal{R}(\bar{\theta}_\alpha)$. This together with the weak lower semicontinuity of the norms involved in R yields

$$\mathcal{R}_\nu(\mu_n^{+,-}) \to \mathcal{R}(\mu^{+,-}), \quad \|\sigma_{\alpha,k_i}^2\|_s \to \|\bar{\sigma^2}_\alpha\|_s \quad \text{and} \quad \|a_{\alpha,k_i}\|_s \to \|\bar{a}_\alpha\|_s.$$

Since for the Hilbert space H_s weak convergence and the above imply strong convergence, we are done. □

The above theorem entails in particular, that the regularized calibration procedure is well-posed for any initial data and that the regularized solution depends in some sense continuously on the data. Furthermore the Tikhonov-regularized parameters can also be shown to converge to the "real solution", if the noise-level tends to 0 and the regularization parameter is chosen accordingly.

Theorem 4. *Under the same assumptions as in Theorem 3 and if a R-minimum norm solution in the sense of* (58) *exists, the regularized solution* $\theta_{\alpha,\delta}$ *converges in the following sense: For a sequence of data* y_{δ_k} *with noise levels* $\delta_k \to 0$ *and regularization parameters* α_k*, such that* $\alpha_k \to 0$ *and* $\delta_k^2/\alpha_k \to 0$*, the sequence of regularized solutions* θ_{α,δ_k} *corresponding to these data has a subsequence, that converges in the same sense as in Theorem 3. Furthermore the limit of any convergent subsequence is an R-minimum solution and if the R-minimum solutions is unique, then* θ_{α,δ_k} *itself converges strongly for* σ^2 *and* a *and weakly for* $\mu^{+,-}$*,* $i = 1, 2$.

Proof. This follows from [27, Theorem 3.5] in the same way as the stability in the proof of Theorem 3. □

Now we want to give some examples of how the above theory can be applied for different choices of $\mathcal{R}_\nu$. In the following $\mu^{+,-}$ are as defined by (56).

Example 1. $\mathcal{R}_\nu(\nu - \nu_*) = \|\mu^+ - \mu_*^+\|_{TV} + \|\mu^- - \mu_*^-\|_{TV}$*, where* TV *stands for total variation.*

Obviously the total variation is weakly lower semi-continuous with respect to the weak topology and hence satisfies (60). However, as $\{\mu^{+,-} | \|\mu^{+,-} - \mu_*^{+,-}\|_{\mathrm{TV}} \leq B\}$ is not weakly closed we have to restrict the domain for the measures a priori to a tight set. This can for instance be done by defining them only on a compact support or by posing some additional moment assumptions. In practice the non-closeness does not pose any problems since numerically the support of the measure has to be discretized anyway.

Example 2. $\mathcal{R}_\nu(\nu - \nu_*) = I(\mu^+|\mu_*^+) + I(\mu^-|\mu_*^-)$*, where* I *is the relative entropy.*

Again (60) is obviously met. Furthermore it is easy to see, that in this case $\{\mathcal{R}_\nu(\nu - \nu_*) \leq B\}$ is weakly closed and hence no extra restriction on the domain $D(H)$ has to be posed.

Both above examples are given at the level of measures, which is slightly inconvenient when one tries to implement the calibration procedure. Thus we now give another possibility motivated by the following proposition

Proposition 12. *Assume that a Lévy measure* ν *is absolutely continuous with respect to the Lebesgue measure and integrates* $xe^x 1_{\{x>1\}}$*. Then it can be written as*

$$\nu(dx) = \begin{cases} \frac{1+x^2}{x^2(1+xe^x)} f^{(1)}(x)dx & \text{for } x > 0 \\ \frac{1+x^2}{x^2} f^{(2)}(x)dx & \text{for } x < 0, \end{cases} \tag{63}$$

where $f^{(1)} \in L^1(\mathbb{R}^+)$*,* $f^{(2)} \in L^1(\mathbb{R}^-)$ *and* $f^{(i)} \geq 0$ *a.e.*
Conversely any measure defined as above is a Lebesgue-absolutely continuous Lévy measure that integrates $xe^x 1_{\{x>1\}}$*.*

Proof. Since ν is a non-negative sigma-finite measure and absolutely continuous with respect to the Lebesgue measure, it admits non-negative densities $\tilde{f}_1$, $\tilde{f}_2$ due to the Radon-Nikodym theorem. Furthermore, since

$$\int_{\mathbb{R}^+} \frac{(1 + x\mathrm{e}^x)x^2}{1 + x^2} \nu(dx) = \int_{\mathbb{R}^+} \frac{(1 + x\mathrm{e}^x)x^2}{1 + x^2} \tilde{f}_1 \, dx < \infty$$

and

$$\int_{\mathbb{R}^-} \frac{x^2}{1 + x^2} \nu(dx) = \int_{\mathbb{R}^-} \frac{x^2}{1 + x^2} \tilde{f}_2 \, dx < \infty$$

$f^{(1)} := \frac{(1+x\mathrm{e}^x)x^2}{1+x^2}\tilde{f}_2$ and $f^{(2)} := \frac{x^2}{1+x^2}\tilde{f}_2$, respectively, are integrable. Because they are also measureable it follows $f^{(1)} \in L^1(\mathbb{R}^+)$, $f^{(2)} \in L^1(\mathbb{R}^-)$ and $f^{(i)} \geq 0$ a.e.
Conversely, one easily observes, that $f^{(1)} \geq 0 \in L^1(\mathbb{R}^+)$ and $f^{(2)} \geq 0 \in L^1(\mathbb{R}^-)$ implies $x\mathrm{e}^x 1_{x>1}$ to be integrable and $\int_{\mathbb{R}} \min\{x^2, 1\}\nu(dx) < \infty$, which in turn entails that ν is a Lévy measure having the specified properties. □

Note that the above assumption is just one on the smoothness, not on the type of ν (i.e. finite or infinite activity and variation, respectively).
As also L^1 is not weakly closed, theoretically the search space has to be narrowed to an equi-integrable set, when the regularization functional is chosen to be the L^1-norm. Again this is merely a theoretic issue, since numerically this will not cause problems. Nevertheless, we propose one possible choice of $\mathcal{R}_\nu$, which fulfills all theoretical requirements and yields a Hilbert space structure for the Lévy measure.

Example 3. *Let $\tilde{f}^{(i)} \geq 0$ be square integrable functions and define $f^{(i)} = \tilde{f}^{(i)}/(1 \vee x^{1/2+\epsilon})$, $i = 1, 2$. Then obviously both $f^{(i)}$ are L^1 functions, which in tern determine a Lévy measure via* (63)*. Then setting $\mathcal{R}(\nu - \nu_*) = \|\tilde{f}^{(1)} - \tilde{f}^{(1)}_*\|_{L^2} + \|\tilde{f}^{(2)} - \tilde{f}^{(2)}_*\|_{L^2}$ all assumptions on $\mathcal{R}$ are fulfilled.*

Turning to the theoretic results, Theorems 3 and 4 give a satisfying answer to the question of the continuous dependence of the calibrated parameters with respect to the input data. Nevertheless, we are mainly interested in the market models associated to these parameters. However, since for every $\theta \in D(H)$ there exists a unique weak solution of the SDE (4), the well-posedness of the Tikhonov-regularized problem is directly implied by Theorem 3. Regarding the convergence issue we state the following lemma. We just prove it for the case that representation (63) holds, since the reasoning in the general case is similar.

Lemma 1. *The weak topology of the probability measures on the space of càdlàg functions (endowed with the Skorokhod-topology) is weaker than the weak topology on U induced by R, i.e. $\theta_n \rightharpoonup \theta$ implies $X^{(n)} \stackrel{\mathcal{L}}{\rightharpoonup} X$, where $X^{(n)}$ and X correspond to the unique weak solutions of* (2) *with parameters θ_n and θ, respectively.*

Proof. Let $\theta_n \rightharpoonup \theta$, i.e. $\sigma_n^2 \rightharpoonup \sigma^2$ in H^s, $a_n \rightharpoonup a$ in H^s and $\mu_n^{+,-} \rightharpoonup \mu^{+,-}$.
Remember that we already have calculated the characteristics (24) of X_t in the proof of Proposition 1. Our aim is to apply [28, Theorem IX.4.8] and we will show that all the conditions of it are satisfied in our case. Note that formally one would have to replace the truncation function in (24) by a continuous one. However, for notational convenience, we discuss the conditions with the truncation function $h = x1_{x\geq -1}$, since all the arguments rest the same for a continuous truncation function, like

$$h(x) = 1_{\{x\geq -1\}}(x)\ x - 1_{\{-2\leq x<-1\}}\ (2+x).$$

We need the modified second characteristic of $X^{(n)}$, that is given by

$$\tilde{\gamma}(X_t^{(n)}, t) = \gamma(X_t^{(n)}) + a(\tilde{X}_t^{(n)}, t)\left(\int_0^\infty \frac{1+x^2}{\mathrm{e}^x+1}\mu_n^+(dx) + \int_{-1}^0 (1+x^2)\mu_n^-(dx)\right),$$

where $\tilde{X}_t^{(n)} = X_t^{(n)} + (r-\eta)t$, as in the proof of Proposition 1.
Note that due to Proposition 1 all $X^{(n)}$, as well as X, satisfy Assumption IX.4.3. Condition

IX.4.9 is implied by the tightness assumption. Condition IX.4.10 is directly implied by the fact that the functions

$$
\begin{aligned}
(x,t) &\mapsto \sigma_n^2(x,t), & (x,t) &\mapsto \sigma^2(x,t),\\
(x,t) &\mapsto a_n(x,t), & (x,t) &\mapsto a(x,t)
\end{aligned}
$$

mapping from $\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ are all continuous, which in turn stems from (55). Thus also

$$b(x,t) = -\frac{1}{2}\sigma^2(x,t) - a(x,t) \int_{\mathbb{R}} \left(\mathrm{e}^z - 1 - z 1_{|z|\leq 1}\right) \nu(dz)$$

is a bounded and continuous function.
Furthermore b, γ, $\tilde{\gamma}$ are uniformly convergent in $\mathbb{R} \times \mathbb{R}^+$, which is a direct consequence of (55). Thus condition IX.4.11 is implied by the weak convergence of $\mu_n^{+,-}$ and the observation that

$$\frac{1+x^2}{x^2(1+x\mathrm{e}^x)}\zeta(x) \in L^\infty(\mathbb{R}^+) \quad \text{and} \quad \frac{1+x^2}{x^2}\zeta(x) \in L^\infty(\mathbb{R}^-)$$

for all functions ζ being continuous, bounded and satisfying $\zeta(x) = o(|x|^2)$ for $|x| \to 0$.
Hence all conditions of the theorem are satisfied.
The theorem is indeed stated for homogeneous processes, but either defining $\hat{X}_t^{(n)} = (X_t^{(n)}, t)$ and $\hat{X}_t = (X_t, t)$ or simply following the proof, it is clear that the uniform convergence in (x,t) is the decisive extra-condition. □

Remark 7. *Theorems 3, 4 and Lemma 1 hold for both specifications of the error terms, i.e. for* (53) *and* (54). *However, it has to be kept in mind that, $\delta \to 0$ in Theorem 4 means, that also the interpolation error has to converge to 0 in the norm specified by* (54). *Therefore the weighting function should be chosen to be positive only in small intervals around the given data and to have suprema at the given strike levels.*

Remark 8. *Lemma 1 also paves the way for the generalization of the theory to pure jump-type processes: Since the payoff of a European put option is a bounded and continuous function, Lemma 1 and an application of the call-put parity imply the weak continuity of the parameter to solution map independently of the PIDE-formulation. Thus, as long as the parameter space is such that the associated asset price models exist (i.e.* (2) *is solvable) for all elements, the results (Theorems 3 and 4) still remain valid.*

Note that Theorem 4 does not contain any information about the speed of the convergence, i.e. whether the convergence has a reasonable asymptotic rate, and in theory the convergence can be arbitrarily slow (see [37]). Thus some work has been devoted to the study of convergence rates of Tikhonov regularization, mostly in the context of Hilbert space operators (see e.g. [22] for an overview). In recent years, however, also results for Banach space operators were obtained (see e.g. [7, 27, 35]). All quantitative results in Hilbert, as well as in, Banach spaces depend on an abstract source condition, which is an assumption on the smoothness of the distance between the real parameter and the initial guess. In general this condition can not be shown to hold, since it depends on the unknown real parameters. Furthermore it would be desireable to translate the convergence rates for the parameters into convergence rates of the market models, which seems to be out of reach, yet. Nevertheless we shall briefly discuss the issue of convergence rates here. In [7, Section 3.3] it was shown that under certain assumptions the Bregmann distance of the calibrated parameters converges with rate $O(\delta)$, if the regularization parameter is chosen, such that $\alpha \sim \delta$ for $\delta \to 0$. In our setting these assumptions are satisfied, if $\|\|_b$ is chosen as norm in the observation space and a source condition involving the adjoint of the Frechet derivative is satisfied.

6 Possible applications and conclusions

The theory developed in this paper has a wide range of applications. Some examples would be to robustly estimate

- the local speed function in a generalized jump-diffusion model, keeping all other parameters fixed (cf. [30]).
- the local volatility function in a Dupire model, that includes jumps (cf [2]).
- a state-dependent time change of a jump-diffusion model. This can be done by setting $\sigma^2(k,T) = \sigma^2\tau(k,T)$ and $a(k,T) = a\tau(k,T)$, where σ^2 and a are constants. Then the regularization should be done by using $\|\tau - \tau_*\|_s$ instead of the norms penalizing $a(k,T)$ and $\sigma^2(k,T)$.
- the jump-process (or certain properties of it) in a generalized jump-diffusion model with state-dependent jump arrival rate. The results suggest to use the $\|\|_s$-norm for the local speed function and a suitable norm for the Lévy measure (cf. the discussion in Section 6).

from given market option prices.
Note that in theory it would be possible to reconstruct all parameters in a stable manner. However, since there is only a limited number of option prices (i.e. input data) available, it is rather questionable, if this can be done in a reasonable way.
Another possible field of application is the identification of an equivalent martingale measure $\mathbb{Q}$, when information about the objective probability measure $\mathbb{P}$ is available. The regularizing functional R should then be chosen in a way, such that it relates to the distance between $\mathbb{P}$ and $\mathbb{Q}$. If R is in accordance with the regularizing functional in Section 5, i.e. if $\mathcal{S}$ in (61) can be continuously embedded in the search space for this problem and it can be shown, that there is at least one equivalent martingale measure $\mathbb{Q}$ with finite distance to $\mathbb{P}$, Theorems 3 and 4 again apply. For example it is easy to see that the framework of Cont & Tankov [10, 11] can be embedded in our framework.
Depending on the application also a numerical procedure to minimize the Tikhonov functional has to be chosen. In [30] a Gauss-Newton type method based on numerical schemes for solving PIDE's was implemented. Proposition 7 also allows for such a kind of optimization routine in the general context. However, in some special cases there might be more direct and thus faster approaches than the discretization of the PIDE's available (cf. e.g. [10]).

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