

# **A model function method in total least squares**

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**RICAM-Report 2008-18**

# A MODEL FUNCTION METHOD IN TOTAL LEAST SQUARES

SHUAI LU, SERGEI V. PEREVERZEV, AND ULRICH TAUTENHAHN

ABSTRACT. In the present paper, we investigate the dual regularized total least squares (dual RTLS) from a computational aspect. More precisely, we propose a strategy for finding two regularization parameters in the resulting equation of dual RTLS. This strategy is based on an extension of the idea of model function originally proposed by Kunisch, Ito and Zou for a realization of the discrepancy principle in the standard one parameter Tikhonov regularization. For dual RTLS we derive a model function of two variables and show its reliability using standard numerical tests.

## 1. Introduction

Let  $A_0 : X \rightarrow Y$  be a bounded linear operator between infinite dimensional real Hilbert spaces  $X$  and  $Y$  with norms  $\|\cdot\|$  and inner products  $\langle \cdot, \cdot \rangle$ . If the range  $\mathcal{R}(A_0)$  is not closed, the equation

$$A_0 x = y_0 \tag{1.1}$$

is ill-posed. Through the context, we assume that the operator  $A_0$  is injective and  $y_0$  belongs to  $\mathcal{R}(A_0)$  so that there exists a unique solution  $x^\dagger \in X$  of the equation (1.1). In real applications an ideal problem as (1.1) is seldomly available, and the following situation is more realistic:

- (1) Instead of  $y_0 \in \mathcal{R}(A_0)$ , noisy data  $y_\delta \in Y$  are given with

$$\|y_0 - y_\delta\| \leq \delta. \tag{1.2}$$

- (2) Instead of  $A_0 \in \mathcal{L}(X, Y)$ , an approximate operator  $A_h$  is given with

$$\|A_0 - A_h\| \leq h. \tag{1.3}$$

The problem (1.1) with noisy data  $y_\delta$  instead of  $y_0$  has extensively been discussed in the last several decades, see, e.g., [8]. In this paper we extend our research on the problem (1.1)-(1.3) in [13] and discuss parameter choice issues for dual regularized total least squares (dual RTLS).

**Regularized total least squares.** In the classical total least squares problem (TLS) an estimate  $(\hat{x}, \hat{y}, \hat{A})$  for  $(x^\dagger, y_0, A_0)$  from known data  $(y_\delta, A_h)$  is determined by solving the constrained minimization problem [7, 10]

$$\|A - A_h\|^2 + \|y - y_\delta\|^2 \rightarrow \min \quad \text{subject to} \quad Ax = y. \tag{1.4}$$

Due to the ill-posedness of the problem (1.1) it may happen that there does not exist a solution  $\hat{x}$  of the TLS problem (1.4) in the space  $X$ . Furthermore,

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*Date:* June 20, 2008.

*2000 Mathematics Subject Classification.* 65J20, 65M30.

if there exists a solution  $\hat{x}$  of the TLS problem (1.4), this solution may be far away from the desired solution  $x^\dagger$ . Therefore, it is quite natural to restrict the set of admissible solutions by searching for an approximation  $\hat{x}$  that belong to some prescribed set  $K$ , which is the philosophy of regularized total least squares (RTLS), see [6]. The simplest case occurs when the set  $K = \{x \in X \mid \|Bx\| \leq R\}$  is a ball with prescribed radius  $R$ . This leads us to the regularized total least squares problem in which some estimate  $(\hat{x}, \hat{y}, \hat{A})$  for  $(x^\dagger, y_0, A_0)$  is determined by solving the constrained minimization problem

$$\|A - A_h\|^2 + \|y - y_\delta\|^2 \rightarrow \min \quad \text{subject to} \quad Ax = y, \quad \|Bx\| \leq R. \quad (1.5)$$

The special case when  $A_h = A_0$  leads to the method of quasi-solution of Ivanov [11], also called  $B$ -constrained least squares solution. In real applications, a good choice of  $R$  is usually based on the knowledge of the exact solution  $x^\dagger$  (i.e.  $R = \|Bx^\dagger\|$ ) [14], which causes inconvenience for RTLS since  $x^\dagger$  is unknown. Numerical tests showing instability of RTLS with a misspecified bound  $R$  can be found in the paper [13].

**Dual regularized total least squares.** To avoid the above mentioned disadvantage of RTLS, it makes sense to look for approximations  $(\hat{x}, \hat{y}, \hat{A})$  which satisfy the constraints

$$Ax = y, \quad \|y - y_\delta\| \leq \delta \quad \text{and} \quad \|A - A_h\| \leq h.$$

The solution set characterized by these three side conditions is non-empty. Selecting from the solution set the element which minimizes  $\|Bx\|$  leads to a problem in which the triple  $(\hat{x}, \hat{y}, \hat{A})$  for  $(x^\dagger, y_0, A_0)$  is determined by solving the constrained minimization problem

$$\|Bx\| \rightarrow \min \quad \text{subject to} \quad Ax = y, \quad \|y - y_\delta\| \leq \delta, \quad \|A - A_h\| \leq h. \quad (1.6)$$

This problem can be considered as the dual one of (1.5). The following theorem has been proven in [13] and will be important for our further discussion.

**Theorem 1.1.** *If the two constraints  $\|y - y_\delta\| \leq \delta$  and  $\|A - A_h\| \leq h$  of the dual RTLS problem (1.6) are active, then its solution  $x = \hat{x}$  satisfies the equation*

$$(A_h^T A_h + \alpha B^T B + \beta I)x = A_h^T y_\delta \quad (1.7)$$

with the parameters  $\alpha, \beta$  solving the system

$$\|A_h x^{\delta, h}(\alpha, \beta) - y_\delta\| = \delta + h \|x^{\delta, h}(\alpha, \beta)\|, \quad \beta = -\frac{h(\delta + h \|x^{\delta, h}(\alpha, \beta)\|)}{\|x^{\delta, h}(\alpha, \beta)\|}, \quad (1.8)$$

where  $x^{\delta, h}(\alpha, \beta)$  is the solution of (1.7) for fixed  $\alpha, \beta$ .

It can be seen from this theorem that a realization of dual RTLS is related with solving a system of highly nonlinear equations (1.8), and it may raise computational difficulties. For example, an application of Newton-type methods would require partial derivatives of  $x^{\delta, h}(\alpha, \beta)$ , which in turn, can be obtained by solving a linear system whose size is the same as that of the regularized problem (1.7).

At the same time, one might observe that the resulting equation (1.7) of dual RTLS looks similar to multi-parameter Tikhonov regularization studied in [3, 4],

and therefore, some methods for determination of multiple regularization parameters, such as a generalized L-curve [3], or a multi-parameter version of the balancing principle [1], might be used in RTLS. But the point is that in the context of dual RTLS one of regularization parameters, namely  $\beta$ , is necessarily negative, while in [3, 1] all regularization parameters are assumed to be positive. On the other hand, the first equation of (1.8) is really similar to one appearing in the discrepancy principle [15] for determining a regularization parameter in one-parameter regularization methods applied to equations with noisy operators. For the standard one-parameter Tikhonov method it has been proposed in [12, 16] to realize the discrepancy principle approximating the discrepancy  $\|A_0 x^\delta(\alpha) - y_\delta\|$  locally by means of some simple model function  $m(\alpha)$ . The goal of the present paper is to extend the idea of [12] to the case of multiple regularization parameters.

In the next section we derive an appropriate form of a model function of two variables and discuss how it can be used for solving the nonlinear system (1.8) appearing in dual RTLS. Then using some standard numerical tests from Matlab Regularization toolbox [9] we demonstrate the efficiency of our proposed model function approach.

## 2. Model function for dual regularized total least squares

In this section we assume that a domain  $\Sigma \subset \mathbb{R}^2$  is given such that for any  $(\alpha, \beta) \in \Sigma$  the equation (1.7) is uniquely solvable, and its solution  $x = x^{\delta,h}(\alpha, \beta)$  is continuously differentiable as a function of  $\alpha$  and  $\beta$ . Moreover, we assume that  $\Sigma$  contains both parameters  $\alpha, \beta$  solving (1.8).

This assumption is not restrictive. For example, in the theory of  $B$ -constrained least squares solutions, source conditions of the form  $x^\dagger = (A_0^T A_0)^{r/2} v$  with  $v \in X$  and some  $r > 0$  are assumed [8]. In this case the choice  $B = (A_0^T A_0)^{-r/2}$  in the dual RTLS problem may be appropriate. Moreover, in some applications (see e.g. [5]) it is assumed that both  $A_0$  and  $A_h$  admit singular value decomposition in the same basis system  $\{v_i\}$ . It means that

$$A_0 = \sum_i a_i u_i \langle v_i, \cdot \rangle, \quad A_h = \sum_i a_{h,i} u_{h,i} \langle v_i, \cdot \rangle, \quad B = \sum_i a_i^{-r} v_i \langle v_i, \cdot \rangle,$$

where  $a_i, a_{h,i} \rightarrow 0$  as  $i \rightarrow \infty$ . Then the solution  $x^{\delta,h}(\alpha, \beta)$  of (1.7) can be represented as

$$x^{\delta,h}(\alpha, \beta) = \sum_i \frac{a_{h,i} \langle u_{h,i}, y_\delta \rangle v_i}{a_{h,i}^2 + \alpha a_i^{-2r} + \beta},$$

and differentiability is guaranteed almost everywhere in  $\mathbb{R}^2$ .

From Theorem 1.1 we know that in case of active constraints the dual RTLS solution  $\hat{x} = x^{\delta,h}(\alpha, \beta)$  of problem (1.6) can be obtained by solving the minimization problem

$$\min_{x \in X} J_{\alpha,\beta}(x), \quad J_{\alpha,\beta}(x) = \|A_h x - y_\delta\|^2 + \alpha \|Bx\|^2 + \beta \|x\|^2$$

with regularization parameters  $(\alpha, \beta)$  chosen by following *a posteriori* rule:

**DRTLS rule:** Choose  $(\alpha, \beta)$  by solving the system (1.8).

Clearly, our DRTLs rule is a special multi-parameter choice rule of a *posteriori* type for choosing the both regularization parameters  $(\alpha, \beta)$  in Tikhonov's functional  $J_{\alpha, \beta}(x)$ . For fixed  $\alpha, \beta \in \Sigma$  the solution  $x^{\delta, h}(\alpha, \beta)$  of the minimization problem  $J_{\alpha, \beta}(x) \rightarrow \min$  is equivalent to the solution of the regularized equation (1.7), or equivalent to the solution of the variational equation

$$\langle A_h x, A_h g \rangle + \alpha \langle Bx, Bg \rangle + \beta \langle x, g \rangle = \langle y_\delta, A_h g \rangle \quad \forall g \in X. \quad (2.1)$$

Substituting  $x^{\delta, h}(\alpha, \beta)$  into  $J_{\alpha, \beta}(x)$  provides the following minimal cost function

$$F(\alpha, \beta) = \|A_h x^{\delta, h}(\alpha, \beta) - y_\delta\|^2 + \alpha \|Bx^{\delta, h}(\alpha, \beta)\|^2 + \beta \|x^{\delta, h}(\alpha, \beta)\|^2. \quad (2.2)$$

Formulas for the partial derivatives of  $F$  are given in the next lemma.

**Lemma 2.1.** *For  $(\alpha, \beta) \in \Sigma$ , the partial derivatives of  $F(\alpha, \beta)$  with respect to  $\alpha$  and  $\beta$  are given by*

$$F'_\alpha(\alpha, \beta) = \|Bx^{\delta, h}(\alpha, \beta)\|^2, \quad F'_\beta(\alpha, \beta) = \|x^{\delta, h}(\alpha, \beta)\|^2.$$

*Proof.* We prove the lemma for  $F'_\alpha$ . The proof for  $F'_\beta$  is similar. Note that if some  $u(\alpha) \in X$  is a differentiable function with respect to  $\alpha$  and  $u'(\alpha) \in X$  then

$$\begin{aligned} \frac{d}{d\alpha} \|u(\alpha)\|^2 &= \lim_{t \rightarrow 0} \frac{\langle u(\alpha + t) - u(\alpha), u(\alpha) \rangle + \langle u(\alpha + t) - u(\alpha), u(\alpha + t) \rangle}{t} \\ &= 2\langle u(\alpha), u'(\alpha) \rangle. \end{aligned}$$

Therefore, for  $x := x^{\delta, h}(\alpha, \beta)$  we have

$$F'_\alpha(\alpha, \beta) = 2\langle A_h x - y_\delta, A_h x'_\alpha \rangle + 2\alpha \langle Bx, Bx'_\alpha \rangle + \|Bx\|^2 + 2\beta \langle x, x'_\alpha \rangle.$$

Moreover, using (2.1) with  $g = x'_\alpha$  we arrive at

$$\langle A_h x - y_\delta, A_h x'_\alpha \rangle + \alpha \langle Bx, Bx'_\alpha \rangle + \beta \langle x, x'_\alpha \rangle = 0,$$

that immediately gives us the result  $F'_\alpha(\alpha, \beta) = \|Bx\|^2$ .  $\square$

In view of Lemma 2.1, for  $x = x^{\delta, h}(\alpha, \beta)$  we have

$$\|A_h x - y_\delta\|^2 = F(\alpha, \beta) - \alpha F'_\alpha(\alpha, \beta) - \beta F'_\beta(\alpha, \beta).$$

Hence, the first equation in (1.8) can be rewritten as

$$F(\alpha, \beta) - \alpha F'_\alpha(\alpha, \beta) - \beta F'_\beta(\alpha, \beta) = (\delta + h\sqrt{F'_\beta(\alpha, \beta)})^2. \quad (2.3)$$

Now the idea is to approximate  $F(\alpha, \beta)$  by a simple model function  $m(\alpha, \beta)$  such that one could easily solve the corresponding approximate equation

$$m(\alpha, \beta) - \alpha m'_\alpha(\alpha, \beta) - \beta m'_\beta(\alpha, \beta) = (\delta + h\sqrt{m'_\beta(\alpha, \beta)})^2$$

for  $\alpha$  or  $\beta$ . To derive an equation for such a model function, we note that for  $g = x = x^{\delta, h}(\alpha, \beta)$  the variational form (2.1) gives us

$$\|A_h x\|^2 + \alpha \|Bx\|^2 + \beta \|x\|^2 = \langle A_h x, y_\delta \rangle.$$

Then, for  $x = x^{\delta, h}(\alpha, \beta)$ ,

$$\begin{aligned} F(\alpha, \beta) &= \langle A_h x - y_\delta, A_h x - y_\delta \rangle + \alpha \|Bx\|^2 + \beta \|x\|^2 \\ &= \|A_h x\|^2 + \|y_\delta\|^2 - 2\langle A_h x, y_\delta \rangle + \alpha \|Bx\|^2 + \beta \|x\|^2 \\ &= \|y_\delta\|^2 - \|A_h x\|^2 - \alpha \|Bx\|^2 - \beta \|x\|^2. \end{aligned}$$

Now, following [12, 16] for  $x = x^{\delta,h}(\alpha, \beta)$ , we approximate the term  $\|A_h x\|^2$  by  $T\|x\|^2$ , where  $T$  is a positive constant to be determined. This approximation together with Lemma 2.1 gives us an approximate formula

$$F(\alpha, \beta) \approx \|y_\delta\|^2 - (\beta + T)F'_\beta(\alpha, \beta) - \alpha F'_\alpha(\alpha, \beta).$$

By a model function we mean a parameterized function  $m(\alpha, \beta)$  for which this formula is exact, that is,  $m(\alpha, \beta)$  should solve the differential equation

$$m(\alpha, \beta) + \alpha m'_\alpha(\alpha, \beta) + (\beta + T)m'_\beta(\alpha, \beta) = \|y_\delta\|^2.$$

It is easy to check that a simple parametric family of the solutions of this equation is given by

$$m(\alpha, \beta) = \|y_\delta\|^2 + \frac{C}{\alpha} + \frac{D}{T + \beta} \quad (2.4)$$

where  $C, D, T$  are constants to be determined. Now we are ready to present an algorithm for the computation of the regularization parameters  $(\alpha, \beta)$  according to our DRTLS rule. That is, we present an algorithm for the approximate solution of the equations (1.8) by a special three-parameter model function approach.

**Model function approach for the DRTLS rule.** Given  $\alpha_0, \beta_0, y_\delta, A_h, \delta$  and  $h$ . Set  $k := 0$ .

- (1) Solve (1.7) with  $(\alpha_k, \beta_k)$  to get  $x^{\delta,h}(\alpha_k, \beta_k)$ . Compute  $F_1 = F(\alpha_k, \beta_k)$ ,  $F_2 = F'_\alpha = \|Bx^{\delta,h}(\alpha_k, \beta_k)\|^2$  and  $F_3 = F'_\beta = \|x^{\delta,h}(\alpha_k, \beta_k)\|^2$ . In (2.4) set  $C = C_k$ ,  $D = D_k$ ,  $T = T_k$  such that

$$\begin{cases} m(\alpha_k, \beta_k) = \|y_\delta\|^2 + \frac{C}{\alpha_k} + \frac{D}{\beta_k + T} = F_1, \\ m'_\alpha(\alpha_k, \beta_k) = -\frac{C}{\alpha_k^2} = F_2, \\ m'_\beta(\alpha_k, \beta_k) = -\frac{D}{(\beta_k + T)^2} = F_3. \end{cases}$$

Then,

$$\begin{cases} C_k = -F_2 \alpha_k^2, \\ D_k = -\frac{(\|y_\delta\|^2 - F_1 - F_2 \alpha_k)^2}{F_3}, \\ T_k = \frac{\|y_\delta\|^2 - F_1 - F_2 \alpha_k}{F_3} - \beta_k. \end{cases} \quad (2.5)$$

Update  $\beta = \beta_{k+1}$  using the second equation (1.8) as

$$\beta_{k+1} = -\frac{h(\delta + h\|x^{\delta,h}(\alpha_k, \beta_k)\|)}{\|x^{\delta,h}(\alpha_k, \beta_k)\|}$$

and update  $\alpha = \alpha_{k+1}$  as the solution of the linear algebraic equation

$$m(\alpha, \beta_{k+1}) - \alpha m'_\alpha(\alpha, \beta_{k+1}) - \beta_{k+1} m'_\beta(\alpha, \beta_{k+1}) = \left(\delta + h\sqrt{m'_\beta(\alpha, \beta_{k+1})}\right)^2.$$

This equation is an approximate version of (1.8), (2.3), where  $F(\alpha, \beta)$  is approximated by a model function  $m(\alpha, \beta)$  given by (2.4), (2.5).

- (2) STOP if the stopping criteria  $|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k| \leq \epsilon$  is satisfied; otherwise set  $k := k + 1$ , GOTO (1).

The proposed algorithm is a special fixed point iteration for realizing the DRTLs rule in the method of Tikhonov regularization with two regularization parameters  $(\alpha, \beta)$ . Although this algorithm works well in experiments, we don't have any convergence results. We even don't know conditions under which the algorithm is well defined. The only result which we have is following: If the iteration converges, then the limit  $(\alpha^*, \beta^*) = \lim_{k \rightarrow \infty} (\alpha_k, \beta_k)$  is a solution of the nonlinear system (1.8).

### 3. Numerical tests

At first we recall a theoretical error bound obtained in [13] for dual RTLs under the assumption that the exact operator  $A_0$  is related with a densely defined unbounded selfadjoint strictly positive operator  $B : X \rightarrow X$  by a so-called link condition.

**Theorem 3.1.** *Assume that there exist positive constants  $a, m, p, E$  such that*

$$m\|B^{-a}x\| \leq \|A_0x\| \quad \text{for all } x \in X \quad (3.1)$$

*and assume in addition the solution smoothness*

$$\|B^p x^\dagger\| \leq E. \quad (3.2)$$

*Let  $x^{\delta,h}(\alpha, \beta)$  be the dual RTLs solution satisfying (1.7), (1.8). If  $1 \leq p \leq 2 + a$ , then*

$$\|x^{\delta,h}(\alpha, \beta) - x^\dagger\| \leq 2E^{\frac{a}{p+a}} \left( \frac{\delta + h\|Bx^\dagger\|}{m} \right)^{\frac{p}{p+a}} = O((\delta + h)^{\frac{p}{p+a}}).$$

This theorem allows a theoretical comparison of dual RTLs with the one-parameter Tikhonov regularization scheme, where a regularized solution  $x = x^{\delta,h}(\alpha)$  is obtained from the following equation

$$(A_h^T A_h + \alpha B^T B)x = A_h^T y_\delta. \quad (3.3)$$

The regularization theory [15, 8] tells us that under the assumptions (1.2), (1.3), (3.1), (3.2) the order of accuracy  $O((\delta + h)^{\frac{p}{p+a}})$  is optimal, and it can be realized within the framework of one-parameter Tikhonov scheme. If the smoothness index  $p$  in (3.2) is unknown, then this scheme should be equipped with some a posteriori parameter choice rule.

Note that a knowledge of the smoothness index  $p$  is not necessary for performing dual RTLs. From Theorem 3.1 it follows that this two-parameter regularization scheme automatically adapts to the unknown smoothness index and, like one-parameter Tikhonov scheme, gives the optimal order of accuracy, at least for some range of  $p$ . In our numerical tests we compare the performance of dual RTLs with one-parameter Tikhonov regularization equipped with the quasi-optimality criterion for choosing the regularization parameter. This noise-free *a posteriori* rule has recently been advocated in [2]. Our dual RTLs rule will be realized by the algorithm discussed in the previous section while the parameter  $\alpha$  in the Tikhonov scheme (3.3) will be chosen in accordance with the quasi-optimality criterion as follows: Consider a set of regularization parameters  $\Gamma_N^p = \{\alpha_i : \alpha_i = \alpha_0 p^i, i = 1, 2, \dots, N\}$ ,  $p > 1$ . The quasi-optimality criterion selects such  $\alpha^* = \alpha_m \in \Gamma_N^p$  for

which the quantity  $v(\alpha_i) = \|x^{\delta,h}(\alpha_i) - x^{\delta,h}(\alpha_{i-1})\|$  has the minimum value  $v(\alpha_m)$  in the chosen set  $\Gamma_N^p$ .

In our numerical experiments we consider Fredholm integral equations

$$Kf(s) \equiv \int_a^b K(s,t)f(t)dt = g(s), \quad s \in [c,d] \quad (3.4)$$

with known solutions  $f(t)$ . Then, operators  $A_0$  and solutions  $x^\dagger$  are obtained in the form of  $n \times n$ -matrices and  $n$ -dimensional vectors by discretizing the corresponding integral operators  $K$  and solutions  $f$ . As in [6, 13], noisy data  $A_h, y_\delta$  are simulated by

$$A_h = A_0 + h\|E\|_F^{-1}E, \quad y_\delta = A_h x^\dagger + \sigma\|e\|^{-1}e$$

where  $E, e$  are  $n \times n$ -random matrices and  $n$ -dimensional random vectors from a normal distribution with zero mean and unit standard deviation, which were generated 50 times, so that each integral equation gives rise to 50 noisy matrix equations.

In our experiments with dual RTLS, for  $B$  we choose the  $(n-1) \times n$ -matrix

$$D = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix},$$

which is a discrete approximation of the first derivative on a regular grid with  $n$  points. The Tikhonov method (3.3) has been implemented with  $B = I$  and  $B = D$ . Moreover, in the quasi-optimality criterion the set of regularization parameters is chosen as

$$\Gamma_{50}^p = \{\alpha_i = 10^{-3} \cdot p^i, i = 1, 2, \dots, 50\}, \quad p = 1.1.$$

The first series of experiments is performed with the *ilaplace*( $n, 1$ ) function from [9]. It is a discretization of the inverse Laplace transformation by means of Gauss-Laguerre quadrature. The kernel  $K$  and the solution  $f$  are given by

$$K(s,t) = \exp(-st), \quad f(t) = \exp(-t/2),$$

both intervals are  $[0, \infty)$ . Noisy data are simulated with  $h = \sigma = 0.2$ . In our algorithm for the DRTLS rule we choose as starting values  $\alpha_0 = 10, \beta_0 = -0.2$ , and for stopping the iteration we choose  $\epsilon = 0.01$ . The results are displayed in Figure 1.

As it can be seen from Figure 1, Tikhonov method with  $B = I$  exhibits poor performance, the relative error is between 0.1 and 0.2. For  $B = D$  the results are better, but the performance is not stable, the relative error varies between 0.01 and 0.16. At the same time, dual RTLS (which has been realized by our model function approach) exhibits stable performance, the relative error is between 0.08 and 0.1.

The second example is performed with the *ilaplace*( $n, 2$ ) function in [9]. Similar to *ilaplace*( $n, 1$ ), we have the same kernel  $K$ , but

$$f(t) = 1 - \exp(-t/2).$$



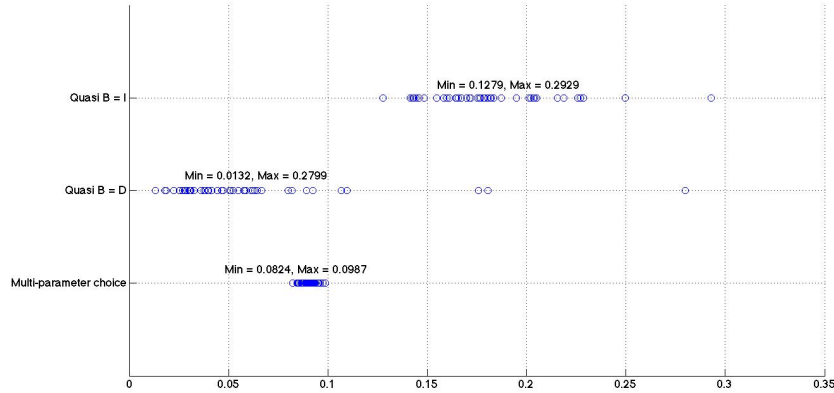


FIGURE 1. Comparison of relative errors in the approximate solutions for  $ilaplace(100, 1)$  computed by Tikhonov method (3.3),  $B = I$ ,  $B = D$ , with quasi-optimality criterion and by dual RTLS which has been realized by our model function approach.

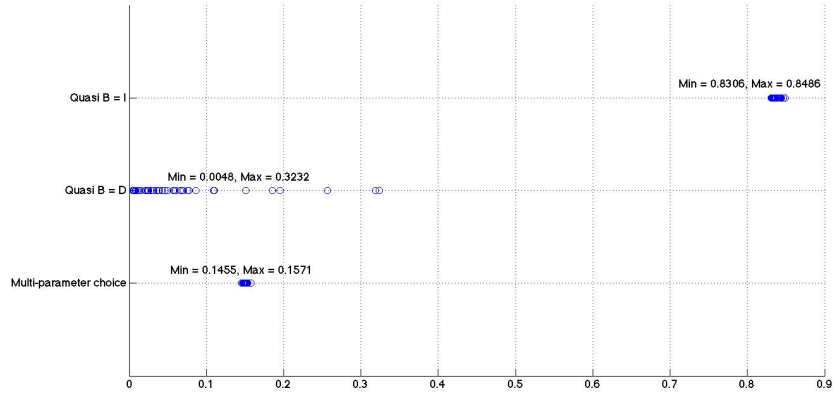


FIGURE 2. Comparison of relative errors in the approximate solutions for  $ilaplace(64, 2)$  computed by Tikhonov method (3.3),  $B = I$ ,  $B = D$ , with quasi-optimality criterion and by dual RTLS which has been realized by our model function approach.

This time we take a discretization level  $n = 64$ , since we would like to compare our results with [6], where the same discretization level has been chosen. In noise simulations,  $h = \sigma = 0.1$  has been used. The results are presented in Figure 2.

It is known [9] that the Tikhonov method with  $B = I$  fails to handle the example  $ilaplace(n, 2)$ . For  $B = D$  the results are reasonable, but again, the performance is not stable. As to dual RTLS, it exhibits stable performance, and it is interesting to note that similar relative errors were reported in [6] for  $ilaplace(64, 2)$  (a slight change in the solution) and RTLS, but the parameters  $\alpha, \beta$  were chosen in [6] "by hand" using the knowledge of the exact solution. In our experiments similar accuracy has been obtained automatically.

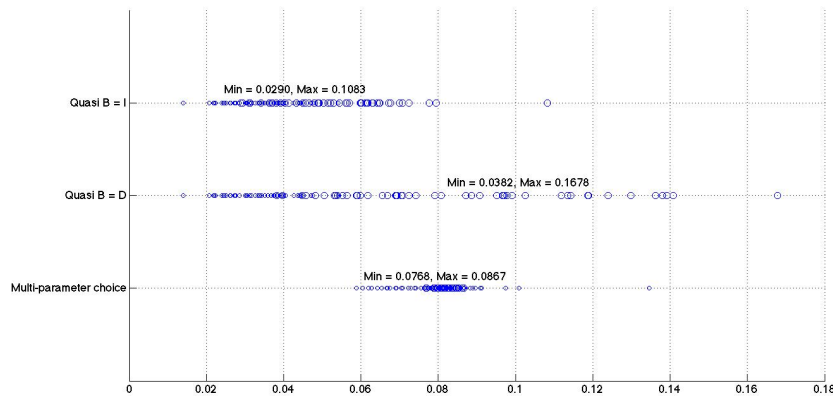


FIGURE 3. Comparison of relative errors in the approximate solutions for  $gravity(100, 1, 0, 1, 0.25)$  computed by Tikhonov method (3.3),  $B = I$ ,  $B = D$ , with quasi-optimality criterion and by dual RTLS which has been realized by our model function approach.

In our final example we use the function  $gravity(n, example, 0, 1, 0.25)$  [9]. It corresponds to the Fredholm equation (3.4) with

$$K(s, t) = 0.25((0.25)^2 + (s - t)^2)^{-3/2}, \quad f(t) = \sin(\pi t) + 0.5 \sin(2\pi t)$$

discretized on the interval  $[0, 1]$  by means of a collocation scheme with  $n = 100$  knots. Noisy data are simulated with  $h = \sigma = 0.2$ . The results are displayed in Figure 3.

In this example, standard Tikhonov method with  $B = I$  produces the best results, while the method with  $B = D$  is extremely unstable. As to dual RTLS (which has been realized by our model function approach), its maximal relative error is almost the same as for standard Tikhonov and it is still the most stable among the tested methods.

Thus, from our tests we conclude that the performance of the Tikhonov method with  $B = I$  and  $B = D$  depends on the problem, while dual RTLS (computed by our model function approach) exhibits a stable reliability independently on the problems under consideration.

### Acknowledgments

The first two authors are supported by the Austrian Fonds Zur Förderung der Wissenschaftlichen Forschung (FWF), Grant P20235-N18.

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