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On the interactions of critical curves, catastrophe points, scale space saddles, and iso-intensity manifolds in Gaussian scale space images under a one-parameter driven deformation

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Abstract

In this work we describe the possible transitions for the hierarchical structure that describes an image in Gaussian scale space. Until now, this structure has only been used for topological segmentation, while image matching and retrieval studies ignored the hierarchy. In order to perform such tasks based on the hierarchical structure, one needs to know which transitions are allowed when the structure is changed under influence of one control parameter.

1 Introduction

In the analysis of images and shapes, descriptors take a prominent place. The first aim of these descriptors is to represent the underlying structure in a simple way that is as invariant as possible, for instance with respect to rotations and scaling. Secondly, they should be robust with respect to (some) noise. Thirdly, they should capture "essential" aspects of the underlying structure, so that efficient and effective comparison tasks can be carried out on the descriptors. Essential for the latter is that the way the descriptor is obtained, is well-understood. This allows the definition of its possible changes.

A nice example of this principle can be found in shape matching using shock graphs [15]. Effective and efficient algorithms [14] are based on allowed changes (transitions) [3] derived from the definition of the descriptor.

For images it is more complicated to define such a descriptor. A starting point is the robustness towards noise. This can be achieved by considering noise as a local perturbation of the structure. One way to accomplish this is by blurring the structure. A Gaussian filter is traditionally used for this purpose. It was pointed out by Koenderink [8], that choosing an a priori width of the kernel relates to observing the image at only one scale. Taking into account all widths (scales), the image is investigated at all small ("noisy") levels and coarse ("structure containing") ones. Doing so, one obtains a scale space image. Secondly, he pointed out that this equals to observing the image dynamically changed by the heat equation, thus linking the kernel based approach to a partially differential equation. Although the scale space image contains an extra dimension, it was shown that it contains a tree-like substructure that serves as a rotation and scale invariant image descriptor [10] that can be used for image segmentation based on topological arguments. However, in order to be able to use the proposed tree structure for image matching, one needs to understand how this tree can change.

The focus of this paper is to describe the possible changes of this tree structure. We restrict ourselves to a one parameter family of perturbations, that is, the changes that occur when one extra introduced parameter changes. The changes can influence the "building blocks" of the tree structures. Such occasions can also occur when an image changes under a one parameter family, like changing camera position.

We will start with an short introduction to scale space and polynomials in it [8, 13], catastrophe theory [2, 1], and the tree structure [10] in section 2. In section 3 we introduce a special type of points that occur in our analysis, namely degenerated scale space saddles. Together with catastrophe points these form the basis of the possible transitions. They are presented in section 4, while the consequences for the tree structure are given in section 5. We give a simple example to illustrate the theory on an MR image in section 6 and we give conclusions in section 7.

2 Theory

Let $L : I\!R^n \rightarrow I\!R(\mathbf{x})$ be an image with $\mathbf{x}$ an n-dimensional spatial variable (point) and $L(x)$ the intensity measured at a point $\mathbf{x}$. For simplicity we will assume that $n = 2$ and, for notational ease $\mathbf{x} \in I\!R^2$, i.e. we assume that the image is embedded in the complete $I\!R^2$. The *Gaussian scale space ($\mathcal{GSS}$) image* $L(\mathbf{x}; t)$ is defined as the convolution of L with a Gaussian:

$$L(\mathbf{x}; t) = \int_{I\!R^n} \frac{1}{\sqrt{4\pi t}^{\,n}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \; L(\mathbf{y}) \; d\mathbf{y} \tag{1}$$

The Gaussian filter is the Greens' function of the diffusion, or heat, equation:

$$\begin{cases} \partial_t L(\mathbf{x}; t) = \Delta L(\mathbf{x}; t) \\ \lim_{t \downarrow 0} L(\mathbf{x}; t) = L(\mathbf{x}) \end{cases} \tag{2}$$

2.1 Scale space polynomials, jets

At each point (x_0, y_0) a Taylor expansion can be made of a function $L(x, y)$ to investigate the local structure:

$$L(x, y) \approx L + iL_i + \frac{ij}{2} L_{ij} + \frac{ijk}{6} L_{ijk} + \ldots, \tag{3}$$

where $L_{(.)}$ denotes the partial derivatives with respect to the variables $i, j, \ldots \in (x, y)$, evaluated at the point of interest.

In Gaussian scale space the same holds for the spacial and scale variable, i.e. $i, j, \ldots \in (x, y, t)$ and all derivatives of $L(x, y, t)$ are evaluated at (x_0, y_0, t_0). This yields a *scale space polynomial*.

Next, due to the heat equation, the scale derivatives in Taylor expression can be expressed in terms of spatial derivatives, since $\partial_t^n = \Delta^n$.

The n^{th} *order scale space jet* is defined as the scale space polynomial with spacial derivatives up to order n.

2.2 Critical curves, scale space germs

Critical curves are curves in scale space that satisfy $\nabla L = 0$. It has been proven by Damon [2] that these curve do not intersect in scale space unless extra constraints (like symmetry) are added. The curves consist of saddle branches and extremum branches that meet pairwise at catastrophe points. At such points the spatial *Hessian matrix*

$$H = \begin{pmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{pmatrix} \tag{4}$$

degenerates and has exactly one eigenvalue equal to zero. Tracing critical points over scale, at such catastrophe points a saddle-extremum pair is created or annihilated. These catastrophe points are also called top points [6, 5], since they occur at local extrema with respect to the scale axis: at local maxima for annihilations and at local minima for creations.

The results of Damon arise from Morse theory [1], that states that the local structure at critical points can be described by a Taylor polynomial of second order; the Hessian matrix contains eigenvalues that are non-zero. With an appropriate transformation one obtains $L = \pm x^2 \pm y^2$. This implies that we generically encounter normal points, extrema, or saddles. If one or more of the eigenvalues of the Hessian at a critical point is zero, the point is degenerated. The Morse lemma stats that at such a point we can distinguish between a Morse part and a non-Morse part. The first has variables with a non-degenerate Hessian matrix, the latter a degenerated one. The multiplicity of the zero eigenvalue(s) determine the order of the polynomial needed to describe the degenerated part. Such a polynomial (of degree 3 or higher) can be perturbed by lower order terms of which the number equal the multiplicity.

In Gaussian scale space, there is one semi-free parameter: scale. Therefore, an eigenvalue of the Hessian matrix can become zero with multiplicity one. The generic catastrophe is thus described by terms x^3 and y^2, called A_2 or cusp [1]. To account for the fact t can only increase during the evolution, two scale space polynomials are needed to describe an annihilation Eq. (5) and a creation Eq. (6) in a small environment of the origin:

$$L^a = x^3 + 6xt + y^2 + 2t \tag{5}$$

$$L^c = x^3 - 6xy^2 - 6xt + y^2 + 2t \tag{6}$$

The critical curves cc occur in the (x, t) plane and are parameterised by $cc^a(x, y, t) = (\pm\sqrt{-2t}, 0, t)$ and $cc^c(x, y, t) = (\pm\sqrt{2t}, 0, t)$. This follows directly from the x derivatives of Eqs. (5 - 6): $L^a_x|_{y=0} = 3x^2 + 6t$ and $L^c_x|_{y=0} = 3x^2 - 6t$.

As an important consequence, critical curves do not intersect (as this requires a higher order catastrophe), but can contain subsequent creation-annihilation points. This implies that we can define *scale space germs* as scale space polynomials that yield generic critical curves. For instance, L^a in Eq. (5) is a valid scale space germ, but L^c in Eq. (6) not, as for $t = \frac{1}{72}$ an intersection of two critical curves occurs. However, $L^c + \epsilon y$, $\epsilon \neq 0$ is a scale space germ.

2.3 Saddle points in scale space

In Gaussian scale space the only type of critical points are saddle points [9]. These scale space saddles appear at critical curves since the spatial derivatives vanish. To investigate these points, consider the Hessian matrix in scale space (the *extended Hessian*):

$$\mathcal{H} = \begin{pmatrix} L_{xx} & L_{xy} & L_{xt} \\ L_{xy} & L_{yy} & L_{yt} \\ L_{xt} & L_{yt} & L_{tt} \end{pmatrix} \tag{7}$$

Since this matrix contains the spatial Hessian, Eq. (4), at least one eigenvalue is positive and one is negative.

At scale space saddles the intensity on a critical curve has a local extremum: Let a curve be parametrised by $L(x(t), y(t), t)$, then

$$\frac{d\ L(x(t), y(t), t)}{dt} = L_x x_t + L_y y_t + L_t. \tag{8}$$

Since the parametrisation takes place at a critical curve, the spatial derivatives are zero, so Eq. (8) reduces to L_t. Next, at a scale space saddle $L_t = 0$.

2.4 A multi-scale image descriptor

The hierarchical structure described in [10] is best understood in analogy with ordinary images. Here, isophotes through spatial saddles in 2D divide an image topologically, in the sense that to each extremum a unique region is assigned. Next, such regions are nested (as the isophotes are nested), and for all extrema regions are obtained.

This idea can be extended to scale space images, where iso-manifolds through scale space saddles can divide the 3D volume into parts. At the initial image such part reduce to areas that correspond to topological segments. Here too a nesting is obtained. In contrast to the 2D case described above, it is possible to discriminate between the two parts connected at the saddle, due to the fact that the scale space saddle is connected to one of the extrema via a critical curve.

A sketch of such a structure is given in Figure 1. On the left one sees critical curves and a iso-manifold through a scale space saddle; a sketch in the (x, y) plane is given in the middle. The critical curve on the right (called "C") contains a saddle branch and an extremum branch. The two branches are connected at the catastrophe (top) point. Via the iso-manifold through the scale space saddle "SSS", this critical curve is connected to the another one on the left (called "D"). This is schematically visualised in the right image,

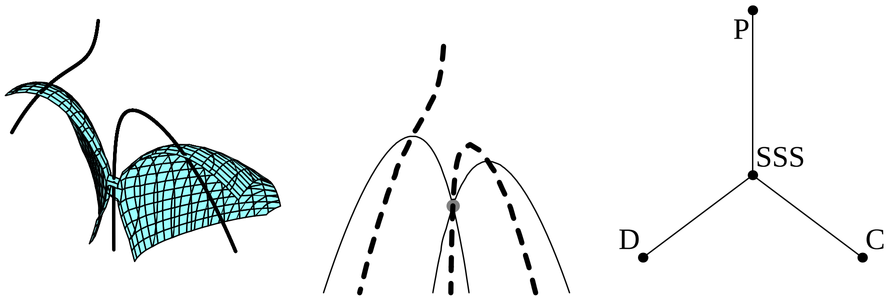

Figure 1: A sketch of the local structure at a scale space saddle in (x, y, t) space, t vertical (left), same in the $y = 0$ plane with the critical curves dashed (middle), and its algebraic tree representation (right).

where the "C" child part is connected to the "D" child part via SSS. This is the "building block" of the hierarchical tree. Each inner node represents a scale space saddle, while the leaves are formed by the extrema in the initial image.

Since on a critical curve multiple scale space saddles can occur, one has to be careful in selecting the right one: the global extremum.

3 Degenerated scale space saddles

The matrix in Eq. (7) is degenerated when at least one of its eigenvalues equal zero, i.e. $\det \mathcal{H} = 0$. Obviously, this is an extra requirement in scale space and thus non-generic.

A degenerated (extended) Hessian implies that the type of the point cannot be resolved. For critical points it means that it is neither a saddle nor an extremum, but merely a combination of both - exactly because two of such points coincide.

For scale space saddles, a zero eigenvalue of the extended Hessian implies that one of the other eigenvalues is positive and one negative. This can be thought of a coincidence of a saddle with two negative eigenvalues and one with two positive eigenvalues. Since the latter denotes a local minimum of $L(x(t), y(t), t)$ and the former a local maximum of $L(x(t), y(t), t)$, such an event is visible as appearing as a point of inflexion of $L(x(t), y(t), t)$.

Theorem 1 *Degenerated scale space saddles coincide with points of inflex-*

ion:

$$\frac{d^2\ L(x(t), y(t), t)}{dt^2} = 0 \Leftrightarrow \det \mathcal{H} = 0 \tag{9}$$

Proof 1 *Firstly, define the points of infection of $L(x(t), y(t), t)$ as scale space saddles with vanishing second order derivative of L:*

$$\begin{array}{rcl} \frac{d^2\ L(x(t),y(t),t)}{dt^2} & = & \frac{d}{dt}(L_x x_t + L_y y_t + L_t) \\ & = & (L_{xx}x_t + L_{xy}y_t + L_{xt})x_t \\ & & +(L_{xy}x_t + L_{yy}y_t + L_{yt})y_t \\ & & +(L_{xt}x_t + L_{yt}y_t + L_{tt}). \end{array} \tag{10}$$

In Eq. (10) we ignored the derivatives of the spatial parametrisations x_t and y_t, as they are accompanied by spatial derivatives. The latter vanish on critical curves. Next, the right hand side of Eq. (10) can be written as

$$(x_t, y_t, 1) \cdot \mathcal{H} \cdot \begin{pmatrix} x_t \\ y_t \\ 1 \end{pmatrix}. \tag{11}$$

That is, the second order derivative vanishes iff $\det \mathcal{H} = 0$. So when two scale space saddles coincide, the resulting point is degenerate.

Theorem 2 *On critical curves, Eq. (10) can be simplified to*

$$\frac{d^2\ L(x(t), y(t), t)}{dt^2} = (L_{xt}x_t + L_{yt}y_t + L_{tt}). \tag{12}$$

Proof 2 *On critical curves, we have $L_x = 0$. Consequently, $\frac{d}{dt}L_x(x(t), y(t), t) = 0$, so $(L_{xx}x_t + L_{xy}y_t + L_{xt}) = 0$. The similar argument holds for L_y.*

In a one-parameter family, only one eigenvalue equals zero. So we assume that the special event takes place in the (x, t) plane, while y is a regular (Morse) variable. Then we may neglect y derivatives and get

$$\frac{d^2\ L(x(t), 0, t)}{dt^2} = (L_{xt}x_t + L_{tt}) \tag{13}$$

and

$$\det \mathcal{H} = L_{xx}L_{tt} - L_{xt}^2 \tag{14}$$

At critical points we obtain $L_{xx}x_t + L_{xt} = 0$.

4 Transitions

When we allow a change driven by one parameter, we expect to see situations that are non-generic for still images. However, for moving images, e.g. films or a sequence to warp one image into another, such situations can become generic. Since the tree structure relies on critical curves, catastrophe points and scale space saddles, we will discuss the effect of the simplest combinations of them:

1. two catastrophe points coincide on a critical curve,
2. two critical curves intersect (necessarily at a catastrophe point),
3. a catastrophe point coincides with scale space saddle,
4. two scale space saddles coincide, and
5. two scale spaces saddles on a critical curve have the same value - either at one critical curve, or at different curves.
6. two scale spaces saddles on different critical curves but on the same iso-manifold have the same value.

For all situation we will describe scale space germs. They are generic in a one-parameter family of perturbations iff there is exactly one parameter that has to be fixed to obtain the described situation.

4.1 Two catastrophe points coincide on a critical curve

The situation that two catastrophe points coincide on a critical curve implies the description of a creation or an annihilation of a pair of creation and annihilations events. Such a pair exists of a critical curve traverses the manifold $\det H = 0$ twice. If the curve is perturbed, it is pulled away from this manifold. Exactly where the curve is tangent to the manifold, this special situation occurs. In [12], this was modelled by using a scale space germ in analogy of Eq. (6) by

$$L^c = x^3 - 6xy^2 - 6xt + y^2 + 2t + \epsilon y, \tag{15}$$

with $\epsilon \neq 0$ a free parameter. For $\epsilon \in (0, \frac{1}{32}\sqrt{6})$ a creation and an annihilation occur, for $\epsilon > \frac{1}{32}\sqrt{6}$ there are zero catastrophes, and for $\epsilon = \frac{1}{32}\sqrt{6}$ the two catastrophes coincide (are created or annihilated, depending on the decrease or increase of ϵ).

So with an additional parameter "wiggles" at a critical curve can be removed, i.e. a smoothing of the critical curve, as shown in Figure 2.

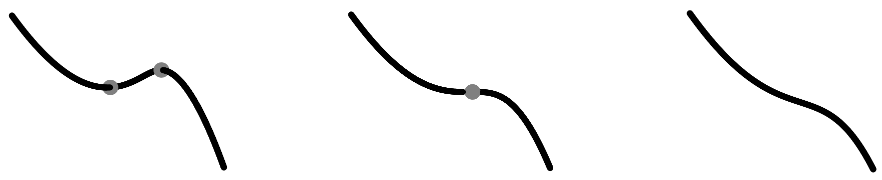

Figure 2: A critical curve in (x, y, t) space, t vertical. From left to right: When ϵ in Eq. (15) increases, a pair of top points is removed.

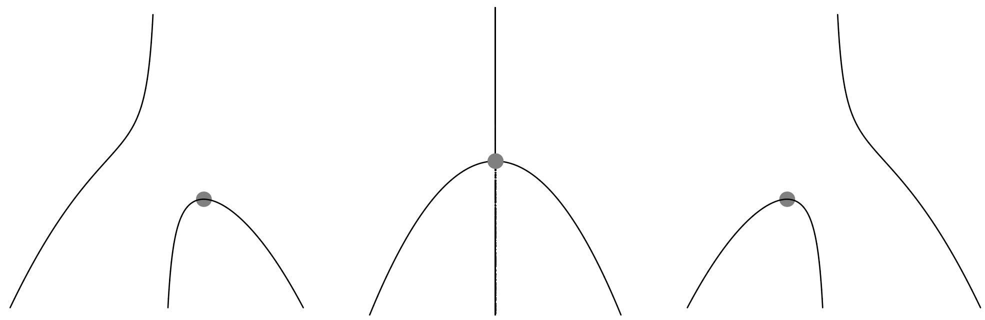

Figure 3: A critical curve in (x, y, t) space, t vertical. From left to right: When the sign of ϵ in Eq. (16) changes, the annihilation takes place with the other extremum.

4.2 two critical curves intersect

The intersection of critical curves occurs at catastrophe points. Therefore, this event is described by an higher order catastrophe.

For an annihilation (Eq. 5) this can be modelled [11] by $L_x = 4x(x^2 + 6t)$, arising from the scale space germ

$$L = x^4 + 12x^2t + 12t^2 + \epsilon x + y^2 + 2t \tag{16}$$

For $\epsilon = 0$ one obtains the so-called A_3 catastrophe in scale space, for $\epsilon \neq 0$ one has the generic A_2 catastrophe, see Figure 3.

For a creation we use the modelling $L_x = 4x(x^2 - 6t)$, arising from the more complicated scale space polynomial

$$L = x^4 - 12x^2t - 12t^2 - 12x^2y^2 + 2y^4 \tag{17}$$

The scale space germ is obtained by adding the perturbation terms

$$\alpha(x^2 + 2t) + \beta(y^2 + 2t) + \gamma x + \delta y. \tag{18}$$

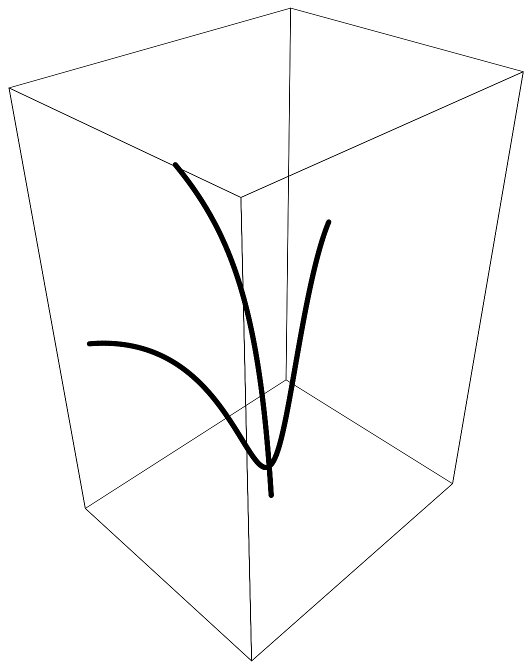

Figure 4: The local situation of the critical curves of Eq. 17 in x, y, t space (t vertical), with perturbations of Eq. 18 for $\delta = 0$. Varying δ yields a scale space germ.

Choosing non-zero values for α, β, and γ, together with $\delta = 0$ (that is, a one parameter degeneration), yields the desired inverse result shown in Figure 4, cf. the middle image in Figure 3.

4.3 A catastrophe point coincides with scale space saddle

When a scale space saddle and a catastrophe point coincide, the following requirements hold: $\det H = 0$ and $trH = 0$. The latter implies that $L_{xx} = -L_{yy}$, so the former reads $-L_{xx}^2 - L_{xy}^2 = 0$. So the complete second order structure has to vanish: $L_{xx} = L_{xy} = L_{yy} = 0$.

A naive example is the A_3 germ with disappearing second direction: $x^3 + 6xt + \epsilon(y^2 + 2t)$. When $\epsilon = 0$ the origin describes the change of a maximum-saddle annihilation to a saddle-minimum one. The local structure looks like a slope with a blob on the positive side changing to one on the negative side. However, in the A_3 catastrophe, only one eigenvalue of the Hessian may vanish. For the situation described above, two eigenvalues vanish.

Such catastrophes are described by the D series [1]. Consider the D_4 as scale space germ: $L = x^3 + \alpha xy^2 + (6+2\alpha)xt + \lambda_1(y^2+2t) + \lambda_2 x + \lambda_3 y$ with $\alpha = \pm 1$. λ_2 is the displacement of t and can be disregarded. For the determinant of the Hessian we get $6x(2\alpha x + 2\lambda_1) - 4\alpha^2 y^2$, for the trace $2x(3+\alpha) + 2\lambda_1$.

The latter is zero for $x = \frac{-\lambda_1}{3+\alpha}$ ($\alpha \neq -3$), and the determinant becomes $-9\lambda_1^2 - \alpha^2(3+\alpha)^2 y^2$ which is non-zero unless $y = 0$ and $\lambda_1 = 0$, yielding $x = 0$. Since $L_y(0,0) = \lambda_3$, also $\lambda_3 = 0$. Thus we get the scale space polynomial

$$L = x^3 + \alpha xy^2 + (6+2\alpha)xt. \tag{19}$$

We needed to set two parameters equal to zero, instead of one. So this situation is not generic in a one parameter family of perturbations. This is in line with intuition, stating that we cannot simple change an extremum into a saddle, vice versa. Note that Eq. 19 describes the scale space version of a (non-generic) monkey-saddle.

4.4 Two scale space saddles coincide

For the situation that two scale space saddles coincide – a degenerated scale space saddle – we take the case that the event takes place in the x, t plane. Then from Eq. (14) it follows that L_{tt} is at least $O(x)$: If $L_{tt} = 0$ than $L_{xt} = 0$, i.e. we are left with an ordinary critical point. If $L_{tt} = O(1)$ we have $L = O(x^4, t^2)$, and similar to the case above the saddle coincides with the catastrophe point, since L_{xx} has to vanish. Therefore, $L_{tt} = O(x)$ and the simplest scale space 5-jet reads

$$L = \frac{x^5}{120} + \frac{tx^3}{6} + \frac{t^2 x}{2} + x^2 - y^2 + \delta\left(x^2 + 2t\right) + \epsilon x \tag{20}$$

Eq. (20) represents an A_4 catastrophe in scale space, where the saddle is located at the origin. For such a catastrophe perturbations are required for the orders x^3, x^2, and x. Note that scale t perturbs x^3. For the three requirements $L_x = 0, L_t = 0$ and $\det \mathcal{H} = 0$ one gets $\delta = \delta(\epsilon)$, that is, one free parameter remains. In Figure 5 a parameterized critical curve is shown for several values of δ and ϵ. Each column shows a sequence with varying ϵ where two scale space saddles (local extrema on the curve) meet and (dis)appear.

Constraining the event to the origin yields $\delta = 0$ and $\epsilon = 0$, i.e. the plot in the middle. Here one clearly sees a horizontal tangent.

Degenerated scale space saddles are generic in a one parameter family of perturbations, which implies that pairs of scale space saddles can be created and annihilated on a critical curve.

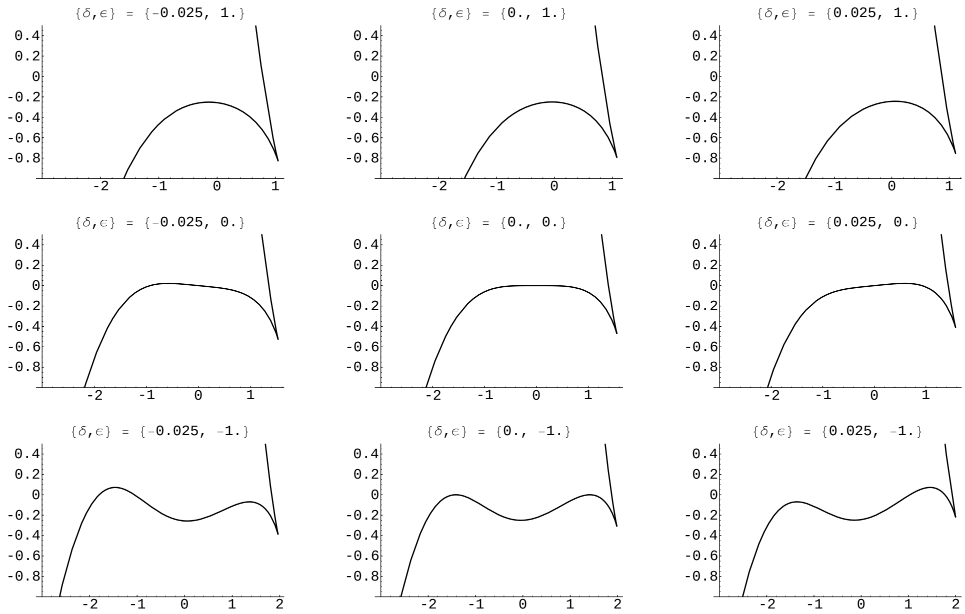

Figure 5: Plots of $L(x(t, y(t)), t)$ along a critical curve of the scale space polynomial of Eq. (20) for several values of δ and ϵ. Each column (changing ϵ, δ fixed) shows a sequence where two scale space saddles (local extrema on the curve) meet and (dis)appear.

4.5 Two scale spaces saddles with the same value

The case that two scale space saddles have the same intensity is easily derived from the previous section. In Figure 5, the plot in the middle of the third row shows exactly this phenomena when in Eq. (20) instead of δ now ϵ is taken fixed. For $\epsilon < 0$ the critical curve contains 3 extrema. When δ varies, their intensities vary and two have equal intensity for $\delta = 0$.

This effect in the (x, t) plane, i.e. for the separation of parts in the scale space image, is shown in Figure 6 (cf. the middle plot in Figure 1). Here the iso-manifold is reduced to an isophote since we do not consider the (Morse) y variable. When $\delta = 0$, the iso-manifolds are connected at two places (middle plot), one of each is taken when $\delta \neq 0$. As one can see, the region enclosed by the iso-manifold remains stable when going through the transition. Only the location of the scale space saddle changes suddenly.

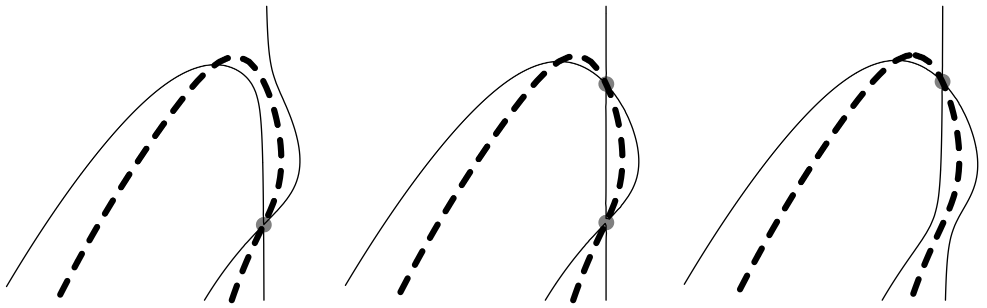

Figure 6: The critical curves (dashed) and iso-manifolds through the scale space saddles for Eq. (20) for several values of δ and $\epsilon = -1$. When $\delta = 0$, the saddles with equal intensity occur.

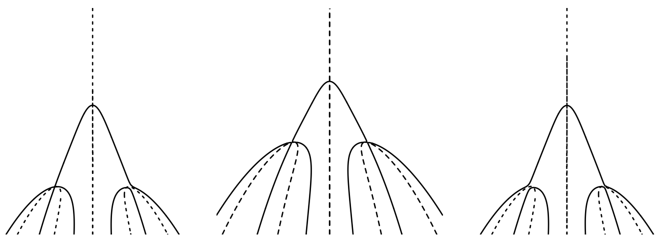

Figure 7: Two scale-space saddles are located at one manifold (middle); Perturbing yields two nested manifolds with each one scale-space saddle (left and right). Critical curves are represented by the dashed curves, theg iso-manifolds by the continuous curves.

4.6 Two saddle points on a manifold with the same intensity

Of course, the scale-space saddles do not have to lie on the same critical curve. In the situation that a iso-manifold contains two scale-space saddles, the local description needs two saddle branches and three extremum ones. Consequently, one needs a polynomial expression of $L_6(x,t) = O(x^6) = x^6 + 30x^4t + 180x^2t^2 + 120t^3$. Perturbations are of orders $L_3(x,t) = x^3 + 6xt$, $L_2(x,t) = x^2 + 2t$, and $L_1(x) = x$. Again, t perturbs the $O(x^4)$ terms. So the simplest description reads

$$L(x,y,t) = x^2 - y^2 + L_6(x,t) + \alpha L_3(x,t) + \beta L_2(x,t) + \gamma L_1(x) \qquad (21)$$

In Fig. 7 one can see the unperturbed situation in the middle, and two perturbed situations on the left and the right. The unstable situation with two scale-space saddles on one iso-manifold is a transition.

5 Consequences for the tree structure

For the tree structure these transitions imply the following results:

1. Two catastrophe points coincide on a critical curve: This has no direct influence, since the complete critical curve is used in the construction of the tree. This event merely describes a smoothing effect allowing one to reduce the number of catastrophes on a critical curve.

2. Two critical curves intersect: This event describes the change in ordering of the two child nodes "C" and "D" in the hierarchy when one catastrophe describes an annihilation.
   In the case that one describes a creation, it can be regarded as handing over a local creation-annihilation from one curve to another, which has no influence on the tree.

3. A catastrophe point coincides with scale space saddle: This requires a total disappearance of second order structure and is a co-dimension two event, i.e. not generic.

4. Two scale space saddles coincide: When going through a degenerated scale space saddle, two scale space saddles are created or annihilated. This does not influence the tree (although it may have some consequences when followed by the following event).

5. Two scale spaces saddles on a critical curve have the same value. In this case, another scale space saddle connects the two parts of the manifold. Although the intensity of the manifold changes continuously, the location of the scale space saddles changes discontinuously. In the tree structure this influences the information stored at the node.

6. If an iso-manifold under perturbation goes through a situation where it contains two scale space saddles on two different curves, the impact on the tree is a change of parent-child nodes, as shown in Fig. 8.

6 Example

We illustrate the theory with the MR image shown in Fig. 9. Normal $[0, 100]$ distributed noise is added, and of both a blurred version is computed.

Using the software courtesy of the authors of [11] we derived the tree structures of both images, as shown in Fig. 10. For visualisation purposes, we used the blurred versions for reference to keep the trees rather simple.

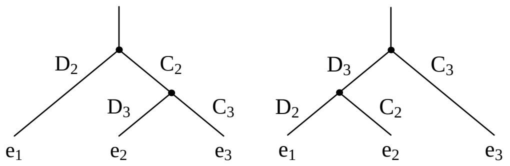

Figure 8: The tree representations of the transition visualised in Fig. 7. The two scale-space saddles swap in hierarchy, which is a simple rotation of a parent-child pair of internal nodes.

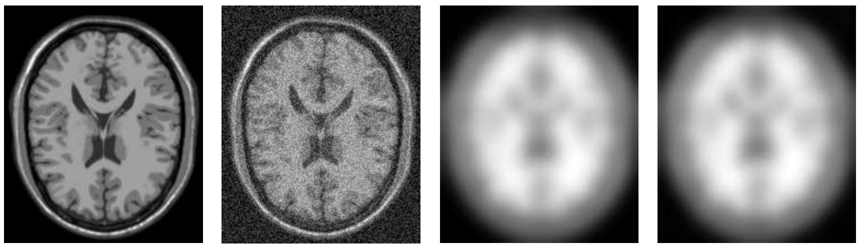

Figure 9: From left to right: An MR image, a noisy MR of it and their blurred versions.

The labels in the trees refer to the extrema in the blurred MR images as shown in Fig. 11. It is clear that a mapping based on the pre-segmentation in these images yields the pair $A1$, $B2$, $C3$, $E5$, $F6$, and $G7$. This is also provided by the locations of spatial locations of these extrema (deviation of maximal one pixel) The differences in the trees are the labels (extrema) D and 4. The operation on D is a simple deletion of a leave. Leave (extremum) 4 is added to the subtree spanned by extremum 1. Its position is found comparing the intensity of the scale space saddle with those that are related to extremum 1. Alternatively, it can be considered as replacing leave 1 by the building block with extrema 1 and 4, and applying the subsequent rotations with the scale space saddle belonging to extremum 4: with the node representing the scale space saddle related to extremum 3, followed by the one related to extremum 2.

7 Summary and discussion

After the introduction of degenerated scale space saddles in section 3, we discussed in section 4 the six possible simple situations that may occur when an extra constraint is posed on the building blocks of the hierarchical structure in scale space, viz. the critical curves, catastrophe points, and (degenerated) scale space saddles. We showed that one of them (described in section 4.3) is

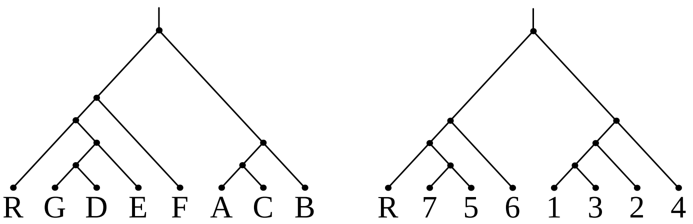

Figure 10: The tree structures of the MR image and the noisy MR image, respectively, starting scale the blurred versions.

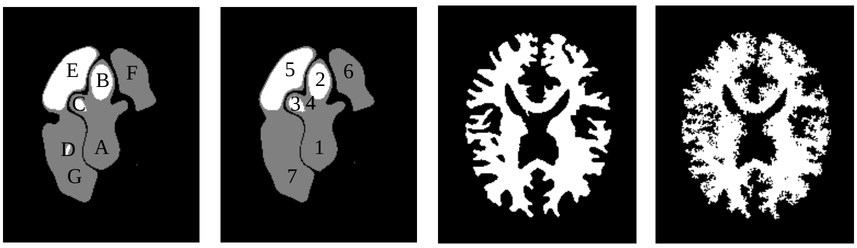

Figure 11: The labelled pre-segmentations, and the segment belonging to the left sub trees in the white matter for both MR images.

a co-dimension two event, requiring two vanishing control parameters. The other cases are co-dimension one events and generic in a one-parameter setting.

These cases describe transitions of structures that are non-generic in scale space, but when allowing an additional constraint they become generic. This is useful when we want to change one image into another, i.e. matching.

The list in section 5 indicates that the hierarchical tree structure changes only with respect to the ordering of children, information stored in the nodes and rotation of a parent-child node combination. The consequences of the standard events in scale space, viz. creation and annihilation of pairs of critical points, are the addition or removal of leave elements.

Together they form the possible changes of the tree under a one parameter family of changes and give the grammar for relevant matching algorithms. A simple example for this was shown in section 6.

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