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Abstract. We study the evolution of the probability density of an asexual, one locus population under natural selection and random evolution. This evolution is governed by a Fokker-Planck equation with degenerate coefficients on the boundaries, supplemented by a pair of conservation laws. It is readily shown that no classical or standard weak solution definition yields solvability of the problem. We provide an appropriate definition of weak solution for the problem, for which we show existence and uniqueness. The solution displays a very distinctive structure and, for large time, we show convergence to a unique stationary solution that turns out to be a singular measure supported at the endpoints. An exponential rate of convergence to this steady state is also proved.

1. Introduction

A classical problem in population genetics is to study the evolution of a mutant gene. A standard approach to this problem is to consider a finite size population and to define a discrete dynamics for the evolution of the probability density of such a population. Usually, such models are Markov chains, in which the only absorbing states are the two pure ones. Therefore, one expects, for large time, convergence to one of these two states and, depending on which state is achieved, one says that the mutation has been either fixed or lost. For large populations, it is natural to ask for a continuous model that approximates this evolution. In a number of different ways, one arrives to a Fokker-Plank equation that describes either the evolution of the probability density (the so-called forward Kolmogorov), or what is sometimes called the transient fixation probability (the backward Kolmogorov equation). From a mathematical point of view, it is interesting to notice that, for the fixation probabilities, it is easy to specify the appropriate non-homogeneous boundary conditions which, after subtraction of an appropriate multiple of a stationary solution, are recast as Dirichlet conditions. Nevertheless,

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this does not seem to be the case for the probability evolution. Since it must conserve mass, in many cases a condition of null probability current at the endpoints is used (e.g. [21]). For a thorough introduction to the several aspects of mathematical population genetics, we refer the reader to the monographs by [3, 11]

For a class of problems, however, these Fokker-Plank equations turn out to have degenerate coefficients at the boundaries, the classical Kimura equation (cf. [16]) being the archetypal example. For the backward one, this is not a problem since the infinitesimal generator is, very generally, self-adjoint. For the forward equation, however, the underlying spectral problem is of the limit-point type and, thus, no boundary conditions can be enforced. In particular, one cannot control the flux of the solutions across the boundary of the corresponding domain, and the existence of conservation laws are not to be expected in general. This is an old issue in the study of diffusions, and it has been tackled by [12], where the so called lateral conditions are derived, in order to ensure that the forward and backward equations are adjoint to each other. With these lateral conditions, however, the forward equation looses its differential character, and this led to a prevalence of the backward equation in the study of diffusions (particularly after [13]). See [14] for a general discussion of degenerate diffusion equations. We shall see below that is possible to ensure the duality of the backward and forward equations, while maintaining the differential character of the forward equation, within the framework of weak solutions.

We shall study the forward Kolmogorov equation

$$\begin{cases} \partial_t p(t,x) = \partial_x^2 \left(F(x)p(t,x)\right) - \partial_x \left(G(x)p(t,x)\right), \quad x \in (0,1), \quad t > 0 \\ p(0,x) = p_0(x) \end{cases} \tag{1}$$

with F positive in $(0,1)$, but with simple zeros at the endpoints, and with G vanishing at the endpoints[1]. Typical examples are $F(x) = G(x) = x(1-x)$ (forward Kimura) and $F(x) = x(1-x)$, $G(x) = x(1-x)(\eta x + \beta)$ (forward Kimura with frequency selection; see [5]).

Equation (1) is supplemented by the following conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p(t,x)\,\mathrm{d}x = 0, \tag{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \psi(x)p(t,x)\,\mathrm{d}x = 0, \tag{2b}$$

where ψ satisfies

$$F(x)\psi'' + G(x)\psi' = 0, \qquad \psi(0) = 0, \quad \text{and} \quad \psi(1) = 1. \tag{3}$$

Remark 1. *In population genetics, the function ψ is referred to as the fixation probability. The condition* (2a) *is usually stated (or assumed), in the literature of population genetics, but*

[1] More precise statements on the hypothesis made upon F and G are deferred to section 2.

condition (2b) *is not. These conditions have been derived in* [5]*, when obtaining the forward Kimura equation, with frequency selection, as a large population limit of the so called Moran process (cf.* [22]*). See also* [26] *for an alternative approach.*

Before we proceed, we want to clarify the nature of the conservation laws given by (2). The backward equation and (formal) adjoint of (1) is given by

$$\begin{cases} \partial_t f = F(x)\partial_x^2 f(t,x) + G(x)\partial_x f(t,x) & x \in (0,1), \quad t > 0 \\ f(0,x) = f^0(x) \end{cases} \tag{4}$$

It is readily seen that any stationary solution to (4) is a linear combination of a constant and $\psi(x)$. Therefore, the conservation laws (2) are related to the kernel of the infinitesimal generator of (4). Finally, it should be mentioned that, if (1) is a correct approximation of the biological process, then one expects that the probability mass accumulates at the endpoints, as t goes to infinity [16].

The goal of this work is to clarify in what sense a solution to (1) that satisfies (2) exists, and how it behaves for large time. In contradistinction with [12], which works in classical function spaces and has to modify equations (1) and (4) in order to obtain the duality relation, we shall always work with these equations, but in more general, non-normed, distributional spaces. This also differs from recent work in degenerate equations, as for instance: the controlability of degenerate heat equations [20], with solutions in weighted Sobolev spaces; entropy solutions of Fokker-Planck from multilane traffic flow [9], where the conditions that might lead to concentration at the end points are explicitly avoided; and from the qualitative studies by [1]. We mention also the classical monographs [8, 4].

Equation (1), with $F(x) = x(1-x)$ and $G(x) = x(1-x)(\eta x+\beta)$ has been studied in reference [5], where a proof of existence and uniqueness in the sense of definition 1 is given, under the assumption of interior regularity. An announcement that also includes other results was made in [6]. More recently, the same problem has been studied through skillful, but formal, calculations in [21], with conditions of null probability current (formally) imposed. Thus, this work can be seen as complementary to the work by [12] by given a differential formulation to the forward-backward duality for degenerate diffusions. Also, it can be regarded as an extension of [5, 6], and as a rigorous proof of the formal calculations in [21].

The main results of the paper can be outlined as follows: let $\mathcal{BM}^+([0,1])$ denote the space of (positive) Radon measures on $[0,1]$. Then we have

Theorem 1 (outline)**.** *For a given $p^0 \in \mathcal{BM}^+([0,1])$, there exists a unique solution p to Equation (1), in a sense to be made precise in definition 1, with $p \in L^\infty\left([0,\infty);\mathcal{BM}^+([0,1])\right)$*

and such that p satisfies the conservations laws (2). The solution can be written as

$$p(t,x) = q(t,x) + a(t)\delta_0 + b(t)\delta_1,$$

where δ_y *denotes the singular measure supported at* y*, and* $q \in C^\infty(\mathbb{R}^+; C^\infty([0,1]))$ *is a classical solution to (1). We also have that* $a(t)$ *and* $b(t)$*, belong to* $C([0,\infty)) \cap C^\infty(\mathbb{R}^+)$*. In particular, we have that*

$$p \in C^\infty(\mathbb{R}^+, \mathcal{BM}^+([0,1])) \cap C^\infty(\mathbb{R}^+, C^\infty((0,1)) \ .$$

For large time, we have that $\lim_{t\to\infty} q(t,x) = 0$*, uniformly, and that* $a(t)$ *and* $b(t)$ *are monotonically increasing functions such that:*

$$a^\infty := \lim_{t\to\infty} a(t) = \int_0^1 (1-\psi(x))p^0(x)\,\mathrm{d}x \quad \text{and}$$

$$b^\infty := \lim_{t\to\infty} b(t) = \int_0^1 \psi(x)p^0(x)\,\mathrm{d}x,$$

Moreover, we have that

$$\lim_{t\to\infty} p(t,\cdot) = a^\infty\delta_0 + b^\infty\delta_1,$$

with respect to the Radon metric. Finally, the convergence rate is exponential.

Remark 2. *The coefficients of the singular measures,* $a(t)$ *and* $b(t)$ *are, respectively, the extinction and the fixation probabilities. Also, notice that the decomposition of p does not follows immediately from the linearity of (1). As a matter of fact, neither of the summands are, per se, a solution to (1) in the sense of definition 1. Heuristically, as (1) is uniformly parabolic in each proper compact set of the unit interval, the parabolic operator erodes the interior density of the initial measure, which is then transferred into the boundaries and absorbed by the singular measures there.*

The outline of the paper is as follows: in section 2, we present background results for the classical (in a broad sense) solutions to (1). In section 3, we introduce an appropriate definition of a weak solution and show that any solution of this type must satisfy the conservation laws (2). We also show that, with this formulation, (5) is indeed the adjoint of (1). In section 4, we present the proofs of existence and uniqueness. Section 5 discusses the convergence to the measures supported at the endpoints as time goes to infinity.

2. Preliminaries

For the convenience of the reader, we present in this section some material that will be useful in the sequel.

Let $F, G : [0,1] \to \mathbb{R}$ be smooth, and assume that

(1) F has single zeros at $x=0$ and at $x=1$, and $F(x)>0$, for $x\in(0,1)$;
(2) G has zeros at $x=0$ and $x=1$.

Hadamard's lemma (cf. [2]) then yields

$$F(x)=x(1-x)\Psi(x),\quad \Psi(x)>0 \text{ for } x\in[0,1] \qquad \text{and } G(x)=x(1-x)\Pi(x)$$

Let us write,

$$\Xi(x)=\frac{\Pi(x)}{\Psi(x)}.$$

Then we can rewrite (4) as

$$\begin{cases} \partial_t f = x(1-x)\Psi(x)\left[\partial_x^2 f(t,x)+\Xi(x)\partial_x f(t,x)\right] & x\in(0,1),\quad t>0\\ f(0,x)=f^0(x) \end{cases} \tag{5}$$

The stationary solutions of (5) are constant and

$$\psi(x)=c^{-1}\int_0^x \mathrm{e}^{-\int_0^s \Xi(r)\,\mathrm{d}r}\,\mathrm{d}s,\quad c=\int_0^1 \mathrm{e}^{-\int_0^s \Xi(r)\,\mathrm{d}r}\,\mathrm{d}s.$$

Let

$$\mathrm{e}^{\frac{1}{2}\int_0^x \Xi(s)\,\mathrm{d}s} w = x(1-x)\Psi(x)p. \tag{6}$$

Then (1) becomes

$$\partial_t w = x(1-x)\Psi(x)\left\{\partial_x^2 w-\frac{1}{4}\left[2\Xi'+\Xi^2\right]w\right\}. \tag{7}$$

Remark 3. *If $p\in L^1_{\mathrm{loc}}([0,1])$, then we have that $w(t,\cdot)$ satisfies Dirichlet boundary conditions at the endpoints. Since, the standard maximum principle holds for $C^{1,2}$ solutions of (7), we find that, if the initial condition is nonnegative, then $w(t,\cdot)$ is also nonnegative. Moreover, this holds also for $p(t,\cdot)$.*

Existence can be established by Fourier series. Consider the associated spectral problem:

$$\begin{gathered} \varphi''+\tfrac{1}{4}\left[2\Xi'+\Xi^2\right]\varphi=\lambda\theta(x)\varphi,\\ \varphi(0)=\varphi(1)=0,\quad \theta(x)=\tfrac{1}{\Psi(x)x(1-x)}. \end{gathered} \tag{8}$$

Sturm-Liouville theory for singular problems allows us to conclude that (8) is a self-adjoint operator in $L^2\left([0,1],\theta(x)\mathrm{d}x\right)$, with a complete set of eigenfunctions.

An important property of (8) is given by

Lemma 1. *The operator defined by (8) is positive-definite.*

Proof. Let

$$v_0 = \mathrm{e}^{\frac{1}{2}\int_0^x \Xi(s)\,\mathrm{d}s}.$$

Then, when $\lambda = 0$, the general solution to (8) is given by

$$\varphi(x) = c_1 v_0(x) + c_2 v_0 \int_0^x v_0^{-2}(s)\,\mathrm{d}s$$

Applying the boundary condition, and using positiveness of v_0 yields $c_1 = c_2 = 0$. Thus, zero cannot be an eigenvalue of (8). Let

$$v = \mathrm{e}^{\frac{1}{2}\int_0^x \Xi(s)\,\mathrm{d}s}\varphi.$$

Then, (8) becomes

$$-\lambda \frac{v}{x(1-x)\Psi(x)} = \partial_x^2 v + \Xi(x)\partial_x, \quad v(0) = v(1) = 0. \tag{9}$$

The eigenfunction φ_0, corresponding to the smallest eigenvalue λ_0, shall not have any zeros inside $(0,1)$. Hence, this will be also true for v. Let us assume, without loss of generality, that $v > 0$ in $(0,1)$. It must have a point of maximum $x = x^*$. In this point $\partial_x v(x^*) = 0$. Hence we must have

$$-\lambda \frac{v(x^*)}{x^*(1-x^*)\Psi(x^*)} = \partial_x^2 v(x^*)$$

Note that $\partial_x^2 v(x^*) \neq 0$, otherwise $\lambda_0 = 0$. Since it is a maximum, we must have $\partial_x^2 v(x^*) < 0$. Since, $v(x^*) > 0$, we should have $\lambda_0 > 0$. □

Remark 4. *Once we know that zero cannot be an eigenvalue of (8), an alternative approach is to consider*

$$\varphi'' + \frac{1}{4}\left[s2\Xi' + \Xi^2\right]\varphi = \lambda\Theta(x)\varphi, \qquad \varphi(0) = \varphi(1) = 0, \tag{10}$$

When $s = 0$, it is straightforward to show that (10) is positive definite. Since (10) is self-adjoint for all s, its eigenvalues are continuous functions of f. Since zero cannot be an eigenvalue for any s, we have the result.

We shall write φ_j, $j = 0, 1, 2, \ldots$, for the eigenfunctions of (8), with corresponding eigenvalue λ_j, and normalization $\|\varphi_j\|_\infty = 1$.

Since Radon measures are distributions of order less or equal to zero, we have—cf. [25] with minor modifications—that:

Proposition 1. *The initial value problem defined by Equation (7) and $w(0,x) = w^0(x)$, with $w^0 \in \mathcal{BM}^+((0,1))$ has the solution*

$$w(t,x) = \sum_{j\geq 0} \widehat{w^0}(j) e^{-t\lambda_j}\varphi_j(x), \quad \widehat{w^0}(j) = \int_0^1 w^0(x)\varphi_j(x)\,\mathrm{d}x, \tag{11}$$

which is unique in the class $C^\infty\left(\mathbb{R}^+; C^\infty([0,1])\right)$.

Remark 5. *It can be shown that, any standard weak solution definition to (7) will lead to the solution above—see for instance* [10, 17]. *Therefore, none of the conservation laws (2) can hold, and no classical-weak solution to (1–2) exists.*

3. Weak solution and duality formulation

We now make precise what we mean by a weak solution to (1).

Definition 1. *A weak solution to (1) will be a function in* $L^\infty\left([0,\infty);\mathcal{BM}^+([0,1])\right)$ *that satisfies*

$$\begin{aligned} -&\int_0^\infty\int_0^1 p(t,x)\partial_t\phi(t,x)\mathrm{d}x\mathrm{d}t \\ &= \int_0^\infty\int_0^1 p(t,x)x(1-x)\Psi(x)\left[\partial_x^2\phi(t,x)+\Xi(x)\partial_x\phi(t,x)\right]\mathrm{d}x\,\mathrm{d}t \\ &\quad+\int_0^1 p^0(x)\phi(0,x)\mathrm{d}x, \end{aligned}$$

where

$$\phi(t,x)\in\mathcal{T}=C_c^\infty\left([0,\infty)\times[0,1]\right).$$

Remark 6. *Notice that the test functions in definition 1 are required to be of compact support in* $[0,1]$ *and not in* $(0,1)$ *as usual. Similar definitions have been given in other contexts; see* [18, 19]*, where they are termed* boundary-coupled weak solutions*.*

Definition 1 can be recast in the framework of usual distribution theory, by identifying a Radon measure with a compactly supported distribution (see [24]*). In this case, the distribution can act in* $C^\infty(\mathbb{R})$*, but it is entirely determined by its behavior in the support; see for instance* [15].

A glance at Definition 1, shows that, on the integral on the right hand side, the test function ϕ is applied to the operator on the right hand side of (4). Thus, one could expect that any solution that satisfies (1) in the sense defined above, also satisfies the conservation laws (2).

Proposition 2. *Let* $p\in L^\infty\left([0,\infty);\mathcal{BM}^+([0,1])\right)$*. If* $\chi(x)$ *is a stationary solution of (4), then the quantity*

$$\int_0^1\chi(x)p(t,x)\,\mathrm{d}x$$

is constant in time.

Proof. Let $\zeta(t)\in C_c^\infty((0,\infty))$. Then, $\phi(t,x)=\zeta(t)\chi(x)$ is an appropriate test function. Let us write

$$\eta(t)=\int_0^1 p(t,x)\chi(x)\,\mathrm{d}x.$$

On substituting $\phi(t,x)$ in Definition 1, we find that

$$-\int_0^\infty \eta(t)\zeta'(t)\mathrm{d}t = 0.$$

Thus $\eta(t)$ is constant almost everywhere. □

Remark 7. *We observe that standard spectral theory shows that both the infinitesimal generators of* (1) *and* (4) *can be appropriately defined in a domain dense in* $L^2((0,1))$ *such that they are adjoints of each other. However, in this case, equation* (1) *will not be the forward Kolmogorov equation associated to* (4)*. On the other hand, in the sense of the pairing used in definition 1, (4) with* $f(t,\cdot)\in C_c^\infty([0,1])$ *is the adjoint of (1) with* $p(t,\cdot)\in\mathcal{E}'$*. Thus, we recover the usual interpretation of the conservation laws given by the kernel of the adjoint.*

4. Existence and uniqueness

In what follows, it will be convenient to decompose a compact distribution, or a Radon measure, as the sum of a distribution without singular support at the endpoints, and two distributions singularly supported at the endpoints.

Lemma 2. *Let* $\nu\in\mathcal{E}'$*, with* $\mathrm{supp}(\nu)=[0,1]$*. Then, the setwise decomposition*

$$[0,1]=\{0\}\cup(0,1)\cup\{1\},$$

yields a decomposition of ν*, namely*

$$\nu=\nu_0+\mu+\nu_1,$$

where ν_i *is a compact distribution supported at* $x=i$*, and sing supp*$(\mu)\subset(0,1)$*. Moreover, if* ν *is a Radon measure, then* $\mu\in\mathcal{BM}^+((0,1))$*, and* $\nu_i=c_i\delta_i$ *, with* $c_i\in\mathbb{R}$*, are singular measures with support at* $x=i$*.*

Proof. Let ζ_i^ϵ, $i=0,1,2$ be a partition of unity of in $[0,1]$, subordinated to the open cover $\{[0,2\epsilon),(1-2\epsilon,1],(\epsilon,1-\epsilon)\}$. Let $\phi\in C_c^\infty([0,1])$. Define ν_i, $i=0,1$ and μ by

$$\int_0^1 \nu_i\phi(x)\,\mathrm{d}x := \lim_{\epsilon\to0}\int_0^1 \zeta_i^\epsilon\nu\phi(x)\,\mathrm{d}x,\quad i=0,1\ ,$$
$$\int_0^1 \mu\phi(x)\,\mathrm{d}x := \lim_{\epsilon\to0}\int_0^1 \zeta_2^\epsilon\nu\phi(x)\,\mathrm{d}x.$$

Then clearly $\nu=\nu_0+\mu+\nu_1$. Also, it is readily seen that sing $\mathrm{supp}(\mu)\subset(0,1)$. Since $\zeta_0^\epsilon(x)=1$ and $\zeta_0^{\epsilon(n)}(x)=0$, $n\geq1$, for $x\in[0,\epsilon)$, we find that and ν_0 is supported at $x=0$, with a similar argument holding for ν_1. Moreover, since a Radon measure is inner regular, the restriction of ν to $(0,1)$ yields a Radon measure in $(0,1)$. Finally, a Radon measure supported in a singleton must be an atomic measure. □

For the initial condition, we will shall write

$$p^0 = a^0\delta_0 + q^0 + b^0\delta_1,$$

to denote the corresponding decomposition. Also, in order to show the existence of a solution to (1) in the sense of definition 1, we shall temporarily consider $p \in L^\infty([0,\infty);\mathcal{E}')$, with support in $[0,1]$. We shall write

$$p = p_0 + q + p_1,$$

for the decomposition of p.

We now show that q must be, as a matter of fact, much more regular.

Proposition 3 (Interior regularity)**.** *Assume that $q^0 \in \mathcal{BM}^+((0,1))$. If a solution to (1) exists, then $q(t,x)$ must be the unique classical solution in the sense of section 2, with $q(0,x) = q^0(x)$, i.e.,*

$$q(t,x) = \sum_{j=0}^{\infty} \widehat{q^0}(j) q_j \mathrm{e}^{-\lambda_j t}, \ , \tag{12}$$

where the q_j are related to the φ_j in section 2 by (6), and $\widehat{q^0}(j)$ is j-th Fourier coefficient of q^0. In particular,

$$q \in C^\infty\left(\mathbb{R}^+; C^\infty([0,1])\right) \ .$$

Proof. Let $\phi \in C_c^\infty([0,\infty),(0,1))$. Applying to definition 1, we find

$$\begin{aligned}
&-\int_0^\infty \int_0^1 q(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_0^\infty \int_0^1 q(t,x)x(1-x)\Psi(x)\left[\partial_x^2\phi(t,x) + \Xi(x)\partial_x\phi(t,x)\right]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+ \int_0^\infty q^0(x)\phi(0,x)\,\mathrm{d}x.
\end{aligned}$$

The result now follows from a standard Galerkin approximation procedure using the, suitably transformed, basis described in section 2. □

Before we proceed, we observe that, since p_0 and p_1 are distributions supported on a singleton, we must have, for some integer M and M', that

$$p(t,x) = \sum_{k=0}^{M} a_k(t)\delta_0^{(k)} + \sum_{k=0}^{M'} b_k(t)\delta_1^{(k)} + q(t,x), \tag{13}$$

where $\delta_{x_0}^{(k)}$ denotes the k-th distributional derivative of the singleton supported measure.

Theorem 2 (Existence and uniqueness)**.** *The unique solution of (1) in the sense of definition 1, with initial condition $p^0 \in \mathcal{BM}^+([0,1])$ is given by*

$$p(t,x) = q(t,x) + a(t)\delta_0 + b(t)\delta_1,$$

with $q(t,x)$ given by (12). Moreover, we have

$$a(t) = \int_0^t q(s,0)\mathrm{d}s + a^0 \quad \textit{and} \quad b(t) = \int_0^t q(s,1)\mathrm{d}s + b^0.$$

Proof. First, we define

$$\widetilde{\mathcal{T}} = \phi \in C_c^\infty\left((0,\infty)\times[0,1]\right).$$

For $l > 0$, we also define,

$$\widetilde{\mathcal{T}}_{l,0} = \left\{\phi \in C_c([0,\infty)\times[0,1)) | \partial_x^n \phi(t,0) = 0, 0 \le n < l\right\},$$

with a similar definition for $\widetilde{\mathcal{T}}_{l,1}$. Notice that, for $r > s$, $\widetilde{\mathcal{T}}_{r,0} \subset \widetilde{\mathcal{T}}_{s,0}$.

On substituting (13) in definition 1, with $\phi \in \widetilde{\mathcal{T}}$, using that q is smooth for $t > 0$ and integrating by parts we obtain that

$$-\int_0^\infty \left[\sum_{k=0}^M a_k(t)\partial_t\partial_x^k\phi(t,0) + \sum_{k=0}^{M'} b_k(t)\partial_t\partial_x^k\phi(t,1)\right]\mathrm{d}t = \int_0^\infty \left[q(t,0)\phi(t,0) + q(t,1)\phi(t,1)\right]\mathrm{d}t +$$

$$+\int_0^\infty \sum_{k=0}^M a_k(t)\partial_x^k\left(x(1-x)\Psi(x)\partial_x^2\phi(t,x)\right)\Big|_{x=0} + \int_0^\infty \sum_{k=0}^M a_k(t)\partial_x^k\left(x(1-x)\Pi(x)\partial_x\phi(t,x)\right)\Big|_{x=0}\mathrm{d}t +$$

$$+\int_0^\infty \sum_{k=0}^{M'} b_k(t)\partial_x^k\left(x(1-x)\Psi(x)\partial_x^2\phi(t,x)\right)\Big|_{x=1} + \int_0^\infty \sum_{k=0}^{M'} b_k(t)\partial_x^k\left(x(1-x)\Pi(x)\partial_x\phi(t,x)\right)\Big|_{x=1}\mathrm{d}t.$$

Restricting somewhat further, for $\phi \in \widetilde{\mathcal{T}}_{M+1,0}$, we find that

$$0 = \int_0^\infty a_M(t)\binom{M}{M-1}\partial_x[x(1-x)\Psi(x)]_{x=0}\partial_x^{M+1}\phi(t,0)\,\mathrm{d}t$$

Thus $a_M(t) = 0$, and the sum can be only up to $M-1$. Repeating the argument inductively yields $M = 0$. An analogous argument yields $M' = 0$. Thus only $a_0(t)$ and $b_0(t)$ can be nonzero. We now drop the subscripts and determine their values. Now applying to $\phi \in \widetilde{\mathcal{T}}$, such that $\phi(t,1) = 0$, we find

$$-\int_0^\infty a(t)\partial_t\phi(t,0)\,\mathrm{d}t = \int_0^\infty q(t,0)\Psi(0)\phi(t,0)\,\mathrm{d}t$$

Integrating by parts, the corresponding relation for $a(t)$, we obtain

$$\int_0^\infty a(t)\partial_t\phi(t,0) = \int_0^\infty \left(\int_0^t q(s,0)\,\mathrm{d}s + a^0\right)\partial_t\phi(t,0)\,\mathrm{d}t$$

Hence

$$a(t) - \int_0^t q(s,0)\,\mathrm{d}s = \mathrm{const},$$

everywhere, in as much as the integral is continuous. Since $a(0) = a^0$, the identity follows. A similar calculation also shows that

$$b(t) = \int_0^t q(s,1)\,\mathrm{d}s + b^0,$$

Uniqueness follows from proposition 3 and from the expressions for $a(t)$ and $b(t)$. Finally, notice that, since $q(t,x) \geq 0$, we have that both a and b are increasing. □

5. Large time behavior

We now present a result for the behavior of the solution in large time limit.

Proposition 4. *Let p be the solution to (1) with initial condition $p^0 \in \mathcal{BM}^+([0,1])$ and let λ_0 be the smallest eigenvalue of (8). Also, let*

$$b^\infty := \int_0^1 \psi(x)p^0(x)\,\mathrm{d}x = b^0 + \int_0^1 \psi(x)q^0(x)\,\mathrm{d}x,$$
$$a^\infty := \int_0^1 p^0(x)\,\mathrm{d}x - b^\infty = a^0 + \int_0^1 (1-\psi(x))q^0(x)\,\mathrm{d}x.$$

Then

(1) *For $t > 0$, we have*

$$||q(t,\cdot)||_1 \leq C\mathrm{e}^{-\lambda_0 t}$$

(2) *For $t \geq 0$, we have*

$$a^0 + \Lambda_a(1-\mathrm{e}^{-\lambda_0 t}) \leq a(t) \leq a^\infty \quad \text{and} \quad b^0 + \Lambda_b(1-\mathrm{e}^{-\lambda_0 t}) \leq b(t) \leq b^\infty \tag{14}$$

where $\Lambda_a = a^\infty - a^0$ and $\Lambda_b = b^\infty - b^0$.

In particular, $\lim_{t\to\infty} q(t,x) = 0$ (uniformly in $[0,1]$), $\lim_{t\to\infty} a(t) = a^\infty$ and $\lim_{t\to\infty} b(t) = b^\infty$.

Proof. We start from the eigenrelation

$$-\lambda_j q_j(x) = \partial_x^2\left[x(1-x)\Psi(x)q_j(x)\right] - \partial_x\left[x(1-x)\Pi(x)q_j(x)\right].$$

Let

$$A_j = \int_0^1 q_j(x)\,\mathrm{d}x \quad \text{and} \quad B_j = \int_0^1 \psi(x)q_j(x)\,\mathrm{d}x.$$

Then, multiply by ψ and integrating the above eigenrelation, and integrating it by itself, we find that

$$\Psi(1)q_j(1) = \lambda_j B_j \quad \text{and} \quad \Psi(0)q_j(0) = \lambda_j(A_j - B_j). \tag{15}$$

Moreover, as $j\to\infty$ we have that (cf. [7, 23])

$$\lim_{j\to\infty}\frac{\lambda_j}{j^2}=K.$$

For the first part, let $\widehat{q^0}(j)$ denotes the j-th coefficient of the Fourier expansion of q^0, we have, for $t>0$, that

$$\|q(t,\cdot)\|_1\le\sum_{j=0}^{\infty}|A_j\widehat{q^0}(j)|\mathrm{e}^{-\lambda_j t}\le C\mathrm{e}^{-\lambda_0 t},$$

since A_j is summable and $q^0\in\mathcal{BM}^+([0,1])$, and thus $|\widehat{q^0}(j)|<C$.

For the second part, we use the relations (15) to write

$$a(t)=\sum_{j=0}^{\infty}\frac{B_j-A_j}{\lambda_j}\widehat{q^0}(j)(1-\mathrm{e}^{-\lambda_j t})+a^0\quad\text{and}\quad b(t)=\sum_{j=0}^{\infty}\frac{B_j}{\lambda_j}\widehat{q^0}(j)(1-\mathrm{e}^{-\lambda_j t})+b^0 \tag{16}$$

Combining that both A_j, B_j and $\widehat{q^0}(j)$ are bounded with the asymptotic behavior of λ_j, we have that the series obtained by formally letting $t=\infty$ in (16) are convergent. Hence

$$\lim_{t\to\infty}|a(t)-a^\infty|=\lim_{t\to\infty}|b(t)-b^\infty|=0$$

Since, for $t\ge0$, we have $1-\mathrm{e}^{-\lambda_j t}<1$ and $1-\mathrm{e}^{-\lambda_0 t}\le 1-\mathrm{e}^{-\lambda_j t}$, for $j=1,2,\ldots$, we have

$$a^0+\Lambda_a(1-\mathrm{e}^{-\lambda_0 t})\le a(t)<a^\infty\quad\text{and}\quad b^0+\Lambda_b(1-\mathrm{e}^{-\lambda_0 t})\le b(t)<b^\infty \tag{17}$$

□

Theorem 3. *Let ρ denote the Radon metric, and let*

$$p^\infty=a^\infty\delta_0+b^\infty\delta_1.$$

Under the same hypothesis of proposition 4, there exists a positive constant C, such that

$$\lim_{t\to\infty}\rho(p,p_\infty)\le C\mathrm{e}^{-\lambda_0 t}. \tag{18}$$

In particular, (18) implies convergence in the Wasserstein metric.

Proof. Recall that

$$\rho(\nu,\mu)=\sup\left\{\int_0^1 f(x)\mathrm{d}(\nu-\mu)\,\middle|\, f\in C([0,1];[-1,1])\right\}.$$

But, for such f we have that

$$\begin{aligned}\left|\int_0^1 f(x)\mathrm{d}(p_\infty - p(t,x))\right| &\leq \int_0^1 |\mathrm{d}(p_\infty - p(t,x))| \\ &\leq \int_0^1 (a^\infty - a(t))\delta_0 \mathrm{d}x + \int_0^1 (b^\infty - b(t))\delta_1 \mathrm{d}x + \int_0^1 |q(t,x)|\mathrm{d}x \\ &= a^\infty - a(t) + b^\infty - b(t) + \|q(t,\cdot)\|_1 \\ &\leq (\Lambda_a + \Lambda_b)\mathrm{e}^{-\lambda_0 t} + C\mathrm{e}^{-\lambda_0 t} \\ &= C\mathrm{e}^{-\lambda_0 t}.\end{aligned}$$

Hence the result. □

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