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J. Liu, M. Sini

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J.J. Liu* and M. Sini†

Abstract

In this paper, we deal with the acoustic inverse scattering problem for detecting obstacles from the far field data. The complex obstacle is characterized by its shape, its type of boundary conditions and the boundary coefficients (surface impedance). We particularly show how one can use the boundary coefficient, distributed on the surface of the obstacle, to design obstacles which can be reconstructed in a more (or less) accurate way from far field measurements using indicator function based methods. We explain this by using the probing methods like the probe or the singular sources methods. However, similar results can be also given by using the sampling methods like the linear sampling method and maybe the factorization methods and the MUSIC algorithms. After computing the indicator functions of the probe method from the far field data, we give their asymptotic analysis near the surface of the obstacle with respect to the used point sources or multipoles. Using multipoles of order two, the first (i.e. highest) order term of the real parts gives the location of this surface and the unit normal vectors on it, while the second order terms involve the curvature coupled, in a clear and simple way, with the imaginary part of the surface impedance. The appearance of the curvature explains the difficulty to reconstruct non-uniform shapes, in particular the non convex ones, without more *a-priori* information. In addition, this relation enables us to use the surface impedance (the coating coefficient) to design obstacles which can be detected in a more (or less) accurate way by using the indicator function based methods.

Key words. Inverse scattering, far-field, impedance boundary conditions, singularity analysis.
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1 Introduction

The impedance boundary condition which gives a relation between electric and magnetic field vectors on a given surface in terms of coefficients called surface impedance is one of the main tools that is used in the solution of electromagnetic scattering problems. The surface impedance is commonly used to model imperfectly coating objects coated with a penetrable or absorbing layer, see [11]. Mathematically, the above scattering phenomenon is described by exterior or transmission problems for the Maxwell system with some mixed boundary conditions on the surface or on the interface of the obstacle.

An application of the impedance boundary condition is the antenna design and analysis. In order to avoid detection by radar, hostile objects are coated by a material designed to reduce the radar cross section of the scattered wave. A different and opposite aim is to design objects which can be detected in a more precise way by exterior measurements. In other words, the question we address is: how to make the reconstruction of these obstacles, from exterior measurements, more

*Department of Mathematics, Southeast University, Nanjing, 210096, P.R.China. email:jjliu@seu.edu.cn.

†Corresponding author. RICAM, Altenbergerstrasse 69, Linz, A-4040, Austria. email: mourad.sini@oeaw.ac.at.

(or less) accurate? The aim of this paper is to give an insight towards an answer to this question under the framework of the scattering models. We restrict our investigations to the Helmholtz models which can be seen as two dimensional approximations of the Maxwell models in case of obstacles of cylindrical forms where the used incident waves are appropriately polarized.

In recent years several non-iterative methods have been proposed to reconstruct obstacles from far field or near field measurements. We can divide them into two families. One family consists of the probing methods like the probe method [14], the singular sources method [25], non-response test [21], range test [27], see [26] for a review of these methods. The second family is composed by the sampling methods, like the linear sampling method [8], the factorization method [16], MUSIC algorithms [10] and the reciprocity-gap method [2]. In this paper, we are interested by such indicator function based imaging methods. Hence the goal is how to make the reconstruction of these obstacles, from exterior measurements, more (or less) accurate using indicator function based methods.

Let us notice that all these methods share two essential points. Firstly they are built on indicator functions which are depending on some parameter. The drastic behavior of these indicator functions when the parameter point approaches the boundary of the obstacle is used to localize and determine the shape of this obstacle. Hence the surface of the obstacle is the boundary of the region where the indicators blowup. This is the main feature used in the literature to localize the obstacles. The second common point of these methods is the fact that the blowup of the corresponding indicator functions is closely related to the singularity of the fundamental solution of the background. Indeed, all these methods use the fundamental solutions to create the needed jump of the indicator functions. One natural and interesting question is to find direct links between these methods. Some results are already known, see [1], [7] and [17] for the the sampling methods, [23] for the probing methods and [12] for links between some sampling methods and some probing methods.

As we already explained it, in all the published works related to the mentioned indicator function based methods, the surface of the obstacle is reconstructed as the boundary of the region where those indicator functions blowup. Unfortunately, this quantitative behavior of these indicator functions is far to be enough to give satisfactory results although in the literature good reconstructed shapes have been shown. From the numerical implementations point of view, it is of a common believe that it is much easier to reconstruct convex shapes than non convex ones. However these cases are just examples and the problem is much deeper. Understanding this point is one of the main reasons why we write this paper. The main issue behind the difficulty of reconstructing non simple shapes, like non-convex obstacles, is that we do not take enough care of the way how these indicator functions blowup near the boundary of the obstacles. To our opinion, to understand this, we need qualitative results on how they blowup and what is the rate of this blowup. To do this we need to analyze the behavior of these indicator functions with respect to the parameter used. In this paper, we take as a pilot one of the probing methods, the probe method (or equally the singular sources method, see [12] and [23]).

Regarding the inverse problem of reconstructing complex obstacles, we wish to cite the work [18] where an iterative method is proposed and [3, 4] where the linear sampling method has been used to reconstruct the shape and give an L^∞ estimate of the boundary term, see also [5] for a review of these results.

Our first works in this direction are in [19] and [22] where we showed that the indicator function can be used to solve the inverse problem of detecting complex obstacles. In [19], we remarked that if the shape of the obstacle is uniform (convex for example) and the material is uniformly distributed on the boundary of the obstacle, then the numerical reconstructions are quite good and the method is trustable. However, for non-uniform shapes or/and for non-uniform distributed coefficients on the boundary, the corresponding numerical results are much less convincing. We wish to explain

theoretically why this is the case. Indeed, using multipoles of order two as sources, we show that the indicator function has two orders in the expansion with respect to the parameter. In the first order, which is the dominant part, the normal vectors appear. In the second order the *curvature* appears coupled in a simple and clear way with the *imaginary part of the surface impedance*. This explains to some extent why in case of high variation of the curvature (even for convex obstacles) and in the case where the curvature changes the sign (near the convex-concave parts of non-convex obstacles), it is hard to see the whole shape from exterior measurements. So, to get more trustable numerical results on the reconstruction, we need to know more *a-priori* information on the unknown obstacle.

Fortunately, this apparent negative point of the reconstruction methods, which are based on indicator functions, can be used in a positive way. This may be helpful in designing objects that can be reconstructed in a more (or less) accurate way from exterior measurements using the already mentioned indicator function based imaging methods. Indeed, we show that, for example, if we choose the surface impedance to be related in an explicit way to the curvature then the obstacle will be reconstructed more (or less) accurately. That is, we can change the effect of the curvature of the shape on the reconstruction accuracy (or the resolution) by introducing complex valued surface impedances. Other ways of weakening or reinforcing the effect of the curvature are possible. In addition, our analysis is due to the use of multipoles up to the order two. However, using multipoles of higher orders would suggest other ways how to design interfaces which can be reconstructed in a more (or less) accurate way from exterior measurements.

Now, we explain the mathematical analysis behind these results. The first main observation is that these probing methods can actually be used to compute the scattered field of multipole sources of any order. However this is not new, see [25]. We wish also to mention the recent result [6] where it is shown that also the linear sampling method can be used to compute the reflected solutions of the multipole sources by the associated and the so-called interior transmission problem for penetrable obstacles or the interior problem for impenetrable obstacles. The second observation, see [22], is that we can give a pointwise analysis of these scattered waves when the source point approaches the boundary. There are two ways of doing this analysis. The first one is by using microlocal analysis techniques, see [24]. However this requires high smoothness assumptions on the coefficients and the obstacle. The second approach is based on two steps. The first step is to compute explicitly the scattered fields in case of flattened boundary and the freezed coefficients. In the second step, we use integral representations and pointwise estimates of the Green's functions to localize the desired dominant part of the scattered fields. It is in this second step that we reduce the required regularity of the obstacle and the coefficients. This approach has been successfully applied in [22] and [19]. The main mathematical advance of this work with respect to [22] and [19] is to have explicitly computed the lower order terms in the asymptotic expansion of the probe indicator function while in [22] and [19] we gave only the higher order terms, which were, of course, enough for the purpose of those works.

Let us finish this introduction by mentioning that, to our opinion, the phenomenon of accuracy (or resolution) of the reconstruction of interfaces, from exterior measurements, is not just due to the method used but it is inherent to the problem itself. The argument that makes us believe on this assertion is the following. As we explained it above, all these methods reconstruct actually the Green's functions of the forward scattering problems. This means that we have at hand all the information that we can obtain from the measurements since those Green's functions characterize completely the forward problem.

The paper is organized as follows. We state mathematically the problem and present the obtained result in section 2. Some comments on these theoretical results are also given. In section 3, we give the proofs of our results.

2 Statement of the problem and the results

Instead of considering the full three dimensional electromagnetism model, we restrict ourself to two dimensional Helmholtz model which is well known to be an approximate model, provided that the obstacle in \mathbb{R}^3 is of a cylinder form and the polarization direction is appropriately chosen, see [5] for the derivation.

Let D be a bounded domain of \mathbb{R}^2 such that $\mathbb{R}^2 \setminus \overline{D}$ is connected. We assume that its boundary ∂D is of class $C^{2,1}$, and has the following form

$$\partial D = \overline{\partial D_I} \cup \overline{\partial D_D}, \quad \partial D_I \cap \partial D_D = \emptyset,$$

where ∂D_D and ∂D_I are open curves in ∂D .

The propagation of time-harmonic acoustic fields in homogeneous cylinder media can be modeled by the Helmholtz equation

$$(2.1) \quad \Delta u + \kappa^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

where $\kappa > 0$ is the wave number. At the part ∂D_I of the obstacle boundary, we assume the total field u to satisfy the impedance boundary condition while on the part ∂D_D to satisfy the Dirichlet boundary condition. That is,

$$(2.2) \quad \frac{\partial u}{\partial \nu} + i\kappa\sigma u = 0 \quad \text{on } \partial D_I$$

with some impedance function σ and

$$(2.3) \quad u = 0 \quad \text{on } \partial D_D,$$

where ν is the outward unit normal of ∂D . We assume that $\sigma(x) = \sigma^r(x) + i\sigma^i(x)$ is a complex valued Hölder continuous function of order $\beta \in (0, 1]$, and its real part σ^r has a uniform lower bound $\sigma^r_- > 0$ on ∂D_I . The part ∂D_I is referred to by the coated part and ∂D_D is the non-coated part.

For a given incident plane wave $u^i(x, d) = e^{ikd \cdot x}$ with incident direction $d \in S^1$, where S^1 is a unit circle in \mathbb{R}^2 , we look for a solution $u := u^i + u^s$ of (2.1), (2.2) and (2.3), where the scattered field u^s satisfies the Sommerfeld radiation condition

$$(2.4) \quad \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - i\kappa u^s \right) = 0$$

with $r = |x|$ and the limit is uniform for all directions $\hat{x} \in S^1$. Moreover, the scattered wave has the following asymptotic expansion

$$(2.5) \quad u^s(x, d) = \frac{e^{i\kappa|x|}}{\sqrt{|x|}} \left(u^\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right) \right), \quad \hat{x} = \frac{x}{|x|} \in S^1$$

as $|x| \rightarrow \infty$. $u^\infty(\hat{x}, d)$ is called as the far-field pattern of scattered wave $u^s(x, d)$.

The mixed problem (2.1) – (2.4) is well posed. More generally, for $h_1 \in H^{\frac{1}{2}}(\partial D_D)$ and $h_2 \in H^{-\frac{1}{2}}(\partial D_I)$, there exists a unique solution of the mixed problem

$$(2.6) \quad \begin{cases} (\Delta + \kappa^2)u = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u = h_1 & \text{on } \partial D_D, \\ \frac{\partial u}{\partial \nu} + i\kappa\sigma u = h_2, & \text{on } \partial D_I, \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - i\kappa u \right) = 0, \end{cases}$$

and the solution satisfies

$$(2.7) \quad \|u\|_{H^1(\Omega_R \cap (\mathbb{R}^2 \setminus \overline{D}))} \leq C_R (\|h_1\|_{H^{1/2}(\partial D_D)} + \|h_2\|_{H^{-1/2}(\partial D_I)})$$

where Ω_R is a disk of radius R and C_R is positive constant depending on R , see [5] for more details.

In this paper, we consider the following

Complex obstacle reconstruction problem. Given $u^\infty(\cdot, \cdot)$ on $S^1 \times S^1$ for the inverse scattering problem (2.1) - (2.5), we need to

- reconstruct the shape of the obstacle D ;
- distinguish the coated part ∂D_I from the non coated one ∂D_D ;
- reconstruct the surface impedance $\sigma(x)$ in ∂D_I .

2.1 Presentation of the results

It is well known [9] that the scattered field associated with the Herglotz incident field $v_g^i := v_g$ defined by

$$(2.8) \quad v_g(x) := \int_{S^1} e^{i\kappa x \cdot d} g(d) ds(d), \quad x \in \mathbb{R}^2$$

with $g \in L^2(S^1)$ is given by

$$(2.9) \quad v_g^s(x) := \int_{S^1} u^s(x, d) g(d) ds(d), \quad x \in \mathbb{R}^2 \setminus D,$$

and its far field is

$$(2.10) \quad v_g^\infty(\hat{x}) := \int_{S^1} u^\infty(\hat{x}, d) g(d) ds(d), \quad \hat{x} \in S^1.$$

We will need the following identity

$$(2.11) \quad u^\infty(\hat{x}, d) = -\gamma_2 \int_{\partial D} \left\{ \frac{\partial u^s(y, d)}{\partial \nu} e^{-i\kappa \hat{x} \cdot y} - \frac{\partial e^{-i\kappa \hat{x} \cdot y}}{\partial \nu} u^s(y, d) \right\} ds(y)$$

with $\gamma_2 = \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}$, see [9]. We set $\Phi(x, z) := \frac{i}{4} H_0^{(1)}(\kappa|x - z|)$, which is the fundamental solution for the Helmholtz equation in \mathbb{R}^2 .

Assume that $\overline{D} \subset\subset \Omega$ for some known Ω with smooth boundary. For $a \in \Omega \setminus D$, denote by $\{z_p\} \subset \Omega \setminus \overline{D}$ a sequence tending to a . For any z_p , set D_a^p a C^2 -regular domain such that $\overline{D} \subset D_a^p$ (resp. $\overline{\partial D} \subset \partial D_a^p$) with $z_q \in \Omega \setminus \overline{D_a^p}$ for every $q = 1, 2, \dots, p$ and that the Dirichlet interior problem on D_a^p for the Helmholtz equation is uniquely solvable. Then the Herglotz wave operator \mathbb{H} defined from $L^2(S^1)$ to $L^2(\partial D_a^p)$ by

$$(2.12) \quad \mathbb{H}[g](x) := v_g(x) = \int_{S^1} e^{i\kappa x \cdot d} g(d) ds(d)$$

is injective, compact with dense range, see [9]. Now we consider the sequence of point sources: poles $\Phi(\cdot, z_p)$, dipoles $\frac{\partial}{\partial x_j} \Phi(\cdot, z)$ and multipoles of order two $\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Phi(\cdot, z)$ with $j = 1, 2$. For every

p fixed, we construct three density sequences $\{g_n^p\}$, $\{f_m^{j,p}\}$ and $\{h_k^{j,p}\}$ in $L^2(S^1)$ by the Tikhonov regularization such that

$$(2.13) \quad \|v_{g_n^p} - \Phi(\cdot, z_p)\|_{L^2(\partial D_a^p)} \rightarrow 0, \quad n \rightarrow \infty$$

$$(2.14) \quad \|v_{f_m^{j,p}} - \frac{\partial}{\partial x_j} \Phi(\cdot, z_p)\|_{L^2(\partial D_a^p)} \rightarrow 0, \quad m \rightarrow \infty,$$

$$(2.15) \quad \|v_{h_k^{j,p}} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Phi(\cdot, z_p)\|_{L^2(\partial D_a^p)} \rightarrow 0, \quad k \rightarrow \infty,$$

Since both $v_{g_n^p}$ and $\Phi(\cdot, z_p)$ satisfy the Helmholtz equation in D_a^p , (2.13) implies that

$$(2.16) \quad \|v_{g_n^p} - \Phi(\cdot, z_p)\|_{H^{\frac{1}{2}}(\partial D)} \rightarrow 0, \quad n \rightarrow \infty$$

$$(2.17) \quad \left\| \frac{\partial}{\partial \nu} v_{g_n^p} - \frac{\partial}{\partial \nu} \Phi(\cdot, z_p) \right\|_{H^{-\frac{1}{2}}(\partial D)} \rightarrow 0, \quad n \rightarrow \infty$$

due to the well-posedness of boundary value problem in D_a^p . Similarly, it follows from (2.14), (2.15) that

$$(2.18) \quad \|v_{f_m^{j,p}} - \frac{\partial}{\partial x_j} \Phi(\cdot, z_p)\|_{H^{\frac{1}{2}}(\partial D)} \rightarrow 0, \quad m \rightarrow \infty,$$

$$(2.19) \quad \|v_{h_k^{j,p}} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Phi(\cdot, z_p)\|_{H^{\frac{1}{2}}(\partial D)} \rightarrow 0, \quad k \rightarrow \infty,$$

and

$$(2.20) \quad \left\| \frac{\partial}{\partial \nu} v_{f_m^{j,p}} - \frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial x_j} \Phi(\cdot, z_p) \right) \right\|_{H^{-\frac{1}{2}}(\partial D)} \rightarrow 0, \quad m \rightarrow \infty,$$

$$(2.21) \quad \left\| \frac{\partial}{\partial \nu} v_{h_k^{j,p}} - \frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Phi(\cdot, z_p) \right) \right\|_{H^{-\frac{1}{2}}(\partial D)} \rightarrow 0, \quad k \rightarrow \infty.$$

Multiplying (2.11) by $g_m^p(d)g_n^p(\hat{x})$ and integrating over $S^1 \times S^1$, we have

$$(2.22) \quad \begin{aligned} & - \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) g_m^p(d) g_n^p(\hat{x}) ds(\hat{x}) ds(d) \\ &= \gamma_2 \int_{\partial D} \left\{ \int_{S^1} \frac{\partial u^s(y, d)}{\partial \nu} g_m^p(d) ds(d) \cdot \int_{S^1} e^{i\kappa \hat{x} \cdot y} g_n^p(\hat{x}) ds(\hat{x}) - \int_{S^1} \frac{\partial e^{i\kappa \hat{x} \cdot y}}{\partial \nu} g_n^p(\hat{x}) ds(\hat{x}) \cdot \int_{S^1} u^s(y, d) g_m^p(d) ds(d) \right\} ds(y) \\ &= \gamma_2 \int_{\partial D} \left\{ \frac{\partial v_{g_m^p}^s}{\partial \nu}(y) v_{g_n^p}^i(y) - \frac{\partial v_{g_n^p}^i}{\partial \nu}(y) v_{g_m^p}^s(y) \right\} ds(y). \end{aligned}$$

From (2.16), (2.17) and (2.22), we have

$$(2.23) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) g_m^p(d) g_n^p(\hat{x}) ds(\hat{x}) ds(d) \\ &= \gamma_2 \int_{\partial D} \left\{ v_{g_m^p}^s \frac{\partial \Phi(y, z_p)}{\partial \nu(y)} - \frac{\partial v_{g_m^p}^s}{\partial \nu(y)} \Phi(y, z_p) \right\} ds(y) = \gamma_2 v_{g_m^p}^s(z_p) \end{aligned}$$

from the Green formula, where $v_{g_m^p}^s(\cdot)$ is the scattered wave corresponding to incident wave $v_{g_m^p}^i(x) = \mathbb{H}[g_m^p](x)$.

Denote by $E_0^s(x, z_p)$ the scattered wave corresponding to the incident wave $\Phi(x, z_p)$, which is well defined for every $x \in \mathbb{R}^2 \setminus \overline{D}$. Then it follows from (2.18), (2.20), the well-posedness of the direct scattering problem and the use of interior estimate that

$$(2.24) \quad E_0^s(x, z_p) = \lim_{m \rightarrow \infty} v_{g_m^p}^s(x), \quad x \in \mathbb{R}^2 \setminus \overline{D}.$$

Hence, from (2.23), we have

$$(2.25) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) g_m^p(d) g_n^p(\hat{x}) ds(\hat{x}) ds(d) = \gamma_2 E_0^s(z_p, z_p).$$

We set

$$(2.26) \quad I^0(z_p) := \frac{1}{\gamma_2} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) g_m^p(d) g_n^p(\hat{x}) ds(\hat{x}) ds(d).$$

Similarly, we denote by $E_{1,j}^s(x, z_p)$ and $E_{2,j}^s(x, z_p)$ the scattered waves corresponding to the incident waves $\frac{\partial \Phi(x, z_p)}{\partial x_j}$ and $\frac{\partial}{\partial x_j} \frac{\partial \Phi(x, z_p)}{\partial x_2}$ respectively. We set

$$(2.27) \quad I_j^1(z_p) := \frac{1}{\gamma_2} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) f_m^{j,p}(d) g_n^p(\hat{x}) ds(\hat{x}) ds(d)$$

which satisfies $I_j^1(x) = \lim_{m \rightarrow \infty} v_{f_m^{j,p}}^s(x)$, $x \in \mathbb{R}^2 \setminus \overline{D}$, where $v_{f_m^{j,p}}^s(x)$ is the scattered wave corresponding to the incident waves $v_{f_m^{j,p}}^i(x) := \mathbb{H}[f_{j,m}^p](x)$.

We also set

$$(2.28) \quad I_j^2(z_p) := \frac{1}{\gamma_2} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{S^1} \int_{S^1} u^\infty(-\hat{x}, d) h_m^{j,p}(d) g_n^p(\hat{x}) ds(\hat{x}) ds(d)$$

which satisfies $I_j^2(x) = \lim_{m \rightarrow \infty} v_{h_m^{j,p}}^s(x)$, $x \in \mathbb{R}^2 \setminus \overline{D}$, where $v_{h_m^{j,p}}^s(x)$ is the scattered wave corresponding to the incident waves $v_{h_m^{j,p}}^i(x) := \mathbb{H}[h_m^{j,p}](x)$.

The main goal of the paper is to give the asymptotic analysis of $E_0^s(x, z)$ and $E_{i,j}^s(x, z)$ with $i, j = 1, 2$ near ∂D in terms of the distance between the points $z \in \mathbb{R}^2 \setminus \overline{D}$ and a given point $a \in \partial D$, i.e. $|(z - a) \cdot \nu(a)|$. Since we are using point sources having at most second order singularities and, then, we are looking for expansions of first and second orders, we need the natural $C^{2,1}$ smoothness assumption on the regularity of ∂D . Precisely, for every point $a \in \partial D$, there exist a rigid transformation of coordinates under which the image of a is $\mathbf{0}$ and some function $f_a \in C^{2,1}(-r, r)$ such that

$$(2.29) \quad f_a(0) = \frac{df_a}{dx_1}(0) = 0, \quad D \cap B(\mathbf{0}, r) = \{(x_1, x_2) \in B(\mathbf{0}, r); x_2 < f_a(x_1)\}$$

in terms of the new coordinates where $B(\mathbf{0}, r)$ is the disc of center $\mathbf{0}$ with radius r .

For the points $a \in \partial D$, we choose the sequence $\{z_p\}_{p \in \mathbb{N}}$ included in $C_{a,\theta}$, where $C_{a,\theta}$ is a cone with center a , angle $\theta \in [0, \frac{\pi}{2})$ and axis $\nu(a)$. The main theoretical result of this paper is as follows.

THEOREM 2.1 *Assume that the boundary ∂D is of class $C^{2,1}$ and the surface impedance σ to be complex valued and Holder continuous function defined on ∂D with a positive real part. Then we have the following formulas:*

I. Using poles $\Phi(x, z)$ as sources, it follows that

$$(2.30) \quad \operatorname{Re}I^0(z_p) = \begin{cases} -\frac{1}{4\pi} \ln(|(z_p - a) \cdot \nu(a)|) + O(1), & a \in \partial D_I, \\ \frac{1}{4\pi} \ln(|(z_p - a) \cdot \nu(a)|) + O(1), & a \in \partial D_D, \end{cases}$$

and

$$(2.31) \quad \operatorname{Im}I^0(z_p) = O(1), \quad a \in \partial D.$$

II. Using dipoles $\frac{\partial}{\partial x_j} \Phi(x, z)$ for $j = 1, 2$ as sources, it holds that

$$(2.32) \quad \operatorname{Re}I_j^1(z_p) = \begin{cases} -\frac{\nu_j(a)}{4\pi|(z_p - a) \cdot \nu(a)|} - \frac{\nu_j(a)(\kappa\sigma^i(a) + \frac{1}{2}f_a''(0))}{\pi} \ln(|(z_p - a) \cdot \nu(a)|) + O(1), & a \in \partial D_I, \\ +\frac{\nu_j(a)}{4\pi|(z_p - a) \cdot \nu(a)|} - \frac{\nu_j(a)}{2\pi} f_a''(0) \ln(|(z_p - a) \cdot \nu(a)|) + O(1), & a \in \partial D_D \end{cases}$$

and

$$(2.33) \quad \operatorname{Im}I_j^1(z_p) = \begin{cases} \frac{\nu_j(a)\kappa\sigma^r(a)}{\pi} \ln(|(z_p - a) \cdot \nu(a)|) + O(1), & a \in \partial D_I, \\ O(1), & a \in \partial D_D. \end{cases}$$

III. Using multipoles of order two $\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Phi(x, z)$, $j = 1, 2$ as sources, it follows that

$$(2.34) \quad \operatorname{Re}I_1^2(z_p) = \begin{cases} \frac{\nu_1(a)\nu_2(a)}{4\pi} \frac{1}{|(z_p - a) \cdot \nu(a)|^2} - \frac{\nu_1(a)\nu_2(a)}{\pi} [\kappa\sigma^i(a) + \frac{3}{4}f_a''(0)] \frac{1}{|(z_p - a) \cdot \nu(a)|} + \\ O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_I, \\ -\frac{\nu_1(a)\nu_2(a)}{4\pi} \frac{1}{|(z_p - a) \cdot \nu(a)|^2} - \frac{3\nu_1(a)\nu_2(a)}{4\pi} f_a''(0) \frac{1}{|(z_p - a) \cdot \nu(a)|} + \\ O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_D \end{cases}$$

and

$$(2.35) \quad \operatorname{Re}I_2^2(z_p) = \begin{cases} +\frac{\nu_2^2(a) - \nu_1^2(a)}{8\pi} \frac{1}{|(z_p - a) \cdot \nu(a)|^2} - \frac{\nu_2^2(a) - \nu_1^2(a)}{2\pi} [\kappa\sigma^i(a) + \frac{3}{4}f_a''(0)] \frac{1}{|(z_p - a) \cdot \nu(a)|} + \\ O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_I, \\ -\frac{\nu_2^2(a) - \nu_1^2(a)}{8\pi|(z_p - a) \cdot \nu(a)|^2} - \frac{3(\nu_2^2(a) - \nu_1^2(a))}{8\pi} f_a''(0) \frac{1}{|(z_p - a) \cdot \nu(a)|} + \\ O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_D. \end{cases}$$

In addition, we have

$$(2.36) \quad \operatorname{Im}I_1^2(z_p) = \begin{cases} \frac{\nu_1(a)\nu_2(a)}{\pi|(z_p - a) \cdot \nu(a)|} \kappa\sigma^r + O(\ln(|(z_p - a) \cdot \nu(a)|)), & a \in \partial D_I, \\ O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_D \end{cases}$$

and

$$(2.37) \quad \operatorname{Im}I_2^2(z_p) = \begin{cases} \frac{\nu_2^2(a) - \nu_1^2(a)}{2\pi|(z_p - a) \cdot \nu(a)|} \kappa\sigma^r + O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_I, \\ O(\ln |(z_p - a) \cdot \nu(a)|), & a \in \partial D_D. \end{cases}$$

The quantity $f_a''(0)$ is the curvature of ∂D at the point a .

REMARK 2.2 In the previous theorem, the terms $O(1)$ and $O(\ln(|(z_p - a) \cdot \nu(a)|))$ satisfy $|O(1)| < \infty$ and $|\frac{O(\ln(|(z_p - a) \cdot \nu(a)|))}{\ln(|(z_p - a) \cdot \nu(a)|)}| < \infty$ for the points z_p in $C_{a,\theta}$ near the point $a \in \partial D$. Since we are studying the inverse scattering problem at the fixed frequency κ and in the resonance regime (using a moderate frequency), then the formulas in Theorem 2.1 are meaningful for distances $|(z_p - a) \cdot \nu(a)|$ of about the order of $o(|D|)$ where $|D|$ measures the surface of D . We would like also to mention that the numerical implementations of these formulas have been realized in [20].

2.2 Comments and Discussions

Now let us give some comments and discussions on the main theorem. We explain how we can use these formulas to solve the problem and give some possibilities how to design obstacles which we can reconstruct in a more (or less) accurate way using the probe indicator function.

(1). **The obstacle is unknown.** If we do not know the complex obstacle, then using these formulas we can *detect the following information*:

- A sample of points on the curve \oplus the normals on these points \oplus the curvatures on these points. The points can be given by numerically solving one of the following level curves equations:

$$(2.38) \quad |\operatorname{Re}I^0(z)| = C, \text{ or } |(\operatorname{Re}I_1^1(z), \operatorname{Re}I_2^1(z))| = C, \text{ or } |(\operatorname{Re}I_1^2(z), \operatorname{Re}I_2^2(z))| = C$$

for constants C large. The normals are obtained as follows

$$\nu(a) = (\pm t \sqrt{\frac{1}{1+t^2}}, \pm \sqrt{\frac{1}{1+t^2}}) \text{ where } t := \lim_{z_p \rightarrow a} \frac{\operatorname{Re}I_1^1(z_p)}{\operatorname{Re}I_2^1(z_p)},$$

where the sign can be fixed by knowing that the normals are oriented towards the exterior of D . The curvature and the imaginary part of the surface impedance are computable as follows. If the point a is on ∂D_I , then we start to compute the two quantities

$$\begin{aligned} & \frac{3}{4} f_a''(0) + \kappa \sigma^i(a) \\ = & -2 \lim_{z_p \rightarrow a} \left[\frac{\pi((2\nu_1(a)\nu_2(a)\operatorname{Re}I_1^2(z_p) + (\nu_2^2(a) - \nu_1^2(a))\operatorname{Re}I_2^2(z_p)))}{|(z_p - a) \cdot \nu(a)|^{-1}} - \frac{1}{8|(z_p - a) \cdot \nu(a)|} \right], \\ \\ & \frac{1}{2} f_a''(0) + \kappa \sigma^i(a) = - \lim_{z_p \rightarrow a} \frac{\pi(\nu_1(a)\operatorname{Re}I_1^1(z_p) + \nu_2(a)\operatorname{Re}I_2^1(z_p)) + \frac{1}{4|(z_p - a) \cdot \nu(a)|}}{\ln(|(z_p - a) \cdot \nu(a)|)} \end{aligned}$$

from which we deduce the values of $f_a''(0)$ and $\sigma^i(a)$.

If a is on ∂D_D , then we have either

$$f_a''(0) = -\frac{8}{3} \lim_{z_p \rightarrow a} \left[\frac{\pi(2\nu_1(a)\nu_2(a)\operatorname{Re}I_1^2(z_p) + (\nu_2^2(a) - \nu_1^2(a))\operatorname{Re}I_2^2(z_p))}{|(z_p - a) \cdot \nu(a)|^{-1}} + \frac{1}{8|(z_p - a) \cdot \nu(a)|} \right]$$

using multipoles of order two as sources or

$$f_a''(0) = -2 \lim_{z_p \rightarrow a} \frac{\pi(\nu_1(a)\operatorname{Re}I_1^1(z_p) + \nu_2(a)\operatorname{Re}I_2^1(z_p)) - \frac{1}{4|(z_p - a) \cdot \nu(a)|}}{\ln(|(z_p - a) \cdot \nu(a)|)}$$

using multipoles of first order as sources.

- Distinguish the parts where we have Dirichlet or Robin type of boundary conditions and reconstruct the surface impedance.

I. Using I_j^1 . For $s \in (0, 1)$, we have:

$$\lim_{z_p \rightarrow a} \frac{\nu_1(a)\operatorname{Im} I_1^1(z_p) + \nu_2(a)\operatorname{Im} I_2^1(z_p)}{|\ln |(z_p - a) \cdot \nu(a)||^s} = \begin{cases} \infty, & a \in D_I \\ 0, & a \in D_D. \end{cases}$$

The real part of the surface impedance can be determined by

$$\frac{\pi}{\kappa} \lim_{z_p \rightarrow a} \frac{\nu_1(a) \operatorname{Im} I_1^1(z_p) + \nu_2(a) \operatorname{Im} I_2^1(z_p)}{\ln |(z_p - a) \cdot \nu(a)|} = \sigma^r(a), \quad a \in D_I$$

II. Using I_j^2 . For $a \in \partial D_I$, we have:

$$\lim_{z_p \rightarrow a} \frac{|2\nu_1(a)\nu_2(a)\operatorname{Im} I_1^2(z_p) + (\nu_2^2(a) - \nu_1^2(a))\operatorname{Im} I_2^2(z_p)|}{|(z_p - a) \cdot \nu(a)|^{-s}} = \begin{cases} \infty, & s \in (0, 1) \\ \frac{\kappa\sigma^r(a)}{2\pi}, & s = 1 \end{cases}$$

while for $a \in \partial D_D$, we have for every $s \geq 0$ that

$$\lim_{z_p \rightarrow a} |2\nu_1(a)\nu_2(a)\operatorname{Im} I_1^2(z_p) + (\nu_2^2(a) - \nu_1^2(a))\operatorname{Im} I_2^2(z_p)| |(z_p - a) \cdot \nu(a)|^s = 0.$$

Hence using I^0 or I_j^1 or I_j^2 , we can compute a sample of points on the surface of the obstacle. Using I_j^1 , we can compute the normal on these points, distinguish between the coated and the non coated parts of the boundary and compute the real parts of the surface impedance on the points living on the coated part. Finally, using both I_j^1 and I_j^2 , we can compute the curvatures of the points and the imaginary part of the surface impedance on the points living on the coated part.

After having computed a sample of points, the usual way is to (linearly) interpolate them to obtain an approximation of the actual obstacle. However, since we can also compute the normals and the curvatures on those points, then we can use data fitting technics (or higher order interpolations, precisely cubic interpolations in this case) to recover a better approximation of the actual obstacle.

(2). **The obstacle is known.** In this case, we can make the reconstruction of the obstacle more (or less) accurate from the measurements by using the probe indicator function. Since the reconstructed points, defining the approximated obstacle, are computed by solving the level curves equations (2.38) for some given constants C , then from the asymptotic of the used indicator functions we see how the lower order terms, which contain properties of the surface of the obstacle, may affect the accuracy of the reconstruction. Hence weakening (or reinforcing) this effect will provide more (or less) accurate reconstructions. Here, we give some examples how this can be done.

- Indeed, if we know the obstacle shape and wish to make its reconstruction more accurate, then we can choose the surface impedance as

$$(2.39) \quad \sigma^i(a) = -\frac{f_a''(0)}{2\kappa} \text{ or } \sigma^i(a) = -\frac{3f_a''(0)}{4\kappa} \text{ for } a \in \partial D_I$$

and take $\partial D_I = \partial D$, i.e. distribute the coating coefficient along all the surface of the obstacle. The blowup of the indicator functions is more uniform around the obstacle which makes the reconstruction of the obstacle from the measurements more accurate. More precisely, we have for $a \in \partial D$ that

$$2\nu_1(a)\nu_2(a)\operatorname{Re} I_1^2(z_p) + (\nu_2^2(a) - \nu_1^2(a))\operatorname{Re} I_2^2(z_p) = \frac{1}{8\pi|(z_p - a) \cdot \nu(a)|^2} + O(\ln |z_p - a|),$$

using multipoles of order two as sources or

$$\nu_1(a)\operatorname{Re} I_1^1(z_p) + \nu_2(a)\operatorname{Re} I_2^1(z_p) = \frac{1}{4\pi|(z_p - a) \cdot \nu(a)|} + O(1),$$

using multipoles of order one as sources.

Of course, the lower order terms $O(\ln|z_p - a|)$ and $O(1)$ also affect the reconstruction. However their effects are less important than the middle terms appearing in (2.32), (2.34) and (2.35). If we wish to increase more the accuracy, then we should take into account those lower order terms and also use higher order point sources.

- If we wish to make the reconstruction of the obstacle by indicator functions based method less accurate, then we choose σ^i on ∂D_I to be non-uniformly distributed (i.e highly oscillating) so that the blowup of the indicator functions would be less uniform around the obstacle. This is true for any of the indicator functions we use, I^i or I_j^i , $i, j = 1, 2$.

In the next section, we will prove Theorem 2.1.

3 Justification of the results

We consider the details for the case III of the theorem concerning multipoles of order two, especially in proving Proposition 3.3. The other cases are similar.

For any given point $a \in \partial D$, we firstly take the rotation R_a and the translation M_a such that $R_a(\nu(a)) = (0, 1)$, $R_a(a) + M_a = \mathbf{0}$ in the new coordinate system \tilde{x} . Under the transform $\tilde{x} := \mathbb{T}(x) := R_a(x) + M_a$, it follows that $\mathbb{T}(\nu(a)) = (0, 1)$, $\mathbb{T}(a) = \mathbf{0}$.

Define $\tilde{\sigma}(\tilde{x}) := \sigma(x)$ and consider the following problems in the coordinate $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ for any given $\tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}_+^2$.

Using poles Φ :

We set $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0(\tilde{x}, \tilde{z})$ and $\tilde{w}_D^0(\tilde{x}, \tilde{z})$ to be two functions satisfying

$$(3.1) \quad \begin{cases} \Delta \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0 = 0, & \tilde{x} \in \mathbb{R}_+^2 \\ (\frac{\partial}{\partial \tilde{x}_2} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0 + i\kappa \tilde{\sigma}(\mathbf{0}) \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0)(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} = -(\frac{\partial}{\partial \tilde{x}_2} + i\kappa \tilde{\sigma}(\mathbf{0}))\Gamma(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0}, \end{cases}$$

$$(3.2) \quad \begin{cases} \Delta \tilde{w}_D^0 = 0, & \tilde{x} \in \mathbb{R}_+^2, \\ \tilde{w}_D^0(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} = -\Gamma(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} \end{cases}$$

respectively, where $\Gamma(\tilde{x}, \tilde{z}) = \frac{1}{2\pi} \ln \frac{1}{|\tilde{x} - \tilde{z}|}$ and the subscript D in $\tilde{w}_D^{1,j}(\tilde{x}, \tilde{z})$ refers to the Dirichlet boundary condition in (3.2).

Using dipoles $\frac{\partial}{\partial x_j} \Phi$:

We set $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j}(\tilde{x}, \tilde{z})$ and $\tilde{w}_D^{1,j}(\tilde{x}, \tilde{z})$ to be two functions satisfying

$$(3.3) \quad \begin{cases} \Delta \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j} = 0, & \tilde{x} \in \mathbb{R}_+^2 \\ (\frac{\partial}{\partial \tilde{x}_2} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j} + i\kappa \tilde{\sigma}(\mathbf{0}) \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j})(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} = -(\frac{\partial}{\partial \tilde{x}_2} + i\kappa \tilde{\sigma}(\mathbf{0}))\nabla \Gamma(\tilde{x}, \tilde{z}) \cdot \tau_j|_{\tilde{x}_2=0}, \end{cases}$$

$$(3.4) \quad \begin{cases} \Delta \tilde{w}_D^{1,j} = 0, & \tilde{x} \in \mathbb{R}_+^2, \\ \tilde{w}_D^{1,j}(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} = -\nabla \Gamma(\tilde{x}, \tilde{z}) \cdot \tau_j|_{\tilde{x}_2=0} \end{cases}$$

respectively, where the vector τ_1 is given by $\tau_1 := R_a(1, 0) = (\nu_2(a), \nu_1(a))$ while the vector τ_2 is given by $\tau_2 := R_a(0, 1) = (-\nu_1(a), \nu_2(a))$.

Using multipoles $\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Phi$:

We denote by $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j}(\tilde{x}, \tilde{z})$ and $\tilde{w}_D^{2,j}(\tilde{x}, \tilde{z})$ two functions satisfying

$$(3.5) \quad \begin{cases} \Delta \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j} = 0, & \tilde{x} \in \mathbb{R}_+^2 \\ \left(\frac{\partial}{\partial \tilde{x}_2} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j} + i\kappa \tilde{\sigma}(\mathbf{0}) \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j} \right) (\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} = -\left(\frac{\partial}{\partial \tilde{x}_2} + i\kappa \tilde{\sigma}(\mathbf{0}) \right) \nabla(\nabla \Gamma(\tilde{x}, \tilde{z}) \cdot \tau_2) \cdot \tau_j|_{\tilde{x}_2=0}, \end{cases}$$

$$(3.6) \quad \begin{cases} \Delta \tilde{w}_D^{2,j} = 0, & \tilde{x} \in \mathbb{R}_+^2, \\ \tilde{w}_D^{2,j}(\tilde{x}, \tilde{z})|_{\tilde{x}_2=0} = -\nabla(\nabla \Gamma(\tilde{x}, \tilde{z}) \cdot \tau_2) \cdot \tau_j|_{\tilde{x}_2=0} \end{cases}$$

respectively.

We give explicit solutions to these problems in the following proposition.

PROPOSITION 3.1 *I. We have the explicit forms of $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0(\tilde{x}, \tilde{z})$ and $\tilde{w}_D^0(\tilde{x}, \tilde{z})$ as follows:*

$$(3.7) \quad \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0(\tilde{x}, \tilde{z}) = \frac{1}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \cdot \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + i\kappa \tilde{\sigma}(\mathbf{0})}{|\xi_1|(|\xi_1| - i\kappa \tilde{\sigma}(\mathbf{0}))} d\xi_1$$

and

$$(3.8) \quad \tilde{w}_D^0(\tilde{x}, \tilde{z}) = -\frac{1}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \cdot \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{1}{|\xi_1|} d\xi_1$$

II. The explicit forms of $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j}(\tilde{x}, \tilde{z})$, $j = 1, 2$, are

$$(3.9) \quad \begin{aligned} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,1}(\tilde{x}, \tilde{z}) &= i \frac{\nu_2(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \cdot \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{\xi_1}{|\xi_1|} \frac{|\xi_1| + i\kappa \tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa \tilde{\sigma}(\mathbf{0})} d\xi_1 + \\ &\frac{\nu_1(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + i\kappa \tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa \tilde{\sigma}(\mathbf{0})} d\xi_1, \end{aligned}$$

and

$$(3.10) \quad \begin{aligned} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,2}(\tilde{x}, \tilde{z}) &= -i \frac{\nu_1(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \cdot \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{\xi_1}{|\xi_1|} \frac{|\xi_1| + i\kappa \tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa \tilde{\sigma}(\mathbf{0})} d\xi_1 + \\ &\frac{\nu_2(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + i\kappa \tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa \tilde{\sigma}(\mathbf{0})} d\xi_1, \end{aligned}$$

while the ones of $\tilde{w}_D^{1,j}(\tilde{x}, \tilde{z})$, $j = 1, 2$, are

$$(3.11) \quad \begin{aligned} \tilde{w}_D^{1,1}(\tilde{x}, \tilde{z}) &= -i \frac{\nu_2(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{\xi_1}{|\xi_1|} d\xi_1 - \\ &\frac{\nu_1(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} d\xi_1 \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} \tilde{w}_D^{1,2}(\tilde{x}, \tilde{z}) &= i \frac{\nu_1(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{\xi_1}{|\xi_1|} d\xi_1 - \\ &\frac{\nu_2(a)}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1) \xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} d\xi_1. \end{aligned}$$

III. We have also the explicit forms of $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j}(\tilde{x}, \tilde{z})$, $j = 1, 2$, as

$$(3.13) \quad \begin{aligned} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,1}(\tilde{x}, \tilde{z}) &= \frac{\nu_1(a)\nu_2(a)}{2\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1 + \\ & i \frac{(\nu_2^2(a) - \nu_1^2(a))}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \xi_1 \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1, \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,2}(\tilde{x}, \tilde{z}) &= \frac{(\nu_2^2(a) - \nu_1^2(a))}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1 - \\ & i \frac{\nu_1(a)\nu_2(a)}{2\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \xi_1 \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1, \end{aligned}$$

while $\tilde{w}_D^{2,j}(\tilde{x}, \tilde{z})$, $j = 1, 2$, have the forms

$$(3.15) \quad \begin{aligned} \tilde{w}_D^{2,1}(\tilde{x}, \tilde{z}) &= -i \frac{\nu_1(a)\nu_2(a)}{2\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| d\xi_1 - \\ & i \frac{(\nu_2^2(a) - \nu_1^2(a))}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \xi_1 d\xi_1 \end{aligned}$$

and

$$(3.16) \quad \begin{aligned} \tilde{w}_D^{2,2}(\tilde{x}, \tilde{z}) &= -i \frac{(\nu_2^2(a) - \nu_1^2(a))}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| d\xi_1 + \\ & i \frac{\nu_1(a)\nu_2(a)}{2\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \xi_1 d\xi_1 \end{aligned}$$

REMARK 3.2 I. If we set $W_{\tilde{\sigma}(\mathbf{0})}^{1,j}$ to be the solutions of (3.3) and (3.4) replacing $\nabla\Gamma(\tilde{x}, \tilde{z}) \cdot \tau_j$ by $\frac{\partial}{\partial x_j}$, then

$$\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,1} = \nu_1(a)W_{\tilde{\sigma}(\mathbf{0})}^{1,2} + \nu_2(a)W_{\tilde{\sigma}(\mathbf{0})}^{1,1}, \quad \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,2} = \nu_2(a)W_{\tilde{\sigma}(\mathbf{0})}^{1,2} - \nu_1(a)W_{\tilde{\sigma}(\mathbf{0})}^{1,1}.$$

II. If we set $W_{\tilde{\sigma}(\mathbf{0})}^{2,j}$ to be the solutions of (3.5) and (3.6) replacing $\nabla(\nabla\Gamma(\tilde{x}, \tilde{z}) \cdot \tau_2) \cdot \tau_j$ by $\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2}$, then

$$\begin{aligned} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,1} &= 2\nu_1(a)\nu_2(a)W_{\tilde{\sigma}(\mathbf{0})}^{2,2} + (\nu_2^2(a) - \nu_1^2(a))W_{\tilde{\sigma}(\mathbf{0})}^{2,1} \\ \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,2} &= (\nu_2^2(a) - \nu_1^2(a))W_{\tilde{\sigma}(\mathbf{0})}^{2,2} - 2\nu_1(a)\nu_2(a)W_{\tilde{\sigma}(\mathbf{0})}^{2,1}. \end{aligned}$$

Similar remarks are valid for $\tilde{w}_D^{i,j}$, $i, j = 1, 2$.

The proposition 3.1 can be proven by expressing the functions $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0$, $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{i,j}$, $i, j = 1, 2$ (respectively \tilde{w}_D^0 , $\tilde{w}_D^{i,j}$, $i, j = 1, 2$) in the form $(U[\tilde{x}_2]\phi)(\tilde{x}_1)$, (respectively in the form $(U[\tilde{x}_2]\psi)(\tilde{x}_1)$) in \mathbb{R}_+^2 with

$$(U[\tilde{x}_2]\phi)(\tilde{x}_1) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tilde{x}_1\xi_1 + \tilde{x}_2|\xi_1|} \hat{\phi}(\xi_1, \tilde{z}) d\xi_1$$

and then compute the density functions ϕ and ψ from the boundary value problems (3.5), (3.6), where $\hat{\phi}$ is the 1-dimensional Fourier transform of ϕ , see [22] for explicit computations.

We define

$$w_{\sigma(a)}^0(x, z) = \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^0(\mathbb{T}x, \mathbb{T}z), \quad w_{\sigma(a)}^{i,j}(x, z) = \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{i,j}(\mathbb{T}x, \mathbb{T}z), \quad i, j = 1, 2$$

and

$$w_D^0(x, z) = \tilde{w}_D^0(\mathbb{T}x, \mathbb{T}z), \quad w_D^{i,j}(x, z) = \tilde{w}_D^{i,j}(\mathbb{T}x, \mathbb{T}z), \quad i, j = 1, 2$$

for $x, z \in \mathbb{R}^2 \setminus \overline{D}$ near a . They are well-defined by the definition of \mathbb{T} . The next proposition gives the relation between $E_0^s(z, z)$, $E_{i,j}^s(z, z)$ and $w_{\sigma(a)}^0(z, z)$, $w_{\sigma(a)}^{i,j}(z, z)$, respectively, near the point a .

PROPOSITION 3.3 *If $a \in \partial D_I$, then there exists $\delta(a) > 0$ and $C > 0$ such that*

$$(3.17) \quad |E_0^s(z, z) - w_{\sigma(a)}^0(z, z)| \leq C,$$

$$(3.18) \quad |\operatorname{Im} E_{1,j}^s(z, z) - \operatorname{Im} w_{\sigma(a)}^{1,j}(z, z)| \leq C, \quad (j = 1, 2)$$

$$(3.19) \quad |\operatorname{Re} E_{1,j}^s(z, z) - \operatorname{Re} w_{\sigma(a)}^{1,j}(z, z) + \frac{\nu_j(a)f_a''(0)}{2\pi} \ln(|(z-a) \cdot \nu(a)|)| \leq C, \quad (j = 1, 2)$$

$$(3.20) \quad |\operatorname{Im} E_{2,j}^s(z, z) - \operatorname{Im} w_{\sigma(a)}^{2,j}(z, z)| \leq C |\ln(|(z-a) \cdot \nu(a)|)|, \quad (j = 1, 2)$$

$$(3.21) \quad |\operatorname{Re} E_{2,j}^s(z, z) - \operatorname{Re} w_{\sigma(a)}^{2,j}(z, z) - \frac{3A_j(a)}{8\pi|(z-a) \cdot \nu(a)} f_a''(0)| \leq C |\ln(|(z-a) \cdot \nu(a)|)|, \quad (j = 1, 2)$$

for $z \in B(a, \delta(a)) \cap C_{a,\theta}$, where

$$A_1(a) := 2\nu_1(a)\nu_2(a) \text{ and } A_2(a) := \nu_2^2(a) - \nu_1^2(a).$$

In these estimate we set $B_+(a, \delta(a)) := B(a, \delta(a)) \cap (\mathbb{R}^2 \setminus D)$ and $B(a, \delta(a))$ is the ball of center a and radius $\delta(a)$. Similarly, if $a \in \partial D_D$, we obtain (3.17), (3.19), (3.20) and (3.21) by replacing $w_{\sigma(a)}^0, w_{\sigma(a)}^{i,j}$ by $w_D^0, w_D^{i,j}$ respectively and for $i, j = 1, 2$.

3.1 Proof of Theorem 2.1

The proof of Theorem 2.1 is a combination of Proposition 3.1 and Proposition 3.3.

We recall the notation $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ and assume that $\tilde{x}_2, \tilde{z}_2 > 0$. We do the computations for $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j}$ and $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j}$. The ones of $\tilde{w}_D^{2,j}$ are similar and easier.

A.) We start with $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j}$. As we mentioned it in Remark 3.2, we have just to deal with $W_{\tilde{\sigma}(\mathbf{0})}^{1,j}$, $j = 1, 2$. We first consider $W_{\tilde{\sigma}(\mathbf{0})}^{1,2}(\tilde{x}, \tilde{z})$.

$$\begin{aligned} W_{\tilde{\sigma}(\mathbf{0})}^{1,2}(\tilde{x}, \tilde{z}) &= -\frac{1}{4\pi} \int_R e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1 \\ &= -\frac{1}{4\pi} \int_R e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} d\xi_1 - \frac{1}{4\pi} \int_R e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{2i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1 \end{aligned}$$

$$= -\frac{1}{2\pi(\tilde{x}_2 + \tilde{z}_2)} - \frac{1}{2\pi}(i\kappa\tilde{\sigma}(\mathbf{0})) \int_R \frac{e^{-(\tilde{x}_2 - \tilde{z}_2)|\xi_1|}}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1$$

However

$$\begin{aligned} \int_R \frac{e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|}}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1 &= \int_R \frac{e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|}}{|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0}) - i\kappa\tilde{\sigma}^r(\mathbf{0})} d\xi_1 \\ &= \int_R e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0}) + i\kappa\tilde{\sigma}^r(\mathbf{0})}{(|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0}))^2 + (\kappa\tilde{\sigma}^r(\mathbf{0}))^2} d\xi_1 \\ &= \int_R e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0})}{(|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0}))^2 + (\kappa\tilde{\sigma}^r(\mathbf{0}))^2} d\xi_1 + O(1). \end{aligned}$$

Now we have

$$\begin{aligned} &\int_R e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \frac{|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0})}{(|\xi_1| + \kappa\tilde{\sigma}^i(\mathbf{0}))^2 + (\kappa\tilde{\sigma}^r(\mathbf{0}))^2} d\xi_1 \\ &= 2 \int_0^\infty e^{-(\tilde{x}_2 + \tilde{z}_2)\xi_1} \frac{\xi_1 + \kappa\tilde{\sigma}^i(\mathbf{0})}{(\xi_1 + \kappa\tilde{\sigma}^i(\mathbf{0}))^2 + (\kappa\tilde{\sigma}^r(\mathbf{0}))^2} d\xi_1 \\ &= 2 \int_{\kappa\tilde{\sigma}^i(\mathbf{0})}^\infty e^{-(\tilde{x}_2 + \tilde{z}_2)(r - \kappa\tilde{\sigma}^i(\mathbf{0}))} \frac{r}{r^2 + (\kappa\tilde{\sigma}^r(\mathbf{0}))^2} d\xi_1 \end{aligned}$$

and

$$\int_{\kappa\tilde{\sigma}^i(\mathbf{0})}^\infty e^{-(\tilde{x}_2 + \tilde{z}_2)r} \frac{r}{r^2 + (\kappa\tilde{\sigma}^i(\mathbf{0}))^2} d\xi_1 = -\ln(\tilde{x}_2 + \tilde{z}_2) e^{-(\tilde{x}_2 + \tilde{z}_2)\kappa\tilde{\sigma}^i(\mathbf{0})} + O(1).$$

Hence

$$W_{\tilde{\sigma}(\mathbf{0})}^{1,2}(\tilde{x}, \tilde{z}) = \frac{-1}{2\pi(\tilde{x}_2 + \tilde{z}_2)} + \frac{i\kappa\tilde{\sigma}(\mathbf{0})}{\pi} \ln(\tilde{x}_2 + \tilde{z}_2) + O(1).$$

Concerning $W_{\tilde{\sigma}(\mathbf{0})}^{1,1}$, we have $W_{\tilde{\sigma}(\mathbf{0})}^{1,1}(\tilde{x}, \tilde{z}) = 0$ for $\tilde{x}_1 = \tilde{z}_1$ since, in this case, the integrand of $W_{\tilde{\sigma}(\mathbf{0})}^{1,1}$ is symmetric with respect to ξ_1 .

Arguing similarly, we prove that for $\tilde{x}_1 = \tilde{z}_1$ we have $\tilde{W}_D^{1,2}(\tilde{x}, \tilde{z}) = \frac{1}{2\pi(\tilde{x}_2 + \tilde{z}_2)} + O(1)$ and $\tilde{W}_D^{1,1}(\tilde{x}, \tilde{z}) = 0$.

Combining these results, we deduce that for $\tilde{x}_1 = \tilde{z}_1$, we have:

$$\begin{cases} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{1,j}(\tilde{x}, \tilde{z}) = \frac{-\nu_j(a)}{2\pi(\tilde{x}_2 + \tilde{z}_2)} + \frac{i\nu_j(a)\kappa\tilde{\sigma}(\mathbf{0})}{\pi} \ln(\tilde{x}_2 + \tilde{z}_2) + O(1) \\ \text{and} \\ \tilde{w}_D^{1,j}(\tilde{x}, \tilde{z}) = \frac{\nu_j(a)}{2\pi(\tilde{x}_2 + \tilde{z}_2)} + O(1). \end{cases}$$

Coming back to the original coordinates (x, z) , we obtain:

$$w^{1,j}(z, z) = -\frac{\nu_j(a)}{4\pi|(z-a) \cdot \nu(a)|} + \frac{i\nu_j(a)\kappa\sigma(a)}{\pi} \ln(|(z-a) \cdot \nu(a)|) + O(1)$$

and

$$w_D^{1,j}(z, z) = \frac{\nu_j(a)}{4\pi(|(z-a) \cdot \nu(a)|)} + O(1).$$

B.) Now, we deal with $\tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,j}$. We start by $W_{\tilde{\sigma}(\mathbf{0})}^{2,2}$.

$$W_{\tilde{\sigma}(\mathbf{0})}^{2,2}(\tilde{x}, \tilde{z}) = \frac{1}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1$$

Taking $\tilde{x}_1 = \tilde{z}_1$ in this expression, we get

$$\begin{aligned} W_{\tilde{\sigma}(\mathbf{0})}^{2,2}(\tilde{x}, \tilde{z}) &= \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1 \\ &= \frac{1}{2\pi} \int_0^\infty e^{-(\tilde{x}_2 + \tilde{z}_2)r} r dr + \frac{i\kappa\tilde{\sigma}(\mathbf{0})}{\pi} \int_0^\infty \frac{r e^{-(\tilde{x}_2 + \tilde{z}_2)r}}{r - i\kappa\tilde{\sigma}(\mathbf{0})} dr \end{aligned}$$

Hence after some computations, we obtain the expansion:

$$W_{\tilde{\sigma}(\mathbf{0})}^{2,2}(\tilde{x}, \tilde{z}) = \frac{1}{2\pi|\tilde{x}_2 + \tilde{z}_2|^2} + \frac{i\kappa\tilde{\sigma}(\mathbf{0})}{\pi(\tilde{x}_2 + \tilde{z}_2)} - \frac{\kappa^2\tilde{\sigma}(\mathbf{0})^2}{\pi} \ln(\tilde{x}_2 + \tilde{z}_2) + O(1).$$

Similarly, we have:

$$W_{\tilde{\sigma}(\mathbf{0})}^{2,1}(\tilde{x}, \tilde{z}) = -\frac{1}{4\pi} \int_{\mathbb{R}} e^{i(\tilde{x}_1 - \tilde{z}_1)\xi_1} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} \xi_1 \frac{|\xi_1| + i\kappa\tilde{\sigma}(\mathbf{0})}{|\xi_1| - i\kappa\tilde{\sigma}(\mathbf{0})} d\xi_1$$

Hence for $\tilde{x}_1 = \tilde{z}_1$, we have $W_{\tilde{\sigma}(\mathbf{0})}^{2,1}(\tilde{x}, \tilde{z}) = 0$.

Combining these two values, for $\tilde{x}_1 = \tilde{z}_1$ and recalling that $\tilde{\sigma}(\mathbf{0}) := \sigma(a)$, we obtain:

$$\begin{cases} \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,1}(\tilde{x}, \tilde{z}) = \frac{\nu_1(a)\nu_2(a)}{\pi|\tilde{x}_2 + \tilde{z}_2|^2} - \frac{i2\kappa\nu_1(a)\nu_2(a)\sigma(a)}{\pi|\tilde{x}_2 + \tilde{z}_2|} + O(\ln(|\tilde{x}_2 + \tilde{z}_2|)) \\ \tilde{w}_{\tilde{\sigma}(\mathbf{0})}^{2,2}(\tilde{x}, \tilde{z}) = \frac{\nu_2^2(a) - \nu_1^2(a)}{2\pi|\tilde{x}_2 + \tilde{z}_2|^2} - \frac{i\kappa(\nu_2^2(a) - \nu_1^2(a))\sigma(a)}{\pi|\tilde{x}_2 + \tilde{z}_2|} + O(\ln(|\tilde{x}_2 + \tilde{z}_2|)). \end{cases}$$

Now, for $\tilde{x}_1 = \tilde{z}_1$, we have after similar calculations:

$$\begin{cases} \tilde{w}_D^{2,1}(\tilde{x}, \tilde{z}) = -\frac{\nu_1(a)\nu_2(a)}{2\pi} \int_{\mathbb{R}} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| d\xi_1 = -\frac{\nu_1(a)\nu_2(a)}{\pi|\tilde{x}_2 + \tilde{z}_2|^2} \\ \tilde{w}_D^{2,2}(\tilde{x}, \tilde{z}) = -\frac{\nu_2^2(a) - \nu_1^2(a)}{4\pi} \int_{\mathbb{R}} e^{-(\tilde{x}_2 + \tilde{z}_2)|\xi_1|} |\xi_1| d\xi_1 = -\frac{\nu_2^2(a) - \nu_1^2(a)}{2\pi|\tilde{x}_2 + \tilde{z}_2|^2} \end{cases}$$

Coming back to the original coordinates (x, z) , taking $x = z$, we have:

$$\begin{cases} w_{\sigma(a)}^{2,1}(z, z) = \frac{\nu_1(a)\nu_2(a)}{2\pi|(z-a)\cdot\nu(a)|^2} - \frac{i\kappa\sigma(a)\nu_1(a)\nu_2(a)}{\pi|(z-a)\cdot\nu(a)|} + O(\ln(|(z-a)\cdot\nu(a)|)), \\ w_{\sigma(a)}^{2,2}(z, z) = \frac{\nu_2^2(a) - \nu_1^2(a)}{4\pi|(z-a)\cdot\nu(a)|^2} - \frac{i\kappa\sigma(a)(\nu_2^2(a) - \nu_1^2(a))}{2\pi|(z-a)\cdot\nu(a)|} + O(\ln(|(z-a)\cdot\nu(a)|)), \end{cases}$$

and

$$\begin{cases} w_D^{2,1}(z, z) = -\frac{\nu_2^2(a) - \nu_1^2(a)}{4\pi|(z-a)\cdot\nu(a)|^2}, \\ w_D^{2,2}(z, z) = -\frac{\nu_1(a)\nu_2(a)}{2\pi|(z-a)\cdot\nu(a)|^2}. \end{cases}$$

We end the proof by using Proposition 3.3. \square

3.2 Proof of Proposition 3.3

We consider the case where $a \in \partial D_I$, The proof for $a \in \partial D_D$ is similar. We give the details for the case where we used the multipoles $\frac{\partial^2}{\partial x_2^2} \Phi$. The other case $\frac{\partial^2}{\partial x_1 \partial x_2} \Phi$ is also similar. We explain the differences for the poles and the dipoles, i.e. in Lemma 3.4 and Lemma 3.7, at the end of the corresponding proofs. We set $E^s := E_{2,2}^s$.

Let $\tilde{E}^s(x, z)$ be the solution of

$$(3.22) \quad \begin{cases} (\Delta + \kappa^2)\tilde{E}^s(x, z) = 0 \text{ in } \mathbb{R}^2 \setminus \bar{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(x))\tilde{E}^s(x, z) = -(\partial_\nu + i\sigma(x))\frac{\partial^2}{\partial x_2^2} \Phi(x, z) \text{ on } \partial D \\ \tilde{E}^s(\cdot, z) \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

Hence $(E^s - \tilde{E}^s)(x, z)$ satisfies the exterior problem

$$(3.23) \quad \begin{cases} (\Delta + \kappa^2)(E^s - \tilde{E}^s)(x, z) = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(x))(E^s - \tilde{E}^s)(x, z) = 0 \text{ on } \partial D_I \\ (E^s - \tilde{E}^s)(x, z) = (E^s - \tilde{E}^s)(x, z) \text{ on } \partial D_D \\ (E^s - \tilde{E}^s)(\cdot, z) \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

We set $H_\sigma(x, z) := \tilde{E}(x, z) + \frac{\partial}{\partial x_2^2}\Phi(x, z)$, hence H_σ satisfies

$$(3.24) \quad \begin{cases} (\Delta + \kappa^2)H_\sigma(x, z) = -\frac{\partial}{\partial x_2^2}\delta(x, z) \cdot (0, 1) \text{ in } \mathbb{R}^2 \setminus \overline{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(x))H_\sigma(x, z) = 0 \text{ on } \partial D \\ H_\sigma(\cdot, z) \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

Let also G_σ be the associated Green's function of the impedance boundary problem (3.22). We have the following estimates, see [13, 28, 29]:

$$(3.25) \quad \begin{cases} |G_\sigma(x, z)| \leq c|\ln|x-z|| \\ |\nabla G_\sigma(x, z)| \leq c|x-z|^{-1} \\ |H_\sigma(x, z)| \leq c|x-z|^{-2} \\ |\nabla H_\sigma(x, z)| \leq c|x-z|^{-3} \end{cases} \text{ in } \mathbb{R}^2 \setminus D \text{ where } c \text{ is a positive constant.}$$

From these estimates, we deduce that $\tilde{E}(\cdot, z)$ and its derivatives are bounded for $x \in \partial D_D$ and z near $a \in \partial D_I$. The well posed-ness of (3.23) implies that $(E^s - \tilde{E}^s)(\cdot, z)$ is bounded in $H_{loc}^1(\mathbb{R}^2 \setminus \overline{D})$ for z near a . Introducing a cutoff function near the point a and using (3.23), we deduce that $(E^s - \tilde{E}^s)(\cdot, z)$ is bounded for x near ∂D_I and z near a .

This means that we can replace in Proposition 3.3 E^s by \tilde{E}^s . Hence, we need to analyze \tilde{E}^s near the point a . We introduce $\tilde{E}_{\sigma(a)}^s(\cdot, z)$ as the solution of

$$(3.26) \quad \begin{cases} (\Delta + \kappa^2)\tilde{E}_{\sigma(a)}^s(x, z) = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(a))\tilde{E}_{\sigma(a)}^s(x, z) = -(\frac{\partial}{\partial \nu} + i\sigma(a))\frac{\partial^2}{\partial x_2^2}\Phi(\cdot, z) \text{ on } \partial D \\ \tilde{E}_{\sigma(a)}^s(\cdot, z) \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

We have the following lemma:

LEMMA 3.4 *There exists $\delta(a) > 0$ such that*

$$|(\tilde{E}^s - \tilde{E}_{\sigma(a)}^s)(x, z)| = O(\ln(|z - a|)^2),$$

for $z \in B(a, \delta(a)) \cap C_{a, \theta}$ and $x \in (\mathbb{R}^2 \setminus D) \cap B(0, R)$, for any $R > 0$ fixed.

Let $\tilde{E}_{\sigma(a), \Phi}^s(\cdot, z)$ be the solution of

$$(3.27) \quad \begin{cases} (\Delta + \kappa^2)\tilde{E}_{\sigma(a), \Phi}^s(x, z) = 0 \text{ in } \Omega \setminus \overline{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(a))\tilde{E}_{\sigma(a), \Phi}^s(x, z) = -(\frac{\partial}{\partial \nu} + i\kappa\sigma(a))\frac{\partial^2}{\partial x_2^2}\Phi(x, z) \text{ on } \partial D \\ \tilde{E}_{\sigma(a), \Phi}^s(\cdot, z) = -\frac{\partial^2}{\partial x_2^2}\Phi(x, z) \text{ on } \partial \Omega \end{cases}$$

and $\tilde{E}_{\sigma(a), \Gamma}^s(\cdot, z)$ be the solution of (3.27) replacing Φ by Γ . Then we have

LEMMA 3.5 *There exists $C > 0$ such that*

$$|(\tilde{E}_{\sigma(a)}^s - \tilde{E}_{\sigma(a), \Phi}^s)(x, z)| \leq C, \quad |(\tilde{E}_{\sigma(a), \Phi}^s - \tilde{E}_{\sigma(a), \Gamma}^s)(x, z)| \leq C$$

for $z \in \Omega \setminus D$ near D and $x \in \Omega \setminus D$.

We define $\tilde{E}_{\sigma(a)}^{s,0}$ to be the solution of (3.27) replacing Φ by Γ and the Helmholtz equation by the Laplace equation. Then we have

LEMMA 3.6 *There exists $C > 0$ such that $|(\tilde{E}_{\sigma(a),\Gamma}^s - \tilde{E}_{\sigma(a)}^{s,0})(x, z)| \leq C$, for $z \in \Omega \setminus D$ near D and $x \in \Omega \setminus D$.*

Recall that the frequency κ is fixed. Hence, regarding Lemma 3.6, the frequency κ does not play a role in comparing singular solutions of the Helmholtz and the Laplace equations as it is at a small frequency in this 2-dimensional case.

Finally, we have

LEMMA 3.7 *There exist $C > 0, \delta(a) > 0$ such that*

1. $|(\text{Im } \tilde{E}_{\sigma(a)}^{s,0} - \text{Im } w_{\sigma(a)}^0)(z, z)| \leq C |\ln(|z - a|)|$ for $z \in B(a, \delta(a)) \cap C_{a,\theta}$.
2. $|(\text{Re } \tilde{E}_{\sigma(a)}^{s,0} - \text{Re } w_{\sigma(a)}^{2,2})(z, z) - \frac{3(\nu_2^2(a) - \nu_1^2(a))}{8\pi|(z-a)\cdot\nu(a)} f_a''(0)| \leq C |\ln(|z - a|)|$ for $z \in B(a, \delta(a)) \cap C_{a,\theta}$.

By combining all these lemmas above, we end the proof. \square

3.3 Proofs of the lemmas 3.4, 3.5, 3.6 and 3.7

Proof of Lemma 3.4.

We set $R(x, z) := \tilde{E}^s(x, z) - \tilde{E}_{\sigma(a)}^s(x, z)$. Then it satisfies

$$(3.28) \quad \begin{cases} (\Delta + \kappa^2)R(x, z) = 0 \text{ in } \mathbb{R}^2 \setminus \bar{D}, \\ \frac{\partial R(x, z)}{\partial \nu} + i\kappa\sigma(a)R(x, z) = -i\kappa(\sigma(x) - \sigma(a))(\tilde{E}^s(x, z) + \frac{\partial^2}{\partial y_2^2}\Phi(x, z)) \text{ on } \partial D, \\ R(\cdot, z) \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

From (3.28), we have the following representation of $R(x, z)$ for $(x, z) \in \mathbb{R}^2 \setminus \bar{D}$:

$$(3.29) \quad R(x, z) = - \int_{\partial D} i\kappa(\sigma(y) - \sigma(a))G_{\sigma(a)}(y, x)(\tilde{E}^s + \frac{\partial^2}{\partial x_2^2}\Phi)(y, z)ds(y)$$

where $G_{\sigma(a)}$ is the Greens' function corresponding to G_σ , i.e. the Green's function of the impedance boundary problem (3.22), replacing σ by $\sigma(a)$. We know that $\tilde{E}^s + \frac{\partial^2}{\partial x_2^2}\Phi = H_\sigma(x, z)$ has the estimate $|H_\sigma(x, z)| \leq \frac{C}{|x-z|^2}$, then from (3.29) and the Lipschitz regularity of $\sigma(x)$, we deduce that

$$|R(x, z)| \leq c \int_{\partial D} |y - a| |\ln(|y - x|)| |z - y|^{-2} ds(y).$$

From the inequality $|y - a| \leq c(\theta)|y - z|$ for $y \in \partial D$ and $z \in C_{a,\theta} \cap B(a, \delta(a))$ we have $\frac{|y-a|}{|y-z|^2} \leq \frac{c(\theta)}{|y-z|}$, which implies

$$|R(x, z)| \leq \int_{\partial D} \frac{Cc(\theta) |\ln |y - x||}{|y - z|} dy$$

and therefore $|R(x, z)| = O(\ln(d(x, \partial D)) \ln(d(z, \partial D)))$ for $x \in \mathbb{R}^2 \setminus D, z \in C_{a,\theta} \cap B(0, R)$. \square

Similar arguments based on the use of integral equation representation and pointwise estimates of corresponding Green's functions, can be applied to prove Lemma 3.5 and Lemma 3.6.

REMARK 3.8 *In case of poles and dipoles, the integrands of $R(x, z)$ are of log type. Hence $R(x, z)$ is bounded.*

Proof of Lemma 3.7. We recall that $\tilde{E}_{\sigma(a)}^{s,0}$ satisfies

$$(3.30) \quad \begin{cases} \Delta \tilde{E}_{\sigma(a)}^{s,0}(x, z) = 0 \text{ in } \Omega \setminus \bar{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(a))(\tilde{E}_{\sigma(a)}^{s,0}(\cdot, z)) = -(\frac{\partial}{\partial \nu} + i\kappa\sigma(a))\frac{\partial^2 \Gamma}{\partial x_2^2} \text{ on } \partial D, \\ \tilde{E}_{\sigma(a)}^{s,0}(\cdot, z) = -\frac{\partial}{\partial x_2}(\Gamma) \text{ on } \partial\Omega. \end{cases}$$

Using the rigid transformation of coordinates $T := [R_a + M_a]$, $\tilde{E}_{\sigma(a)}^{s,0} \circ T^T$ satisfies:

$$(3.31) \quad \begin{cases} \Delta(\tilde{E}_{\sigma(a)}^{s,0} \circ T^T)(x, z) = 0 \text{ in } \Omega \setminus \bar{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(a))(\tilde{E}_{\sigma(a)}^{s,0} \circ T^T(\cdot, z)) = -(\frac{\partial}{\partial \nu} + i\kappa\sigma(a))\nabla(\nabla\Gamma \cdot \tau_2) \cdot \tau_2 \text{ on } \partial D, \\ \tilde{E}_{\sigma(a)}^{s,0} \circ T^T(\cdot, z) = -\nabla(\nabla\Gamma \cdot \tau_2) \cdot \tau_2 \text{ on } \partial\Omega. \end{cases}$$

We write

$$\begin{aligned} \nabla(\nabla\Gamma \cdot \tau_2) \cdot \tau_2 &= \nu_1^2(a)\frac{\partial^2 \Gamma}{\partial x_1^2} + \nu_2^2(a)\frac{\partial^2 \Gamma}{\partial x_2^2} - \nu_1(a)\nu_2(a)\frac{\partial^2 \Gamma}{\partial x_1 \partial x_2} - \nu_1(a)\nu_2(a)\frac{\partial^2 \Gamma}{\partial x_2 \partial x_1} \\ &= (\nu_2^2(a) - \nu_1^2(a))\frac{\partial^2}{\partial x_2^2}\Gamma - 2\nu_1(a)\nu_2(a)\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\Gamma. \end{aligned}$$

We set $v_{\sigma(a)}^j$ to be the solution of the following problem:

$$(3.32) \quad \begin{cases} \Delta v_{\sigma(a)}^j(x, z) = 0 \text{ in } \Omega \setminus \bar{D}, \\ (\frac{\partial}{\partial \nu} + i\kappa\sigma(a))(v_{\sigma(a)}^j(\cdot, z)) = -(\frac{\partial}{\partial \nu} + i\kappa\sigma(a))\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_2}\Gamma \text{ on } \partial D, \\ v_{\sigma(a)}^j(\cdot, z) = -\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_2}\Gamma \text{ on } \partial\Omega, \end{cases}$$

then

$$(3.33) \quad \tilde{E}_{\sigma(a)}^{s,0} \circ T^T = (\nu_2^2(a) - \nu_1^2(a))v_{\sigma(a)}^2 - 2\nu_1(a)\nu_2(a)v_{\sigma(a)}^1$$

Let $\xi = F(x)$ be the local change of variables

$$(3.34) \quad \xi_1 = x_1, \quad \xi_2 = x_2 - f_a(x_1),$$

where f_a is the function defined in the introduction. F depends on the point a , but we do not mention this to avoid more complicated notations. We have the following properties:

$$(3.35) \quad \begin{cases} c_1|x - z| \leq |F(x) - F(z)| \leq c_2|x - z|, \\ |F(x) - x| \leq c_3|x|^2, \\ |DF(x) - I| \leq c_4|x| \end{cases}$$

for x, z near the point 0, where $c_i (i = 1, 2, 3, 4)$ are positive constants, which is due to hypothesis on the regularity of ∂D .

Let x, z be points near 0. We set $\tilde{v}_{\sigma(a)}^j(\xi, \eta) := v_{\sigma(a)}^j(x, z)$ where $\xi := F(x)$ and $\eta := F(z)$, $B := JJ^T$ and $J := \frac{\partial \xi}{\partial x}(F^{-1}(\xi))$ and $\tilde{\nu} := (0, 1)$ is the unit normal to $\partial\mathbb{R}_+^2$. We have from (3.35) that $|B(\xi) - B(0)| \leq c|\xi|$ and $B(0) = I$.

We set also $\Gamma_{\sigma(a)}^j(x, z) := (W_{\sigma(a)}^{2,j} + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_2}\Gamma)(x, z)$ and write $\tilde{R}^j(\xi, \eta) := \tilde{v}_{\sigma(a)}^j(\xi, \eta) - W_{\sigma(a)}^{2,j}(\xi, \eta)$. Then the function $\tilde{R}(\cdot, \eta)$ satisfies

$$(3.36) \quad \begin{cases} \nabla_\xi \cdot B(\xi)\nabla_\xi \tilde{R}^j = \nabla_\xi \cdot (I - B)\nabla_\xi W_{\sigma(a)}^j, \\ B(\xi)\nabla_\xi \tilde{R}^j \cdot \tilde{\nu} + i\kappa\sigma(a)\tilde{R}^j = (I - B)\nabla_\xi \Gamma_{\sigma(a)}^j \cdot \tilde{\nu} + i\kappa\sigma(a)(1 - |J^{-T}\nu|^{-1})\tilde{G}_{\sigma(a)}^j(\xi, \eta), \end{cases}$$

where the first relation holds in \mathbb{R}_+^2 near $F(0)$, while the second one is satisfied on $\partial\mathbb{R}_+^2$ near $F(0)$.

Let G (not to be confused with G_σ) be the Neumann Green's function associated to the expression $\nabla \cdot B \nabla$ on $B(0, r)$. Let $B_r^+ := B(0, r) \cap \mathbb{R}_+^2$. We write $\partial B_r^+ = S_r \cup S_r^c$ with $S_r := \partial B_r^+ \cap \partial F(D)$.

Integrating by parts in (3.36), we obtain

$$(3.37) \quad \begin{aligned} \tilde{R}^j(\xi, \eta) &= \int_{B_r^+} (I - B) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,j}(z, \eta) dz + \int_{S_r} (I - B) \frac{\partial W_{\sigma(a)}^{2,j}}{\partial \nu}(z, \eta) G(z, \xi) ds(z) - \\ &\quad i\kappa\sigma(a) \int_{S_r} \tilde{R}^j(z, \eta) G(z, \xi) ds(z) + i\kappa\sigma(a) \int_{S_r} (1 - |J^{-T}\nu|)(\tilde{v}_{\sigma(a)}^j + \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \Gamma)(z, \eta) G(z, \xi) ds(z) - \\ &\quad \int_{S_r^c} (I - B) \nabla W_{\sigma(a)}^{2,j} \cdot \tilde{v}(z, \eta) G(z, \xi) ds(z) + \int_{S_r^c} B \nabla \tilde{R}^j(z, \eta) \cdot \tilde{v} G(z, \eta) ds(z). \end{aligned}$$

Notice that ν is the normal on ∂B_r^+ oriented into inside of B_r^+ since it is outside of the obstacle D if we restrict it to S_r , i.e. $\nu = (0, 1)$ on S_r .

Due to the estimates of the elements defining $\tilde{R}^j(z, \eta)$ and $G(z, \xi)$, we see that the fourth term of (3.37) has a singularity of log type for ξ and η near 0 and since $|1 - |J^{-T}\nu|(x, z)|$ behaves like $|x - z|$ near a due to the property (3.34), we deduce that the third term also has a singularity of log type for ξ and η near 0. The fifth and the sixth terms are obviously bounded since the point 0 is away from S_r^c . In the next step, we shall compute the dominant parts of the first and the second terms.

We use the explicit form of B , i.e.

$$B(z) := \begin{bmatrix} 1 & -f'_a(z_1) \\ -f'_a(z_1) & 1 + (f'_a)^2(z_1) \end{bmatrix}$$

to write

$$B(z) - B(\eta) = \begin{bmatrix} 0 & -f''_a(\eta_1)(z_1 - \eta_1) \\ -f''_a(\eta_1)(z_1 - \eta_1) & 2f'_a(\eta_1)f''_a(\eta_1)(z_1 - \eta_1) \end{bmatrix} + O(z_1 - \eta_1)^2$$

and then

$$(3.38) \quad B(z) - B(\eta) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} f''_a(\eta_1)(z_1 - \eta_1) + O(\eta_1)O(z_1 - \eta_1) + O(z_1 - \eta_1)^2$$

I. The first term in (3.37).

Using the upper estimates of ∇G and $\nabla W_{\sigma(a)}^{2,j}(z, \eta)$, we obtain

$$(3.39) \quad \begin{aligned} &\int_{B_r^+} (B(\eta) - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,j}(z, \eta) dz \\ &= f''_a(\eta_1) \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} G(z, \xi) \frac{\partial}{\partial z_1} W_{\sigma(a)}^{2,j}(z, \eta) + \frac{\partial}{\partial z_1} G(z, \xi) \frac{\partial}{\partial z_2} W_{\sigma(a)}^{2,j}(z, \eta) \right] dz + O(1). \end{aligned}$$

We use the following estimates

$$(3.40) \quad G(z, \xi) = \Gamma(z, \xi) + \Gamma(z, \xi^*) + O(1), \quad \nabla G(z, \xi) = \nabla \Gamma(z, \xi) + \nabla \Gamma(z, \xi^*) + O(\ln |\xi - z|),$$

$$(3.41) \quad W_{\sigma(a)}^{2,j}(z, \eta) = \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_2} \Gamma(z, \eta^*) + O(\ln |z - \eta|)$$

and

$$(3.42) \quad \nabla W_{\sigma(a)}^{2,j}(z, \eta) = \nabla \left(\frac{\partial}{\partial z_j} \frac{\partial}{\partial z_2} \Gamma(z, \eta^*) \right) + O(|z - \eta|).$$

The estimates of G can be proved as in [28, 29] and [13], see also [22], while the ones of $W_{\sigma(a)}^{2,j}$ can be deduced directly from Proposition 3.1. Hence it is enough to compute the integrals using $\Gamma(z, \xi), \Gamma(z, \xi^*)$ and $\frac{\partial}{\partial z_j} \Gamma(z, \eta^*)$ which have explicit forms.

We consider first the case $j = 2$. We have

$$\begin{aligned} & \int_{B_r^+} (B(\eta) - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz \\ &= f_a''(\eta_1) \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial \Gamma}{\partial z_2}(z, \xi) \frac{\partial}{\partial z_1} \left(\frac{\partial^2 \Gamma}{\partial z_2^2} \right)(z, \eta^*) + \frac{\partial \Gamma}{\partial z_1}(z, \xi) \frac{\partial}{\partial z_2} \left(\frac{\partial^2 \Gamma}{\partial z_2^2} \right)(z, \eta^*) \right] dz + \\ & f_a''(\eta_1) \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial \Gamma}{\partial z_2}(z, \xi^*) \frac{\partial}{\partial z_1} \left(\frac{\partial^2 \Gamma}{\partial z_2^2} \right)(z, \eta^*) + \frac{\partial \Gamma}{\partial z_1}(z, \xi^*) \frac{\partial}{\partial z_2} \left(\frac{\partial^2 \Gamma}{\partial z_2^2} \right)(z, \eta^*) \right] dz + O(1). \end{aligned}$$

The integrand of the first integral is obtained from the integrand of the second one replacing ξ by ξ^* . We write a more computable form of the second integrand as follows:

$$\begin{aligned} & (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} \Gamma(z, \xi^*) \frac{\partial}{\partial z_1} \left(\frac{\partial^2}{\partial z_2^2} \Gamma \right)(z, \eta^*) + \frac{\partial}{\partial z_1} \Gamma(z, \xi^*) \frac{\partial}{\partial z_2} \left(\frac{\partial^2}{\partial z_2^2} \Gamma \right)(z, \eta^*) \right] \\ &= -\frac{1}{4\pi^2} \left[\frac{-2(z_2 + \xi_2)(z_1 - \eta_1)^2 - 6(z_1 - \xi_1)(z_1 - \eta_1)(z_2 + \eta_2)}{|z - \xi^*|^2 |z - \eta^*|^4} + \right. \\ & \quad \left. \frac{8(z_1 - \eta_1)^8 (z_2 + \xi_2)(z_2 + \eta_2)^2 + 8(z_1 - \xi_1)(z_1 - \eta_1)(z_2 + \eta_2)^3}{|z - \xi|^2 |z - \eta^*|^6} \right] \end{aligned}$$

The integral of the first term can be computed directly:

$$\begin{aligned} & \int_0^{r_2} \int_{-r_1}^{r_1} \frac{(z_2 + \xi_2)(z_1 - \eta_1)^2}{|z - \xi^*|^2 |z - \eta^*|^4} dz_1 dz_2 \\ &= \int_0^{r_2} \frac{1/8}{|z_2 + \eta_2|^2} \left[\arctan \frac{r_1 - \eta_1}{z_2 + \eta_2} - \arctan \frac{-r_1 - \eta_1}{z_2 + \eta_2} \right] dz_2 - \\ & \quad \frac{1}{4} \left[\sin 4 \arctan \frac{r_1 - \eta_1}{z_2 + \eta_2} - \sin 4 \arctan \frac{-r_1 - \eta_1}{z_2 + \eta_2} \right] \\ &= \frac{\pi}{8} \frac{1}{\eta_2} + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right) + O(\ln \eta_2). \end{aligned}$$

Similarly, we have

$$\int_0^{r_2} \int_{-r_1}^{r_1} \frac{(z_1 - \xi_1)(z_1 - \eta_1)(z_2 + \xi_2)}{|z - \xi^*|^2 |z - \eta^*|^4} dz_1 dz_2 = \frac{\pi}{8} \frac{1}{\eta_2} + O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right).$$

With more, but straight computations, we get

$$\int_0^{r_2} \int_{-r_1}^{r_1} \frac{(z_1 - \xi_1)(z_1 - \eta_1)(z_2 + \eta_2)^3}{|z - \xi^*|^2 |z - \eta^*|^6} dz_1 dz_2 = \frac{\pi}{16} \frac{1}{\eta_2} + O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right).$$

and

$$\int_0^{r_2} \int_{-r_1}^{r_1} \frac{(z_1 - \eta_1)^2 (z_2 + \xi_2)(z_2 + \eta_2)^2}{|z - \xi|^2 |z - \eta^*|^2} = \frac{\pi}{16} \frac{1}{\eta_2} + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right) + O(\ln(\eta_2)).$$

Gathering all these integrals, we obtain:

$$\begin{aligned} & \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} \Gamma(z, \xi^*) \frac{\partial}{\partial z_1} \left(\frac{\partial^2}{\partial z_2^2} \Gamma \right)(z, \eta^*) + \frac{\partial}{\partial z_1} \Gamma(z, \xi^*) \frac{\partial}{\partial z_2} \left(\frac{\partial^2}{\partial z_2^2} \Gamma \right)(z, \eta^*) \right] dz \\ &= O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right). \end{aligned}$$

Also, changing ξ by ξ^* , we get the asymptotic of the first integral:

$$\begin{aligned} & \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} \Gamma(z, \xi) \frac{\partial}{\partial z_1} \left(\frac{\partial^2}{\partial z_1^2} \Gamma \right) (z, \eta^*) + \frac{\partial}{\partial z_1} \Gamma(z, \xi) \frac{\partial}{\partial z_2} \left(\frac{\partial^2}{\partial z_2^2} \Gamma \right) (z, \eta^*) \right] dz \\ &= O(\ln \eta_2) + O\left(\frac{|\xi^* - \eta^*|}{\eta_2}\right) + O\left(\frac{(|\xi^* - \eta^*|^2)}{\eta_2^2}\right). \end{aligned}$$

Hence

$$\begin{aligned} & \int_{B_r^+} (B(\eta) - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz \\ &= O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{(|\xi - \eta^*|^2)}{\eta_2^2}\right) + O\left(\frac{|\xi^* - \eta^*|}{\eta_2}\right) + O\left(\frac{(|\xi^* - \eta^*|^2)}{\eta_2^2}\right). \end{aligned}$$

We consider now the case $j = 1$. Taking $j = 1$ in (3.39), (3.40), (3.41) and (3.42), we have:

$$\begin{aligned} & \int_{B_r^+} (B(\eta) - B(z)) \nabla G(z, \xi) \cdot \nabla W^{2,1}(z, \eta) dz \\ &= f_a''(\eta_1) \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial \Gamma}{\partial z_2}(z, \xi) \frac{\partial}{\partial z_1} \left(\frac{\partial^2 \Gamma}{\partial z_1 \partial z_2} \right) (z, \eta^*) + \frac{\partial \Gamma}{\partial z_1}(z, \xi) \frac{\partial}{\partial z_2} \left(\frac{\partial^2 \Gamma}{\partial z_1 \partial z_2} \right) (z, \eta^*) \right] dz + \\ & f_a''(\eta_1) \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial \Gamma}{\partial z_2}(z, \xi^*) \frac{\partial}{\partial z_1} \left(\frac{\partial^2 \Gamma}{\partial z_1 \partial z_2} \right) (z, \eta^*) + \frac{\partial \Gamma}{\partial z_1}(z, \xi^*) \frac{\partial}{\partial z_2} \left(\frac{\partial^2 \Gamma}{\partial z_1 \partial z_2} \right) (z, \eta^*) \right] dz \\ & + O(1). \end{aligned}$$

After some computations, we write the second integrand as follows:

$$\begin{aligned} & (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} \Gamma(z, \xi^*) \frac{\partial}{\partial z_1} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \Gamma \right) (z, \eta^*) + \frac{\partial}{\partial z_1} \Gamma(z, \xi^*) \frac{\partial}{\partial z_2} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \Gamma \right) (z, \eta^*) \right] \\ &= \frac{(z_1 - \eta_1)}{4\pi^2} \left[-2 \frac{(z_2 + \xi_2)(z_2 + \eta_2) + (z_1 - \xi_1)(z_1 - \eta_1)}{|z - \xi^*|^2 |z - \eta^*|^4} - \right. \\ & \quad \left. 4 \frac{(z_2 + \xi_2)(z_2 + \eta_2)(z_1 - \eta_1)^2 + (z_1 - \xi_1)(z_1 - \eta_1)(z_2 + \eta_2)^2}{|z - \xi^*|^2 |z - \eta^*|^6} \right] \end{aligned}$$

Since the numerators of these two terms are antisymmetric with respect to z_1 in the limit case $\eta = \xi = 0$, we prove that:

$$\begin{aligned} & \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} \Gamma(z, \xi) \frac{\partial}{\partial z_1} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \Gamma \right) (z, \eta^*) + \frac{\partial}{\partial z_1} \Gamma(z, \xi) \frac{\partial}{\partial z_2} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \Gamma \right) (z, \eta^*) \right] dz \\ &= O(\ln(\eta_2)) + O\left(\frac{|\xi^* - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi^* - \eta^*|^2}{\eta_2^2}\right) \end{aligned}$$

Dealing similarly with the first integral, we obtain:

$$\begin{aligned} & \int_{B_r^+} (z_1 - \eta_1) \left[\frac{\partial}{\partial z_2} \Gamma(z, \xi^*) \frac{\partial}{\partial z_1} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \Gamma \right) (z, \eta^*) + \frac{\partial}{\partial z_1} \Gamma(z, \xi^*) \frac{\partial}{\partial z_2} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \Gamma \right) (z, \eta^*) \right] dz \\ &= O(\ln(\eta_2)) + O\left(\frac{|\xi^* - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi^* - \eta^*|^2}{\eta_2^2}\right) \end{aligned}$$

Hence:

$$\int_{B_r^+} (B(\eta) - B(z)) \nabla G(z, \xi) \cdot \nabla W^{2,1}(z, \eta) dz = O(\ln(\eta_2)) + O\left(\frac{|\xi^* - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi^* - \eta^*|^2}{\eta_2^2}\right).$$

II. The second term in (3.37)

Let us consider the boundary integral terms.

Since

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \nabla W_{\sigma(a)}^{2,1}(z, \eta) \cdot \nu = \frac{\partial^2}{\partial z_1^2} \frac{\partial}{\partial z_2} \Gamma(z, \eta^*), \text{ on } S_r$$

because ν is given by the vector $(0, 1)$ on the flat S_r and

$$(z_1 - \eta_1) \frac{\partial^2}{\partial z_1^2} \frac{\partial}{\partial z_2} \Gamma(z, \eta^*) = -2 \frac{(z_2 + \eta_2)(z_1 - \eta_1)}{|z - \eta^*|^4} + 8 \frac{(z_1 - \eta_1)^3 (z_2 + \eta_2)}{|z - \eta^*|^6}$$

is antisymmetric with respect to z_1 near $\eta_1 = 0$, then we show that:

$$\int_{S_r} (B(\eta) - B(z)) \nabla W^{2,1}(z, \eta) \cdot \nu G(z, \xi) ds(z) = O(\ln(\eta_2)) + O\left(\frac{|\xi^* - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi^* - \eta^*|^2}{\eta_2^2}\right).$$

Let us now consider the last term.

$$\int_{S_r} (B(\eta) - B(z)) \nabla W^{2,2}(z, \eta) \cdot \nu G(z, \xi) ds(z).$$

Using (3.38), we can show that

$$\int_{S_r} (B(\eta) - B(z)) \nabla W^{2,2}(z, \eta) \cdot \nu G(z, \xi) ds(z) = f_a''(\eta_1) \int_{-r_1}^{r_1} \frac{\partial}{\partial z_1} W_{\sigma(a)}^{2,2}(z, \eta) G(z, \xi) (z_1 - \eta_1) dz_1 + O(1).$$

From (3.41), we obtain

$$\int_{-r_1}^{r_1} \frac{\partial}{\partial z_1} W_{\sigma(a)}^{2,2}(z, \eta) G(z, \xi) (z_1 - \eta_1) dz_1 = \int_{-r_1}^{r_1} \frac{\partial}{\partial z_1} \left(\frac{\partial^2}{\partial z_2^2} \right) \Gamma(z, \eta^*) [\Gamma(z, \xi) + \Gamma(z, \xi^*)] (z_1 - \eta_1) dz_1 + O(1).$$

Integrating by parts and recalling that we are interested in ξ and η near zero, we obtain:

$$\begin{aligned} & \int_{-r_1}^{r_1} \frac{\partial}{\partial z_1} \left(\frac{\partial^2}{\partial z_2^2} \right) (z, \eta^*) [\Gamma(z, \xi) + \Gamma(z, \xi^*)] (z_1 - \eta_1) dz_1 = \\ & - \int_{-r_1}^{r_1} \frac{\partial^2}{\partial z_2^2} (z, \eta^*) \left[\frac{\partial}{\partial z_1} \Gamma(z, \xi) + \frac{\partial}{\partial z_1} \Gamma(z, \xi^*) \right] (z_1 - \eta_1) dz_1 - \int_{-r_1}^{r_1} \frac{\partial^2}{\partial z_2^2} (z, \eta^*) [\Gamma(z, \xi) + \Gamma(z, \xi^*)] dz_1 + O(1) \end{aligned}$$

Since η^* is out of B_r^+ , then $\frac{\partial}{\partial z_2} \Gamma(z, \eta^*)$ is harmonic in B_r^+ , hence an integration by parts in B_r^+ gives

$$\begin{aligned} & - \int_{-r_1}^{r_1} \frac{\partial^2}{\partial z_2^2} (z, \eta^*) [\Gamma(z, \xi) + \Gamma(z, \xi^*)] dz_1 \\ & = \int_{\partial B_r^+} \frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial z_2} \Gamma(z, \eta^*) \right) [\Gamma(z, \xi) + \Gamma(z, \xi^*)] ds(z) + O(1) \\ & = \int_{B_r^+} \nabla \left(\frac{\partial}{\partial z_2} \Gamma(z, \eta^*) \right) \cdot [\nabla \Gamma(z, \xi) + \nabla \Gamma(z, \xi^*)] dz + O(1) \\ & = \frac{1}{4\pi^2} \left[\int_{B_r^+} \frac{-2(z_1 - \eta_1)(z_2 + \eta_2)}{|z - \eta^*|^4} \left(\frac{z_1 - \xi_1}{|z - \xi|^2} + \frac{z_1 - \xi_1}{|z - \xi^*|^2} \right) dz + \right. \\ & \quad \left. \int_{B_r^+} \left(\frac{1}{|z - \eta^*|^2} - 2 \frac{(z_2 + \eta_2)^2}{|z - \eta^*|^4} \right) \left(\frac{z_2 - \xi_2}{|z - \xi|^2} + \frac{z_2 + \xi_2}{|z - \xi^*|^2} \right) dz \right] + O(1) \end{aligned}$$

After some calculations, we obtain:

$$\begin{cases} \int_{B_r^+} \frac{(z_1 - \eta_1)(z_2 + \eta_2)(z_1 - \xi_1)}{|z - \eta^*|^4 |z - \xi|^2} dz_1 dz_2 = \frac{\pi}{8} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right), \\ \int_{B_r^+} \frac{(z_1 - \eta_1)(z_2 + \eta_2)(z_1 - \xi_1)}{|z - \eta^*|^4 |z - \xi^*|^2} dz_1 dz_2 = \frac{\pi}{8} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right) \\ \int_{B_r^+} \frac{z_2 - \xi_2}{|z - \eta^*|^2 |z - \xi|^2} dz_1 dz_2 = \frac{\pi}{2} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right) = \int_{B_r^+} \frac{z_2 + \xi_2}{|z - \eta^*|^2 |z - \xi^*|^2} dz_1 dz_2, \\ \int_{B_r^+} \frac{(z_2 + \eta_2)^2 (z_2 - \xi_2)}{|z - \eta^*|^4 |z - \xi|^2} dz_1 dz_2 = \frac{3\pi}{8} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right) = \int_{B_r^+} \frac{(z_2 + \eta_2)^2 (z_2 + \xi_2)}{|z - \eta^*|^4 |z - \xi^*|^2} dz_1 dz_2, \end{cases}$$

Hence

$$\begin{aligned} & - \int_{-r_1}^{r_1} \frac{\partial^2}{\partial z_2^2} \Gamma(z, \eta^*) [\Gamma(z, \xi) + \Gamma(z, \xi^*)] dz_1 \\ &= \frac{1}{4\pi^2} \left[-2 \left(\frac{\pi}{8} + \frac{\pi}{8} \right) + \pi - 2 \left(\frac{3\pi}{4} \right) \right] \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right) \\ &= -\frac{1}{4\pi} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right) \end{aligned}$$

In a similar way, we obtain:

$$- \int_{-r_1}^{r_1} \frac{\partial^2}{\partial z_2^2} \Gamma(z, \eta^*) \left[\frac{\partial}{\partial z_1} \Gamma(z, \xi) + \frac{\partial}{\partial z_1} \Gamma(z, \xi^*) \right] (z_1 - \eta_1) ds(z) = -\frac{1}{8\pi} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right)$$

Gathering these two integrals, we end up with:

$$\int_{S_r} (B(\eta) - B(z)) \nabla W^{2,2}(z, \eta) \cdot \nu G(z, \xi) ds(z) = -\frac{3}{8\pi} \frac{1}{\eta_2} + O(\ln(\eta_2)) + O\left(\frac{|\xi - \eta|}{\eta_2}\right)$$

III. End of the proof.

Now we write

$$\begin{aligned} & \int_{B_r^+} (I - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz \\ &= \int_{B_r^+} (I - B(\eta)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz + \int_{B_r^+} (B(\eta) - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz. \end{aligned}$$

We have

$$\begin{aligned} & \left| \int_{B_r^+} (I - B(\eta)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz \right| \leq |I - B(\eta)| \int_{B_r^+} |\nabla G(z, \xi)| |\nabla W_{\sigma(a)}^{2,2}(z, \eta)| dz \\ & \leq c |\eta_1| \int_{B_r^+} |z - \xi|^{-1} |z - \eta^*|^{-2} dz \leq c \frac{|\eta_1|}{|\eta|} \int_{B_r^+} |z - \xi|^{-1} |z - \eta^*|^{-1} dz = O(1) \end{aligned}$$

for ξ and $\eta \in C_{F(a), \theta}$, hence it implies that

$$\int_{B_r^+} (I - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,2}(z, \eta) dz = O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right).$$

Similarly, we have

$$\int_{S_r} (I - B(z)) \frac{\partial W^{2,2}}{\partial \bar{\nu}}(z, \eta) G(z, \xi) ds(z) = -\frac{3}{8\pi} \frac{f_a''(0)}{\eta_2} + O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right).$$

Doing the same computations for $j = 1$, we obtain

$$\int_{B_r^+} (I - B(z)) \nabla G(z, \xi) \cdot \nabla W_{\sigma(a)}^{2,1}(z, \eta) dz = O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right)$$

and

$$\int_{S_r} (I - B(z)) \frac{\partial W_{\sigma(a)}^{2,1}}{\partial \tilde{\nu}}(z, \eta) G(z, \xi) ds(z) = O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right)$$

Hence

$$\tilde{R}^1(\xi, \eta) = O(\ln(\eta)) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right)$$

and

$$\tilde{R}^2(\xi, \eta) = -\frac{3}{8\pi} f_a''(0) \frac{1}{\eta_2} + O(\ln(\eta)) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right)$$

Using the relation (3.33) together with (3.37), we get the following asymptotic:

$$\tilde{R}^2(\xi, \eta) = -\frac{3(\nu_2^2(a) - \nu_1^2(a))}{8\pi} \frac{f_a''(0)}{\eta_2} + O(\ln \eta_2) + O\left(\frac{|\xi - \eta^*|}{\eta_2}\right) + O\left(\frac{|\xi - \eta^*|^2}{\eta_2^2}\right)$$

hence

$$\tilde{R}^2(\xi, \eta) = -\frac{3(\nu_2^2(a) - \nu_1^2(a))}{8\pi} \frac{f_a''(0)}{\eta_2} + O(\ln \eta_2), \text{ for } \xi = \eta \in C_{F(a)}(\theta).$$

We go back to $R^2(x, z) := \tilde{E}_{\sigma(a)}^{s,0}(x, z) - w_{\sigma(a)}^{2,2}(x, z)$ and we write it as

$$R^2(x, z) = w_{\sigma(a)}^{s,0}(x, z) - w_{\sigma(a)}^{2,2}(F(x), F(z)) + w_{\sigma(a)}^{2,2}(F(x), F(z)) - w_{\sigma(a)}^{2,2}(x, z),$$

and then

$$(3.43) \quad \begin{aligned} R^2(x, z) &= \tilde{R}^2(F(x), F(z)) + [w_{\sigma(a)}^{2,2}(F(x), F(z)) - w_{\sigma(a)}^{2,2}(F(x), z)] + \\ &[w_{\sigma(a)}^{2,2}(F(x), z) - w_{\sigma(a)}^{2,2}(x, z)]. \end{aligned}$$

From the previous computations, we have:

$$\tilde{R}^2(F(z), F(z)) = -\frac{3}{8\pi} \frac{\nu_2^2(a) - \nu_1^2(a)}{F(z)_2} f_a''(0) + O(\ln(F(z)_2))$$

recalling that $F(z) = (F(z)_1, F(z)_2) := (z_1, z_2 - f_a(z_1))$. Arguing as in [22] by using the properties (3.35) of F , we show that the second and the third terms in (3.43) are bounded. Finally, going back to the original coordinates, we obtain:

$$R^2(z, z) = -\frac{3}{8\pi} \frac{\nu_2^2(a) - \nu_1^2(a)}{|(z-a) \cdot \nu(a)|} f_a''(0) + O(\ln |(z-a) \cdot \nu(a)|) \text{ for } z \in B(a, \delta(a))$$

for some $\delta(a) > 0$. □

REMARK 3.9 1.) In case of the pole Φ , the integrand in \tilde{R} is of the order of $\frac{1}{|z-\eta|}$ for the part $\int_{B_r^+} \dots dz$ and $\ln(|z-\eta|)$ for $\int_{S_r} \dots ds(z)$, hence $R(x, z)$ is bounded. In case of dipoles $\frac{\partial}{\partial x_j} \Phi$, the integrand are more singular, however, doing similar computations, as for the multipoles, we show that the dominant part is of order of $\ln(|z-\eta|)$ and the corresponding multiplicative coefficient of $f_a''(0)$ is exactly $-\frac{1}{4\pi}$.

2.) The part 1.) of Lemma 3.7 is also justified by knowing that the imaginary parts of the Green's functions are less singular than the real parts. We have one degree less for the power of the singularity.

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