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Optimal Portfolio Policies Under Bounded Expected Loss and Partial Information*

Jörn Sass† and Ralf Wunderlich‡

Summary: In a market with partial information we consider the optimal selection of portfolios for utility maximizing investors under joint budget and shortfall risk constraints. The shortfall risk is measured in terms of expected loss. Stock returns satisfy a stochastic differential equation. Under general conditions on the corresponding drift process we provide the optimal trading strategy using Malliavin calculus. We give extensive numerical results in case of a model for the drift as a Markov process with finitely many states. To deal with the problem of time-discretization when applying the results to market data, we propose a method to detect and correct possible tracking errors.

Key Words: Portfolio optimization, utility maximization, expected shortfall, risk constraint, tracking error.

1 Introduction

In this paper we consider a financial market model which consists of a bank account with stochastic interest rates and n stocks whose returns $(R_t)_{t\in[0,T]}$ satisfy

$$R_t = \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s,$$

where $W = (W_t)_{t\in[0,T]}$ is a n-dimensional standard Brownian motion. We allow for a stochastic drift process $\mu = (\mu_t)_{t\in[0,T]}$ which may be independent of the driving Brownian motion. The investor's objective is to maximize the expected utility of terminal wealth for a finite time horizon T. We add two special features to this standard problem in mathematical finance: We assume that only the stock returns can be observed (partial information) and impose a constraint for the shortfall risk motivated by the extreme (risky) positions which typically arise in these models.

In detail, the stochastic drift process μ leads to a model with *partial information* since an investor can only observe the stock prices or returns, but neither the underlying Brownian motion nor the drift process directly. Then investment decisions have to be based on

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the knowledge of the stock prices only. In this context there are two popular models for the drift. The first uses some linear Gaussian dynamics (GD), see e.g. [21, 24], while the second models the drift as a continuous time Markov chain with finitely many states. The latter model was proposed in [6] and we refer to it as hidden Markov model (HMM). It satisfies a lot of stylized facts observed in stock markets, cf. [30], and efficient algorithms for estimating the parameters of this model are available, cf. [15] and the references therein. It turns out that the optimal investment strategies depend on the filter – the conditional expectation of the drift given the observation – and its dynamics. The filter for the first model is called Kalman filter and for the second model HMM filter. The filters can be described as solutions of one stochastic differential equation (SDE) and an ordinary differential equation for the second moment in the Kalman case and of one SDE in the HMM case. For more models that allow for finite dimensional filters we refer to [33] and the references therein.

To find the optimal terminal wealth in such a model with partial information is quite straightforward after a change of measure to the risk neutral probability measure which coincides with the reference measure for filtering. Then the market model can be transformed into a complete one with full information and martingale representation arguments guarantee existence and uniqueness of an optimal trading strategy. For GD explicit solutions for the problem of optimizing the terminal wealth are provided e.g. in [3, 21, 26, 27]. Utility maximization under a HMM model is investigated e.g. in [16, 23, 29, 32]. These approaches are generalized in [2]. We allow for non-constant volatility, cf. [16, 17]. So we start with a quite general model for μ and σ and only in the second part we shall concentrate on the HMM with constant σ to present explicit strategies and numerical examples.

When optimizing under partial information the optimal strategies are very risky since they lead to extreme long and short positions. This can result in a poor performance if we trade only daily, even bankruptcy might occur, cf. [27, 32]. On the other hand it is very convenient to use a continuous time model which allows to derive optimal policies explicitly. So we have to look for continuous time strategies which are more robust with respect to this discretization. This can be done by using so-called convex constraints on the strategy see e.g. [31], dynamic risk constraints see [28] or static constraints which we have used in [8, 9, 10] and which play the key role in the present paper. For a broader picture and more references see [7]. Static risk constraints are an appropriate tool if we want to control the risk profile of the terminal wealth.

We investigate utility maximization under an additional *constraint for the shortfall risk*. Imposing such a constraint is also motivated by the observation that without risk constraint the distribution of the optimal terminal wealth often is quite skew: There is a high probability for the terminal wealth falling short a prescribed benchmark. This is an undesirable and unacceptable property e.g. from the viewpoint of a pension fund manager. On the other hand, imposing a strict restriction to portfolio values above a benchmark leads to a considerable decrease in the portfolio's expected utility. Thus it seems reasonable to allow for shortfall and restrict only some shortfall risk measure.

A very popular measure to quantify the shortfall risk is value at risk (VaR) which takes the probability of a shortfall into account. For utility maximization under bounded VaR it is known (see e.g. [1]), that the losses may be larger than for the unconstrained optimization problem, since the magnitude of the losses below the benchmark plays no role for the risk measure. Therefore we use the so-called expected loss criterion resulting from averaging the magnitude of the losses. More precisely, for terminal portfolio value X_T and shortfall level $q > 0$ we consider the risk constraint

$$E_Q[\gamma(X_T^\pi - q)^-] \leq \varepsilon,$$

where γ is some discounting factor and Q a probability measure equivalent to the reference measure P. If we drop the risk constraint ($\varepsilon = \infty$) , the utility maximization problem is the well-known Merton problem while the limiting case $\varepsilon = 0$ corresponds to the portfolio insurer problem, cf. [12, 13, 22]. If $\varepsilon > 0$ and the risk constraint is binding, there are two typical choices for Q. First $Q = \widetilde{P}$, where $\widetilde{P}$ denotes the unique equivalent martingale or risk neutral measure: If we choose γ as the discount factor corresponding to the money market, the expected loss corresponds to the price of a derivative to hedge against the shortfall. So we may call this choice present expected loss (PEL), since by paying ε now, one can hedge against the risk to fall short of q. This criterion is called limited expected loss in [1] and analyzed thoroughly showing that the distribution of the resulting optimal terminal wealth has more desirable properties than a VaR based risk measure. Another choice would be $Q = P$, termed as future expected loss (FEL), since it corresponds to the (discounted) amount we have to pay at T to cover the loss. Utility maximization under bounded FEL has been studied for a Black Scholes model with constant parameters in [9]. In addition any other measure Q equivalent to P might be used.

In [10] we study in detail the existence and uniqueness of the optimal terminal wealth for the constrained utility maximization problem in a general complete market model and give conditions under which the optimal Lagrange multipliers exist. Here we show that a model with partial information satisfies the conditions in [10] and then compute the optimal trading strategies quite explicitly. A special feature of this paper is the extensive numerical part which also illustrates the findings of [10]. In addition we discuss an updating procedure for the trading strategy for the case that the current wealth deviates from the optimal wealth which happens quite often due to time discretization when applying the strategies. This updating is necessary, since under risk constraints the optimal fraction invested in the stock also depends on the wealth, even for logarithmic and power utility.

The paper is organized as follows. Section 2 gives a detailed description of the financial market model and of the basic filtering results which are used to transform the model with partial information into a complete market model. Section 3 formulates the utility maximization problem under risk constraints and classifies several cases arising for certain values of the bound ε and the measure Q. In Section 4 we give for a quite general drift process explicit representations of the optimal terminal wealth and the optimal trading strategies in terms of the filter for the martingale density and its Malliavin derivative. For the HMM model in Section 5 we describe the dynamics of this filter and its Malliavin derivative as solutions of SDEs and derive the optimal strategies in the PEL and FEL case. Section 6 is devoted to comprehensive numerical experiments illustrating the findings of the paper. We present efficient frontiers for visualizing the dependence of the optimal expected utility on the bound ε for the shortfall risk and study the influence of several parameters such as the benchmark q, the risk aversion parameter of the utility function, and the type of the risk measure (PEL, FEL). Moreover, we investigate the PEL-optimal terminal wealth as a function of the conditional martingale density and study its probability density function. Finally we are concerned with the problem of time-discretization which is of great practical importance when the optimal strategies are applied to market data. We propose a method for computing optimal strategies, where possible errors due to the time-discretization are detected and corrected.

NOTATION The symbol $^\top$ will denote transposition. For a vector v, $\mathrm{Diag}(v)$ is the diagonal matrix with diagonal v. For a matrix M, $\mathrm{diag}(M)$ is the vector consisting of the diagonal of the matrix M. We use the symbol $\mathbf{1}_n$ for the n-dimensional vector whose entries all equal 1. Moreover, $\mathcal{F}^X = (\mathcal{F}_t^X)_{t\in[0,T]}$ stands for the filtration of augmented σ-algebras generated by the $\mathcal{F}$-adapted process $X = (X_t)_{t\in[0,T]}$. We write x^- for the negative part of x, i.e. $x^- = \max\{-x, 0\}$.

2 A market model with partial information

In this section we introduce for terminal trading time $T > 0$ a market consisting of one money market with interest rates $r_t \geq 0$, $t \in [0,T]$, and n stocks. The return process $R = (R_t)_{t\in[0,T]}$ of the stock prices is given by

$$R_t = \int_0^t \mu_s\,ds + \int_0^t \sigma_s\,dW_s,$$

where $W = (W_t)_{t\in[0,T]}$ is a n-dimensional standard Brownian motion on a suitable filtered probability space $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $\mathcal{F}$ satisfies the usual conditions. The n-dimensional drift process $\mu = (\mu_t)_{t\in[0,T]}$, the interest rates, and the $n \times n$-dimensional volatility process $\sigma = (\sigma_t)_{t\in[0,T]}$ are progressively measurable. For the latter σ_t is non-singular for all $t \in [0,T]$. The processes satisfy

$$\int_0^T \left(|r_t| + \|\mu_t\| + \|\sigma_t\|^2\right) dt < \infty, \tag{2.1}$$

where $\|\sigma_t\| = (\sum_{i,j=1}^n (\sigma_t^{i,j})^2)^{1/2}$, as well as

$$\int_0^T \|\sigma_t^{-1}(\mu_t - r_t\mathbf{1}_n)\|^2\,dt < \infty, \tag{2.2}$$

all inequalities understood to hold almost surely. The stock prices $S = (S_t)_{t\in[0,T]}$ are defined by $dS_t^i = S_t^i dR_t^i$ with constant $S_0^i > 0$, and thus evolve according to

$$dS_t^i = S_t^i \mu_t^i\,dt + S_t^i \sum_{j=1}^n \sigma_t^{i,j}\,dW_t^j, \quad i = 1,\dots,n.$$

We can identify investment in the money market with investment in a riskless asset S^0,

$$S_t^0 = \exp\left\{\int_0^t r_s\,ds\right\}, \quad t \in [0,T].$$

By $\beta_t = 1/S_t^0$, $t \in [0,T]$, we shall denote the corresponding discount factor. Further we introduce the *excess return process* $\widetilde{R} = (\widetilde{R}_t)_{t\in[0,T]}$ by $d\widetilde{R}_t = dR_t - r_t\mathbf{1}_n dt$, i.e.

$$d\widetilde{R}_t = (\mu_t - r_t\mathbf{1}_n)dt + \sigma_t dW_t.$$

Under partial information $\mathcal{F}^{r,S}$ models the information available to the investor, so trading strategies have to be adapted w.r.t. $\mathcal{F}^{r,S}$. The case of partial information accounts for a realistic approach since information about stock prices and interest rates is publicly available while μ_t and W_t are not directly observable. Since $(\sigma_t\sigma_t^\top)_{ij}\, t = [S^i, S^j]_t/(S_t^i S_t^j) = [R^i, R^j]_t$ and σ_t can be obtained by a fixed algebraic scheme, $(\sigma_t)_{t\in[0,T]}$ is adapted to $\mathcal{F}^S$ as well as to $\mathcal{F}^R$. Here $[\,\cdot\,]$ denotes the quadratic variation. This yields $\mathcal{F}^S = \mathcal{F}^R$. For the interest rates we have to make the following strong assumption. It will allow us to show in Proposition 2.4 that the observation filtration $\mathcal{F}^{r,S}$ is a Brownian filtration.

Assumption 2.1 *r is adapted to $\mathcal{F}^S$.*

Hence we have $\mathcal{F}^{r,S} = \mathcal{F}^S$ and as a direct consequence we obtain by the definitions of R and $\widetilde{R}$

$$\mathcal{F}^{r,S} = \mathcal{F}^S = \mathcal{F}^R \supseteq \mathcal{F}^{\widetilde{R}}. \tag{2.3}$$

The market price of risk is defined as $\vartheta_t = \sigma_t^{-1}(\mu_t - r_t\mathbf{1}_n)$, $t \in [0,T]$. Due to (2.2) we can introduce the density process $Z = (Z_t)_{t\in[0,T]}$,

$$Z_t = \exp\left\{-\int_0^t \vartheta_s^\top dW_s - \frac{1}{2}\int_0^t \|\vartheta_s\|^2 ds\right\}.$$

Assumption 2.2 $E[Z_T] = 1$.

Then Z is a martingale and we can introduce a new probability measure $\widetilde{P}$ – the *risk neutral measure* – by

$$\widetilde{P}(A) = E[Z_T 1_A], \quad A \in \mathcal{F}_T\,.$$

So $\widetilde{P}$ is equivalent to P. Girsanov's theorem guarantees that $\widetilde{W} = (\widetilde{W}_t)_{t\in[0,T]}$ with

$$\widetilde{W}_t = W_t + \int_0^t \vartheta_s\, ds$$

is a Brownian motion under the risk neutral measure $\widetilde{P}$. Thus also the excess return process

$$d\widetilde{R}_t = (\mu_t - r_t\mathbf{1}_n)dt + \sigma_t dW_t = \sigma_t d\widetilde{W}_t \tag{2.4}$$

is a martingale under $\widetilde{P}$; and the price process has under $\widetilde{P}$ dynamics

$$dS_t = \mathrm{Diag}(S_t)(r_t\mathbf{1}_n dt + \sigma_t d\widetilde{W}_t). \tag{2.5}$$

We will model the dynamics of the volatility σ and the interest rate r under $\widetilde{P}$ using an $\mathcal{F}^S$-adapted process ξ which leads to a Markovian structure.

Assumption 2.3 *Assume that the m-dimensional* factor process $\xi = (\xi_t)_{t\in[0,T]}$ *satisfies*

$$d\xi_t = \nu(\xi_t)dt + \tau(\xi_t)d\widetilde{W}_t, \tag{2.6}$$

where ν and τ are $\mathbb{R}^m$-valued, and that $\sigma_t = \tilde\sigma(\xi_t)$, $r_t = \tilde r(\xi_t)$. Further we demand that ν, τ, $\tilde r$, and $\tilde\sigma$ are measurable and satisfy the usual Lipschitz and linear growth conditions.

Assumption 2.3 implies the following result which shows that the observation filtration is a Brownian filtration and hence martingale representation results can be used to find optimal trading strategies in Section 4.

Proposition 2.4 *It holds* $\mathcal{F}^S = \mathcal{F}^R = \mathcal{F}^{\widetilde{W}}$.

Proof: By (2.3) and (2.4) we have $\mathcal{F}^{\widetilde{W}} \subseteq \mathcal{F}^{S,\widetilde{R}} \subseteq \mathcal{F}^S$. On the other hand, the conditions of Assumption 2.3 guarantee that a strong solution (S,ξ) of the system of SDEs (2.5) and (2.6) exists, in particular it follows that S is $\mathcal{F}^{\widetilde{W}}$-adapted, hence $\mathcal{F}^S \subseteq \mathcal{F}^{\widetilde{W}}$. □

We shall denote the filter for μ_t, the conditional expectation given information $\mathcal{F}^S_t$, by

$$\hat\mu_t = E[\mu_t \,|\, \mathcal{F}^S_t], \quad t\in[0,T].$$

Then we can define the innovation process

$$V_t = \widetilde{W}_t - \int_0^t \sigma_s^{-1}(\hat\mu_s - r_s\mathbf{1}_n)ds, \quad t\in[0,T],$$

which is a $\mathcal{F}^S$-Brownian motion under P. So under P we now have dynamics

$$dR_t = \hat\mu_t dt + \sigma_t dV_t. \tag{2.7}$$

Note that in general we have the strict inclusion $\mathcal{F}^V_t \subset \mathcal{F}^{\widetilde{W}}_t$ and only for special dynamics of μ the equality. But this imposes no problems for our analysis.

Remark 2.5 All processes in (2.7) are $\mathcal{F}^S$-adapted. Since under partial information trading strategies have to be adapted to $\mathcal{F}^S$, we can think of starting with this representation on a filtered probability space $(\Omega, \mathcal{F}^S_T, \mathcal{F}^S, P|_{\mathcal{F}^S_T})$ and thus work in a model with full information. In this model we could introduce a change of measure by $d\overline{P} = \zeta_T dP|_{\mathcal{F}^S_T}$, where

$$d\zeta_t = -\left(\sigma_t^{-1}(\hat\mu_t - r_t\mathbf{1}_n)\right)^\top dV_t, \quad \zeta_0 = 1.$$

Assuming that $\zeta = (\zeta_t)_{t\in[0,T]}$ is a martingale, we get the corresponding Brownian motion

$$\overline{W}_t = V_t + \int_0^t \sigma_s^{-1}(\hat\mu_s - r_s\mathbf{1}_n)ds.$$

One can show that $\overline{P} = \widetilde{P}|_{\mathcal{F}^S_T}$, $\overline{W}_t = \widetilde{W}_t$ and

$$\zeta_t = E[Z_t \,|\, \mathcal{F}^S_t], \quad t\in[0,T]. \tag{2.8}$$

Starting with (2.7) we thus end up with the same risk neutral measure on $\mathcal{F}^S_T$.

From now on we will only write $\widetilde{P}$ and $\widetilde{W}$ for the risk neutral measure and the corresponding Brownian motion and use the density process $\zeta = (\zeta_t)_{t\in[0,T]}$ given by (2.8). Only for computing the filters we have to deal with the underlying model.

We describe the self-financing trading of an investor by his initial capital $x_0 > 0$ and his n-dimensional $\mathcal{F}^S$-adapted trading strategy $\pi = (\pi_t)_{t\in[0,T]}$ where π_t^i is the amount of money invested in stock i at time t. The corresponding wealth process $X^\pi = (X_t^\pi)_{t\in[0,T]}$ satisfies

$$dX_t^\pi = \left(X_t^\pi - \pi_t^\top \mathbf{1}_n\right) r_t\, dt + \pi_t^\top dR_t = \left(X_t^\pi r_t + \pi^\top(\hat{\mu}_t - r_t \mathbf{1}_n)\right) dt + \pi_t^\top \sigma_t dV_t. \quad (2.9)$$

So for the discounted wealth process we find

$$d(\beta_t X_t^\pi) = \beta_t \pi_t^\top(\hat{\mu}_t - r_t\mathbf{1}_n)dt + \beta_t\pi_t^\top\sigma_t dV_t = \beta_t\pi_t^\top\sigma_t d\widetilde{W}_t. \quad (2.10)$$

Since the interest rates are positive, β_t is uniformly bounded. Hence it is enough for (2.9) and (2.10) to be well defined that we require

$$\int_0^T \left(\|\pi_t^\top(\hat{\mu}_t - r_t\mathbf{1}_n)\| + \|\pi_t^\top\sigma_t\|^2\right) dt < \infty. \quad (2.11)$$

A trading strategy satisfying this condition and $X_T^\pi \geq 0$ will be called admissible. By $\mathcal{A}(x_0)$ we denote the corresponding class of admissible trading strategies for initial capital $x_0 > 0$. Note that X_T^π as well as $\beta_T X_T^\pi$ are $\mathcal{F}_T^S$-measurable.

Remark 2.6 If all assumptions can be verified and the filter $\hat{\mu}$ can be computed, we do not have to rely on the underlying model. The transformed model (2.7) is a complete market model with respect to $\mathcal{F}^S$. So we can transfer corresponding results to our case as we will do in the next two sections.

3 Utility maximization under bounded shortfall risk

Following Remark 2.6 we will cite in this section results from [10] and provide some more interpretation. Substituting $\hat{\mu}$ for μ, $\mathcal{F}^S$ for $\mathcal{F}$, and ζ for Z in [10], the results carry over to our situation, since (2.1), (2.2) and Assumption 2.2 imply (4), (5) and Assumption 1 in [10]. First we have to introduce the optimization problem.

A utility function $U : [0,\infty) \to \mathbb{R}\cup\{-\infty\}$ is strictly increasing, strictly concave, twice continuously differentiable and satisfies the Inada conditions

$$\lim_{x\to\infty} U'(x) = 0 \quad \text{and} \quad \lim_{x\to 0} U'(x) = \infty.$$

The inverse function of U' is denoted by I. The function I is defined on $(0,\infty)$, continuously differentiable and strictly decreasing with limits

$$\lim_{x\to\infty} I(x) = 0 \quad \text{and} \quad \lim_{x\to 0} I(x) = \infty.$$

Given initial capital $x_0 > 0$ any terminal wealth X_T^π satisfies the so-called budget constraint $\widetilde{E}[\beta_T X_T^\pi] = E[\beta_T\zeta_T X_T^\pi] \leq x_0$, because $(\beta_t X_t^\pi)_{t\geq 0}$ is a $\widetilde{P}$-supermartingale due to (2.10) and the admissibility requirement $X_T^\pi \ge 0$. Here, $\widetilde{E}$ denotes the expectation w.r.t. the risk neutral measure $\widetilde{P}$.

As motivated in the introduction for shortfall level $q > 0$ we measure the risk by averaging the loss $(X_T^\pi - q)^-$ w.r.t. some probability measure Q which is equivalent to P. By Z^Q we denote the Radon-Nikodym derivative of Q w.r.t. P. Further we use a strictly positive, $\mathcal{F}_T^S$-measurable factor γ for discounting the loss and call the non-negative number $E_Q[\gamma(X_T^\pi - q)^-] = E[\gamma\, Z^Q(X_T^\pi - q)^-]$ expected loss.

For a given bound $\varepsilon \ge 0$ the dynamic optimization problem under risk constraints is

$$\begin{aligned} &\text{maximize} && E[U(X_T^\pi)] && \text{for } \pi \in \mathcal{A}(x_0) \\ &\text{subject to} && \widetilde{E}[\beta_T X_T^\pi] \le x_0 && \text{(budget constraint)}, \\ & && E_Q[\gamma(X_T^\pi - q)^-] \le \varepsilon && \text{(risk constraint).} \end{aligned} \tag{3.1}$$

The dynamic portfolio optimization problem can be splitted into two problems - the static and the representation problem. While the static problem is concerned with the form of the optimal terminal wealth the representation problem consists in the computation of the optimal trading strategy.

We shall first consider the static problem and use for convenience a shorter notation: We write simply X_T for the $\mathcal{F}_T^S$-measurable terminal wealth X_T^π, $Z_1 = \beta_T \zeta_T$ and $Z_2 = \gamma \zeta_T^Q$, where $\zeta_t^Q = E[Z_t^Q \,|\, \mathcal{F}_t^S]$ is the filter for Z_t^Q. Then the static problem reads as

$$\begin{aligned} &\text{maximize} && E[U(X_T)] && \text{for all } \mathcal{F}_T^S\text{-measurable } X_T \ge 0 \\ &\text{subject to} && E[Z_1 X_T] \le x_0 && \text{(budget constraint)}, \\ & && E[Z_2(X_T - q)^-] \le \varepsilon && \text{(risk constraint).} \end{aligned} \tag{3.2}$$

We impose the following technical conditions.

Assumption 3.1 *For all $y > 0$ it holds*

(A1) $E[Z_1 I(yZ_1)] < \infty$,

(A2) $E[Z_1^2 |I'(yZ_1)|] < \infty$,

(A3) $Z_1 = Z_2$ *or* $P(a + \lambda Z_1 = Z_2) = 0$ *for all* $\lambda > 0$, $a \in \mathbb{R}$; *further* $P(Z_1 > c) > 0$ *for all* $c > 0$.

They are not very restrictive, in particular (A3) comprises the special cases of (3.2) which are considered in the literature and which are introduced below.

Without risk constraint ($\varepsilon = \infty$), we face the *Merton problem*

$$\max_{X_T \ge 0} E[U(X_T)] \qquad \text{subject to} \quad E[Z_1 X_T] \le x_0. \tag{3.3}$$

The solution of the Merton problem is given by $X_T^M = I(y^M Z_1)$, where $y^M > 0$ solves $E[Z_1 I(yZ_1)] = x_0$ uniquely. Let us denote the risk of the Merton portfolio by

$$\overline{\varepsilon} := E[Z_2(X_T^M - q)^-].$$

If we allow an expected loss $\varepsilon \geq \overline{\varepsilon}$, then X_T^M would still be optimal with risk $\overline{\varepsilon}$ and thus the risk constraint not binding. In this sense we can use $\overline{\varepsilon}$ as an upper bound for ε. On the other hand there is no admissible solution of the problem, if we choose ε too small, because the risk constraint cannot be satisfied. So we need to find the smallest value $\underline{\varepsilon}$ of the shortfall risk measure for given initial capital x_0, shortfall level q and measures Q and $\widetilde{P}$.

If $qE[Z_1] \leq x_0$ the bound $\varepsilon \geq 0$ can be chosen arbitrarily small since $X_T \equiv q$ satisfies the budget constraint $E[Z_1 X_T] \leq x_0$ and yields a risk measure of zero. Hence, for $qE[Z_1] \leq x_0$ we can set the minimal shortfall risk to $\underline{\varepsilon} = 0$.

Optimization problem (3.2) for $\varepsilon = 0$ is known as the *portfolio insurer problem*

$$\max_{X_T \geq 0} E[U(X_T)] \qquad \text{subject to} \quad E[Z_1 X_T] \leq x_0 \quad \text{and} \quad X_T \geq q. \tag{3.4}$$

The portfolio insurer problem admits for $qE[Z_1] > x_0$ no admissible solution. If $qE[Z_1] = x_0$, the only admissible choice is $X_T = q$. For $qE[Z_1] < x_0$ its optimal solution is known to be $X_T^{PI} = f_{PI}(Z_1) = f_{PI}(Z_1; y^{PI})$, where

$$f_{PI}(x;y) = \begin{cases} I(yx) & \text{for} \quad yx \in (0, U'(q)], \\ q & \text{for} \quad yx \in (U'(q), \infty). \end{cases} \tag{3.5}$$

E.g. by [10, Theorem 4] we know that $y^{PI} > 0$ solves $E[Z_1 f_{PI}(Z_1; y)] = x_0$ uniquely with $y^{PI} \uparrow \infty$ for $q \uparrow x_0/E[Z_1]$. For $qE[Z_1] = x_0$ the solution of the portfolio insurer problem is $X_T^{PI} = q$. Thus we can set $y^{PI} = \infty$ for $qE[Z_1] = x_0$ in order to incorporate this case. In case of $qE[Z_1] > x_0$, where the portfolio insurer problem has no admissible solution, there is a strictly positive minimal shortfall risk $\underline{\varepsilon}$ which can be found from the solution of the following risk minimization problem

$$\underline{\varepsilon} := \inf_{X_T \geq 0} E[Z_2 (X_T - q)^-] \ \text{subject to} \quad E[Z_1 X_T] \leq x_0.$$

In [10, Theorems 6,7] we derived

$$\underline{\varepsilon} = \begin{cases} qE[Z_1] - x_0, & \text{if } Z_1 = Z_2, \\ qE[Z_2 \mathbf{1}_{\{\lambda^* Z_1 > Z_2\}}] & \text{if } P(\lambda Z_1 = Z_2) = 0 \text{ for all } \lambda > 0, \end{cases}$$

where λ^* is the unique solution of $qE[Z_1 \mathbf{1}_{\{\lambda^* Z_1 \leq Z_2\}}] = x_0$. The corresponding wealth is given in the first case by any $\underline{X}_T$ with values in $[0, q]$ and $E[Z_1 \underline{X}_T] = x_0$ and in the second case uniquely by $\underline{X}_T = q \mathbf{1}_{\{\lambda^* Z_1 \leq Z_2\}}$.

For $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$ where the risk constraint can be satisfied and is binding, there are two typical choices for Q. First we use $\gamma = \beta_T$ and $Q = \widetilde{P}$ (i.e. $Z_1 = Z_2$), then (3.2) reads as

$$\max_{X_T \geq 0} E[U(X_T)] \qquad \text{subject to} \quad \widetilde{E}[\beta_T X_T] \leq x_0 \quad \text{and} \ \widetilde{E}[\beta_T (X_T - q)^-] \leq \varepsilon. \tag{3.6}$$

The risk measure $\widetilde{E}[\beta_T (X_T - q)^-]$ is referred to as *present expected loss* (PEL). It corresponds to the price of a derivative to hedge against the shortfall. By paying ε now, one can hedge against the risk to fall short of q.

Remark 3.2 If an investor really splits his initial capital in $x_0 - \varepsilon$ which is invested according to PEL to get terminal wealth $X_T^{x_0-\varepsilon}$ and buys the corresponding option with price ε and payout $C = (X_T^{x_0-\varepsilon} - q)^-$, then the combined payout corresponds to a portfolio insurer and maximizing expected utility of this total payout (including C) yields the same terminal wealth as (3.5). But that is only possible for $qE[Z_1] \leq x_0$, otherwise the budget constraint cannot be fulfilled in the portfolio insurer problem. So PEL becomes interesting when $qE[Z_1] > x_0$ which corresponds to a guaranteed level q which is higher (after discounting) than the initial capital. So the investor won't buy the option and use the price ε only as a measure of the risk he is willing to take to reach at least level q.

It is also reasonable to average the possible losses using the real world measure P. Therefore, another choice would be $Q = P$, termed as *future expected loss* (FEL),

$$\max_{X_T \geq 0} E[U(X_T)] \qquad \text{subject to} \quad \widetilde{E}[\beta_T X_T] \leq x_0 \quad \text{and } E[\gamma(X_T - q)^-] \leq \varepsilon. \tag{3.7}$$

This risk measure corresponds to the (discounted) amount we have to pay at T to cover the loss. In addition any other measure Q equivalent to P might be used in (3.1).

4 Optimal trading

In this section we present the form of the optimal terminal wealth which is the solution of the static problem (3.2) and denoted by X_T^*. Moreover we derive the optimal trading strategy $\pi^* = (\pi_t^*)_{t\in[0,T]}$ generating $X_T^{\pi^*} = X_T^*$, i.e. we solve the dynamic portfolio optimization problem (3.1).

The following theorem gives the form of the optimal terminal wealth X_T^* depending on the bound of shortfall risk ε. This is Theorem 9 of [10]. Part (iii) corresponds to [1, Proposition 4] where it was shown under the assumption that an optimal solution exists. The different cases are similar to those derived in [14, Theorem 3.3] for risk measures based on strictly convex loss functions. We call the risk constraint binding, if it holds with equality for the optimal terminal wealth.

Theorem 4.1 *The optimal terminal wealth X_T^* of problem (3.2) satisfies:*

(i) If $\varepsilon \geq \overline{\varepsilon}$, then $X_T^* = X_T^M$ (Merton optimal, see (3.3)) and if $E[|U(X_T^*)|] < \infty$, then X_T^* is the unique optimal solution. For $\varepsilon > \overline{\varepsilon}$ the risk constraint is not binding.

(ii) If $0 \leq \varepsilon < \underline{\varepsilon}$, then there is no solution.

(iii) If $\underline{\varepsilon} < \varepsilon < \overline{\varepsilon}$, then the optimal terminal wealth is given by $X_T^* = f(y_1^* Z_1, y_2^* Z_2)$ where

$$f(x_1, x_2) = \begin{cases} I(x_1) & \text{for} \quad x_1 \leq U'(q), \\ q & \text{for} \quad U'(q) < x_1 \leq U'(q) + x_2, \\ I(x_1 - x_2) & \text{for} \quad x_1 > U'(q) + x_2. \end{cases}$$

Here, $y_1^*, y_2^* > 0$ solve the following system of equations

$$E[Z_1 f(y_1 Z_1, y_2 Z_2)] = x_0$$
$$E[Z_2 (f(y_1 Z_1, y_2 Z_2) - q)^-] = \varepsilon.$$

There exist unique solutions of this system of equations and if $E[|U(X_T^*)|] < \infty$*, then* X_T^* *is the unique optimal solution. The risk constraint is binding.*

(iv) Let $\varepsilon = \underline{\varepsilon}$*. For* $qE[Z_1] \le x_0$ *it holds* $\underline{\varepsilon} = 0$ *and*

$$X_T^* = \begin{cases} X_T^{PI} & \text{for} \quad qE[Z_1] < x_0, \\ q & \text{for} \quad qE[Z_1] = x_0 \end{cases}$$

where X_T^{PI} *is the portfolio insurer optimal terminal wealth, see (3.4), (3.5).*

For $qE[Z_1] > x_0$ *it holds* $\underline{\varepsilon} > 0$ *and*

(a) if $Z_1 = Z_2$ *(in particular for* $Q = \widetilde{P}$, $\gamma = \beta_T$*), then* $X_T^* = f_0(Z_1) = f_0(Z_1; y_0^*)$*, where*

$$f_0(x; y) = \begin{cases} q & \text{for} \quad yx \le U'(q), \\ I(yx) & \text{for} \quad yx > U'(q), \end{cases}$$

and $y_0^* > 0$ *is the unique solution of* $E[Z_1 f_0(Z_1; y_0)] = x_0$*. If* $E[|U(X_T^*)|] < \infty$*, then* X_T^* *is unique;*

(b) if $P(\lambda Z_1 = Z_2) = 0$ *for all* $\lambda > 0$ *, then*

$$X_T^* = \underline{X}_T = q\mathbf{1}_{\{\lambda^* Z_1 \le Z_2\}}$$

where λ^* *is the unique solution of* $qE[Z_1 \mathbf{1}_{\{\lambda Z_1 \le Z_2\}}] = x_0$*.*

We shall derive optimal trading strategies for the most interesting case, i.e. we assume $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$. Then by Theorem 4.1 (iii) we have $X_T^* = f(y_1^* Z_1, y_2^* Z_2)$, where $Z_1 = \beta_T \zeta_T$, $Z_2 = \gamma\, \zeta_T^Q$ and y_1^*, y_2^* are the optimal unique parameters given by

$$E[Z_1 X_T^*] = x_0, \quad E[Z_2 (X_T^* - q)^-] = \varepsilon.$$

For random variables $Y \in \mathbb{D}_{1,p} \subset L^p(\widetilde{P}, \mathcal{F}_T^{\widetilde{W}})$, $p \in [1, \infty)$, we introduce the Malliavin derivatives $\mathrm{D}Y = (\mathrm{D}_t Y)_{t \in [0,T]}$ w.r.t. $\widetilde{W}$ as introduced in [25]. For details and suitable chain rules we also refer to [32]. For $Y \in \mathbb{D}_{1,1}$ Clark's formula holds, cf. [19]:

$$Y = \widetilde{E}[Y] + \int_0^T \widetilde{E}[\mathrm{D}_t Y \,|\, \mathcal{F}_t^{\widetilde{W}}]^\top d\widetilde{W}_t.$$

Note that we use the convention that for m-dimensional Y, the matrix $\mathrm{D}_t Y$ is $n \times m$-dimensional with $(\mathrm{D}_t Y)_{i,j} = \mathrm{D}_t^i Y^j$, where D^i denotes the operator w.r.t. $\widetilde{W}^i$.

Strategies can be computed by comparing martingale representation

$$\beta_T X_T^* = x_0 + \int_0^T \beta_t (\pi_t^*)^\top \sigma_t d\widetilde{W}_t$$

and Clark's formula for $Y = \beta_T X_T^*$. This is the idea of Theorem 4.2 below which provides conditions that this approach works. We cannot apply standard chain rules directly since f is not differentiable on the set $\{(z_1, z_2) \,:\, z_1 = U'(q) \text{ or } z_1 - z_2 = U'(q)\}$. But similar as in Section 5.1 of [22] a chain rule can be derived if we use the following piecewise derivatives.

Theorem 4.2 is based on [10, Theorem 18] using other conditions than e.g. [22, Proposition 5.2]. As substitute for the derivative of $f(z_1, z_2)$ w.r.t. the first variable we use

$$f_1(z_1,z_2) = \begin{cases} I'(z_1) & \text{for} \quad z_1 \le U'(q), \\ 0 & \text{for} \quad U'(q) < z_1 \le U'(q) + z_2, \\ I'(z_1 - z_2) & \text{for} \quad z_1 > U'(q) + z_2, \end{cases}$$

and w.r.t. the second variable

$$f_2(z_1,z_2) = \begin{cases} 0 & \text{for} \quad z_1 \le U'(q), \\ 0 & \text{for} \quad U'(q) < z_1 \le U'(q) + z_2, \\ -I'(z_1 - z_2) & \text{for} \quad z_1 > U'(q) + z_2. \end{cases}$$

Moreover, we set

$$F_1 = f_1(y_1 Z_1, y_2 Z_2), \quad F_2 = f_2(y_1 Z_1, y_2 Z_2).$$

Theorem 4.2 *Suppose that the conditions on the coefficients r, μ, σ in Section 2 are satisfied and that*

(a) β_T^{-1} *is bounded,*

(b) $\beta_T, Z_1, Z_2 \in \mathbb{D}_{1,\frac{p}{p-1}}$ *and* $I'(y_1 Z_1), I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_{\{y_1 Z_1 - y_2 Z_2 > U'(q)\}} \in L^p(\widetilde{P})$ *for some* $p \in (1,2]$ *and all* $y_1 > 0$, $y_2 \ge 0$,

(c) $\beta_T X_T^* \in L^2(\widetilde{P})$.

Then an optimal trading strategy is given by

$$\pi_t^* = \beta_t^{-1}(\sigma_t^\top)^{-1} \widetilde{E}[(D_t\beta_T)X_T^* + y_1\beta_T(D_t Z_1)F_1 + y_2\beta_T(D_t Z_2)F_2 \,|\, \mathcal{F}_t^S].$$

Proof: Since by Proposition 2.4 $\mathcal{F}^S = \mathcal{F}^{\widetilde{W}}$, this follows directly from [10, Theorem 18]. This theorem is applicable because by the arguments in Remarks 2.5, 2.6 and at the beginning of Section 3 we are in a complete market setting with respect to $\mathcal{F}^S$. □

We shall now provide the optimal terminal wealth for two important cases discussed above. First we look at PEL, corresponding to a bound on the hedging price for the shortfall.

Corollary 4.3 *If* $Q = \widetilde{P}$ *and* $\gamma = \beta_T$ *(i.e.* $Z_2 = Z_1$*) and the conditions of Theorem 4.2 are satisfied we have for the PEL-optimal terminal wealth* $X_T^* = X_T^P = f_P(Z_1)$, *where*

$$f_P(z) = f_P(z; y_1^P, y_2^P) = \begin{cases} I(y_1^P z) & \textit{for } z \le \frac{U'(q)}{y_1^P}, \\ q & \textit{for } \frac{U'(q)}{y_1^P} < z \le \frac{U'(q)}{y_1^P - y_2^P}, \\ I((y_1^P - y_2^P)z) & \textit{for } z > \frac{U'(q)}{y_1^P - y_2^P}, \end{cases}$$

and $y_1^P > 0$, $y_2^P > 0$ *are given as the unique solution of*

$$\widetilde{E}[\beta_T f_P(Z_1)] = x_0, \quad \widetilde{E}[\beta_T (f_P(Z_1) - q)^-] = \varepsilon.$$

The PEL-optimal trading strategy is given by

$$\pi_t^P = \beta_t^{-1}(\sigma_t^\top)^{-1}\widetilde{E}[X_T^P\, D_t\beta_T + \beta_T f_P'(Z_1)\, D_t Z_1 \,|\, \mathcal{F}_t^S]$$
$$\text{where } f_P'(z) = y_1^P I'(y_1^P z)\mathbf{1}_{\{y_1^P z \le U'(q)\}} + (y_1^P - y_2^P) I'((y_1^P - y_2^P)z)\mathbf{1}_{\{(y_1^P - y_2^P)z > U'(q)\}}.$$

Next we look at FEL, corresponding to putting a bound on the future expected shortfall.

Corollary 4.4 *If $Q = P$ and $\gamma = 1$ (i.e. $Z_2 = 1$) and the conditions of Theorem 4.2 are satisfied, then it holds for the FEL-optimal terminal wealth $X_T^* = X_T^F = f_F(Z_1)$, where*

$$f_F(z) = f_F(z; y_1^F, y_2^F) = \begin{cases} I(y_1^F z) & \text{for } y_1^F z \le U'(q), \\ q & \text{for } U'(q) < y_1^F z \le U'(q) + y_2^F, \\ I(y_1^F z - y_2^F) & \text{for } y_1^F z > U'(q) + y_2^F, \end{cases}$$

and $y_1^F > 0$, $y_2^F > 0$ are given as the unique solution of

$$\widetilde{E}[\beta_T f_F(Z_1)] = x_0, \quad E[(f_F(Z_1) - q)^-] = \varepsilon.$$

The FEL-optimal trading strategy is given by

$$\pi_t^F = \beta_t^{-1}(\sigma_t^\top)^{-1}\widetilde{E}[X_T^F\, D_t\beta_T + \beta_T f_F'(Z_1)\, D_t Z_1 \,|\, \mathcal{F}_t^S]$$
$$\text{where } f_F'(z) = y_1^F I'(y_1^F z)\mathbf{1}_{\{y_1^F z \le U'(q)\}} + y_1^F I'(y_1^F z - y_2^F)\mathbf{1}_{\{y_1^F z > U'(q) + y_2^F\}}.$$

For a general discount factor γ we simply had to substitute $y_2^F\gamma$ for y_2^F in Corollary 4.4. Then f_F above had to be written as a function of z and γ.

To compute a strategy explicitly we have to determine the filters and therefore have to specify a model for μ. There are two typical examples for a process μ independent of W, a continuous time Markov chain or some linear Gaussian dynamics. Both models allow for the computation of finite dimensional filters and thus closed form solutions can be derived, cf. the introduction. In the next section we look at the Markov chain example.

5 A hidden Markov model for the drift

We look at a drift process given by $\mu_t = BY_t$, i.e. at returns

$$R_t = \int_0^t B\, Y_s\, ds + \sigma\, W_t\,, \quad t \in [0, T], \tag{5.1}$$

where $(Y_t)_{t\in[0,T]}$ is a stationary, irreducible, continuous time Markov chain independent of the n-dimensional Brownian motion W. Y has state space $\{e_1, \ldots, e_d\}$, the standard unit vectors in $\mathbb{R}^d$. The columns of the state matrix $B \in \mathbb{R}^{n\times d}$ contain the d possible states of μ_t. Further Y is characterized by its rate matrix $G \in \mathbb{R}^{d\times d}$, where $\lambda_k = -G_{kk} = \sum_{l=1, l\neq k}^d G_{kl}$ is the rate of leaving e_k and G_{kl}/λ_k is the probability that the chain jumps to e_l when leaving e_k. We assume a constant volatility matrix σ and a money market with constant interest rate r. For the treatment (without shortfall constraints) of non-constant r and σ we refer to [32] and [16, 17], respectively.

Due to the constant parameters r, σ, B and the boundedness of Y (2.1), (2.2) and Assumptions 2.1, 2.2, 2.3 hold. In particular the density processes Z and ζ are martingales. We have to compute the filter $\hat{Y}_t$ for Y_t,

$$\hat{Y}_t = E[Y_t \,|\, \mathcal{F}_t^S], \quad t \in [0,T].$$

Note that $\hat{\mu}_t = B\hat{Y}_t$. In view of (5.1) we are in the classical situation of HMM filtering with signal Y and observation R, where we want to determine the filter $\hat{Y}_t = E[Y_t \mid \mathcal{F}_t^R] = E[Y_t \mid \mathcal{F}_t^S]$. By Theorem 4 in [5], Bayes' law, and using $\mathbf{1}_d^\top Y_t = 1$ we get

Theorem 5.1 *The filters $\hat{Y}_t$ for Y_t, ζ_t for Z_t and the unnormalized filter for Y_t, $\mathcal{E}_t = \widetilde{E}[Z_T^{-1} Y_t \,|\, \mathcal{F}_t^S]$, satisfy $\hat{Y}_t = \zeta_t \, \mathcal{E}_t$, $\zeta_t^{-1} = \mathbf{1}_d^\top \mathcal{E}_t$, and*

$$\mathcal{E}_t = E[Y_0] + \int_0^t G^\top \mathcal{E}_s \, ds + \int_0^t \mathrm{Diag}(\mathcal{E}_s) B^\top (\sigma\sigma^\top)^{-1} \, dR_s, \quad t \in [0,T]. \tag{5.2}$$

Furthermore, $\zeta_t^{-1} = \widetilde{E}[Z_t^{-1} \mid \mathcal{F}_t^S]$ and $\zeta_t^{-1} = 1 + \int_0^t (B\mathcal{E}_s)^\top (\sigma\sigma^\top)^{-1} \, dR_s, t \in [0,T]$.

We shall look at PEL ($Q = \widetilde{P}$) and FEL with $\gamma = 1$ ($Q = P$). Then we only have to compute the Malliavin derivative of ζ_T. This is done in [32, Proposition 4.2, Lemma 4.4]:

Proposition 5.2 *For all $p > 1$ we have $\zeta_T \in \mathbb{D}_{1,p}$ and $\mathcal{E}_t \in (\mathbb{D}_{1,p})^d$, $t \in [0,T]$, and for $s > t$*

$$\begin{aligned}
D_t\zeta_T &= -\zeta_T^2 \, \sigma^{-1} \left(B\, \mathcal{E}_t + \int_t^T (\sigma D_t \mathcal{E}_u) B^\top (\sigma\sigma^\top)^{-1} dR_u \right), \\
\sigma\, D_t \mathcal{E}_s &= B \, \mathrm{Diag}(\mathcal{E}_t) + \int_t^s (\sigma D_t \mathcal{E}_u) G \, du + \int_t^s (\sigma D_t \mathcal{E}_u) \mathrm{Diag}\left(B^\top (\sigma\sigma^\top)^{-1} dR_u \right).
\end{aligned}$$

The main assertions of the following theorem are that the conditions of Theorem 4.2 are fulfilled or follow easily from the conditions, and that the strategy can be computed based on the state of the unnormalized filter. Example 5.4 will show that all conditions hold for logarithmic and power utility.

Theorem 5.3 *Suppose $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$ and that $I(y\zeta_T)$, $I(y\zeta_T - y_0)\mathbf{1}_{\{y\zeta_T - y_0 > U'(q)\}} \in L^2(\widetilde{P})$ and $I'(y\zeta_T)$, $I'(y\zeta_T - y_0)\mathbf{1}_{\{y\zeta_T - y_0 > U'(q)\}} \in L^p(\widetilde{P})$ for some $p > 1$ and all $y > 0$, $y_0 \geq 0$. Then the PEL- and FEL-optimal terminal wealth and the optimal trading strategy are given as in Corollaries 4.3 and 4.4, respectively. In particular, the optimal strategy reads*

$$\pi_t^{P/F} = \beta_t^{-1} (\sigma^\top)^{-1} \widetilde{E}[\beta_T^2 f'_{P/F}(\beta_T \zeta_T) \, D_t \zeta_T \,|\, \mathcal{E}_t],$$

where $f_P(z)$ and $f_F(z)$ are defined in Corollaries 4.3 and 4.4, respectively, and the derivatives are defined piecewise. Moreover, $D_t\zeta_T$ is given in Proposition 5.2.

Proof: Due to the constant parameters r, σ, B and the boundedness of Y, (2.1), (2.2) and Assumptions 2.1, 2.2, 2.3 and condition (a) of Theorem 4.2 hold. Now $Z_1 = \beta_T \zeta_T$ and $Z_2 = \beta_T \zeta_T$ for PEL and $Z_2 = 1$ for FEL. Thus $Z_1, Z_2 \in \mathbb{D}_{1,\frac{p}{1-p}}$ for any $p > 1$ by Proposition 5.2, and by the conditions on I' in this theorem condition (b) of Theorem 4.2 is satisfied. From the representation in Corollaries 4.3 and 4.4 we get that $0 \leq X_T^{P/F} = f_{P/F}(\beta_T \zeta_T) \leq I(y\, \beta_T \zeta_T) + q$ for suitable $y > 0$. Thus also (c) holds and we can use the representations of the trading strategies in Corollaries 4.3, 4.4 which simplify further due to constant β_T. Since $\mathcal{F}_t^S \supseteq \mathcal{F}_t^{\mathcal{E}}$ and $\zeta_T = (\mathbf{1}_d^\top \mathcal{E}_T)^{-1}$, exploiting the Markov property of $\mathcal{E}$, Theorem 5.1 implies that we can write the conditional expectation w.r.t. $\mathcal{E}_t$. □

Example 5.4 We consider logarithmic utility $U(x) = \log(x)$ and power utility $U(x) = \frac{x^\alpha}{\alpha} - \frac{1}{\alpha}$, $\alpha < 1, \alpha \neq 0$. Suppose $\alpha < 1$ is fixed with $\alpha = 0$ corresponding to logarithmic utility. Then $I(y) = y^{\frac{1}{\alpha-1}}$, in particular $I(x\,y) = I(x)I(y)$ which allows us to verify the integrability conditions of Theorem 5.3 easily, and to derive for PEL and FEL

$$X_T^{P/F} = \frac{X_t^{P/F} G_{t,T}^{P/F}}{\widetilde{E}[\beta_{t,T} G_{t,T}^{P/F} \,|\, \mathcal{E}_t]}, \quad t \in [0,T],$$

where $\beta_{t,T} = \beta_T/\beta_t$ and $G_{t,T}^{P/F} = f_{P/F}(\beta_T\zeta_T)/I(y_1^{P/F}\beta_t\zeta_t)$, e.g.

$$G_{t,T}^P = \begin{cases} I(\beta_{t,T}\zeta_{t,T}), & \beta_T\zeta_T \leq \frac{U'(q)}{y_1^P}, \\ q/I(y_1^P\beta_t\zeta_t), & \frac{U'(q)}{y_1^P} < \beta_T\zeta_T < \frac{U'(q)}{y_1^P - y_2^P}, \\ I((1 - y_2^P/y_1^P)\beta_{t,T}\zeta_{t,T}), & \beta_T\zeta_T \geq \frac{U'(q)}{y_1^P - y_2^P}, \end{cases}$$

where $\zeta_{t,T} = \zeta_T/\zeta_t$. A not always straightforward computation yields

$$\begin{aligned} \frac{\pi_t^{P/F}}{X_t^{P/F}} = \frac{(\sigma\sigma^\top)^{-1}}{1-\alpha} \bigg(& \widetilde{E}[C_{t,T}^{P/F} \,|\, \mathcal{E}_t, X_t^{P/F}]\, B\, \hat{Y}_t \\ & + \widetilde{E}\Big[C_{t,T}^{P/F} \int_t^T (\sigma D_t \mathcal{E}_{t,s}) B^\top (\sigma\sigma^\top)^{-1} dR_s \,\Big|\, \mathcal{E}_t, X_t^{P/F} \Big] \bigg), \end{aligned}$$

where $\mathcal{E}_{t,s} = \mathcal{E}_s\,\zeta_t$ and the correction factors are given by

$$\begin{aligned} C_{t,T}^P &= \frac{\beta_{t,T}\zeta_{t,T}G_{t,T}}{\widetilde{E}[\beta_{t,T}G_{t,T} \,|\, \mathcal{E}_t]} - \beta_{t,T}\zeta_{t,T}\,\frac{q}{X_t^P}\mathbf{1}_{\left\{\frac{U'(q)}{y_1^P} < \beta_T\zeta_T \leq \frac{U'(q)}{y_1^P - y_2^P}\right\}} \\ C_{t,T}^F &= \frac{\beta_{t,T}\zeta_{t,T}G_{t,T}}{\widetilde{E}[\beta_{t,T}G_{t,T} \,|\, \mathcal{E}_t]} - \beta_{t,T}\zeta_{t,T}\,\frac{q}{X_t^F}\mathbf{1}_{\left\{\frac{U'(q)}{y_1^F} < \beta_T\zeta_T \leq \frac{U'(q)+y_2^F}{y_1^F}\right\}} \\ & \quad + (1-\alpha)\beta_{t,T}\zeta_{t,T}(y_1^F\beta_T\zeta_T - y_2^F)^{-\frac{2-\alpha}{1-\alpha}}\,\frac{y_2^F}{y_1^F}\mathbf{1}_{\left\{\beta_T\zeta_T > \frac{U'(q)+y_2^F}{y_1^F}\right\}}. \end{aligned}$$

Note that the conditional expectations are w.r.t. the σ-algebra generated by the unnormalized filter $\mathcal{E}_t$ and the current optimal wealth $X_t^{P/F}$ which are known to the investor at time t. This allows to use Monte Carlo methods for their computation.

Remark 5.5 The correction factor $C_{t,T}$ replaces $(\widetilde{E}[\zeta_{t,T}^{\frac{1}{\alpha-1}} \mid \mathcal{F}_t^S])^{-1}\zeta_{t,T}^{\frac{\alpha}{\alpha-1}}$ in Proposition 4.10 of [32]. In particular this shows that the solutions coincide for $q \to 0$. For constant drift ($d = 1$, $\hat{Y} = Y \equiv 1$) we are in the classical Black Scholes model with constant drift. The computation of the optimal trading strategy obtained in Example 5.4 leads to the optimal trading strategy obtained for PEL in [1, Proposition 5] and for FEL in [9, Proposition 9].

6 Numerical examples

In this section we illustrate the findings of the previous sections. These numerical experiments are based on a financial market model where the drift follows a continuous time Markov chain with $d = 5$ states, volatility σ is constant and the interest rate r equals zero. For simulated stock prices we consider the maximization of expected power utility under bounded expected loss. The complete setting is described in Sections 6.1 and 6.2.

Section 6.3 gives an overview of the numerical procedures we use to compute the required filter, optimal terminal wealth and the optimal strategy based on discrete-time observations of the stock prices. Section 6.4 is devoted to the description of the optimal terminal wealth. We present efficient frontiers for visualizing the dependence of the optimal expected utility on the bound ε for the shortfall risk and study the influence of several parameters such as the benchmark q, the risk aversion parameter α of the utility function, and the type of the risk measure (PEL, FEL). We compare the probability density function of X_T^P, its expectation and expected utility with the corresponding results for the pure stock portfolio and the optimal portfolios for maximum and minimum values for the risk bound ε, i.e. for $\varepsilon = \overline{\varepsilon}^P$ and $\varepsilon = \underline{\varepsilon}^P$, respectively. The optimal strategies generating the optimal terminal wealth are presented in Section 6.5 for two simulated paths of the stock price. In Section 6.6 we propose a method for computing optimal strategies, for which possible errors due to the time-discretization are detected and corrected.

6.1 Parameters of the financial market

We consider a market consisting of a risk-free asset with interest rate $r = 0$ and one stock ($n = 1$) with constant volatility $\sigma = 0.25$ and drift process μ modeled as a continuous-time Markov chain, cf. Section 5, with $d = 5$ states and state matrix B, rate matrix G given by

$$B = (0.9, 0.4, 0.1, -0.3, -0.8) \quad \text{and} \quad G = \begin{pmatrix} -80 & 20 & 60 & 0 & 0 \\ 8 & -40 & 28 & 4 & 0 \\ 2 & 8 & -20 & 8 & 2 \\ 0 & 3 & 21 & -30 & 6 \\ 0 & 0 & 45 & 15 & -60 \end{pmatrix}.$$

This Markov chain has stationary distribution $\frac{1}{102}(3, 15, 60, 20, 4)$ and $E[\mu_t]$ converges for $t \to \infty$ to its ergodic mean $\overline{\mu} = \frac{11}{204} \approx 0.054$. The transition probabilities are such that switching to the extreme states is less likely. For a time horizon of $T = 1$ year consisting of $M = 250$ trading days Figure 6.1 shows a simulated path of the drift process μ (top left) and the probabilities of the stationary distribution (top right). For the investor the drift

Figure 6.1 Drift μ, filter $\widehat{\mu}$ and ergodic mean $\overline{\mu}$ (top left), stationary distribution (top right), stock prices S (bottom left), daily returns ΔR_t and $\mu_t \Delta t$ for $\Delta t = 1/250$ (bottom right).

process is not observable since he only observes the daily stock prices (bottom left) or – equivalently – the daily returns (bottom right). The investor estimates the unknown drift μ_t from the available stock prices resp. returns using the filter

$$\widehat{\mu}_t = E[\mu_t | \mathcal{F}_t^S] = B\widehat{Y}_t = B\zeta_t \mathcal{E}_t.$$

Here, we get the unnormalized filter $\mathcal{E}_t$ solving SDE (5.2) using M time-steps. From $\mathcal{E}_t$ the filter ζ_t for the martingale density Z_t can be obtained via $\zeta_t = (\mathbf{1}_d^\top \mathcal{E}_t)^{-1} = (\mathcal{E}_t^1 + \ldots + \mathcal{E}_t^d)^{-1}$. The resulting filter $\widehat{\mu}$ can be seen in the top left plot of Figure 6.1. Comparing $\widehat{\mu}$ with the states of the drift we observe that $\widehat{\mu}$ is rather close to the ergodic mean $\overline{\mu}$ (dashed line). However, the subsequent results indicate, that the investor can benefit from this information contained in the filter by choosing an appropriate investment strategy.

6.2 Parameters for the portfolio optimization

We consider an investor with terminal trading time $T = 1$ (year) and initial capital $x_0 = 1$. The investor's utility function U, by which the terminal wealth is evaluated, is taken from the family of CRRA-utilities:

$$U(x) = \begin{cases} \frac{x^\alpha}{\alpha} - \frac{1}{\alpha} & \text{for } \alpha \in (-\infty, 1) \setminus \{0\} \qquad \text{power utility} \\ \log x & \text{for } \alpha = 0 \qquad \text{logarithmic utility.} \end{cases}$$

Benchmark	q	0.95	1.05
Bound	ε	0.05	0.1
PEL Minimal risk	$\underline{\varepsilon} = \underline{\varepsilon}^P$	0	0.05
Maximal risk	$\overline{\varepsilon} = \overline{\varepsilon}^P$	0.183	0.248
FEL Minimal risk	$\underline{\varepsilon} = \underline{\varepsilon}^F$	0	0.031
Maximal risk	$\overline{\varepsilon} = \overline{\varepsilon}^F$	0.142	0.195

Table 6.1 Parameters for the risk constraints

Here, $1-\alpha$ is the Arrow Pratt index of relative risk aversion. The subsequent examples compare investors with different relative risk aversions, in particular we set $\alpha = 0.5, 0, -1$ standing for risk aversion lower, the same, higher as for logarithmic utility. The shift by $-\frac{1}{\alpha}$ in the definition of U does not affect the optimization. It is sometimes convenient since for the shifted utility it holds $U(1) = 0$ for arbitrary α and $\lim\limits_{\alpha \to 0} U(x) = \log x$.

We measure the risk that the terminal wealth falls short of benchmark q by present and future expected loss PEL and FEL, respectively. Since we assume $r = 0$, it holds

$$\beta_T = \gamma = 1, \quad Z_1 = \beta_T \zeta_T = \zeta_T \quad \text{and for PEL} \quad Z_2 = \gamma \zeta_T = \zeta_T \tag{6.1}$$

while for FEL we have $Z_2 = 1$. We consider two cases for the benchmark: $q = 0.95$ and $q = 1.05$. Note that for $q = 0.95$ portfolio insurance is possible since $x_0 \geq E[Z_1]q = q$ by (6.1). The minimal shortfall risk is $\underline{\varepsilon} = 0$. For $q = 1.05 > x_0$ there is no admissible solution for the portfolio insurer problem and we have a strictly positive $\underline{\varepsilon}$. Table 6.1 shows for the considered benchmarks q the bounds ε chosen in the examples below and the minimal and maximal shortfall risks $\underline{\varepsilon}^P, \overline{\varepsilon}^P$ and $\underline{\varepsilon}^F, \overline{\varepsilon}^F$ for PEL and FEL, respectively.

Using (6.1), it turns out that the optimal terminal wealth for the different optimization problems can be written as a function $f(\zeta_T)$ of the conditional martingale density and of one or two Lagrange multipliers. In particular, by Corollaries 4.3 and 4.4

$$X_T^P = f_P(\zeta_T) = f_P(\zeta_T; y_1^P, y_2^P), \quad \text{and} \quad X_T^F = f_F(\zeta_T) = f_F(\zeta_T; y_1^F, y_2^F).$$

Moreover, we recall and introduce the following notation

$$\begin{aligned}
X_T^M &= f_M(\zeta_T) = f_M(\zeta_T; y^M) &&= I(y^M \zeta_T) \\
X_T^{PI} &= f_{PI}(\zeta_T) = f_{PI}(\zeta_T; y^{PI}) &&= \begin{cases} I(y^{PI}\zeta_T) & \text{for } y^{PI}\zeta_T \leq U'(q) \\ q & \text{for } y^{PI}\zeta_T > U'(q) \end{cases} \\
X_T^{PM} &= f_{PM}(\zeta_T) = f_{PM}(\zeta_T; y^{PM}) &&= \begin{cases} q & \text{for } y^{PM}\zeta_T \leq U'(q) \\ I(y^{PM}\zeta_T) & \text{for } y^{PM}\zeta_T > U'(q) \end{cases} \\
X_T^{FM} &= f_{FM}(\zeta_T) = f_{FM}(y^{FM}\zeta_T) &&= q\mathbf{1}_{\{y^{FM}\zeta_T \leq 1\}}.
\end{aligned}$$

The optimal terminal wealth X_T^M, X_T^{PI} for Merton and portfolio insurer problem are introduced in Section 3 while X_T^{PM} and X_T^{FM} denote PEL- and FEL-optimal terminal wealth for minimal risk $\varepsilon = \underline{\varepsilon}$ for $x_0 < q$, see Theorem 4.1 (iv). The parameters $y^M, y^{PI}, y^{PM}, y^{FM}$ are the unique solutions of the corresponding budget equations $\widetilde{E}[f(\zeta_T; y)] = x_0$.

6.3 Numerical Approximations

To determine PEL- and FEL-optimal terminal wealth in Corollaries 4.3, 4.4 we have to solve the following system of equations to determine the optimal Lagrange multipliers y_1^*, y_2^*:

$$g_1(y_1, y_2) = x_0 \quad \text{and} \quad g_2(y_1, y_2) = \varepsilon \tag{6.2}$$

where $g_1(y_1, y_2) = \widetilde{E}[f_{P/F}(\zeta_T; y_1, y_2)]$ and $g_2(y_1, y_2) = \widetilde{E}[\ (f_P(\zeta_T; y_1, y_2) - q)^-\ ]$ for PEL while for FEL we have $g_2(y_1, y_2) = E[(f_F(\zeta_T; y_1, y_2) - q)^-] = \widetilde{E}[\zeta_T^{-1}(f_F(\zeta_T; y_1, y_2) - q)^-]$. The functions g_1 and g_2 of y_1, y_2 are nonlinear. In [10, Section 6] we prove that under suitable conditions there exists a unique solution (y_1^*, y_2^*) of the equations (6.2). Moreover that paper studies properties of the functions $g_1(y_1, y_2)$ and $g_2(y_1, y_2)$, in particular of their partial derivatives. Based on these properties we propose a nested Newton iteration for the numerical solution of the two equations, see [10, Remark 17]. For the other optimization problems, we find $X_T^M, X_T^{PI}, X_T^{PM}, X_T^{FM}$ by solving numerically (Newton iteration) the corresponding budget equations $g_1(y) = \widetilde{E}[f(\zeta_T; y)] = x_0$ to determine the optimal Lagrange multipliers $y^M, y^{PI}, y^{PM}, y^{FM}$.

It turns out that the expectations defining the functions g_1 and g_2 cannot be evaluated explicitly but have to be approximated by Monte-Carlo methods based on a sample of N realizations of ζ_T. The same holds for the partial derivatives of g_1 and g_2 which are needed for the Newton iteration. To this end we use the relation $\zeta_T = (\mathbf{1}_d^\top \mathcal{E}_T)^{-1}$ and generate N realizations of the unnormalized filter $\mathcal{E}_T$ from N paths of the Wiener process $(\widetilde{W}_t)_{t\in[0,T]}$ by solving SDE (5.2) using $dR_t = \sigma d\widetilde{W}_t$. The initial value $\mathcal{E}_0 = E[Y_0]$ is computed w.r.t. the stationary distribution of the Markov chain. For the solution we apply the Euler scheme with M time-steps and use robust filter techniques to reduce discretization errors, see [4, 18, 27]. The PEL- and FEL-optimal strategies π_t^P, π_t^F in Theorem 5.3 simplify for $r = 0$ to

$$\pi_t^{P/F} = (\sigma^\top)^{-1} \widetilde{E}[f'_{P/F}(\zeta_T)\, D_t\zeta_T \,|\, \mathcal{E}_t], \tag{6.3}$$

where $f_P(z) = f_P(z; y_1^P, y_2^P)$ and $f_F(z) = f_F(z; y_1^F, y_2^F)$, respectively. We obtain the optimal strategies $\pi_t^M, \pi_t^{PI}, \pi_t^{PM}$ for the Merton, the portfolio insurer problem and the PEL-constrained optimization for minimal risk $\varepsilon = \underline{\varepsilon}^P$ for $x_0 < q$, if we replace in (6.3) $f_{P/F}(z)$ by $f_M(z; y^M)$, $f_{PI}(z; y^{PI})$ and $f_{PM}(z; y^{PM})$, respectively. This can be proven by similar arguments as in Section 4 and 5. For FEL-constrained optimization for minimal risk $\varepsilon = \underline{\varepsilon}^F$ representation (6.3) does not hold for the optimal strategy π_t^{FM}, since $f_{FM}(z; y^{FM})$ is not continuous. There is a jump of size q at $z = 1/y^{FM}$.

The conditional expectation in (6.3) cannot be evaluated explicitly but has to be approximated by a Monte-Carlo estimate. The aim is to compute at time t the optimal strategy based on the observation of the stock prices up to time t. From $(S_u)_{u\in[0,t]}$ we get the unnormalized filter $\mathcal{E}_t$, which can be computed numerically as solution of (5.2). By (6.3),

$$\pi_t^{P/F} = h^{P/F}(t, \mathcal{E}_t), \quad \text{where} \quad h^{P/F}(t, x) = (\sigma^\top)^{-1} \widetilde{E}[f'_{P/F}(\zeta_T)\, D_t\zeta_T \,|\, \mathcal{E}_t = x]. \tag{6.4}$$

For the Monte-Carlo estimate of the conditional expectation given $\mathcal{E}_t = x$ we generate N realizations of ζ_T and $D_t\zeta_T$ starting at t with $\mathcal{E}_t = x$. We denote them by $\zeta_T^{t,x}$ and

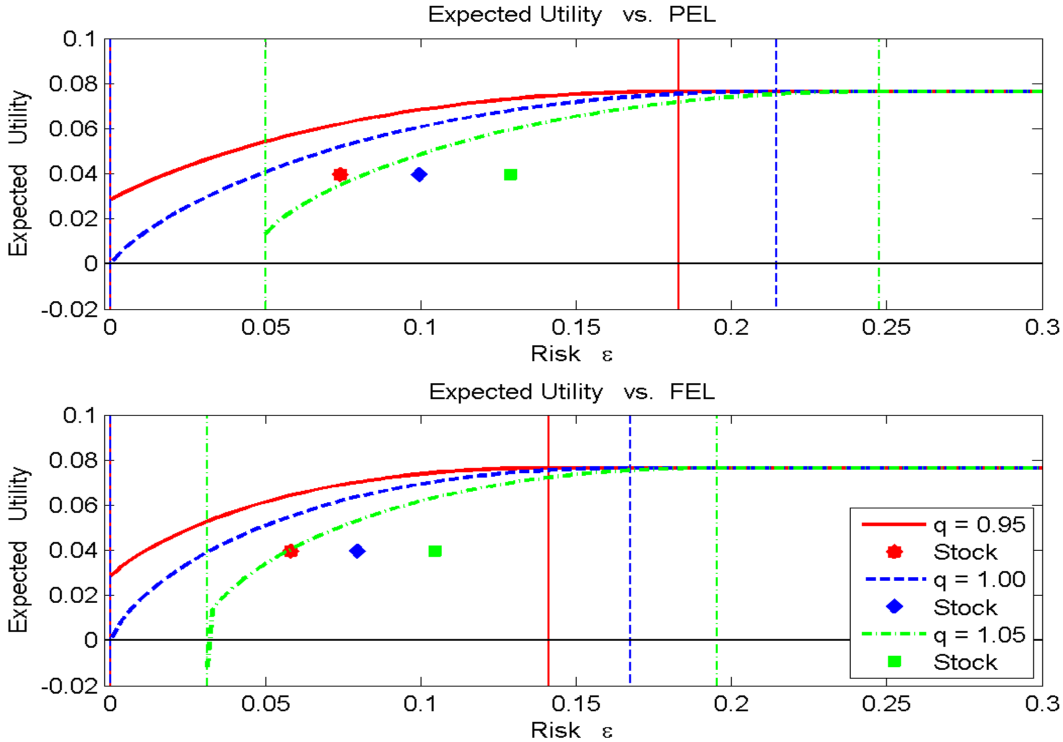

Figure 6.2 Efficient frontier for PEL-optimal (top) and FEL-optimal (bottom) portfolios, power utility with $\alpha = 0.5$, benchmarks $q = 0.95, 1, 1.05$

$(D_t\zeta_T)^{t,x}$. To this end we simulate N paths of $\widetilde{W}$ under $\widetilde{P}$. From these we get N paths of the unnormalized filter $\mathcal{E}_s^{t,x}$, $s \in [t,T]$, as solution of

$$\mathcal{E}_s^{t,x} = x + \int_t^s G^\top \mathcal{E}_u^{t,x}\, du + \int_t^s \mathrm{Diag}(\mathcal{E}_u^{t,x})(\sigma^{-1}B)^\top d\widetilde{W}_u, \quad s \in [t,T], \tag{6.5}$$

which results from SDE (5.2) using $dR_t = \sigma d\widetilde{W}_t$. From the terminal value $\mathcal{E}_T^{t,x}$ we get N realizations of the conditional martingale density via $\zeta_T^{t,x} = (\mathbf{1}_d^\top \mathcal{E}_T^{t,x})^{-1}$.

Observing $dR_t = \sigma d\widetilde{W}_t$, the realizations for the Malliavin derivative $(D_t\zeta_T)^{t,x}$ can be obtained from the SDEs in Proposition 5.2 using the same N paths of $\widetilde{W}$. The SDEs are discretized using the Euler scheme with M time steps. For the second SDE describing the dynamics of the Malliavin derivative $(D_t\mathcal{E}_s)^{t,x}$ of the unnormalized filter we apply robust filter techniques, see [27]. Finally, given $\mathcal{E}_t = x$ the optimal strategy is given by

$$\pi_t^{P/F} = h^{P/F}(t,x) = (\sigma^\top)^{-1}\widetilde{E}[f'_{P/F}(\zeta_T)\, D_t\zeta_T \,|\, \mathcal{E}_t = x] = (\sigma^\top)^{-1}\widetilde{E}[f'_{P/F}(\zeta_T^{t,x})\,(D_t\zeta_T)^{t,x}]$$

and can be approximated using the sample mean of $f'_{P/F}(\zeta_T^{t,x})\,(D_t\zeta_T)^{t,x}$. The same procedure can be applied for the evaluation of the optimal strategies $\pi_t^M, \pi_t^{PI}, \pi_t^{PM}$.

6.4 Optimal terminal wealth

Next we illustrate properties of the PEL- and FEL-optimal terminal wealth. We start with efficient frontiers by which the influence of several parameters can be studied and visualized. Efficient frontiers are known from the Markowitz portfolio theory where the "return" of a portfolio is measured in terms of the expected terminal wealth and the "risk" by its variance. Every admissible portfolio can be plotted in the "risk-return space", and the collection of all such possible portfolios defines a region in this space. The curve along the upper edge of this region is known as efficient frontier. It represents those portfolios which have the lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier offers the best possible return.

We adopt this approach replacing "return" by expected utility of terminal wealth and "risk" by expected loss. Given the bound ε for the expected loss (measured by PEL or FEL) the optimal portfolio maximizes the expected utility $E[U(X_T)]$ among all admissible portfolios whose shortfall risk does not exceed ε. Denoting this optimal terminal wealth by ${}^{\varepsilon}X_T^*$, the efficient frontier plots the maximal expected utility $E[U({}^{\varepsilon}X_T^*)]$ against ε.

For $\varepsilon \geq \overline{\varepsilon}$ we have $E[U({}^{\varepsilon}X_T^*)] = E[U({}^{\overline{\varepsilon}}X_T^*)] = E[U(X_T^M)] = \text{const}$. In case of $\varepsilon < \underline{\varepsilon}$ there are no admissible portfolios and for $\varepsilon = \underline{\varepsilon}$ it holds ${}^{\underline{\varepsilon}}X_T{}^* = X_T^{PM}$ (PEL) or ${}^{\underline{\varepsilon}}X_T{}^* = X_T^{FM}$ (FEL). Thus the efficient frontier can be drawn for $\varepsilon \geq \underline{\varepsilon}$ only. Figure 6.2 shows efficient frontiers for the risk measures PEL and FEL. Each plot shows the corresponding frontiers for power utility with parameter $\alpha = 0.5$ and benchmarks $q = 0.95, 1$ and 1.05. For each frontier 50 points $(\varepsilon, E[U({}^{\varepsilon}X_T^*)])$ for equispaced $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$ are computed, and $N = 10^7$ realizations of ζ_T are used to estimate the expectations in (6.2) for the computation of the optimal Lagrange multipliers y_1^*, y_2^* and to estimate $E[U({}^{\varepsilon}X_T^*)]$.

The points on and below the frontier correspond to admissible portfolios with risk measure bounded by ε. The pure stock portfolio is represented by a marker for each value of q. While the expected utility of the pure stock portfolio does not depend on the benchmark q, the corresponding values for the risk measures increase with increasing q. Thus the 3 markers are on a horizontal line. The higher we choose the benchmark for fixed ε, the lower is the maximum expected utility. For $q \leq x_0 = 1$ the minimal risk $\underline{\varepsilon}$ is zero since portfolio insurance is possible. For $q > x_0$ we have positive $\underline{\varepsilon}$ which is for $q = 1.05$ in case of PEL $\underline{\varepsilon} = q - x_0 = 0.05$ and for FEL $\underline{\varepsilon} = qP(y^{FM}\zeta_T > 1) \approx 0.030$. Note that for each efficient frontier the values of $\underline{\varepsilon}$ and $\overline{\varepsilon}$ are visualized by vertical lines.

Figure 6.3 shows the corresponding efficient frontiers for fixed benchmark $q = 1.05$ and different utility functions ($\alpha = 0.5, 0, -1$). Note that $\alpha = 0$ corresponds to logarithmic utility. For fixed ε the maximum expected utility decreases if the relative risk aversion $1-\alpha$ increases. Since X_T^{FM} equals zero with positive probability, the expected utility of X_T^{FM} for $\alpha \leq 0$ is $-\infty$. The risk of the pure stock portfolio does not depend on the parameter α. Thus the 3 markers are on a vertical line.

Next, we have a closer look at the form and the distribution of the optimal terminal wealth. For power utility with $\alpha = 0.5$ and risk measured by PEL, we consider two cases:

(A) $q = 0.95$ and $\varepsilon = 0.05$

(B) $q = 1.05$ and $\varepsilon = 0.1$.

In case (A) portfolio insurance is possible, i.e. the minimal shortfall risk is $\underline{\varepsilon} = 0$ while the maximal risk is $\overline{\varepsilon} \approx 0.183$. In case (B) we have strictly positive minimal shortfall risk $\underline{\varepsilon} = q - x_0 = 0.05$, the maximal risk is $\overline{\varepsilon} \approx 0.248$, see Table 6.1.

Figure 6.4 shows for case (A) and (B) the function $f_P(z; y_1^P, y_2^P)$ given in Corollary 4.3 by which the PEL-optimal terminal wealth can be written as a function of the conditional

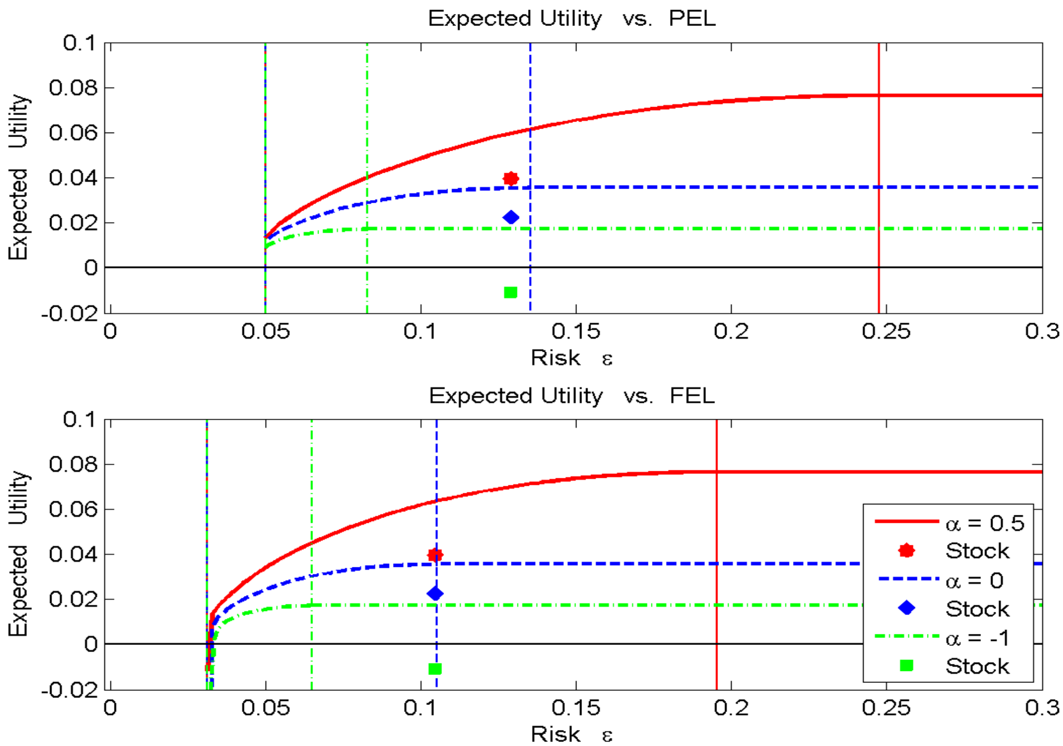

Figure 6.3 Efficient frontier for PEL-optimal (top) and FEL-optimal (bottom) portfolios, utility functions with parameters $\alpha = 0.5, 0, -1$ and benchmark $q = 1.05$

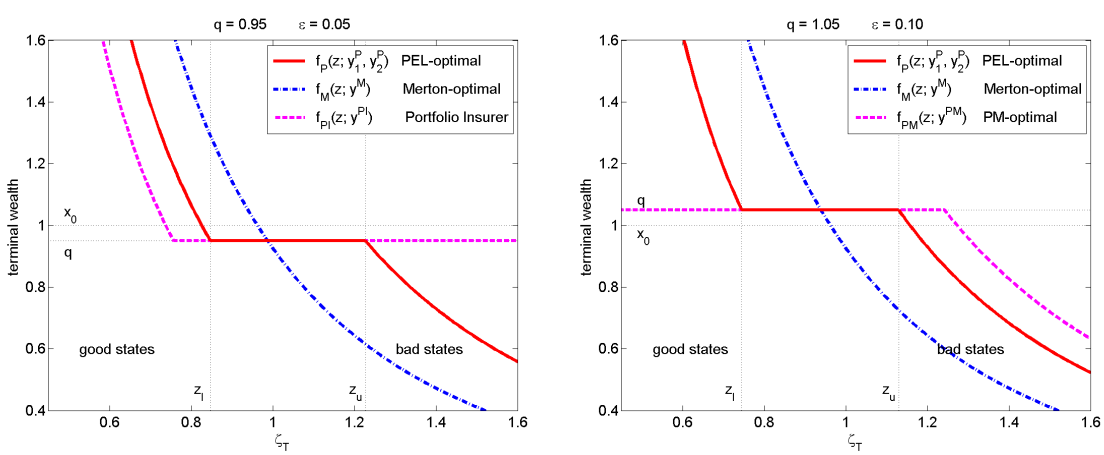

Figure 6.4 Functions $f(z)$ describing the form of the optimal terminal wealth by $X^* = f(\zeta_T)$, risk measure PEL, power utility with $\alpha = 0.5$; left: case (A), right: case (B).

martingale density ζ_T, i.e. $X_T^P = f_P(\zeta_T; y_1^P, y_2^P)$. The optimal Lagrange multipliers y_1^P, y_2^P are determined as described in Section 6.3:

$$\text{(A)} \quad y_1^P \approx 1.210 \quad \text{and} \quad y_2^P \approx 0.374$$
$$\text{(B)} \quad y_1^P \approx 1.309 \quad \text{and} \quad y_2^P \approx 0.446\,.$$

For comparison, Figure 6.4 also shows the functions $f_M(z; y^M) = I(y^M z)$ for the Merton problem as well as $f_{PI}(z; y^{PI})$ in case (A) and $f_{PM}(z; y^{PM})$ in case (B) for minimal shortfall risk $\varepsilon = \underline{\varepsilon}$. For the corresponding Lagrange multipliers we find $y^M \approx 1.038$, in case (A) $y^{PI} \approx 1.355$ and in case (B) $y^{PM} \approx 0.785$. The plots of $f_P(z)$ show three different regions separated by $z_l = \frac{U'(q)}{y_1^P}$ and $z_u = \frac{U'(q)}{y_1^P - y_2^P}$. The region $(0, z_l]$ of small values of the conditional martingale density ζ_T corresponds to terminal wealth $X_T^P = I(y_1^P \zeta_T) \geq q$ (good states). It exceeds the benchmark and its form is similar to the Merton case where we have $X_T^M = I(y^M \zeta_T)$. For large values of ζ_T in (z_u, ∞) we have $X_T^P = I((y_1^P - y_2^P)\zeta_T) < q$ (bad states), i.e. the portfolio falls short. Again the form of the terminal wealth corresponds to that of the Merton portfolio but with multiplier $y_1^P - y_2^P$ instead of y^M. In the intermediate region between z_l and z_u we have $X_T^P = q$, i.e. the investor manages the portfolio such that the benchmark q is reached exactly and thus prevents a shortfall.

Comparing with the Merton-optimal portfolio it can be observed that the PEL-optimal terminal wealth is smaller in the good states but larger in the bad states. There are less cases where the portfolio falls short and if the shortfall occurs, then there are smaller losses. For the portfolio insurer problem in case (A), where $q = 0.95 < x_0$, we have $X_T^{PI} \geq q$ in all states, i.e. no shortfall at all. It turns out that X_T^P is larger than or equal to X_T^{PI} in the good and intermediate states but smaller in the bad states. For case (B) where $q = 1.05 > x_0$ the PM-optimal portfolio X_T^{PM} for the minimal risk ($\varepsilon = \underline{\varepsilon} = 0.05$) never exceeds q. Only for large ζ_T the portfolio falls short and X_T^{PM} is strictly smaller than the benchmark but always larger than X_T^P, where the shortfall risk constraint is less restrictive.

Next we look at the distribution of the terminal wealth. For both cases (A) and (B) Figure 6.5 shows the probability density functions (for the absolutely continuous part) of the distribution of the terminal wealth of the PEL-optimal portfolio. For comparison we also plot the densities for the Merton-, the PI- resp. PM-optimal portfolios, and the density for the pure stock portfolio where the investor invests all his money in the stock.

Table 6.2 gives the expected terminal wealth $E[X_T]$, expected utility $E[U(X_T)]$, and the present expected loss $\widetilde{E}[(X_T - q)^-]$ for the considered portfolios. The Merton strategy generates (by definition) the maximum expected utility but among the considered portfolios it also exhibits the largest shortfall risk. There is a large probability for values in the "shortfall region" $[0, q)$ leading to large values for PEL. On the other hand there are considerable tail probabilities leading to a high value for the expected (utility of) terminal wealth.

Imposing the risk constraint, i.e bounding PEL by ε, results in a shift of probability mass from that "shortfall region" as well as from the upper tail of the distribution to the benchmark q. This atom at q carries a probability mass of size $P(X_T^P = q) = P(z_l < \zeta_T \leq z_u)$ which is about 0.516 in case (A) and 0.529 in case (B), i.e. more than one half of the total mass. In the density plot the atom is marked by a vertical line at q. Another consequence is the decrease of the expectation for the terminal wealth and its utility. However, it can be observed that the PEL-optimal portfolio outperforms the pure stock portfolio w.r.t. both

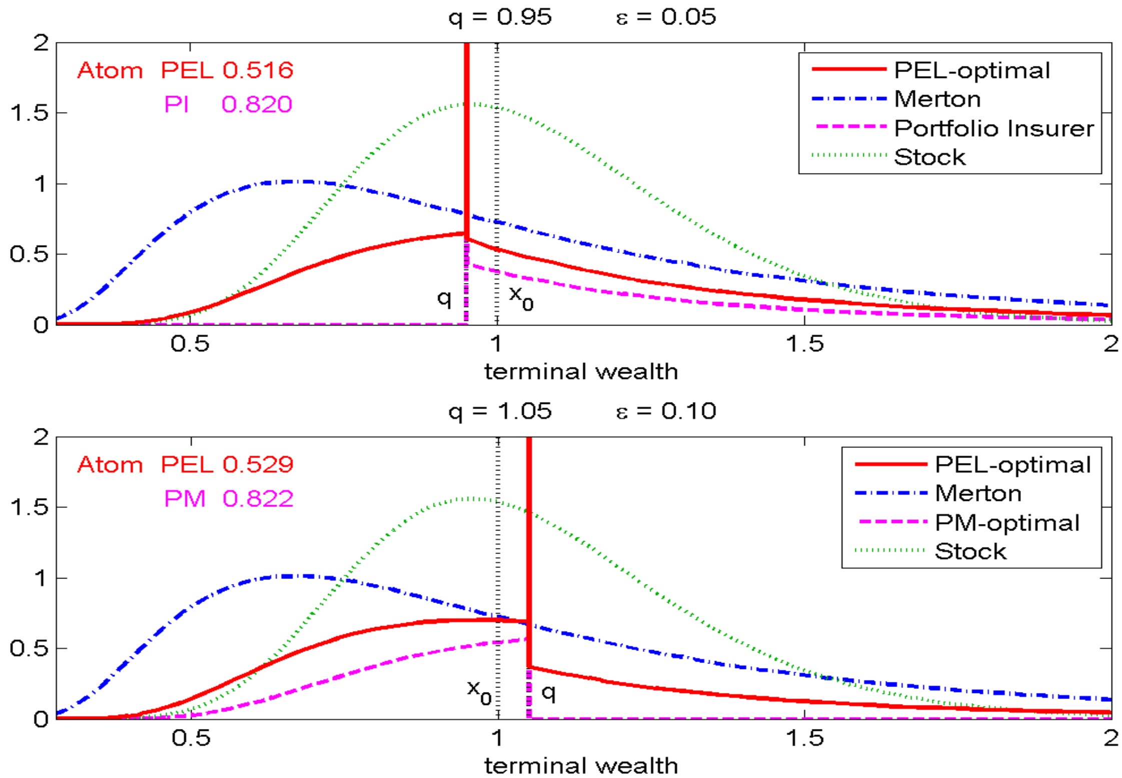

Figure 6.5 Probability density functions for the terminal wealth of PEL-, Merton-, PI-optimal and pure stock portfolio; power utility with $\alpha = 0.5$; top: case (A), bottom: case (B).

expected utility and PEL.

Decreasing the bound ε to the smallest possible value $\underline{\varepsilon}$ increases the probability mass of the atom at q and further decreases the expected utility. In case (A) portfolio insurance is possible and we have $\underline{\varepsilon} = 0$. Here, the PI-optimal portfolio never falls short and it holds $P(X_T^{PI} > q) = 1 - P(X_T^{PI} = q)$ where the atom at q is $P(X_T^{PI} = q) \approx 0.820$. For case (B) it holds $\underline{\varepsilon} = q - x_0 = 0.05$. Here the terminal wealth X_T^{PM} does not exceed the benchmark q. For the atom we find $P(X_T^{PM} = q) \approx 0.822$ and for the shortfall probability we find $P(X_T^{PM} < q) = 1 - P(X_T^{PM} = q) \approx 0.178$.

6.5 Optimal strategy

For illustrating properties of the optimal trading strategies we restrict to case (B) where $q = 1.05 > x_0$ and $\varepsilon = 0.1$ and consider two scenarios for the stock prices. Based on the parameters in Section 6.1 we have simulated several paths of S and out of these paths we have selected a "good path" and a "bad path" where the latter together with the corresponding path of the drift and its filter can be seen in Figure 6.1.

Figures 6.6 and 6.7 show in the bottom plots (among other curves) the paths of the stock prices for $S_0 = x_0 = 1$. For the "good path" the pure stock and the Merton-optimal portfolio end above the benchmark while for the "bad path" the opposite is true. The top plots show the corresponding strategies in terms of fraction of wealth π_t / X_t invested in

$q = 0.95$	Expected terminal wealth $E[X_T]$	Expected utility $E[U(X_T)]$	Present expected loss (PEL) $\widetilde{E}[(X_T - q)^-]$
Risk constraint	1.092	0.054	$0.050 = \varepsilon$
Merton	1.178	0.076	$0.183 = \overline{\varepsilon}^P$
Portfolio Insurer	1.046	0.029	$0 \quad = \underline{\varepsilon}^P$
Pure stock	1.058	0.040	0.074

$q = 1.05$	Expected terminal wealth $E[X_T]$	Expected utility $E[U(X_T)]$	Present expected loss (PEL) $\widetilde{E}[(X_T - q)^-]$
Risk constraint	1.075	0.049	$0.100 = \varepsilon$
Merton	1.178	0.076	$0.248 = \overline{\varepsilon}^P$
Minimal Risk ($\varepsilon = \underline{\varepsilon}^P$)	1.016	0.013	$0.050 = \underline{\varepsilon}^P = q - x_0$
Pure stock	1.058	0.040	0.129

Table 6.2 Expected terminal wealth, expected utility and PEL for PEL-, Merton-, PI-optimal and pure stock portfolio; power utility with $\alpha = 0.5$; top: case (A), bottom: case (B).

the stock. For the pure stock portfolio we have a buy-and-hold strategy, i.e. this fraction equals 1. The Merton-, PEL- and PM-optimal strategies have been computed by estimating for each time step the conditional expectation in (6.3) from $N = 1000$ realizations of ζ_T and $D_t\zeta_T$. Obviously trading according to the optimal Merton strategy is quite risky and often requires extreme positions in the stock. For example, following the optimal strategy requires that the investor has to borrow an amount of four to six times the initial capital for investing in the risky stock.

For the "good path" we are faced with a considerable smaller terminal wealth compared to the Merton-optimal portfolio but we reach nearly the terminal wealth of the pure stock portfolio and we end above the benchmark. Bounding the shortfall risk by imposing a risk constraint decreases the extreme positions of the Merton strategy. However, if the PEL-optimal wealth X_t^P is above the benchmark and especially when t approaches T the PEL-optimal strategy is rather close to the Merton-strategy and thus just as risky. Since the wealth is above the benchmark, shortfall seems to be "unlikely" and this allows for riskier investments. The PM-optimal portfolio strategy is close to the bond strategy. After the wealth X_t^{PM} reaches the benchmark q at $t \approx 0.4$ the wealth stays at this level and the strategy turns over to the pure bond strategy.

For the "bad path" we also observe a decrease of the short positions for the PEL-optimal strategy compared to the Merton-optimal strategy. Contrary to the "good path" these differences are much stronger and do not vanish close to T. Moreover, it can be seen that the PEL-optimal strategy finally approaches the pure bond strategy $\pi_t \equiv 0$. By this strategy the PEL-optimal wealth is driven exactly to the benchmark q. So this strategy prevents the portfolio from falling short of q which is the case for the Merton-optimal as well as for the pure stock portfolio. Comparing the PM-optimal (for $\varepsilon = \underline{\varepsilon} = 0.05$) with the PEL-optimal strategy (for $\varepsilon = 0.1$) it can be observed, that the PM-strategy is less riskier. This strategy approaches the pure bond strategy earlier and remains there until horizon time T while the wealth stays at the benchmark q. Note that the PM-optimal wealth approaches for $t \to T$

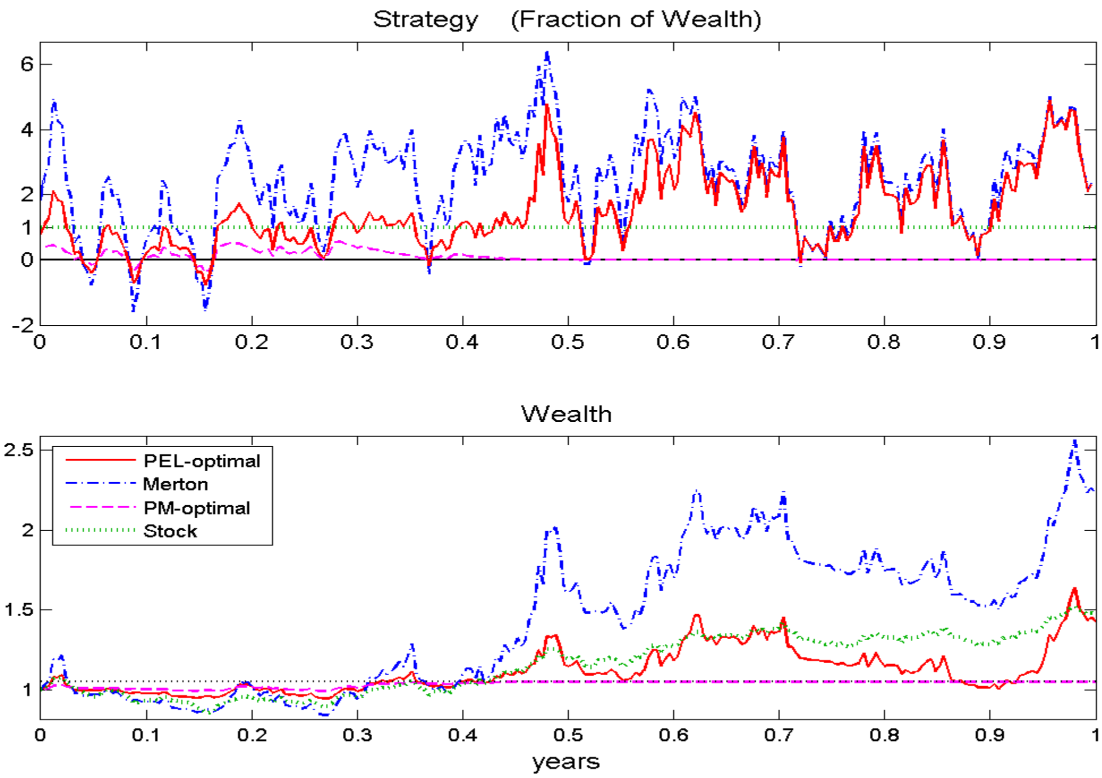

Figure 6.6 "Good path": strategies in terms of fraction of wealth π_t/X_t (top) and wealth X_t (bottom) for the PEL-, Merton-, PM-optimal and pure stock portfolio

a constant which is slightly smaller than the benchmark q. This deviation results from discretization errors which are investigated in the next section.

6.6 Updating

In practice and even in a simulation experiment we have to take into account that contrary to our model assumptions the portfolio cannot be readjusted continuously but only at M discrete trading times $t_i = i\Delta t$, $i = 0, \ldots, M-1$, $\Delta t = T/M$. Moreover, instead of the theoretical optimal strategy π_t^* we only know an approximation ${}^A\pi_t^*$ since we have to approximate $\mathcal{E}$ and ζ, which we need for its computation, based on discrete-time observations of the stock prices. Further, the conditional expectations we need are evaluated using Monte-Carlo methods. Therefore, instead of the theoretical optimal wealth $X_{t_i}^*$ resulting from the wealth equation (2.10), we have in practice the actual wealth

$$ {}^AX_{t_i}^* = x_0 + \sum_{j=0}^{i-1} ({}^A\pi_{s_j}^*)^\top \Delta R_{s_j} \quad \text{instead of} \quad X_{t_i}^* = x_0 + \int_0^{t_i} (\pi_s^*)^\top dR_s $$

and are faced with a tracking error ${}^AX_{t_i}^* - X_{t_i}^*$.

Remark 6.1 The examples in Section 6.5 show that optimal strategies based on a HMM model for the drift can be very risky. There are extreme long and short positions in the

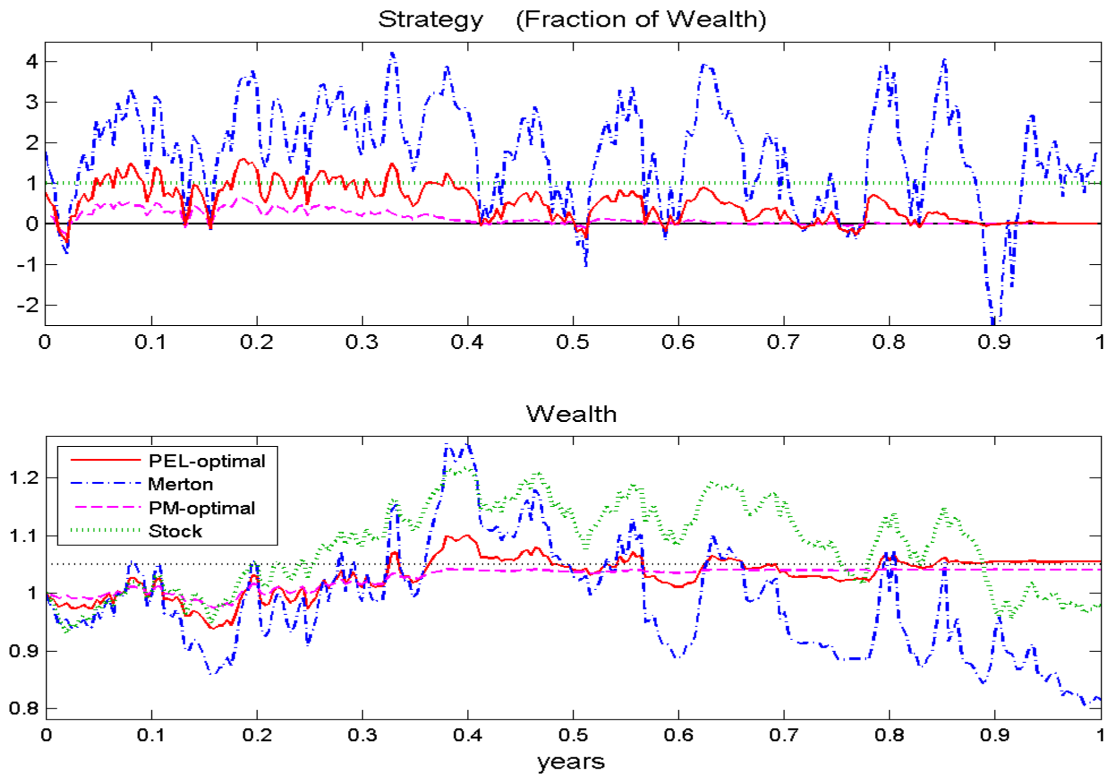

Figure 6.7 "Bad path": strategies in terms of fraction of wealth π_t/X_t (top) and wealth X_t (bottom) for the PEL-, Merton-, PM-optimal and pure stock portfolio

stocks. Indeed, imposing a risk constraint decreases these extreme positions if we compare with the unconstrained (Merton) case but they can still be considerable. Application of ${}^A\pi_t^*$ or even π_t^* with large absolute values may lead to serious deviations ${}^AX_t^* - X_t^*$ (tracking errors) due to time discretization. Moreover, these errors accumulate in time.

Due to the tracking error, trading at t according to the optimal strategy π_t^* may no longer be optimal since it depends implicitly on the wealth. Note that for risk constraints also the risky fraction π_t^*/X_t^* is wealth dependent, cf. Example 5.4. Therefore, the application of π_t^* (or of ${}^A\pi_t^*$) may not generate the maximum expected utility of terminal wealth and/or violate the risk constraint. So we are faced with the following questions.

1. How can we detect tracking errors?
2. How can we correct or update the optimal strategy π_t^* if we detect a tracking error?

To answer the first question we use an alternative representation for the theoretical optimal wealth X_t^*. Since $X_T^* = f(\zeta_T)$ we get with the same argument as in the proof of Theorem 5.3 that $X_t^* = \widetilde{E}[X_T^*|\mathcal{F}_t^S] = \widetilde{E}[f(\zeta_T)|\mathcal{E}_t]$, hence

$$X_t^* = g(t, \mathcal{E}_t), \qquad \text{where} \qquad g(t,x) = \widetilde{E}[f(\zeta_T) \,|\, \mathcal{E}_t = x].$$

In the notation of Section 6.3 we can generate N realizations of $\zeta_T^{t,x}$, i.e. of ζ_T given $\mathcal{E}_t = x$, from N paths of the Wiener process $(\widetilde{W}_u)_{u\in[0,T]}$ by solving SDE (6.5) numerically. Then we approximate X_t^* by the sample mean of $f(\zeta_T^{t,x})$ which we denote by ${}^NX_t^*$.

Remark 6.2 Note, that for the computation of the approximation ${}^N X_t^*$ of the theoretical wealth X_t^* we do not need to know the optimal strategy π_t^* which would require the rather time consuming computation of Malliavin derivatives $D_t \zeta_T$. Thus, errors in the approximation of the optimal strategy do not affect ${}^N X_t^*$ and reliable results can be obtained faster.

A tracking error can be detected comparing the approximation ${}^N X_t^*$ of the theoretical optimal wealth with the actual wealth ${}^A X_t^*$. Given some threshold $\delta > 0$ we use a relative criterion and call at time $t \in (0, T]$ the deviation critical if $\frac{|{}^N X_t^* - {}^A X_t^*|}{{}^N X_t^*} > \delta$. The threshold δ controls the sensitivity of the detection procedure.

After detecting a critical tracking error we want to update or correct the portfolio strategy. This leads to the second question. A natural idea for this updating is to set up at time t a new optimization problem. The corresponding initial capital is the actual wealth ${}^A X_t^*$. This is the amount of money we would receive if we sell the portfolio. As pointed out above, due to the deviation ${}^A X_t^* \neq X_t^*$ an application of ${}^A \pi_t^*$ or even π_t^* would typically not yield the same maximum expected utility $E[U(X_T^*)]$ and satisfy the risk constraint $E_Q[(X_T^* - q)^-] \leq \varepsilon$. We propose a correction for which the risk constraint holds with equality on average while we have to accept a smaller expected utility. More precisely, we use as new bound for the risk

$$\varepsilon_t^* = E_Q[(X_T^* - q)^- \,|\, \mathcal{F}_t^S]$$

for which $E_Q[\varepsilon_t^*] = \varepsilon$. This conditional shortfall risk is the conditional expectation of the loss of the theoretical optimal terminal wealth X_T^* given the observed stock prices up to time t. For PEL it can be rewritten as $\varepsilon_t^* = \widetilde{E}[(f_P(\zeta_T) - q)^- \,|\, \mathcal{E}_t]$ and for FEL as $\varepsilon_t^* = \widetilde{E}[\zeta_T^{-1}(f_F(\zeta_T) - q)^- \,|\, \mathcal{E}_t]$. These conditional expectations w.r.t. $\widetilde{P}$ can be approximated quite accurately by Monte-Carlo estimates from a sample of N realizations of $\zeta_T^{t,x}$ (as described before Remark 6.2 for ${}^N X_t^*$ and X_t^*). We denote this approximation by ${}^N \varepsilon_t^*$.

Remark 6.3 In the PEL-case $\varepsilon_t^* = \widetilde{E}[(X_T^P - q)^- \,|\, \mathcal{F}_t^S]$ can be considered as the price at time t of an option paying the loss $(X_T^P - q)^-$ at T. At time 0 the price of this option equals ε, cf. Remark 3.2. Suppose an investor buys this option to insure the portfolio against shortfall. If the investor at time t detects a tracking error and decides to update the portfolio strategy by restarting the portfolio optimization, he can sell not only the portfolio earning the actual wealth ${}^A X_t^*$ for the new initial capital. He can also sell the option receiving ε_t^* and use this money for financing the purchase of a new option paying at time T the loss of the updated portfolio.

Restarting the optimization means to solve the following optimization problem

$$\begin{aligned} &\text{maximize} && E[U(X_T^\pi) \,|\, \mathcal{F}_t^S] && \text{for } \pi \in \mathcal{A}_t({}^A X_t^*) \\ &\text{subject to} && \widetilde{E}[X_T^\pi \,|\, \mathcal{F}_t^S] \leq {}^A X_t^* && \\ & && E_Q[(X_T^\pi - q)^- \,|\, \mathcal{F}_t^S] \leq {}^N \varepsilon_t^*, && \end{aligned}$$

where the admissibility set $\mathcal{A}_t(x)$ is defined as $\mathcal{A}$ in Section 2 but for processes starting at time t and given $X_t = x$. The corresponding static problem leads to equations

$$\widetilde{E}[f(\zeta_T; y_1^t, y_2^t) \,|\, \mathcal{E}_t] = {}^A X_t^* \quad \text{and} \quad E_Q[(f(\zeta_T; y_1^t, y_2^t) - q)^- \,|\, \mathcal{E}_t] = {}^N \varepsilon_t^* \qquad (6.6)$$

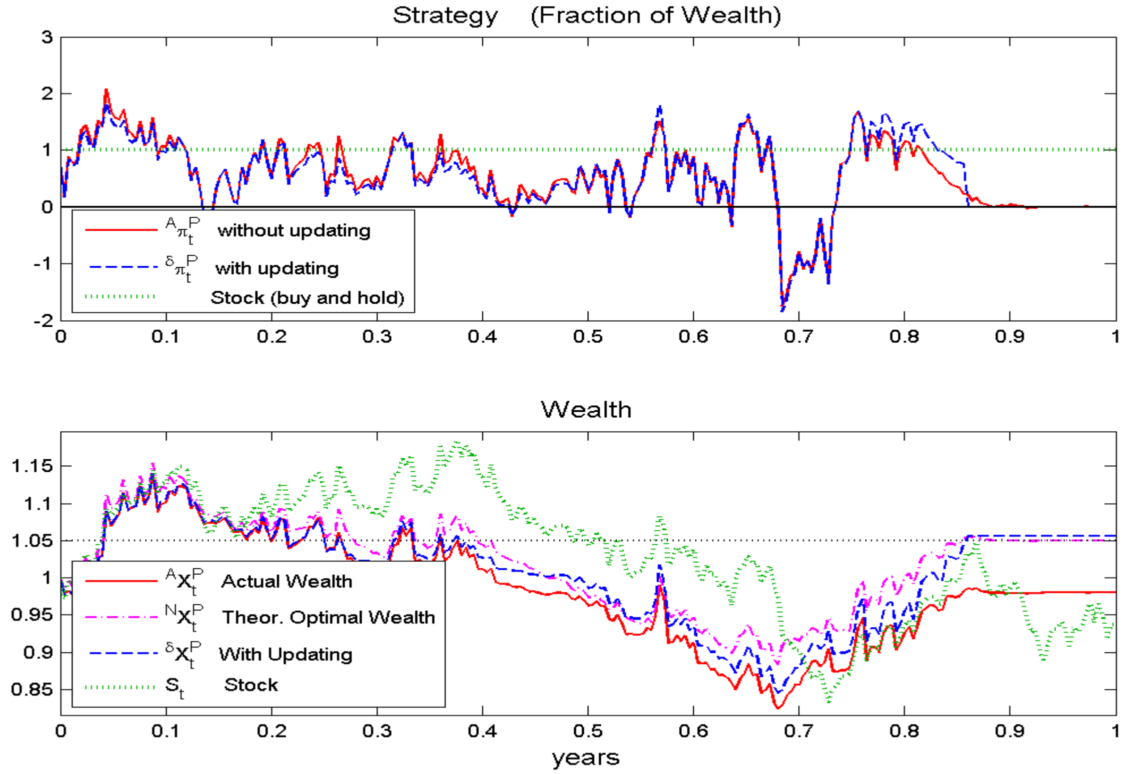

Figure 6.8 Top: PEL-optimal strategy with updating for threshold $\delta = 0.01$ and without updating ($\delta = \infty$); bottom: corresponding wealth processes and theoretically optimal wealth.

for the updated optimal Lagrange multipliers y_1^t, y_2^t. With these updated Lagrange multipliers we compute the corrected strategies as described in Section 6.3. We only have to replace $f(\zeta_T) = f(\zeta_T; y_1^*, y_2^*)$ by $f(\zeta_T; y_1^t, y_2^t)$ to get the updated form of the optimal terminal wealth and to compute the optimal strategy according to (6.6). We reset the theoretical optimal wealth X_t^* to the new initial capital ${}^AX_t^*$ and follow the updated strategy until the next time $t' > t$ when a tracking error is detected. Then we repeat the updating steps described above. We denote the resulting strategy by ${}^\delta\pi_t^*$ and the wealth by ${}^\delta X_t^*$.

Remark 6.4 If we replace in (6.6) the actual wealth ${}^AX_t^*$ and the approximation ${}^N\varepsilon_t^*$ by their respective theoretical values X_t^* and ε_t^* then the resulting equations are fulfilled for the optimal Lagrange multipliers y_1^*, y_2^* computed at time 0. Therefore, updating yields $y_1^t = y_1^*$ and $y_2^t = y_2^*$, i.e. the updating procedure is consistent with the optimal strategy.

Remark 6.5 In case of ${}^AX_t^* < q$ it may happen that there is no admissible solution of the above optimization problem because of the deviations of ${}^AX_t^*$ and ${}^N\varepsilon_t^*$ from their respective theoretical values. Then the bound ${}^N\varepsilon_t^*$ for the risk is smaller than the corresponding minimal risk $\underline{\varepsilon}_t$ of a portfolio starting at time t with initial capital ${}^AX_t^*$. For the theoretical values this is impossible, so we assume that the difference ${}^N\varepsilon_t^* - \underline{\varepsilon}_t$ is negative but close to zero. In this case one may follow the optimal strategy for the risk minimizing case.

Figure 6.8 illustrates the updating procedure for an example with benchmark $q = 1.05$, bound $\varepsilon = 0.1$ for PEL, power utility with $\alpha = 0.5$, for $T = 1$ year consisting of $M = 250$

trading days. The top plot shows the approximation of the PEL-optimal strategy ${}^A\pi^P$ and the updated strategy ${}^\delta\pi^P$ for threshold $\delta = 0.01$. The bottom plot shows the actual wealth ${}^AX^P$ together with the corresponding approximation ${}^NX^P$ for the theoretical optimal wealth X^P and the updated wealth ${}^\delta X^P$.

For this example we have chosen a path of the stock prices which is at T below its initial price $S_0 = 1$ and in particular below the benchmark $q = 1.05$. In Figure 6.8, when t approaches T, the approximation ${}^A\pi_t^P$ turns into the pure bond strategy $\pi_t \equiv 0$ ensuring a theoretical terminal wealth $X_T^P = q$ and preventing a shortfall. An investor who trades daily and readjusts the portfolio weights according to ${}^A\pi^P$ does not generate the theoretical terminal wealth $X_T^P = q$. Instead the investor's portfolio incurs a shortfall.

Contrary to ${}^A\pi^P$, for the updated strategy ${}^\delta\pi^P$ the terminal wealth (nearly) reaches the benchmark q which is in this example the theoretical terminal wealth for the original optimal Lagrange multipliers y_1^P, y_2^P. Note, that in general the updated actual wealth does not reach the (not updated) theoretical optimal wealth, it can be larger as well as smaller.

Finally, we want to investigate the impact of the updating procedure on the distribution of the terminal wealth. For parameters as above we generate $L = 500$ paths of μ and W and simulate the resulting stock prices S and stock returns R which we use as observations. For each of these 500 observation paths we compute the corresponding filters and the terminal wealth ${}^\delta X_T^P$ of portfolios following the PEL-optimal strategies

(i) without updating ($\delta = \infty$)

(ii) with updating for threshold $\delta = 0.01$.

In both cases we get a sample of L values for the terminal wealth ${}^\delta X_T^P$ from which we estimate the mean value, the expected utility, the risk measure PEL, the shortfall probability and the probability density function. These quantities we want to compare with the corresponding theoretical values.

The distribution of the theoretical optimal terminal wealth X_T^P is absolutely continuous on $\mathbb{R} \setminus \{q\}$ and contains an atom at the point $\{q\}$. This causes problems in the density plots. Therefore we use a histogram like representation for the interval $(q-c, q+c]$, where $c = 0.02$. Consistently we compute the shortfall probability as $P(X_T^P < q - c)$.

Figure 6.9 compares the estimates for the probability density function of the terminal wealth ${}^\infty X_T^P$ without updating ($\delta = \infty$) and the corresponding wealth ${}^\delta X_T^P$ generated by the updated strategy with the "theoretical" density of $X_T^P = f_P(\zeta_T)$, obtained from a sample of $N = 10^7$ realizations of ζ_T. Note that there is a logarithmic scaling of the vertical axis. The density plot indicates, that without updating the distribution of the terminal wealth ${}^\infty X_T^P$ is considerably different from the theoretical distribution. Probability mass from the atom at the benchmark q is shifted to

(a) the shortfall region $(0, q)$,
this increases the shortfall probability as well as the expected loss;

(b) the region (q, ∞),
this compensates partly the "loss" of expected utility and terminal wealth resulting from the high probability for values smaller than q.

Updating clearly improves the approximation of the theoretical density, in particular in the region (q, ∞). Since a part of the probability mass of the atom at q is still shifted

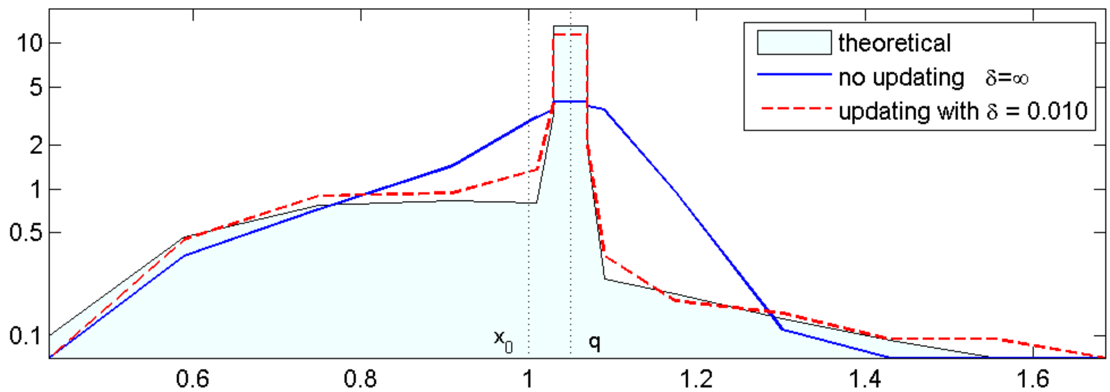

Figure 6.9 Estimates for the probability density functions of ${}^{\infty}X_T^P$, ${}^{\delta}X_T^P$ for $\delta = 0.01$ and X_T^P

	Expected terminal wealth $E[X_T]$	Expected utility $E[U(X_T)]$	Present expected loss (PEL) $\widetilde{E}[(X_T - q)^-]$	Shortfall probability $P(X_T < q - c)$
$\delta = \infty$	1.01 (0.02)	−0.01 (0.02)	0.13 (0.01)	0.53 (0.02)
$\delta = 0.01$	1.02 (0.03)	0.00 (0.03)	0.13 (0.01)	0.41 (0.02)
theoretical	1.08	0.05	$0.10 = \varepsilon$	0.38

Table 6.3 Expected terminal wealth, expected utility, risk measure PEL and shortfall probability for the PEL-optimal portfolios with updating for threshold $\delta = 0.01$, without updating ($\delta = \infty$) and corresponding theoretical values.

to the shortfall region $(0, q)$, we observe only slight improvements in the expected values given in Table 6.3. This table gives estimates for expected (utility of) terminal wealth, PEL and shortfall probability for applying the PEL-optimal strategies with and without updating. The last row contains the corresponding estimates for $X_T^P = f_P(\zeta_T)$ using $N = 10^7$ realizations of ζ_T which we call "theoretical" values. For assessing the estimation error we give in parentheses the tolerance values Δ of asymptotic confidence intervals of the form $[m - \Delta, m + \Delta]$ for error probability 5%. Here m denotes the point estimate for the above expectations. The portfolios following the approximated optimal strategies (with and without) updating do not reach the theoretical maximum expected utility $E[U(X_T^P)] \approx 0.05$ and they also fail to fulfill the risk constraint since their expected loss is larger than $\varepsilon = 0.1$. Updating results in an improvement but there is still a considerable gap to the theoretical values. This has to be expected since once we lose track of the optimal portfolio we can only do the best given the current wealth. The updating avoids a worse scenario, but one cannot make up for the losses made so far.

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