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A POSTERIORI ERROR ESTIMATES FOR NONCONFORMING APPROXIMATION OF ELLIPTIC PROBLEMS BASED ON HELMHOLTZ TYPE DECOMPOSITION OF THE ERROR

SERGEY REPIN AND SATYENDRA TOMAR

Abstract. We present new a posteriori error estimates for nonconforming approximations of elliptic problems. Our analysis is based on Helmholtz type decomposition of the error expressed in terms of fluxes. Such a decomposition of the nonconforming error has been suggested and studied in [2, 7] where it was shown that the error can be presented as the sum of two functions (a gradient term and a divergence-free term), which are the exact solutions of two auxiliary problems. We follow a similar decomposition of the error but use a different procedure for obtaining computable two-sided bounds of the norms of these solutions. For this purpose, we use the method suggested in [23] for conforming approximations. A posteriori estimates obtained in this paper differ from those which are based on a different type of decomposition of the error and the projection of the nonconforming approximation to the conforming space (see [18, 28]). Numerical experiments confirm that these new estimates provide very accurate error bounds and can be efficiently exploited in practical computations.

1. Introduction

In this paper we derive a posteriori estimates for the nonconforming approximations of the problem

$$\mathbf{div}\, A\nabla u + f = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = u_0 \quad \text{on } \Gamma_1, \tag{1.2}$$

$$A\nabla u \cdot n = F \quad \text{on } \Gamma_2, \tag{1.3}$$

where $\Omega \in \mathbb{R}^d$ is a bounded connected domain with Lipschitz continuous boundary that consists of two measurable nonintersecting parts Γ_1 and Γ_2. We assume that $\text{meas}_{d-1}\{\Gamma_1\} > 0$, and the matrix $A = \{a_{ij}\}$ is symmetric and satisfies the relation

$$c_1^2|\xi|^2 \leq A\xi \cdot \xi \leq c_2^2|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d. \tag{1.4}$$

We also assume that $u_0 \in H^1(\Omega)$, $f \in L^2(\Omega)$ and $F \in L^2(\Gamma_2)$. Further, let

$$V_0 + u_0 := \{v = w + u_0 \mid w \in V_0(\Omega)\},$$

where

$$V_0 := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\}.$$

We will also need the following space in the analysis

$$H_{0,\Gamma_2} := \left\{ \tau \in L^2(\Omega, \mathbb{R}^d) \mid \int\limits_\Omega \tau \cdot \nabla w \, dx = 0, \quad \forall w \in V_0 \right\}.$$

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Note that, if $\tau_0 \in H_{0,\Gamma_2}$ then $\mathbf{div}\ \tau_0 = 0$ and $\tau_0 \cdot n = 0$ on Γ_2. Further, when $\Gamma_2 = \emptyset$ then H_{0,Γ_2} will be replaced by H_0 for simplicity reasons.

The generalized solution u of (1.1)-(1.3) is a function in $V_0 + u_0$ that meets the integral identity

$$\int\limits_{\Omega} A\nabla u \cdot \nabla w \; dx = \int\limits_{\Omega} fw \; dx + \int\limits_{\Gamma_2} Fw \; ds, \quad \forall w \in V_0(\Omega). \tag{1.5}$$

It is well known that the solution u exists and is unique.

We analyze errors of the nonconforming approximations that belong to a functional class $\widehat{V}$ wider than $H^1(\Omega)$. More precisely, we assume a partition $\mathcal{T}_h$ of Ω into a collection of subdomains Ω_i, $i = 1, 2, \ldots, N$, such that

$$\overline{\Omega} = \bigcup_i \overline{\Omega}_i.$$

We consider approximations that may violate the continuity on the boundaries of subdomains Ω_i and boundary conditions on Γ_1. The respective functions are marked by "hats" and form a *broken* Sobolev space

$$\widehat{V} = \widehat{H}^1(\Omega) := \left\{ \widehat{v} \in L^2(\Omega) \; : \; \widehat{v} \in H^1(\Omega_i) \quad i = 1, 2 \ldots, N \right\},$$

with the norm defined by the relation

$$|[\widehat{v}]|^2 = \sum_i \int\limits_{\Omega_i} A\widehat{\nabla}\widehat{v} \cdot \widehat{\nabla}\widehat{v} \; dx. \tag{1.6}$$

Finite-dimensional subspace related to the nonconforming partition $\mathcal{T}_h$ is defined as

$$\widehat{V}_h := \left\{ v \in L^2(\Omega) \; : \; v|_{\Omega_i} \in P_r(\Omega_i), \quad i = 1, 2 \ldots, N \right\},$$

where P_r is the set of polynomials of degree $r \geq 1$. We now define the following norms which will be used in the analysis:

$$\| q \|^2_{\Omega_i} := \int\limits_{\Omega_i} Aq \cdot q \; dx, \qquad \|[q]\|^2 := \sum_i \| q \|^2_{\Omega_i},$$

$$\| q \|^2_{*,\Omega_i} := \int_{\Omega_i} A^{-1} q \cdot q \; dx, \qquad \|[q]\|^2_* := \sum_i \| q \|^2_{*,\Omega_i} .$$

Note that Ω_i will be omitted from the notations $\| \cdot \|_{\Omega_i}$ and $\| \cdot \|_{*,\Omega_i}$ whenever $\Omega_i = \Omega$. Note also that $|[\widehat{v}]| = \|[\widehat{\nabla}\widehat{v}]\|$.

A posteriori error estimates for conforming approximations of various boundary value problems have been thoroughly studied and a huge literature can be found. However, a posteriori error estimates for nonconforming approximations, in particular, DG approximations, is comparatively a new topic and has a significant interest these days. In mesh-dependent energy-type norm a residual based error estimator has been used in, e.g., [1, 2, 4, 9, 13, 14]. Similar estimates for the *local* DG (LDG) approximation of linear and nonlinear diffusion problems were derived in [5]. A posteriori error analysis for locally conservative mixed methods, with applications to P^1 nonconforming FEM and interior penalty DG (IPDG) as well as the mixed finite element method, has been studied in [16, 17] (the latter for nonlinear elliptic problems). Recently, a new form of the a posteriori estimate with an advanced structure of the residual terms has been proposed in [9]. Estimates (upper bounds) obtained in, e.g., [2, 9] are of guaranteed nature, i.e. they contain no un-determined constants. Guaranteed functional type both-sided bounds of approximation errors for any of the DG schemes can be found in [18, 28]. For linear elliptic boundary value problems a different approach based on the Helmholtz decomposition of the error is presented in [2, 4, 5]. L^2-norm a posteriori error estimates for elliptic boundary value problems are derived for the DG approximation in [26], and for the LDG approximation in [6], respectively. A posteriori error estimates for DG approximations were also obtained for other classes of problems, in particular, in [27] time-dependent (transport) equations and in [12] elliptic problems of the Maxwell type, were considered. In [8, 9] a posteriori estimates were obtained for DG approximations of the convection-diffusion equation.

Guaranteed functional type both-sided bounds of approximation errors generated by the nonconforming approximations can be derived in two ways. The first approach consists of projecting a nonconforming approximation into the energy space, with the help of a suitable post-processing procedure, and using the functional error majorant/minorant for the post-processed approximation. This is always possible because the latter is valid for any conforming approximation. This approach was presented in [18] within the framework of Discontinuous Galerkin (DG) methods. The paper [28] continues the analysis and discusses a series of numerical experiments that have shown practical applicability of this computational technology. The second approach is based on the Helmholtz decomposition of vector-valued fields. The Helmholtz decomposition is applied to the error function expressed in terms of fluxes, where the approximate flux is considered as a piecewise continuous function defined at almost all points of a domain. Such a decomposition of the error has been suggested and studied in [1, 2, 7] where it was shown that the error can be decomposed into two orthogonal parts ("gradient" and "divergence-free") which are defined as the solutions of the auxiliary problems. We also use a similar decomposition of the error into the "gradient" and "divergence-free" parts and obtain computable two-sided bounds for them. For this purpose, we use the same technique which was earlier proposed for the derivation of a posteriori error estimates for conforming approximations (see [21, 22, 23] and the book [20]). The estimates thus obtained differ from those derived using a projection to the conforming space in [18, 28], see also [9, 16] for a similar decomposition of the error. Since the evaluation of errors is performed on the functional level, the estimates are applicable to the nonconforming approximations of all the types, e.g. discontinuous Galerkin, mortar elements, versions of the Trefftz method, etc. The numerical experiments confirm the practical efficiency of the proposed error estimation method and that these estimates are qualitatively same as those obtained from the projection onto the conforming space [18, 28].

The outline of the paper is as follows. In Section 2, we present basic results related to the Helmholtz type decomposition of the error. Upper and lower bounds of the error are derived in Sections 3, and 4, respectively. Main result is given by the estimates/Theorems 3.1 and 4.1. In Section 5 we consider the application of the a posteriori estimates to the Trefftz type approximations and obtain new estimates for this case. The numerical experiments are presented in Section 6.

2. Helmholtz decomposition of the error

A posteriori estimates based on Helmholtz type decomposition of the error were studied in [1, 2, 7] for nonconforming approximations, and in [25] for conforming approximations. In our context, we follow [1, 2, 7] but use a different approach.

For function v, we define "broken" gradient $\widehat{\nabla}\widehat{v}$ at almost all points of Ω by the relation $\widehat{\nabla}\widehat{v}(x) := \nabla\widehat{v}(x)$ for $x \in \Omega_i$, $i = 1, 2, \ldots, N$. Then, the error function

$$\eta := \nabla u - \widehat{\nabla}\widehat{v}$$

can be associated with a vector-valued function in $L^2(\Omega, \mathbb{R}^d)$, for which the well known Helmholtz decomposition applies. For our purposes, it is more convenient to use a similar orthogonal decomposition of $A\eta$, namely

$$A\eta = A\nabla u_\eta + \tau_0, \tag{2.1}$$

where $u_\eta \in V_0$ and $\tau_0 \in H_{0,\Gamma_2}$. In general, these relations should be understood in a generalized sense and they hold in the classical sense if τ_0 is a sufficiently regular function.

The decomposition (2.1) is motivated by the problem:
Find $u_\eta \in V_0$ such that

$$\int_\Omega A\nabla u_\eta \cdot \nabla w \; dx = \int_\Omega A\eta \cdot \nabla w \; dx, \quad \forall w \in V_0, \tag{2.2}$$

which is uniquely solvable. From (2.2) it follows that

$$\int_\Omega \tau_0 \cdot \nabla w \; dx = 0, \quad \forall w \in V_0, \qquad \text{where } \tau_0 := A(\eta - \nabla u_\eta). \tag{2.3}$$

This implies that $\tau_0 \in H_{0,\Gamma_2}$.

Now following [1, 2, 7], τ_0 can be presented as $\mathbf{curl}\ \psi$ where the scalar-valued function $\psi \in \mathcal{H}$ is defined by the relation

$$\int_\Omega A^{-1}\mathbf{curl}\ \psi \cdot \mathbf{curl}\ v \; dx = \int_\Omega \eta \cdot \mathbf{curl}\ v \; dx \quad \forall v \in \mathcal{H}. \tag{2.4}$$

Here $\mathcal{H}$ denotes the subspace of $H^1(\Omega)/\mathbb{R}$ which contains functions with zero values of tangential derivatives on Γ_2.

By the definition of H_{0,Γ_2} we know that τ_0 is orthogonal to ∇w for any $w \in V_0$. Therefore, we arrive at the following important relation

$$\begin{aligned} \|[\eta]\|^2 = \sum_i \int_{\Omega_i} A\eta \cdot \eta \; dx &= \int_\Omega \left(A\nabla u_\eta \cdot \nabla u_\eta + A^{-1}\tau_0 \cdot \tau_0 + 2\nabla u_\eta \cdot \tau_0 \right) \; dx \\ &= \| \nabla u_\eta \|^2 + \| \tau_0 \|_*^2 \, . \end{aligned} \tag{2.5}$$

Hence, the overall error is presented by (2.5), where u_η and τ_0 are the solutions of the auxiliary boundary-value problems (2.2) and (2.4), respectively. The relation (2.5) plays a crucial role in the subsequent analysis.

As in [2], we now directly compute two-sided bounds for the terms $\| \nabla u_\eta \|$ and $\| \tau_0 \|_*$, respectively. However, our approach and the results are essentially different than those in [2]. For $\| \nabla u_\eta \|$ we use the same procedure that has been used in the process of deriving the so-called functional both-sided error bounds for the conforming approximations (originally exposed in [23], see also [24]). Then, we show that the value of $\| \tau_0 \|_*$ is estimated from above by the broken norm of the difference $\nabla v - \widehat{\nabla} \widehat{v}$ (the nonconforming error), for some $v \in V_0 + u_0 + \mathbb{R}$. The sum of these estimates gives a directly computable bound of the error expressed in the broken energy norm.

3. Upper bound of the error

We first find computable upper bounds for the norms $\| \nabla u_\eta \|$ and $\| \tau_0 \|_*$.

3.1. **Upper bound of $\|\!| \nabla u_\eta \|\!|$.**

Lemma 3.1. *Let us assume that $u_\eta \in V_0$, $\widehat{v} \in \widehat{V}$, and $y \in H(\Omega, \mathbf{div})$. Also, let the positive constant $\lambda_1(\Omega, \Gamma_2)$ be such that*

$$\lambda_1^2(\Omega, \Gamma_2) = \inf_{w \in V_0} \frac{\|\!| \nabla w \|\!|^2}{\|w\|^2 + \|w\|^2_{\Gamma_2}}. \tag{3.1}$$

Then, the following relation holds for an arbitrary positive constant $C > \lambda_1^{-1}(\Omega, \Gamma_2)$

$$\|\!| \nabla u_\eta \|\!| \leq \|[y - A\widehat{\nabla}\widehat{v}]\|_* + C\left(\|\mathbf{div}\ y + f\|^2 + \|y \cdot n - F\|^2_{\Gamma_2}\right)^{1/2}. \tag{3.2}$$

Further, if instead of C we use the constants C_{F,Γ_1} and C_{T,Γ_2} then another upper bound is obtained of the form

$$\|\!| \nabla u_\eta \|\!| \leq \|[y - A\widehat{\nabla}\widehat{v}]\|_* + C_{F,\Gamma_1}\|\mathbf{div}\ \mathrm{y} + f\| + C_{T,\Gamma_2}\|y \cdot n - F\|_{\Gamma_2}, \tag{3.3}$$

where C_{F,Γ_1} and C_{T,Γ_2} denote the constants in the following Friedrich's type inequality and the trace inequality, respectively,

$$\|w\| \leq C_{F,\Gamma_1}\|\nabla w\| \qquad \forall w \in V_0, \tag{3.4}$$

$$\|w\|_{\Gamma_2} \leq C_{T,\Gamma_2}\|\nabla w\| \qquad \forall w \in V_0. \tag{3.5}$$

Proof. From (2.1) and the definition of the error function we have $\forall w \in V_0$

$$\begin{aligned}
\int_\Omega A\nabla u_\eta \cdot \nabla w\, dx &= \sum_i \int_{\Omega_i} A\eta \cdot \nabla w\, dx - \int_\Omega \tau_0 \cdot \nabla w\, dx \\
&= \int_\Omega A\nabla u \cdot \nabla w\, dx - \sum_i \int_{\Omega_i} A\widehat{\nabla}\widehat{v} \cdot \nabla w\, dx \\
&= \int_\Omega fw\, dx + \int_{\Gamma_2} Fw\, d\Gamma - \sum_i \int_{\Omega_i} A\widehat{\nabla}\widehat{v} \cdot \nabla w\, dx.
\end{aligned} \tag{3.6}$$

We rearrange the right-hand side of (3.6) by introducing a vector-valued function $y \in H(\Omega, \mathbf{div})$ and obtain

$$\int_\Omega A\nabla u_\eta \cdot \nabla w\, dx = \int_\Omega (\mathbf{div}\ y + f)w\, dx + \int_{\Gamma_2} (F - y \cdot n)w\, d\Gamma + \sum_i \int_{\Omega_i} (y - A\widehat{\nabla}\widehat{v}) \cdot \nabla w\, dx. \tag{3.7}$$

Now, it is easy to see that

$$\begin{aligned}
\sum_i \int_{\Omega_i} (y - A\widehat{\nabla}\widehat{v}) \cdot \nabla w\, dx &\leq \sum_i \|\!| y - A\widehat{\nabla}\widehat{v} \|\!|_{*,\Omega_i} \|\!| \nabla w \|\!|_{\Omega_i} \leq \|[y - A\widehat{\nabla}\widehat{v}]\|_* \|[\nabla w]\| \\
&= \|[y - A\widehat{\nabla}\widehat{v}]\|_* \|\!| \nabla w \|\!| .
\end{aligned} \tag{3.8}$$

Besides,

$$\left| \int_\Omega (\mathbf{div}\ y + f)w\, dx + \int_{\Gamma_2} (F - y \cdot n)w\, ds \right| \leq C\left(\|\mathbf{div}\ y + f\|^2 + \|F - y \cdot n\|^2_{\Gamma_2}\right)^{1/2} \|\!| \nabla w \|\!|, \tag{3.9}$$

where $C > \lambda_1^{-1}(\Omega, \Gamma_2)$ is any constant. Then, by setting $w = u_\eta$ in (3.7) and using (3.8) and (3.9) we get (3.2). (3.3) follows immediately. □

Remark 3.1. *We observe that the upper bound (3.2) is valid for* any $\widehat{v} \in \widehat{V}$ *and involves only one global constant C, the value of which can be estimated by minimizing the quotient (3.1) on a sufficiently rich finite-dimensional subspace.*

3.2. **Upper bound of $\| \tau_0 \|_*$.**

Lemma 3.2. *Let us assume that $\widehat{v} \in \widehat{V}$, $v \in V_0 + u_0 + \mathbb{R}$, and $\tau_0 \in H_{0,\Gamma_2}$. Then, the following relation holds*

$$\| \tau_0 \|_* \leq \| [\nabla v - \widehat{\nabla} \widehat{v}] \| \, . \tag{3.10}$$

Proof. Again, from (2.1) and the definition of the error function we have

$$\begin{aligned}\int\limits_\Omega A^{-1}\tau_0 \cdot \tau_0 \, dx &= \sum_i \int\limits_{\Omega_i} (\eta - \nabla u_\eta) \cdot \tau_0 \, dx = \sum_i \int\limits_{\Omega_i} (\nabla u - \widehat{\nabla} \widehat{v}) \cdot \tau_0 \, dx \\ &= \sum_i \int\limits_{\Omega_i} (\nabla v - \widehat{\nabla} \widehat{v}) \cdot \tau_0 \, dx. \end{aligned} \tag{3.11}$$

Hence,

$$\| \tau_0 \|_*^2 \leq \left(\sum_i \int\limits_{\Omega_i} A(\nabla v - \widehat{\nabla} \widehat{v}) \cdot (\nabla v - \widehat{\nabla} \widehat{v}) \, dx \right)^{1/2} \| \tau_0 \|_*,$$

and this gives the result. □

3.3. **Upper bound of the error.** We are now in a position to give the upper bound of the error in the broken energy norm.

Theorem 3.1. *Let us assume that $\widehat{v} \in \widehat{V}$ and $v \in V_0 + u_0 + \mathbb{R}$. Then, the upper bounds of the error are given by the estimates*

$$\| [\eta] \|^2 \leq \inf_{v \in V_0 + u_0 + \mathbb{R}} \| [\nabla v - \widehat{\nabla} \widehat{v}] \|^2 + \left(\| [y - A \widehat{\nabla} \widehat{v}] \|_* + C \left(\| \mathbf{div}\, y + f \|^2 + \| y \cdot n - F \|_{\Gamma_2}^2 \right)^{1/2} \right)^2, \tag{3.12}$$

and

$$\| [\eta] \|^2 \leq \inf_{v \in V_0 + u_0 + \mathbb{R}} \| [\nabla v - \widehat{\nabla} \widehat{v}] \|^2 + \left(\| [y - A \widehat{\nabla} \widehat{v}] \|_* + C_{F,\Gamma_1} \| \mathbf{div}\, y + f \| + C_{T,\Gamma_2} \| y \cdot n - F \|_{\Gamma_2} \right)^2, \tag{3.13}$$

where y is an arbitrary vector-valued function in $H(\Omega, \mathbf{div})$. Further, if y is subject to the boundary condition $y \cdot n = F$, then we have the following relation

$$\| [\eta] \|^2 \leq \| [\nabla v - \widehat{\nabla} \widehat{v}] \|^2 + \left(\| [y - A \widehat{\nabla} \widehat{v}] \| + C \, \| \mathbf{div}\, y + f \| \right)^2 . \tag{3.14}$$

Proof. Follows directly from the relations (2.5), (3.2), (3.3), and (3.10). □

Remark 3.2. *The right hand side of (3.12) presents a natural decomposition of the overall error into three terms: nonconforming error, error in the duality relation for fluxes, and error in the equilibrium equation and boundary condition for fluxes.*

Let us now set $v = \mathsf{P}(\widehat{v})$, where $\mathsf{P} : \widehat{V} \to V_0 + u_0 + \mathbb{R}$ denotes the projection operator, see, e.g., [18, Section 5.1.2] for orthogonal projection, and [28, Algorithm 1] for Oswald interpolation operator. Then, denoting $\|[\nabla v - \widehat{\nabla}\widehat{v}]\|$ by η_{P} (the *projection error* which is directly computable), we get

$$\|[\eta]\|^2 \leq \eta_{\mathsf{P}}^2 + (1+\beta)\, \|[y - A\widehat{\nabla}\widehat{v}]\|^2 + C^2\left(1+\frac{1}{\beta}\right)\|\mathbf{div}\, y + f\|^2\,, \tag{3.15}$$

where β is an arbitrary positive number. Minimization with respect to y is now reduced to a quadratic problem.

4. Lower bound of the error

To derive a lower bound of the error we again use (2.5). Our goal now is to find lower bounds for the norms in the right-hand side of (2.5).

4.1. Lower bound of $\|\nabla u_\eta\|$.

Lemma 4.1. *Let us assume that $u_\eta \in V_0$ and $\widehat{v} \in \widehat{V}$. Then, the following relation holds $\forall w \in V_0$*

$$\|\nabla u_\eta\|^2 \geq 2\ell(w) - \|\nabla w\|^2, \tag{4.1}$$

where $\ell : V_0 \to \mathbb{R}$ is a linear functional defined as

$$\ell(w) = \int\limits_{\Gamma_2} Fw\, d\Gamma + \int\limits_{\Omega} fw\, dx - \sum_i \int\limits_{\Omega_i} A\widehat{\nabla}\widehat{v}\cdot\nabla w\, dx. \tag{4.2}$$

Proof. We first rewrite (3.6) in the form

$$\int_\Omega A\nabla u_\eta \cdot \nabla w\, dx = \ell(w). \tag{4.3}$$

Note that (4.3) is the Euler's equation of the variational problem

$$\min_{w\in V_0} J_\ell(w), \quad \text{where } J_\ell(w) = \frac{1}{2}\|\nabla w\|^2 - \ell(w). \tag{4.4}$$

From (4.3) it follows that

$$\|\nabla u_\eta\|^2 = \langle \ell, u_\eta\rangle, \tag{4.5}$$

and, therefore,

$$J_\ell(u_\eta) = -\frac{1}{2}\|\nabla u_\eta\|^2\,. \tag{4.6}$$

From (4.4) and (4.6), we obtain

$$\|\nabla u_\eta\|^2 = -2J_\ell(u_\eta) = -2\inf_{w\in V_0}\left\{\frac{1}{2}\|\nabla w\|^2 - \ell(w)\right\} = \sup_{w\in V_0}\left\{2\ell(w) - \|\nabla w\|^2\right\}.$$

The result is then an immediate consequence. □

For the second term in (2.5), we proceed analogously in the next section.

4.2. **Lower bound of $\| \tau_0 \|$.**

Lemma 4.2. *Let us assume that $\widehat{v} \in \widehat{V}$, $v \in V_0 + u_0 + \mathbb{R}$, and $\tau_0 \in H_{0,\Gamma_2}$. Then the following relation holds $\forall \zeta_0 \in H_{0,\Gamma_2}$*

$$\| \tau_0 \|_*^2 \geq 2\mu(\zeta_0) - \| \zeta_0 \|_*^2, \tag{4.7}$$

where the linear functional $\mu : H_{0,\Gamma_2} \to \mathbb{R}$ is defined as

$$\mu(\zeta_0) := \sum_i \int_{\Omega_i} (\nabla v - \widehat{\nabla}\widehat{v}) \cdot \zeta_0 \, dx. \tag{4.8}$$

Proof. In view of (2.1), τ_0 satisfies the following relation

$$\int_\Omega A^{-1}\tau_0 \cdot \zeta_0 \, dx = \int_\Omega (\eta - \nabla u_\eta) \cdot \zeta_0 \, dx = \sum_i \int_{\Omega_i} (\nabla u - \widehat{\nabla}\widehat{v}) \cdot \zeta_0 \, dx = \sum_i \int_{\Omega_i} (\nabla v - \widehat{\nabla}\widehat{v}) \cdot \zeta_0 \, dx$$

$$= \mu(\zeta_0). \tag{4.9}$$

Then, τ_0 is a minimizer of the variational problem

$$\min_{\zeta_0 \in H_{0,\Gamma_2}} I_\mu(\zeta_0), \quad \text{where } I_\mu(\zeta_0) := \left\{ \frac{1}{2} \| \zeta_0 \|_*^2 - \mu(\zeta_0) \right\}. \tag{4.10}$$

By the similar arguments as used before we conclude that

$$\| \tau_0 \|_*^2 = -2I_\mu(\tau_0) = \sup_{\zeta_0 \in H_{0,\Gamma_2}} \left\{ 2\mu(\zeta_0) - \| \zeta_0 \|_*^2 \right\}. \tag{4.11}$$

The result is then an immediate consequence. □

4.3. **Lower bound of the error.** We now give the lower bound of the error in the broken energy norm.

Theorem 4.1. *With $\ell(w)$ defined by (4.2) and $\mu(\zeta_0)$ defined by (4.8), a lower bound of the error is given by the relation*

$$\|[\eta]\|^2 \geq 2\ell(w) + 2\mu(\zeta_0) - \| \nabla w \|^2 - \| \zeta_0 \|_*^2 . \tag{4.12}$$

where w and ζ_0 are arbitrary functions in V_0 and H_{0,Γ_2}, respectively.

Proof. Combining (4.1) and (4.7) we immediately obtain the result. Note that, however, getting a realistic estimate certainly requires a proper selection of these functions. □

Remark 4.1. *If $\widehat{v} \in V_0 + u_0 + \mathbb{R}$, then $\mu(\zeta_0) \equiv 0$, and we should take $\zeta_0 = 0$ to make the right-hand side of (4.12) maximal. In this case, (4.12) is transformed to the lower bound that was derived for conforming approximations in* [23].

Remark 4.2. *Note that, both-sided estimates similar to those in (3.12) and (4.12) can be easily derived if instead of the norm (1.6) we use the full DG norm, see e.g.* [3, Eqs. (4.1–4.2)].

5. Accuracy of approximations obtained by the method of Trefftz

In this method (see, e.g., [19, 15, 11]), approximations are constructed as a series $\sum_i \alpha_i \phi_i(x)$, where ϕ_i exactly satisfy the given differential equation. The coefficients α_i are selected in order to approximate the Dirichlet boundary condition as accurately as possible (in the sense of least squares). Hence, for the problem (1.1)–(1.3) the respective approximate solution $\widehat{v}$ satisfies the equation

$$\mathbf{div}\, A\nabla\widehat{v} + f = 0, \quad \text{in } \Omega, \tag{5.1}$$

but violates the condition $u = u_0$ on Γ_1.

5.1. Upper bound of the error.

Theorem 5.1. *Let us assume that $\widehat{v} \in \widehat{V}$ and $v \in V_0 + u_0 + \mathbb{R}$. Then, the upper bound of the error is given by the estimate*

$$\| \nabla u - \nabla\widehat{v} \|^2 \leq \inf_{v \in V_0 + u_0 + \mathbb{R}} \| \nabla(v - \widehat{v}) \|^2 + C^2 \|A\nabla\widehat{v} \cdot n - F\|^2_{\Gamma_2}. \tag{5.2}$$

Further, if $\Gamma_2 = \emptyset$, then the error is estimated by a simple projection estimate

$$\| \nabla u - \nabla\widehat{v} \| \leq \| \nabla(\mathsf{P}(\widehat{v}) - \widehat{v}) \| . \tag{5.3}$$

Proof. We apply (3.12) to Trefftz type approximations. Note that, in this case, $\widehat{V} = H^1(\Omega)$, and $\widehat{\nabla} = \nabla$. Thus,

$$\|[\nabla u - \widehat{\nabla}\widehat{v}]\| = \| \nabla(u - \widehat{v}) \| . \tag{5.4}$$

Now in (3.12) setting $y = A\nabla\widehat{v}$ and using (5.1) gives (5.2). Further, (5.3) follows by taking $v = \mathsf{P}(\widehat{v})$ in (5.2). □

5.2. Lower bound of the error.

Theorem 5.2. *Let us assume that $\widehat{v} \in \widehat{V}$ and $v \in V_0 + u_0 + \mathbb{R}$. Then, with $\ell(w)$ defined by (4.2) and $\mu(\zeta_0)$ defined by (4.8), a lower bound of the error is given by the relation*

$$\| \nabla(u - \widehat{v}) \|^2 \geq 2 \int_{\Gamma_2} (F - A\nabla\widehat{v} \cdot n) w \, d\Gamma + 2 \int_{\Gamma_1} (v - \widehat{v}) n \cdot \zeta_0 \, d\Gamma - \| \nabla w \|^2 - \| \zeta_0 \|^2_*, \tag{5.5}$$

where w and ζ_0 are arbitrary functions in V_0 and H_{0,Γ_2}, respectively. Further, if $\Gamma_2 = \emptyset$, then the lower bound is given by a simple relation

$$\| \nabla(u - \widehat{v}) \|^2 \geq \sup_{\bar{\zeta}_0 \in H_0, \|\bar{\zeta}_0\|_* = 1} \int_{\Gamma} (v - \widehat{v}) n \cdot \bar{\zeta}_0 \, d\Gamma. \tag{5.6}$$

Proof. Since $\widehat{\nabla} = \nabla$, we can write $\ell(w)$ and $\mu(\zeta_0)$ as

$$\ell(w) = \int_{\Gamma_2} Fw \, d\Gamma + \int_{\Omega} fw \, dx - \int_{\Omega} A\nabla\widehat{v} \cdot \nabla w \, dx, \tag{5.7}$$

$$\mu(\zeta_0) = \int_{\Omega} (\nabla v - \nabla\widehat{v}) \cdot \zeta_0 \, dx. \tag{5.8}$$

Using (5.1) after integration by parts in (5.7) gives

$$\ell(w) = \int_{\Gamma_2} (F - A\nabla\widehat{v} \cdot n) w \; d\Gamma. \tag{5.9}$$

Integration by parts on (5.8) gives

$$\mu(\zeta_0) = \int_{\Gamma_1} (v - \widehat{v}) n \cdot \zeta_0 \; d\Gamma. \tag{5.10}$$

By using (5.9-5.10) in (4.12) we get (5.5).

Next, if $\Gamma_2 = \emptyset$, then $w = 0$ gives the following simple relation for the lower bound

$$\| \nabla(u - \widehat{v}) \|^2 \geq 2 \int_{\Gamma} (v - \widehat{v}) n \cdot \zeta_0 \; d\Gamma - \| \zeta_0 \|_*^2, \quad \forall \zeta_0 \in H_0. \tag{5.11}$$

The vector-valued functions in H_0 can be presented in the form $\alpha\bar{\zeta}_0$, where $\alpha \in \mathbb{R}$ and $\| \bar{\zeta}_0 \|_* = 1$. Then, by taking α as follows we obtain (5.6).

$$\alpha = \int_{\Gamma} (v - \widehat{v}) n \cdot \bar{\zeta}_0 \; d\Gamma. \tag{5.12}$$

□

Analogous estimates are easily obtained for a generalized version of the method, in which the differential equation is approximately satisfied in subdomains Ω_i, i.e.,

$$\mathbf{div}\; A\nabla\widehat{v} + f = \epsilon_i(x), \quad x \in \Omega_i, \tag{5.13}$$

where $\epsilon_i(x)$ is a small residual.

6. Numerical results

In this Section we present the numerical results for upper bound of the error. As a particular case of nonconforming approximations we consider the symmetric interior penalty DG scheme, however, as mentioned earlier, the estimates are applicable to any of the DG scheme as well as other nonconforming approximations. The numerical examples demonstrate the efficiency and robustness of the proposed estimates.

The conforming function $v \in \mathcal{V}_0 + u_0$ is computed using Oswald's interpolation operator [9, 28]. The free function $y \in H(\Omega, \mathrm{div})$ is computed by the minimization of the right hand side of (3.15), and thus, involves solving a linear system of equations for a vector-valued function. The majorant for the nonconforming approximation is defined as the right hand side of (3.15) and denoted by $M_{\oplus}^{\mathrm{DG}}$. The efficiency of the majorant can then be defined as

$$\mathrm{I}_{\mathrm{eff}} = \frac{M_{\oplus}^{\mathrm{DG}}}{\|[\eta]\|}. \tag{6.1}$$

Further, by p_u and p_y we denote the polynomial degree for the DG approximation $\widehat{v}$ (and the conforming approximation v) and y, respectively.

We consider the following examples which are chosen on the basis of the difficulty level in computing an upper bound of the approximation error. For all the examples we consider the Poisson problem on the unit square. We choose f and the Dirichlet boundary condition (assuming

$\Gamma_D = \partial\Omega$) such that the exact (analytic) solution of the problem, which is shown in Figure 1, is given as follows.

Example 6.1. $u = x(x-1)y(y-1)$.

Example 6.2. $u = x(x-1)y(y-1)\sin(17xy)\exp^{(x+y)}$. [1]

Example 6.3. $u = x(1-x)y(1-y)e^{-1000((x-.5)^2+(y-.117)^2)}$. [2]

Further, we also consider the following example with a jump in the coefficients.

Example 6.4. *Consider the elliptic problem*

$$-\mathbf{div}\ A\nabla u = 1$$

on the unit square with homogenous Dirichlet boundary condition. The coefficient A has the jumps as follows: $A = 1$ *in* $(0, 0.5] \times (0, 0.5] \cup (0.5, 1) \times (0.5, 1)$ *and* $A = \varepsilon(= .0001)$ *in the remaining domain.*

TABLE 1. Various errors for Example 6.1

DOF	$\lVert[\nabla v - \widehat{\nabla}\widehat{v}]\rVert$	$M_{\oplus}^{\mathrm{DG}}$	$\lVert[\eta]\rVert$
400	2.956e-04	1.526e-02	1.497e-02
1600	1.460e-04	7.607e-03	7.463e-03
6400	7.257e-05	3.800e-03	3.729e-03
25600	3.617e-05	1.900e-03	1.864e-03

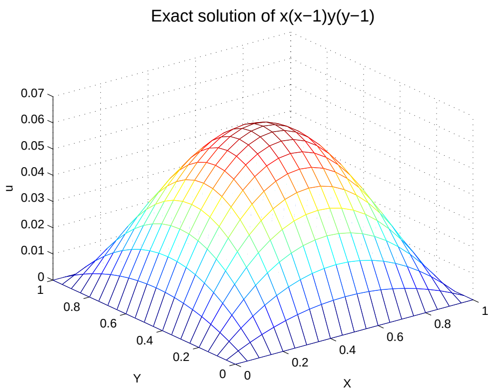
TABLE 2. Efficiency of the majorant for Example 6.1, $p_u = 1, p_y = 1$

DOF	β	$\lVert[\nabla v - \widehat{\nabla}\widehat{v}]\rVert^2$	$\lVert[y - A\widehat{\nabla}\widehat{v}]\rVert_*^2$	$\lVert\mathrm{div}\ y + f\rVert^2$	Ieff
400	0.116	8.736e-08	2.176e-04	5.784e-05	1.119
1600	0.054	2.133e-08	5.529e-05	3.212e-06	1.070
6400	0.026	5.267e-09	1.387e-05	1.874e-07	1.045
25600	0.013	1.309e-09	3.471e-06	1.128e-08	1.032

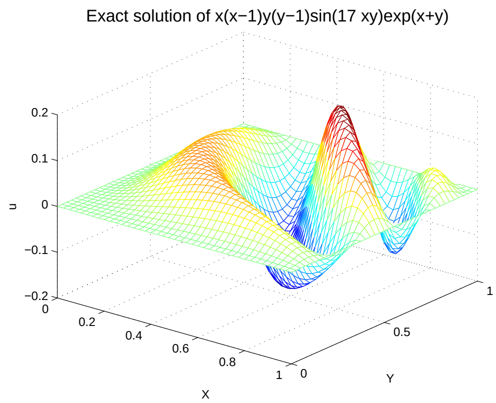
We first present the numerical results for Example 6.1. With $p_u = p_y = 1$, various components of the errors are presented in Table 1. This shows that the true error $\lVert\,[\eta]\,\rVert$ is bounded by the majorant $M_{\oplus}^{\mathrm{DG}}$. The nonconforming error is shown in the second column. In Table 2 we present the efficiency of the majorant. For a reasonably accurate DG approximation the nonconforming error is very small as compared to the other two terms of the majorant. Moreover, with $p_y = 1$, the **div** y approximation term results in a constant which is sufficient to approximate the constant function $f(= 1)$. Thus, the flux equilibration term $\lVert\mathrm{div}\ y + f\rVert^2$ is also of low order as compared to the duality error term $\lVert[y - A\widehat{\nabla}\widehat{v}]\rVert_*^2$. However, this may not be the case in general, as clear from the other examples.

As a remedy to the above-mentioned difficulty, we first try increasing both of the p_u and p_y. With $p_u = p_y = 2$, this is shown in the numerical results for Example 6.2 in Table 3. Though the efficiency

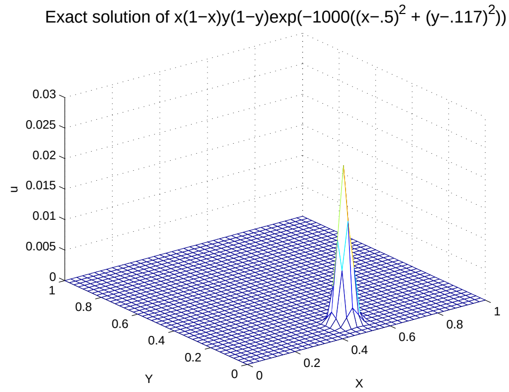
[1] Based on [29, Example 6.2].
[2] From [10].



(a) Example 6.1



(b) Example 6.2



(c) Example 6.3

FIGURE 1. Illustration of the analytic solutions and the jump in the coefficients.

TABLE 3. Efficiency of the majorant for Example 6.2, $p_u = p_y = 2$

DOF	β	$\|[\nabla v - \widehat{\nabla}\widehat{v}]\|^2$	$\|[y - A\widehat{\nabla}\widehat{v}]\|_*^2$	$\|\mathrm{div}\ y + f\|^2$	Ieff
900	1.283	4.939e-06	1.702e-02	5.530e-01	3.785
3600	0.647	8.144e-07	1.203e-03	9.936e-03	2.896
14400	0.447	1.084e-07	7.612e-05	2.998e-04	2.574
57600	0.385	1.367e-08	4.774e-06	1.397e-05	2.491

of the majorant is still very good, we note however that, unlike the results for Example 6.1, the flux equilibration term dominates in the error majorant. This is because the approximation of div y is performed by polynomials of the same order as for the DG (and conforming) approximation. To alleviate this difference of approximation order we increase p_y accordingly (see Table 4 with

TABLE 4. Efficiency of the majorant for Example 6.2, $p_u = 2, p_y = 3$

DOF	β	$\|[\nabla v - \widehat{\nabla}\widehat{v}]\|^2$	$\|[y - A\widehat{\nabla}\widehat{v}]\|_*^2$	$\|\text{div } y + f\|^2$	Ieff
900	0.140	4.939e-06	6.698e-03	2.590e-03	1.205
3600	0.043	8.144e-07	4.123e-04	1.525e-05	1.102
14400	0.013	1.084e-07	2.547e-05	8.102e-08	1.081
57600	0.004	1.367e-08	1.583e-06	3.831e-10	1.093

$p_u = 2, p_y = 3$) [3]. This makes the flux equilibration term asymptotically negligibly small as compared to the duality error term and we get an excellent efficiency index. The convergence of the true error and the majorant, with $p_u = 2, p_y = 3$, is shown in Figure 2. Further, the computing cost for $M_{\oplus}^{\mathrm{DG}}$ with respect to the cost of the DG solution, alongwith the efficiency index Ieff, is shown in Table 5. For given DOF if we denote the computing time for the DG solution by t_{DG}, and the computing time for $M_{\oplus}^{\mathrm{DG}}$ by $t_{M_{\oplus}}$, then the relative cost of $M_{\oplus}^{\mathrm{DG}}$ can be defined as $R_{M_{\oplus}}^{T} = t_{M_{\oplus}}/t_{\mathrm{DG}}$. Since the computational cost of Oswald interpolation (to obtain v) is negligible as compared to other computations we do not report it here. Obviously, the cost of computing $M_{\oplus}^{\mathrm{DG}}$ will depend on the polynomial degree used. For equal degree the computing time for $M_{\oplus}^{\mathrm{DG}}$ is proportional to that of finding the DG solution whereas its $\sim$2.3 times with one degree higher.

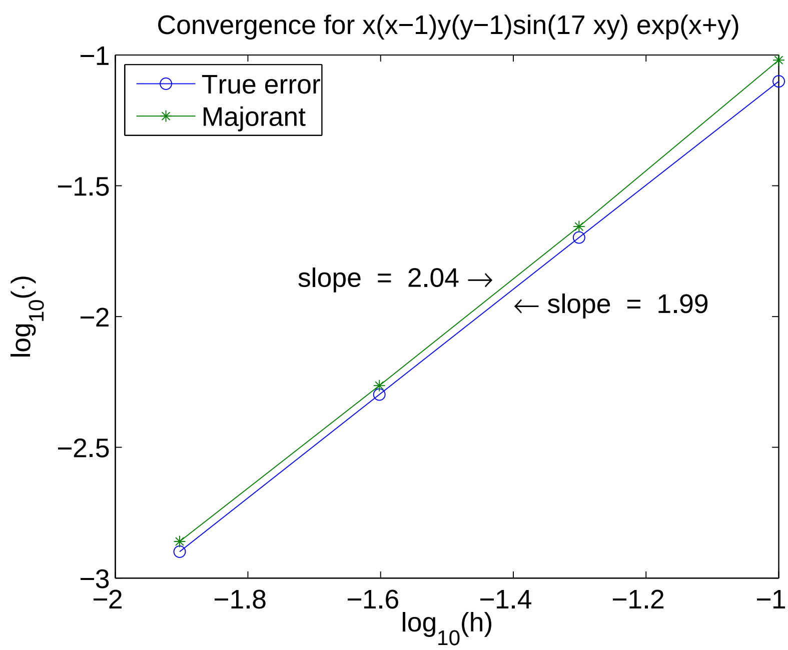


FIGURE 2. Convergence of the true error and the majorant for Example 6.2 with $p_u = 2, p_y = 3$.

[3]Having computed $\widehat{v}$ for a given polynomial degree one can easily compute its value at any number of points in a given element and use it to compute y for higher polynomial degree.

TABLE 5. Relative computing cost of $M_{\oplus}^{\mathrm{DG}}$ for Example 6.2, $p_u = 2$

DOF	$R^T_{M_\oplus}$(Ieff) $p_y = 2$	$R^T_{M_\oplus}$(Ieff) $p_y = 3$
900	0.938 (3.785)	2.188 (1.205)
3600	0.946 (2.896)	2.209 (1.102)
14400	0.965 (2.574)	2.276 (1.081)
57600	0.956 (2.491)	2.285 (1.093)

TABLE 6. Effect of p_y on the majorant for Example 6.3, $\|[\nabla v - \widehat{\nabla}\widehat{v}]\|^2 = 2.826e-08$

p_y	β	$\|[y - A\widehat{\nabla}\widehat{v}]\|_*^2$	$\|\mathrm{div}\, y + f\|^2$	Ieff	$R^T_{M_\oplus}$
2	3.654	2.524e-03	6.653e-01	15.630	0.903
3	2.048	6.175e-04	5.110e-02	5.070	2.210
4	0.664	3.699e-04	3.217e-03	2.149	4.364
5	0.251	2.309e-04	2.880e-04	1.281	8.173
6	0.047	2.260e-04	9.669e-06	1.062	14.937

We now present the results for Example 6.3, which exhibits a sharp peak, in Table 6. We also aim to see how the variation of p_y alone effects the efficiency. For this, we fix $p_u = 2$, $h = 0.05$ (somewhat coarse mesh for this problem), and vary $p_y = 2, \ldots, 6$, by extending the idea discussed in the previous example. We see that even for this admittedly difficult problem, with f having very large gradients, one obtains very promising results and the effectivity index can be brought as close to 1 as one pleases, though with associated high cost. Thus, this is a trade-off between the efficiency and the associated cost. As the efficiency index between 2 to 5 is considered very good, the upper bound of the true error can be *computed with guarantee* at a reasonable cost.

TABLE 7. Efficiency of the majorant for Example 6.4, $p_u = 1, p_y = 1$

DOF	β	$\|[\nabla v - \widehat{\nabla}\widehat{v}]\|^2$	$\|[y - A\widehat{\nabla}\widehat{v}]\|_*^2$	$\|\mathrm{div}\, y + f\|^2$	Ieff
400	0.000	6.236e-07	1.974e+01	3.925e-13	2.949
1600	0.001	1.685e-07	1.332e+01	3.666e-12	4.807
6400	0.005	4.434e-08	1.057e+01	4.679e-11	8.574
25600	0.019	1.145e-08	9.282e+00	6.794e-10	16.28

TABLE 8. Efficiency of the majorant for Example 6.4, $p_u = 1, p_y = 2$

DOF	β	$\|[\nabla v - \widehat{\nabla}\widehat{v}]\|^2$	$\|[y - A\widehat{\nabla}\widehat{v}]\|_*^2$	$\|\mathrm{div}\, y + f\|^2$	Ieff
400	0.000	6.236e-07	2.632e+00	1.872e-18	1.077
1600	0.000	1.685e-07	6.588e-01	2.047e-20	1.068
6400	0.000	4.434e-08	1.656e-01	2.572e-22	1.068
25600	0.000	1.145e-08	4.258e-02	4.223e-24	1.083

Finally, we present the results for Example 6.4. Since $f = 1$, as in Example 6.1, we first present the results for the combination of $p_u = 1$ and $p_y = 1$ in Table 7. With this choice of the approximation spaces, though the flux equilibration term is negligibly small, however, the duality error term is rather persistent and does not converge as the true error. This shows that the choice of the approximation spaces for the construction of y, in the presence of the jumps in A, is not sufficient. In Table 8 we present the results with $p_u = 1$ and $p_y = 2$. We see that the efficiency index is again excellent and that the estimates are now robust with respect to the jumps in the coefficients.

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