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Functional a posteriori error estimates for incremental models in elasto-plasticity

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FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR INCREMENTAL MODELS IN ELASTO-PLASTICITY

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ABSTRACT. We consider a convex variational problem related to a time-step problem in elasto-plastic models with isotropic hardening. Our goal it to derive a posteriori error estimate of the difference between the exact solution and any function in the admissible (energy) class of the problem considered. The estimates are obtained by a advanced version of the variational approach earlier used for linear boundary-value problems and nonlinear variational problems with convex functionals (see [20, 21] and the monography [18]). They do no contain mesh-dependent constants and are valid for any conforming approximations regardless of the method used for their derivation.

It is shown that the structure of error majorant reflects properties of the exact solution so that the majorant vanishes only if an approximate solution coincides with the exact one. Moreover, it possesses necessary continuity properties, so that any sequence of approximations converging to the exact solution in the energy space generates a sequence of positive numbers (explicitly computable by the majorant functional) that tends to zero.

1. INTRODUCTION

Incremental models in the theory of elasto-plasticity are among the most widely used in the numerical analysis of processes that include plasticity phenomenon. These typically include memory effect and exhibit hysteresis behavior which are described by time-dependent variational inequalities. If an implicit Euler scheme is used, then the evolutionary variational inequality is approximated by a sequence of stationary variational inequalities of the second kind [14] in which the unknown functions are displacement u and plastic strain p. Each of these inequalities is equivalent to a minimization problem with a convex but non-smooth energy functional, $J(u,p) \rightarrow \min$. There exist various

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methods for solving this minimization problems such a return mapping algorithm [25], an alternating minimization [2] or quasi-Newton methods [15, 7]. The main focus of this paper is not to develop new methods for solving the minimization problems itself, but to provide a guaranteed a posteriori estimate to measure the quality of the numerical solution computed.

A posteriori estimates of approximation errors are intended to (a) give a presentation on the overall accuracy of an approximate solution and (b) serve as an error indicator that show regions with excessively high errors (typically a new finite dimensional space constructed on the basis of this information has extra trial functions in each regions). There exist various approaches to the construction of a posteriori error estimates (a discussion of them can be found in e. g., in monographs [3, 1, 4, 18] or in the recent overview [22]. In application of a posteriori techniques to elasto-plasticity, let us mention the works [19, 9].

In this paper we use the method exposed it the book [18], in the framework of which the estimates are derived by purely functional analysis of the respective variational problem and its dual counterpart. Hence, a computable upper bound of the error is obtained on a purely functional level without exploitation of specific properties of an approximation or a method used for its computing. Therefore, estimates of such a type are often called “functional a posteriori estimates”. One of the first publications presenting this method was [24] where the a posteriori estimates were derived for a deformation plasticity model with hardening. Recently, the method was applied to the Ramberg-Osgood model (sometimes also called Norton-Hoff) in the theory of nonlinear solid media, see [6]. Also, we note close publications [5] and [12] where such estimates were derived for nonlinear viscous flow problems. Finally, we mention [11], where the method was applied to the variational functional arising in the theory of Reissner- Mindlin plates, which from the mathematical point of view has common features with the one considered in the present paper.

Based on their experience in implementation of functional a posteriori estimates for problems with nonlinear boundary conditions [23], authors plan in the forthcoming paper to concentrate on numerical implementation and verification of estimates derived here.

2. Minimization problem

Let us consider the first plastic time-step problem in the isotropic hardening case of elasto-plasticity [16, 2]. It can be described as a minimization problem related to the functional

$$J(v,q) := \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v)-q):(\boldsymbol{\varepsilon}(v)-q)+\sigma_y H|q|^2\,dx + \int_\Omega \sigma_y|q|\,dx - \int_\Omega fv\,dx \to \min. \tag{2.1}$$

The free variables are displacement

$$v \in V_0 + u_0,$$

where $V_0 := H^1_0(\Omega;\mathbb{R}^d), u_0 \in H^1(\Omega;\mathbb{R}^d)$ satisfied the prescribed boundary condition $u_0 = u_D$ (in the sense of traces) on the Dirichlet boundary Γ_D and the plastic strain

$$q \in Q_0 := \{q \in L^2(\Omega;\mathbb{R}^{d\times d}_{sym}) : \operatorname{tr} q = 0 \;\; a.\ e.\ \text{in } \Omega\}.$$

Note that $\operatorname{tr}\cdot$ denotes the trace operator defined by $\operatorname{tr} A = A : \mathbb{I}$ for all $A \in \mathbb{R}^{d\times d}$ with $\mathbb{I}$ denoting the identity matrix . The positive constants H and σ_y represent an isotropic hardening parameter and a yield stress. The external forces f are assumed to satisfy

$$f \in L^2(\Omega;\mathbb{R}^d)$$

and $\mathbb{C} \in \mathcal{L}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})$ denotes the fourth-order elastic stiffness tensor which satisfies a relation (for known positive constants c_1, c_2)

$$c_1|q| \le Cq : q \le c_2|q| \tag{2.2}$$

for all $q \in \mathbb{R}^{d\times d}$. Finally, the linearized Green-St. Venant strain tensor is defined as

$$\boldsymbol{\varepsilon}(v) := \frac{1}{2}\left(\nabla v + (\nabla v)^T\right). \tag{2.3}$$

3. Variational inequality

Theorem 1. *The pair* $(u,p) \in (V_0+u_0)\times Q_0$ *that solves* (2.1) *satisfies the variational inequality*

$$a(u,p;v-u,q-p) + \Psi(q) - \Psi(p) - l(v-u) \ge 0, \tag{3.1}$$

where

$$a(u,p;v,q) := \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(u)-p):(\boldsymbol{\varepsilon}(v)-q)+\sigma_y Hp:q\,dx,$$
$$\Psi(q) := \int_\Omega \sigma_y|q|\,dx,$$
$$l(v) := \int_\Omega fv\,dx$$

for all $(v,q)\in(V_0+u_0)\times Q_0$.

Proof. Due to the assumption (2.2), the ellipticity and boundedness of the bilinear form $a(u,p;v,q)$ can be proved. Then the variational inequality follows from the Lions-Stampacchia Theorem [17]. □

4. Basic estimate of the deviation from exact solution

Theorem 2. *For any* $(v,q)\in(V_0+u_0)\times Q_0$, *an estimate*

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \le J(v,q)-J(u,p) \tag{4.1}$$

holds, where a norm $|||\cdot|||$ *is defined as*

$$|||(u-v),(p-q)||| := \|\boldsymbol{\varepsilon}(u-v)-p+q\|^2 + H\,\|p-q\|^2\,.$$

Proof. The direct calculation shows

$$\begin{aligned} J(v,q)-J(u,p) &= \frac{1}{2}a(v,q;v,q)-\frac{1}{2}a(u,p;u,p)+\Psi(q)-\Psi(p)-l(v)+l(u)\\ &= \frac{1}{2}|||(u-v),(p-q)|||^2\\ &\quad +a(u,p;v-u,q-p)+\Psi(q)-\Psi(p)-l(v-u)\\ &\ge \frac{1}{2}|||(u-v),(p-q)|||^2, \end{aligned}$$

where we used the inequality (3.1). □

5. Perturbed problem and Lagrangian

Let us introduce a perturbed problem

$$J_\lambda(v,q) := \frac{1}{2}a(v,q;v,q)-l(v)+\int_\Omega \sigma_y\lambda:q\,dx$$

and Lagrangian

$$
\begin{aligned}
L_\lambda(v,q;\tau,\xi) : &= \int_\Omega \left(\tau : (\boldsymbol{\varepsilon}(v) - q) - \frac{\mathbb{C}^{-1}}{2}\tau : \tau + \xi : q - \frac{1}{2\sigma_y H}|\xi|^2 - fv\right) dx \\
&\quad + \int_\Omega \sigma_y \lambda : q \, dx, \\
&= \int_\Omega (\tau : \boldsymbol{\varepsilon}(v) - fv)\, dx + \int_\Omega q : (\xi - \tau + \sigma_y\lambda)\, dx \\
&\quad - \int_\Omega \left(\frac{1}{2}\mathbb{C}^{-1}\tau : \tau + \frac{1}{2\sigma_y H}|\xi|^2\right) dx
\end{aligned}
$$

where the multiplier

$$\lambda \in \Lambda := \{\lambda \in L^\infty(\Omega, \mathbb{R}^{d\times d}) : |\lambda| \leq 1 \ \ a.\ e. \text{ in } \Omega\}$$

and

$$\tau, \xi \in Q := L^2(\Omega; \mathbb{R}^{d\times d}_{sym}).$$

Note that

$$\sup_\lambda J_\lambda(v,q) = J(v,q) \quad \text{and} \quad \sup_{\tau,\xi} L_\lambda(v,q;\tau,\xi) = J_\lambda(v,q)$$

for all $(v,q) \in (V_0 + u_0) \times Q_0$. Thus it holds

$$
\begin{aligned}
J(u,p) = \inf_{v,q} J(v,q) &\geq \inf_{v,q} J_\lambda(v,q) = \inf_{v,q}\sup_{\tau,\xi} L_\lambda(v,q;\tau,\xi) \\
&\geq \sup_{\tau,\xi}\inf_{v,q} L_\lambda(v,q;\tau,\xi) \\
&\geq \inf_{v,q} L_\lambda(v,q;\tau,\xi)
\end{aligned}
$$

and therefore the substitution in (4.1) yields an estimate

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq J(v,q) - \inf_{v,q} L_\lambda(v,q;\tau,\xi) \tag{5.1}$$

valid for all $\tau, \xi \in Q$. Using the substitution $w = v - u_0 \in H^1_0(\Omega; \mathbb{R}^d)$, we can reformulate

$$
\begin{aligned}
L_\lambda(w,q;\tau,\xi) &= \int_\Omega (\tau : \boldsymbol{\varepsilon}(w) - fw)\, dx + \int_\Omega q : (\xi - \tau + \sigma_y\lambda)\, dx \\
&\quad - \int_\Omega \left(\frac{1}{2}\mathbb{C}^{-1}\tau : \tau + \frac{1}{2\sigma_y H}|\xi|^2 - \tau : \boldsymbol{\varepsilon}(u_0) + fu_0\right) dx.
\end{aligned}
$$

Note that

$$\inf_{w,q} \int_\Omega (\tau : \boldsymbol{\varepsilon}(w) - fw)\, dx = \begin{cases} 0 & \text{if } \operatorname{div} \tau + f = 0 \ \ a.\ e. \text{ in } \Omega, \\ -\infty & \text{otherwise}, \end{cases}$$

and

$$\inf_{w,q} \int_\Omega q : (\xi - \tau + \sigma_y \lambda)\, dx = \begin{cases} 0 & \text{if } \operatorname{dev} \tau = \operatorname{dev} \xi + \sigma_y \operatorname{dev} \lambda \ \ a.\ e. \text{ in } \Omega, \\ -\infty & \text{otherwise}, \end{cases}$$

where $\operatorname{dev} \cdot$ defines a deviatoric operator $\operatorname{dev} A = A - \frac{\operatorname{tr}(A)}{d}\mathbb{I}$ for all $A \in \mathbb{R}^{d\times d}$. In summary, we can conclude

$$\inf_{w,q} L_\lambda(w,q;\tau,\xi) = -\int_\Omega \left(\frac{1}{2}\mathbb{C}^{-1}\tau : \tau + \frac{1}{2\sigma_y H}|\xi|^2 - \tau : \boldsymbol{\varepsilon}(u_0) + f u_0 \right) dx,$$

if the pair (τ, ξ) satisfies the constrain $(\tau,\xi) \in Q_{f_\lambda}$, where

$$Q_{f_\lambda} := \{(\tau,\xi) \in Q \times Q : \\ \operatorname{div} \tau + f = 0, \operatorname{dev} \tau = \operatorname{dev} \xi + \sigma_y \operatorname{dev} \lambda \ \ a.\ e. \text{ in } \Omega\}.$$

If the constrain $(\tau,\xi) \in Q_{f_\lambda}$ is not satisfied, it holds

$$\inf_{w,q} L_\lambda(w,q;\tau,\xi) = -\infty.$$

Now, in order to utilize the estimate (5.1), let us compute the difference

$$\mathcal{M}(v,q,\tau,\xi,\lambda) := J(v,q) - \inf_{v,q} L_\lambda(v,q;\tau,\xi)$$

which defines a functional majorant. Under the validity of the constrain $(\tau,\xi) \in Q_{f_\lambda}$, after the substitution for $J(v,q)$ we obtain

$$\begin{aligned} & J(v,q) - \inf_{v,q} L_\lambda(v,q;\tau,\xi) \\ = \ & \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q) : (\boldsymbol{\varepsilon}(v) - q) + \mathbb{C}^{-1}\tau : \tau\, dx + \int_\Omega \frac{\sigma_y H}{2}|q|^2 + \frac{1}{2\sigma_y H}|\xi|^2\, dx \\ & + \int_\Omega \sigma_y |q|\, dx - \int_\Omega f(v - u_0)\, dx - \int_\Omega \tau : \boldsymbol{\varepsilon}(u_0)\, dx \\ = \ & \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau)\, dx + \frac{1}{2}\int_\Omega \sigma_y H (q - \frac{1}{\sigma_y H}\xi)^2\, dx \\ & + \int_\Omega \sigma_y |q|\, dx - \int_\Omega q : \tau - \xi : q\, dx + \int_\Omega \tau : \boldsymbol{\varepsilon}(v - u_0) - f(v - u_0)\, dx. \end{aligned}$$

The last two terms are simplified due the the constrain $(\tau, \xi) \in Q_{f_\lambda}$ as

$$\int_\Omega q : \tau - \xi : q \, dx = \int_\Omega \sigma_y \lambda : q \, dx,$$

$$\int_\Omega \tau : \boldsymbol{\varepsilon}(v - u_0) - f(v - u_0) \, dx = 0$$

and the functional majorant finally reads

$$\begin{aligned} \mathcal{M}(v, q, \tau, \xi, \lambda) \ &:= \ \frac{1}{2} \int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) \, dx \\ &+ \frac{1}{2} \int_\Omega \sigma_y H (q - \frac{1}{\sigma_y H}\xi)^2 \, dx + \int_\Omega \sigma_y |q| - \sigma_y \lambda : q \, dx. \end{aligned} \tag{5.2}$$

We proved an upper (reliability) estimate

$$\frac{1}{2} |||(u - v), (p - q)|||^2 \ \leq \ \mathcal{M}(v, q, \tau, \xi, \lambda) \tag{5.3}$$

valid for arbitrary $\lambda \in \Lambda, (\tau, \xi) \in Q_{f_\lambda}, v \in V_0 + u_0, q \in Q_0$. For practical implementation, v and q will represent discrete displacement and plastic strain which will be computed numerically, e. g., by the finite element method. Then, in order to keep this estimate as sharp as possible, we analyze an estimate

$$\frac{1}{2} |||(u - v), (p - q)|||^2 \ \leq \ \inf_{(\tau,\xi) \in Q_{f_\lambda}} \mathcal{M}(v, q, \tau, \xi, \lambda) \tag{5.4}$$

which is valid for arbitrary $\lambda \in \Lambda$. The structure of the functional majorant allows for

Theorem 3. *The majorant* (5.2) *attains the zero value if and only if the following conditions hold almost everywhere in* Ω*:*

$$\operatorname{div} \tau + f \ = \ 0, \tag{5.5}$$

$$\boldsymbol{\varepsilon}(v) - q \ = \ \mathbb{C}^{-1}\tau, \tag{5.6}$$

$$q \ = \ \frac{1}{\sigma_y H}\xi, \tag{5.7}$$

$$\lambda \ \in \ \begin{cases} \Lambda & \text{if } q = 0, \\ \frac{q}{|q|} & \text{otherwise.} \end{cases} \tag{5.8}$$

The interpretation of Theorem (3) has a clear meaning: the majorant value $\mathcal{M}(v, q, \tau, \xi, \lambda)$ attains the zeros value if and only if the discrete solution is equal to the exact solution of the minimization problem

(2.1), i.e., $u = v$ and $p = q$. Then τ represents an exact stress which has to satisfy the equilibrium of forces (5.5) and is also compatible with the additive decomposition of strain combined with a Hook's law (5.6). The equation (5.7) then represent a normal flow law for the von Mises criteria [2] under which the minimization problem (2.1) was derived. Finally, the Lagrange multiplier λ activates in (5.8) if the plasticity strain p is present.

6. Basic estimate of the deviation from exact solution

The practical implementation of the estimate (5.3) is not straightforward since the variable τ in $\mathcal{M}(v, q, \tau, \xi, \lambda)$ must satisfy the equilibrium condition

$$\int_\Omega (\tau : \boldsymbol{\varepsilon}(w) - fw)\, dx = 0 \text{ for all } w \in V_0,$$

as a part of the constrain $(\tau, \xi) \in Q_{f_\lambda}$. Using for instance an equilibration technique [8], we might be able to reconstruct τ from ∇v. Then, the estimate (5.3) would reduced to a simpler residual-type estimate

$$|||(u-v),(p-q)|||^2 \le \int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau)\, dx.$$

Here, we derive another estimate from (5.3) to avoid the equilibrium constrain. Let us decompose

$$\tau = \tau - \hat{\tau} + \hat{\tau},$$

where a variable $\hat{\tau} \in Q$ does not need to satisfy the equilibrium condition and rewrite

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\tau)\, dx \\
= &\frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}(\tau - \hat{\tau} + \hat{\tau})) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}(\tau - \hat{\tau} + \hat{\tau})\, dx \\
\le &\frac{1}{2}(1+\beta_1)\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\hat{\tau}) : (\boldsymbol{\varepsilon}(v) - q - \mathbb{C}^{-1}\hat{\tau})\, dx \\
&+\frac{1}{2}(1+\frac{1}{\beta_1})\int_\Omega \mathbb{C}^{-1}(\tau - \hat{\tau}) : (\tau - \hat{\tau})\, dx,
\end{aligned}$$

which is valid for all $\beta_1 > 0$. Note that the last term can be further estimated in order to eliminate the equilibrated unknown τ as

$$
\begin{aligned}
& \inf_{(\tau,\xi)\in Q_{f_\lambda}} \frac{1}{2}\int_\Omega \mathbb{C}^{-1}(\tau-\hat\tau):(\tau-\hat\tau)\,dx \\
= & \inf_{(\tau,\xi)\in Q_{f_\lambda}} \sup_{w\in V_0} \frac{1}{2}\int_\Omega \left(\mathbb{C}^{-1}(\tau-\hat\tau):(\tau-\hat\tau)+\tau:\boldsymbol{\varepsilon}(w)-fw\right)\,dx \\
& \text{the interchange of operators follows e. g., from [10], Theorem 4.1.} \\
= & \sup_{w\in V_0} \inf_{(\tau,\xi)\in Q_{f_\lambda}} \int_\Omega \left(\frac{1}{2}\mathbb{C}^{-1}(\tau-\hat\tau):(\tau-\hat\tau)+\tau:\boldsymbol{\varepsilon}(w)-fw\right)\,dx \\
& \text{(the infimum is attained at the argument } \tau=\hat\tau-\mathbb{C}\boldsymbol{\varepsilon}(w)) \\
= & \sup_{w\in V_0}\left(-\int_\Omega \frac{1}{2}\mathbb{C}\boldsymbol{\varepsilon}(w):\boldsymbol{\varepsilon}(w)\,dx-\int_\Omega \left(-\hat\tau:\boldsymbol{\varepsilon}(w)+fw\right)\,dx\right) \\
= & \sup_{w\in V_0}\left(-\frac{1}{2}\left\|\boldsymbol{\varepsilon}(w)\right\|_{\mathbb{C}}^2-\int_\Omega (\operatorname{div}\hat\tau+f)w\,dx\right) \\
\le & \sup_{w\in V_0}\left(-\frac{1}{2}\left\|\boldsymbol{\varepsilon}(w)\right\|_{\mathbb{C}}^2+\left\|\operatorname{div}\hat\tau+f\right\|\left\|w\right\|\right) \\
\le & \sup_{w\in V_0}\left(-\frac{1}{2}\left\|\boldsymbol{\varepsilon}(w)\right\|_{\mathbb{C}}^2+C\left\|\operatorname{div}\hat\tau+f\right\|\left\|\boldsymbol{\varepsilon}(w)\right\|_{\mathbb{C}}\right)=\frac{1}{2}C^2\left\|\operatorname{div}\hat\tau+f\right\|^2,
\end{aligned}
$$

where the constant $C > 0$ satisfies the inequality

$$\|w\| \le C\,\|\boldsymbol{\varepsilon}(w)\|_{\mathbb{C}}$$

valid for all $w \in V_0$. The existence of such constant follows from the Korn's and Friedrichs' inequalities. Altogether we have formulated the inequality

$$\inf_{(\tau,\xi)\in Q_{f_\lambda}} \mathcal{M}(v,q,\tau,\xi,\lambda) \le \inf_{(\hat\tau,\xi)\in \hat Q_{f_\lambda}} \hat{\mathcal{M}}(v,q,\hat\tau,\xi,\lambda,\beta_1)$$

valid for all $\lambda \in \Lambda$ and $\beta_1 > 0$, where

$$
\begin{aligned}
\hat{\mathcal{M}}(v,q,\hat\tau,\xi,\lambda,\beta_1) \;:=\; & \frac{1}{2}(1+\beta_1)\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(v)-q-\mathbb{C}^{-1}\hat\tau):(\boldsymbol{\varepsilon}(v)-q-\mathbb{C}^{-1}\hat\tau)\,dx \\
& +\frac{1}{2}(1+\frac{1}{\beta_1})C^2\left\|\operatorname{div}\hat\tau+f\right\|^2 \\
& +\frac{1}{2}\int_\Omega \sigma_y H(q-\frac{1}{\sigma_y H}\xi)^2\,dx+\int_\Omega \sigma_y|q|-\sigma_y\lambda:q\,dx
\end{aligned}
$$

and

$$\hat{Q}_{f_\lambda} := \{(\hat{\tau}, \xi) \in Q \times Q : \operatorname{dev} \hat{\tau} = \operatorname{dev} \xi + \sigma_y \operatorname{dev} \lambda \;\; a.\ e. \text{ in } \Omega\}.$$

7. Particular case

Theorem 3 provides optimal choice of parameters

$$(7.1) \quad \hat{\tau} = \mathbb{C}(\boldsymbol{\varepsilon}(v) - q), \quad \xi = \sigma_y H q, \quad \lambda \in \begin{cases} \frac{q}{|q|} & \text{if } q \neq 0, \\ \Lambda & \text{if } q = 0 \end{cases}$$

which annulates the majorant $\hat{\mathcal{M}}(v, q, \hat{\tau}, \xi, \lambda, \beta_1)$ (for any $\beta_1 > 0$) in the case of the exact solutions $v = u$ and $q = p$. However, for the arbitrary approximate solutions v and q, these parameters can not be substituted to the majorant $\hat{\mathcal{M}}(v, q, \hat{\tau}, \xi, \lambda, \beta_1)$, since the constrain

$$(7.2) \qquad (\hat{\tau}, \xi) \in \hat{Q}_{f_\lambda}$$

is not necessary satisfied.

Let us discuss this condition for the isotropic case, which assumes the elasticity tensor in the form $\mathbb{C}_{ijkl} = K\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$. It reads in a matrix notation as follows

$$\mathbb{C}\boldsymbol{\varepsilon}(v) = 2\mu \operatorname{dev}(\boldsymbol{\varepsilon}(v)) + K \operatorname{tr}(\boldsymbol{\varepsilon}(v))\mathbb{I},$$

where the constants K and μ represent the bulk and the shear elastic moduli. Secondly, let us now consider a special discrete solution, where the discrete plastic strain q is related to the discrete displacement v through a pointwise formula

$$(7.3) \quad q = \begin{cases} \frac{2\mu|\operatorname{dev}(\boldsymbol{\varepsilon}(v))| - \sigma_y}{2\mu + \sigma_y H} \frac{\operatorname{dev}(\boldsymbol{\varepsilon}(v))}{|\operatorname{dev}(\boldsymbol{\varepsilon}(v))|} & \text{if } 2\mu|\operatorname{dev}(\boldsymbol{\varepsilon}(v))| \geq \sigma_y, \\ 0 & \text{if } 2\mu|\operatorname{dev}(\boldsymbol{\varepsilon}(v))| \leq \sigma_y. \end{cases}$$

This formula is known due to [2] and allows to express a plastic strain q from a given displacement v in the first plastic time-step, so that the energy $J(v, q)$ in (2.1) is minimal.

Then, to check, whether the substitution of parameters (7.1) does not violate the condition (7.2), it must hold

$$(7.4) \quad 2\mu(\operatorname{dev}(\boldsymbol{\varepsilon}(v) - q) = \sigma_y H q + \sigma_y \frac{\operatorname{dev}(\boldsymbol{\varepsilon}(v))}{|\operatorname{dev}(\boldsymbol{\varepsilon}(v))|} \qquad \text{if } q \neq 0,$$

$$(7.5) \quad 2\mu \operatorname{dev}(\boldsymbol{\varepsilon}(v)) = \sigma_y \lambda \qquad \text{if } q = 0$$

for some $\lambda \in \Lambda$. A careful substitution of (7.3) shows the equality (7.4) in the plastic case $q \neq 0$ holds true. Obviously, the value $\lambda = \frac{2\mu}{\sigma_y} \operatorname{dev}(\boldsymbol{\varepsilon}(v)) \in \Lambda$ validates the equality (7.5). Then, majorant

$\hat{\mathcal{M}}(v, q, \hat{\tau}, \xi, \lambda, \beta_1)$ reduces in both cases $q \neq 0$ and $q = 0$ in the limit case $\beta_1 \to +\infty$ to

$$\hat{\mathcal{M}}(v, q, \hat{\tau}) = \frac{1}{2} C^2 \left\| \operatorname{div} \hat{\tau} + f \right\|^2 .$$

The simplified form of the majorant states, the error of discrete solution can be estimated from above by its stress only.

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