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Efficient computable error bounds for discontinuous Galerkin approximations of elliptic problems

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Efficient computable error bounds for discontinuous Galerkin approximations of elliptic problems

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Abstract

We present *guaranteed and computable* both sided error bounds for the discontinuous Galerkin (DG) approximations of elliptic problems. These estimates are derived in the full DG-norm on purely *functional* grounds by the analysis of the respective differential problem, and thus, are applicable to any qualified DG approximation. Based on the triangle inequality, the underlying approach has the following steps for a given DG approximation: (1) computing a conforming approximation in the energy space using the Oswald's interpolation operator, and (2) application of the existing functional a posteriori error estimates to the conforming approximation. Various numerical examples with varying difficulty in computing the error bounds, from simple problems of polynomial type analytic solution to problems with analytic solution having sharp peaks, or problems with jumps in the coefficients of the partial differential equation operator, are presented which confirm the efficiency and robustness of the estimates.

Key words: a posteriori error estimates, discontinuous Galerkin method, elliptic partial differential equation

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1 Introduction

Discontinuous Galerkin (DG) finite element (FE) methods for elliptic problems, though initially proposed in 70s-80s, see [3,43], have gained much interest in the last decade due to their suitability for *hp*-adaptive techniques. They offer several advantages, e.g. the ease of treatment of meshes with hanging nodes, elements of varying shape and size, polynomials of variable degree, parallelization, preservation of local conservation properties, etc. Their application spans a wide variety of problems, see the review article [14] and the references therein. An excellent overview and a detailed analysis of DG methods for elliptic problems can be found in [4]. Further, in [9] a general framework for the construction and analysis of DG methods have been proposed.

A posteriori error estimates for conforming approximations of various boundary value problems have been thoroughly studied and a huge literature can be found (see, e.g., articles [5,8,13,16,26,38,41] and books [2,6,28,29,42] and the references therein). However, a posteriori error estimates for DG approximations have gained significant interest only in recent years. A residual based error estimator (in mesh-dependent energy-type norm) has been used in [7,22,23]. A posteriori error analysis for locally conservative mixed methods, with applications to P^1 nonconforming FEM and interior penalty DG (IPDG) as well as the mixed finite element method, has been studied in [24,25] (the latter for nonlinear elliptic problems). Similar estimates for the *local* DG (LDG) approximation of linear and nonlinear diffusion problems were derived in [11]. In [37] a posteriori error estimates for a DG approximation of elliptic boundary value problems are derived in L^2 -norm. For the LDG approximation of elliptic boundary value problem a posteriori estimates in L^2 -norm were derived in [12]. A posteriori error estimates for DG approximations were also obtained for other classes of problems, in particular, in [40] time-dependent (transport) equations and in [21] elliptic problems of the Maxwell type, were considered. In [18], a posteriori estimates were obtained for DG approximations of the convection-diffusion equation. Recently, a new form of the a posteriori estimate with an advanced structure of the residual terms has been proposed in [19]. For nonconforming approximations of linear elliptic boundary value problems a different approach based on the Helmholtz decomposition of the error is presented in [1,7,11,15,36].

It is known that the error in the DG approximation, in general, is not in the natural energy space $H^1(\Omega)$. Thus, it is useful to decompose the error into *conforming* and *nonconforming* parts. Two approaches have been followed in this regard, the Helmholtz type, see e.g. [1,7,11,15], and the *direct*, see e.g. [19,24,25]. As noted in [19] the latter approach, which is also used in this article, is simpler than the former. After the decomposition of the error into conforming and nonconforming parts we estimate both the terms separately.

For the conforming part we use the so-called *functional* a posteriori estimates which give *guaranteed and computable* upper and lower bounds of the error in the energy norm for *any conforming approximation*, for details see [32–35], and see [31] for a systematic exposition and overview. Since the nonconforming error, which can be viewed as a *penalty for the nonconformity*, can be explicitly determined, we obtain estimates that are also valid for any nonconforming approximations. As we do not use any specificity of the DG schemes any of the DG methods proposed in the literature can be used.

The organization of the paper is as follows. In Section 2 we introduce the preliminaries, i.e. the problem, notations, triangulation, function spaces and the norms. This is followed by the discontinuous Galerkin formulation in Section 3. The error decomposition and the upper and the lower error estimates are discussed in Section 4. Some of the results of this section were announced in [27]. Finally, in Section 5 we discuss the numerical results which confirm the efficiency and the robustness of the method.

2 Preliminaries

Consider a second order elliptic problem on a bounded Lipschitz domain $\Omega \subset R^2$, with homogeneous Dirichlet boundary data

$$-\operatorname{div}(\underline{A}(x) \nabla u) = f(x) \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \Gamma_D. \quad (1)$$

Here we assume that \underline{A} is a symmetric positive definite matrix such that

$$c_1|\xi|^2 \leq \underline{A}\xi \cdot \xi \leq c_2|\xi|^2 \quad \forall \xi \in R^2, \quad (2)$$

and it has a positive inverse \underline{A}^{-1} . For the sake of simplicity we restrict ourselves to the homogeneous Dirichlet boundary data, however, the method is also applicable with mixed boundary data, see [27] for details.

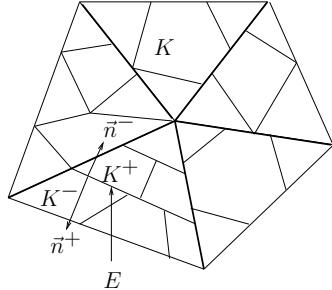


Fig. 1. A possible DG mesh on Ω

Let \mathcal{T}_h be a non-overlapping partition of Ω into shape-regular [19] finite elements K (polygonal or polyhedral shape) with boundaries ∂K . The partition

is assumed to be locally quasi-uniform in the sense that the ratio of the diameters of any pair of neighboring elements is uniformly bounded above and below over the whole partition. We allow finite elements to vary in size and shape for local mesh adaptation and the mesh is not required to be conforming, i.e. elements may possess hanging nodes, see Figure 1. Let h_K denote the size of the element K and $h = \max_{K \in \mathcal{T}_h} h_K$ denote the characteristic mesh size. Further, let E be an interior face shared by two adjacent elements K^+ , and K^- . We denote the set of all such internal faces by \mathcal{E}_0 , set of faces on $\partial\Omega$ by \mathcal{E}_D and set of all faces by $\mathcal{E} : \mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_D$. The face measure h_E is constant on each face $E \in \mathcal{E}$ such that $h_E = |E|$.

We now define the space

$$\mathcal{V}_0 := H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\},$$

where $H^1(\Omega)$ is the usual Sobolev space. Further, on the partition \mathcal{T}_h we define the finite dimensional space:

$$\mathcal{V}_h = \{v \in L^2(\Omega) : v|_K \in P_r(K), \forall K \in \mathcal{T}_h\},$$

where P_r is the set of polynomials of degree $r \geq 1$. Note that on \mathcal{V}_h neither any boundary condition nor any inter-element continuity is enforced. The inter-element continuity will be enforced indirectly via the variational formulation.

To deal with the multivalued traces at the inter-element faces we define the trace operators $\{\cdot\}$ and $[\![\cdot]\!]$, **average** and **jump**, respectively, as follows [4]: Let $E \in \mathcal{E}_0$. Define the unit normal vectors \vec{n}^+ and \vec{n}^- on E pointing exterior to K^+ and K^- , respectively. For $v \in \mathcal{V}_h$ we define $v^{+/-} := v|_{\partial K^{+/-}}$ and set

$$\{v\} = \frac{1}{2} (v^+ + v^-), \quad [\![v]\!] = v^+ \vec{n}^+ + v^- \vec{n}^- \quad \text{on } E \in \mathcal{E}_0.$$

On $E \in \mathcal{E}_D$ the functions v are uniquely defined and are set as

$$\{v\} = v, \quad [\![v]\!] = v \vec{n}.$$

On \mathcal{V}_h we now define the following weighted broken (DG) norm:

$$[\![v_h]\!]^2 = \sum_{K \in \mathcal{T}_h} \int_K \underline{A} \nabla_h v_h \cdot \nabla_h v_h \, dx + \eta h_E^{-1} \sum_{E \in \mathcal{E}} \int_E [\![v_h]\!] \cdot [\![v_h]\!] \, ds, \quad (3)$$

where η is a positive parameter and will be defined later. Finally, we define the following norms on $H^1(\omega)$ (where ω is either the whole domain Ω or a given element K) which will be needed in the sequel

$$\|z\|_{a,\omega}^2 = \int_\omega \underline{A} z \cdot z \, dx, \quad \|z\|_{\bar{a},\omega}^2 = \int_\omega \underline{A}^{-1} z \cdot z \, dx. \quad (4)$$

Together with (2) these norms are equivalent to the L^2 norm $\|z\|$.

3 Discontinuous Galerkin formulation

Before proceeding with the DG formulation we first state the standard variational formulation of the problem (1):

Find $u \in \mathcal{V}_0$ such that

$$\int_{\Omega} \underline{A} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in \mathcal{V}_0. \quad (5)$$

For elliptic boundary value problems a large number of DG methods have been developed, see e.g. [4,9] and the references therein. In this article we consider the interior-penalty (IP) DG method which has a large class in itself. For the problem (1) the IP-DG formulation can be stated as follows:

Find $u_h \in \mathcal{V}_h$ such that for all $v_h \in \mathcal{V}_h$ the following relation holds:

$$\mathcal{B}_h(u_h, v_h) = \mathcal{L}_h(v_h), \quad (6a)$$

where the bilinear form $\mathcal{B}_h(u_h, v_h) : \mathcal{V}_h \times \mathcal{V}_h \rightarrow \mathbb{R}$ and the linear form $\mathcal{L}_h(v_h) : \mathcal{V}_h \rightarrow \mathbb{R}$ are defined as

$$\begin{aligned} \mathcal{B}_h(u_h, v_h) &= \sum_{K \in \mathcal{T}_h} \int_K \underline{A} \nabla_h u_h \cdot \nabla_h v_h \, dx + \eta h_E^{-1} \sum_{E \in \mathcal{E}} \int_E [\![u_h]\!] \cdot [\![v_h]\!] \, ds \\ &\quad - \sum_{E \in \mathcal{E}} \int_E (\{\underline{A} \nabla_h u_h\} \cdot [\![v_h]\!] + [\![u_h]\!] \cdot \{\underline{A} \nabla_h v_h\}) \, ds, \end{aligned} \quad (6b)$$

$$\mathcal{L}_h(v_h) = \int_{\Omega} f v_h \, dx. \quad (6c)$$

Here η is a parameter which is to be defined to guarantee the coercivity of the bilinear form \mathcal{B}_h . The bilinear form \mathcal{B}_h is coercive and bounded in \mathcal{V}_h , equipped with the norm (3), for $\eta > 0$ sufficiently large, see e.g. [4]. For an explicit expression and a lower bound for η see [17,30,39]. Further, for $f \in L^2(\Omega)$ the problem (6) has a unique solution $u_h \in \mathcal{V}_h$. The IP-DG method delivers an optimal order of convergence in the DG-norm (3) as well as, when equipped with elliptic regularity, in L^2 -norm. For proofs see, e.g., [4].

Remark 1 Since we do not need any specific structure of the DG schemes we can use any of the DG methods proposed in the literature instead of IP-DG method given in (6). The DG-norm (3) will then be accordingly chosen.

4 A posteriori error estimates

We first discuss the *direct* decomposition of the error in the conforming and the nonconforming parts. The following result, due to [24, Lemma 4.4], will be useful in this respect.

Lemma 2 Let $\tilde{u}_h \in \mathcal{V}_0$ be such that

$$\int_{\Omega} \underline{A} \nabla \tilde{u}_h \cdot \nabla v \, dx = \sum_{K \in \mathcal{T}_h} \int_K \underline{A} \nabla u_h \cdot \nabla v \, dx, \quad \forall v \in \mathcal{V}_0. \quad (7)$$

Then the decomposition

$$u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h) \quad (8)$$

satisfies the following orthogonal relation

$$\sum_{K \in \mathcal{T}_h} \|\nabla u - \nabla u_h\|_{a,K}^2 = \|\nabla u - \nabla \tilde{u}_h\|_{a,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \|\nabla \tilde{u}_h - \nabla u_h\|_{a,K}^2. \quad (9)$$

PROOF. By taking $v = u - \tilde{u}_h$ in the orthogonality relation obtained from (7) we get

$$\sum_{K \in \mathcal{T}_h} \int_K \underline{A} \nabla(\tilde{u}_h - u_h) \cdot \nabla(u - \tilde{u}_h) \, dx = 0.$$

The conclusion is then straightforward by using this relation in (8). \square

Note that the terms $u - \tilde{u}_h$ and $\tilde{u}_h - u_h$ are referred as *conforming error* and *nonconforming error*, respectively, see e.g. [1,19,25]. Now before (8) can be used in deriving a posteriori estimates we need to suggest some ways to compute \tilde{u}_h such that the *nonconforming* error $\tilde{u}_h - u_h$ is minimal. One such cheap method is based on Oswald interpolation operator, see e.g. [10,19,23]. Here we present only the construction of the Oswald interpolation operator $\Pi_O : \mathcal{V}_h \rightarrow \mathcal{V}_h \cap \mathcal{V}_0$ and refer the reader for a detailed analysis, in particular, the approximation properties, to [10,23].

Algorithm 1 Given a function $v_h \in \mathcal{V}_h$ the value of $\Pi_O(v_h)$ is prescribed at suitable (e.g. Lagrangian) vertices/points of the elements in \mathcal{T}_h as follows:

- (1) For points interior to elements the value is calculated from the corresponding basis functions and the location of the point,
- (2) For points shared by elements (either on the inter-element faces or the common vertices) the value is taken as a simple average, i.e., if \mathcal{N}_v denotes the total number of elements sharing a point V then

$$\Pi_O(v_h)(V) = \frac{1}{\mathcal{N}_v} \sum_{K \in K_V} v_h|_K(V),$$

where K_V denotes the set of elements K which shares the point V .

- (3) For points residing on the boundary of the domain the value is set to zero.

Remark 3 It is also possible to choose \tilde{u}_h by orthogonal projection onto the energy space, see [27, Section 5.1.2]. However, that will require solution of a system of equations and thus, would incur further cost on the total procedure.

We now derive our a posteriori error estimates.

4.1 Global upper bound of the error

Lemma 4 Let u_h be the DG solution obtained from (6) and $\tilde{u}_h \in \mathcal{V}_0$ be its projection obtained by Algorithm 1. Then the following inequality holds.

$$\llbracket u - u_h \rrbracket \leq \|\nabla(u - \tilde{u}_h)\|_{a,\Omega} + \llbracket \tilde{u}_h - u_h \rrbracket. \quad (10)$$

PROOF. By using the triangle inequality on (8) we get

$$\llbracket u - u_h \rrbracket \leq \llbracket u - \tilde{u}_h \rrbracket + \llbracket \tilde{u}_h - u_h \rrbracket. \quad (11)$$

Now for any conforming function $v \in \mathcal{V}_0$ we have $\llbracket v \rrbracket = 0$. Thus, from (3-4) we get

$$\llbracket u - \tilde{u}_h \rrbracket = \|\nabla(u - \tilde{u}_h)\|_{a,\Omega}.$$

By using this relation in (11) we get the result. \square

In [31–35] functional type a posteriori error estimates have been derived for conforming approximations of a wide spectrum of problems. These estimates are valid for *any conforming approximation*, contain *no mesh-dependent constants* and provide *guaranteed and computable* upper and lower bounds of the approximation error. We exploit these properties here and use them for $\|\nabla(u - \tilde{u}_h)\|_{a,\Omega}$. However, before proceeding further we briefly recall some principal results of these estimates to keep this article as self content as possible.

Lemma 5 Let $\tilde{u}_h \in \mathcal{V}_0$ be a certain conforming approximation of the exact solution u of the problem 1. Then the following estimate holds [32–34].

$$\|\nabla(u - \tilde{u}_h)\|_{a,\Omega} \leq \|\underline{A}\nabla\tilde{u}_h - y\|_{\bar{a},\Omega} + C_{\Omega,\underline{A}}\|\operatorname{div} y + f\|, \quad (12)$$

where $y \in H(\Omega, \operatorname{div})$ is a "free" vector-valued function and $C_{\Omega,\underline{A}}$ is a constant in the inequality

$$\|v\| \leq C_{\Omega,\underline{A}}\|\nabla v\|_{a,\Omega}, \quad \forall v \in \mathcal{V}_0. \quad (13)$$

If $\Omega \subset \Omega_\square$, Ω_\square being a square domain with the side l , then $C_{\Omega,\underline{A}} \leq c_2 \frac{l}{\sqrt{d\pi}}$, where the constant c_2 comes from (2). Some of the ways to compute y , which should be chosen to minimize the right hand side, will be discussed later in this section.

We are now in a position to describe our main result for global upper bound of the error.

Theorem 6 *Let u_h be the DG solution obtained from (6) and $\tilde{u}_h \in \mathcal{V}_0$ be its projection obtained by Algorithm 1. Then the following relation holds.*

$$\llbracket u - u_h \rrbracket \leq \|\underline{A} \nabla \tilde{u}_h - y\|_{\bar{a}, \Omega} + C_{\Omega, \underline{A}} \|\operatorname{div} y + f\| + \llbracket \tilde{u}_h - u_h \rrbracket. \quad (14)$$

PROOF. Follows immediately by using (12) in (10). \square

Corollary 7 *It is easy to see that for $\beta > 0$ the relation (14) can also be presented in the form*

$$\llbracket u - u_h \rrbracket \leq M_{\oplus}^{\frac{1}{2}} + \llbracket \tilde{u}_h - u_h \rrbracket, \quad (15)$$

$$\text{where } M_{\oplus} = (1 + \beta) \|\underline{A} \nabla \tilde{u}_h - y\|_{\bar{a}, \Omega}^2 + (1 + 1/\beta) C_{\Omega, \underline{A}}^2 \|\operatorname{div} y + f\|^2. \quad (16)$$

Remark 8 *The upper bound in (14) is sharp since for $y = \underline{A} \nabla u$ and $\tilde{u}_h = u$ the right hand side coincides with the left hand one. Thus, the upper bound computed with the help of these functions is as close to the exact error norm as it is desired.*

We now discuss some ways to define the vector-valued function $y \in H(\Omega, \operatorname{div})$. There are three basic ways to define a suitable flux y :

- (1) A post-processing on the primal mesh
- (2) Flux reconstruction using DG method
- (3) Minimizing the `majorant` M_{\oplus} , a quadratic functional in y and β , in (16)

The first approach is a cheap one but might result in a coarse estimate, see [27] for numerical results. For second the reader can refer to, e.g., [19]. The third approach, which is used in this article, will produce sharp estimates but with the cost comparable to that of finding the DG solution since this will involve solving a linear system of equations for a vector-valued function.

4.2 Global lower bound of the error

Lemma 9 *Let u_h be the DG solution obtained from (6) and $\tilde{u}_h \in \mathcal{V}_0$ be its projection obtained by Algorithm 1. Then the following inequality holds.*

$$\llbracket u - u_h \rrbracket \geq \left| \|\nabla(u - \tilde{u}_h)\|_{a, \Omega} - \llbracket \tilde{u}_h - u_h \rrbracket \right|. \quad (17)$$

PROOF. We first write the decomposition (8) in the following form

$$u - u_h = (u - \tilde{u}_h) - (u_h - \tilde{u}_h). \quad (18)$$

Then, by using the inverse triangle inequality and proceeding as in the proof of Lemma 4 we get the result. \square

Before proceeding further we again recall some results for the lower bound of the error $\|\nabla(u - \tilde{u}_h)\|_{a,\Omega}$.

Lemma 10 *Let $\tilde{u}_h \in \mathcal{V}_0$ be a certain conforming approximation of the exact solution u of the problem 1. Then the following estimate holds (see, e.g.[35]).*

$$\|\nabla(u - \tilde{u}_h)\|_{a,\Omega}^2 \geq -\|\nabla w\|_{a,\Omega}^2 - 2 \int_{\Omega} A \nabla \tilde{u}_h \cdot \nabla w \, dx + 2 \int_{\Omega} f w \, dx, \quad (19)$$

where $w \in \mathcal{V}_0$ is a “free” function.

We now describe our main result for global lower bound of the error.

Theorem 11 *Let u_h be the DG solution obtained from (6) and $\tilde{u}_h \in \mathcal{V}_0$ be its projection obtained by Algorithm 1. Then the following relation holds.*

$$\|u - u_h\| \geq \left| M_{\ominus}^{\frac{1}{2}} - \|\tilde{u}_h - u_h\| \right|, \quad (20)$$

$$\text{where } M_{\ominus} = \left(-\|\nabla w\|_{a,\Omega}^2 - 2 \int_{\Omega} A \nabla \tilde{u}_h \cdot \nabla w \, dx + 2 \int_{\Omega} f w \, dx \right)^{\frac{1}{2}}. \quad (21)$$

PROOF. Follows immediately by using (19) in (17). \square

Remark 12 *The lower bound in (20) is also sharp since for $w = u - \tilde{u}_h$ and $\tilde{u}_h = u$ the right hand side coincides with the left hand one.*

The free function w can be computed in the following ways:

- (1) Maximizing the minorant M_{\ominus} , a quadratic functional in w , in (21)
- (2) Choosing $w = \tilde{u}_k - \tilde{u}_{k-1}$ when a sequence of refined meshes is used

In this article we use the first approach which produces sharp estimates.

Remark 13 *For lower bound of the conforming error the right inequality of [25, Lemma 1] can also be used. Though it will come at no extra cost, however, firstly, it will not be sharp (a factor of 2 will arise), and secondly, it will have an additional dependence on the constants c_1 and c_2 from (2).*

Remark 14 Instead of the DG-norm (3) if we use the broken H^1 -seminorm, defined as follows

$$\|v_h\|^2 = \sum_{K \in T_h} \int_K A \nabla_h v_h \cdot \nabla_h v_h \, dx, \quad (22)$$

then both the estimates (15) and (20) hold without the nonconforming error $\|\tilde{u}_h - u_h\|$. These estimates without nonconforming error are of independent interest in view of the observation that the jump errors in the DGFEM are subordinate to the error in the broken H^1 -seminorm [1], however, not the focus of this article.

4.3 Efficiency of the global bounds of the error

For the DG solution we define the majorant as the right hand side of (15) and denote it by M_{\oplus}^{DG} , and the minorant as the right hand side of (20) and denote it by M_{\ominus}^{DG} . The effectivity indices I_{\oplus} (of the majorant) and I_{\ominus} (of the minorant) are then defined as

$$I_{\oplus} = M_{\oplus}^{\text{DG}} / \|u - u_h\|, \quad I_{\ominus} = M_{\ominus}^{\text{DG}} / \|u - u_h\|. \quad (23)$$

5 Numerical examples

In this Section we present the numerical results of some categorically selected examples to demonstrate the efficiency and robustness of the proposed estimates. The choice of the examples is guided by the difficulty level in computing the bounds of the approximation error.

Now let p_u , p_y , and p_w denote the polynomial degree for the approximation of u_h , y , and w , respectively. For all the examples we consider the Poisson problem on the unit square. We choose f and the Dirichlet boundary condition (assuming $\Gamma_D = \partial\Omega$) such that the exact (analytic) solution of the problem, which is shown in Figure 2, is given as follows:

Example 1 $u = x(x - 1)y(y - 1)$.

Example 2 $u = \sin(2\pi x)\sin(2\pi y)$.

Example 3 $u = x(x - 1)y(y - 1)\sin(17xy)e^{(x+y)}$. ¹

Example 4 $u = x(1 - x)y(1 - y)e^{-1000((x-.5)^2 + (y-.117)^2)}$ [20].

¹ Based on [41, Example 6.2].

Further, we also consider the following example with jump in the coefficients:

Example 5 Consider the elliptic problem $-\operatorname{div} \underline{A} \nabla u = 1$ on the unit square with homogenous Dirichlet boundary condition. The coefficient \underline{A} has the jumps as follows: $\underline{A} = 1$ in $(0, 0.5] \times (0, 0.5] \cup (0.5, 1) \times (0.5, 1)$ and $\underline{A} = \varepsilon (= .0001)$ in the remaining domain.

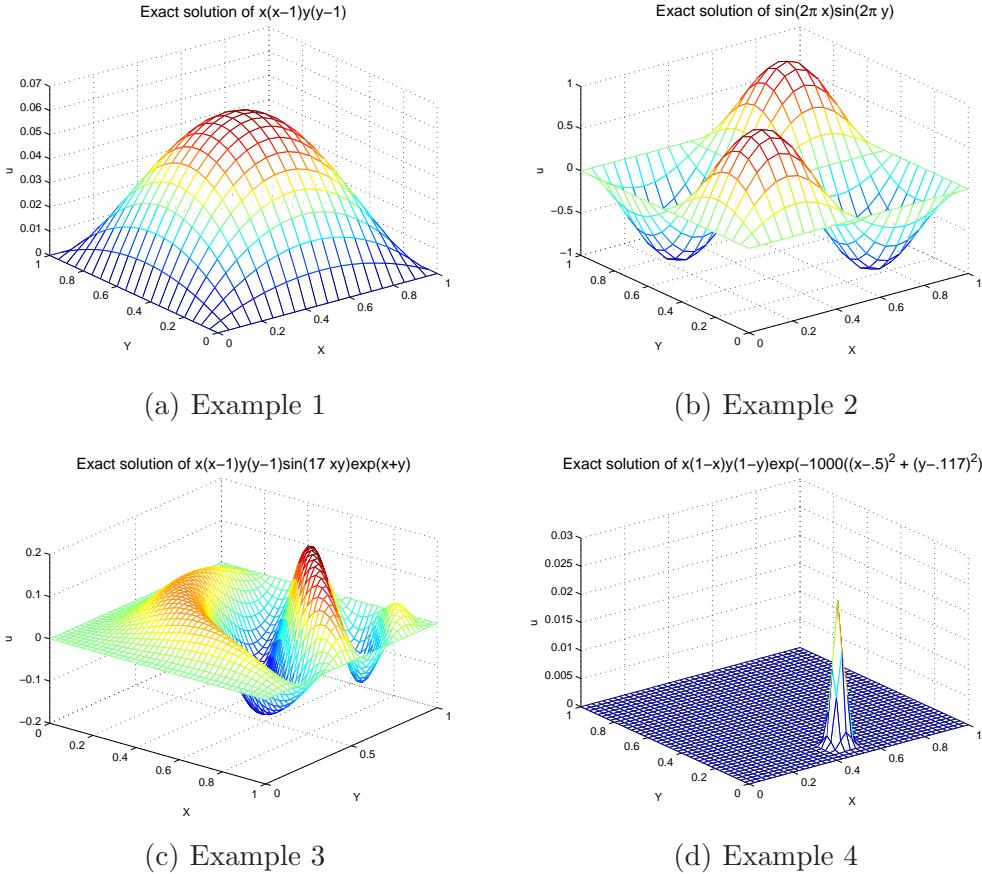


Fig. 2. Illustration of the analytic solutions and the jump in the coefficients.

We first discuss the results on upper bound (15).

5.1 Upper bounds

We first present the numerical results for Example 1. With $p_u = p_y = 1$, various components of the errors are presented in Table 1. This shows that the true error $\|\nabla u - \nabla_h u_h\|_a$ (last column) is bounded by the majorant M_{\oplus}^{DG} (fourth column), which is a sum of the nonconforming error (second column) and the conforming error (third column). In Table 2 we present the efficiency of the majorant. For a reasonably accurate DG approximation the nonconforming error is very small as compared to the other two terms of the majorant. Moreover, with $p_y = 1$ the div y approximation term results in a constant which is

Table 1
Various errors for Example 1

DOF	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a$	$\ \nabla u - \nabla \tilde{u}_h\ _a$	I+II	$\ \nabla u - \nabla_h u_h\ _a$
	I	II		
400	2.956e-04	1.497e-02	1.526e-02	1.497e-02
1600	1.460e-04	7.461e-03	7.607e-03	7.463e-03
6400	7.257e-05	3.728e-03	3.800e-03	3.729e-03
25600	3.617e-05	1.864e-03	1.900e-03	1.864e-03

Table 2
Effectivity of the majorant for Example 1

DOF	β	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$\ y - \underline{A} \nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}
400	0.116	8.736e-08	2.177e-04	5.734e-05	1.119
1600	0.054	2.133e-08	5.529e-05	3.194e-06	1.070
6400	0.026	5.267e-09	1.387e-05	1.868e-07	1.045
25600	0.013	1.309e-09	3.471e-06	1.127e-08	1.032

sufficient to approximate the constant function $f(= 1)$. Thus, the flux equilibration term $\|\operatorname{div} y + f\|^2$ is also of low order as compared to the duality error term $\|y - \underline{A} \nabla_h u_h\|_{\bar{a}}^2$. However, this may not be the case in general, as clear from the other examples.

As a remedy to the above-mentioned difficulty, we first try increasing both of the p_u and p_y . With $p_u = p_y = 2$ this is shown in the numerical results for Example 2 in Table 3. Though the effectivity of the majorant is still very good, we note however that, unlike the results for Example 1, the flux equilibration term is dominating the duality error term. This is because of the approximation of the $\operatorname{div} y$ term with the same polynomial order as for the DG (and conforming) approximation. To alleviate this difference of approximation order we increase p_y while keeping p_u fixed, see Table 4 with $p_u = 1, p_y = 2$ ². This makes the flux equilibration term asymptotically negligibly small as compared to the duality error term and we get an excellent effectivity index.

The convergence of the true error and the majorant with $p_u = 1, p_y = 2$ is shown in Figure 3. Further, the comparison of the computing time for the DG solution and M_{\oplus}^{DG} is shown in Table 5. Since the computational cost of Oswald interpolation (to obtain \tilde{u}_h) is negligible as compared to other computations

² Having computed \tilde{u}_h for a given polynomial degree one can easily compute its value at any number of points in a given element and use it to compute y for higher polynomial degree.

Table 3

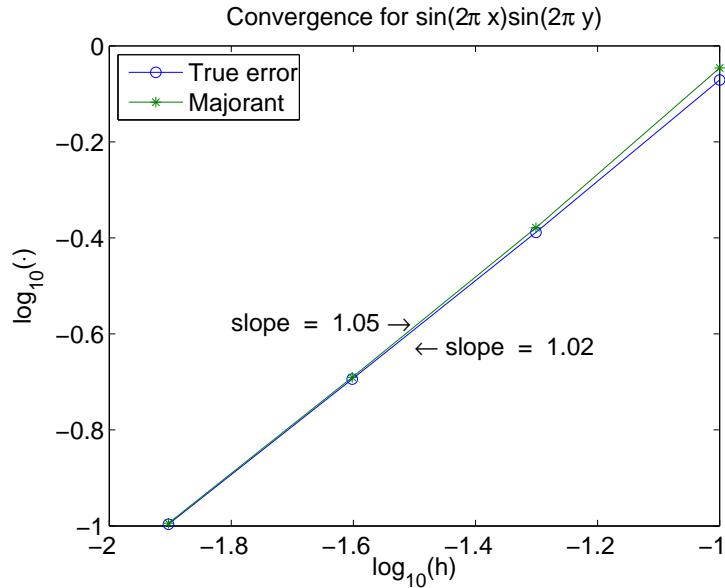
Effectivity of the majorant for Example 2, $p_u = p_y = 2$

DOF	β	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$\ y - \underline{A} \nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}
900	0.329	5.272e-06	7.150e-03	1.528e-02	1.756
3600	0.222	6.292e-07	4.533e-04	4.395e-04	1.639
14400	0.190	7.550e-08	2.836e-05	2.010e-05	1.614
57600	0.181	9.204e-09	1.772e-06	1.144e-06	1.626

Table 4

Effectivity of the majorant for Example 2, $p_u = 1, p_y = 2$

DOF	β	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$\ y - \underline{A} \nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}
400	0.062	5.883e-05	7.052e-01	5.365e-02	1.060
1600	0.017	7.294e-06	1.668e-01	9.411e-04	1.022
6400	0.004	8.800e-07	4.089e-02	1.526e-05	1.009
25600	0.001	1.073e-07	1.017e-02	2.416e-07	1.004

Fig. 3. Convergence of the true error and the majorant for Example 2 with $p_u = 1, p_y = 2$.

we do not report it here. Obviously, the cost of computing M_{\oplus}^{DG} will depend on the polynomial degree used. For equal degree the computing time for M_{\oplus}^{DG} is proportional to that of finding the DG solution whereas its almost double with one degree higher (with the combination of $p_u = 1, p_y = 2$).

We now present the results for Example 3 in Table 6. To counter the large

Table 5

Computing time (sec) for Example 2, $p_u = 2$

$p_u = 2, p_y = 2$			$p_u = 1, p_y = 2$		
DOF	u_h	M_{\oplus}^{DG}	DOF	u_h	M_{\oplus}^{DG}
900	0.623	0.499	400	0.412	0.630
3600	2.137	2.041	1600	1.255	2.515
14400	8.796	8.510	6400	5.001	10.215
57600	36.731	35.283	25600	20.068	42.791

variations in f we choose $p_u = 3$ (and thus avoid a highly refined mesh). Exploiting the technique discussed in previous example we take $p_y = 4$. This again shows an excellent efficiency.

Table 6

Effectivity of the majorant for Example 3, $p_u = 3, p_y = 4$

DOF	β	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$\ y - \underline{A} \nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}
1600	0.088	4.396e-08	1.159e-04	1.761e-05	1.129
6400	0.020	6.073e-10	1.809e-06	1.492e-08	1.044
25600	0.005	7.956e-12	2.823e-08	1.396e-11	1.023
102400	0.001	1.096e-13	4.409e-10	1.347e-14	1.017

We now present the results for Example 4, which exhibits a sharp peak, in Table 7. We again note that the flux equilibration term is dominating the duality error term unless a highly refined mesh is taken. The convergence of the true error and the majorant with $p_u = 2, p_y = 3$ is shown in Figure 4. This, apart from a poor choice of h/p , may be an effect of p_y .

Table 7

Effectivity of the majorant for Example 4, $p_u = 2, p_y = 3$

DOF	β	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$\ y - \underline{A} \nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}
3600	2.047	2.826e-08	6.177e-04	5.110e-02	5.070
14400	0.702	2.984e-08	4.619e-05	4.488e-04	2.700
57600	0.415	7.623e-09	1.693e-06	5.767e-06	1.823
230400	0.199	1.148e-09	8.553e-08	6.655e-08	1.434

Thus, next we see how the variation of p_y alone effects the efficiency. In Table 8 we fix $p_u = 2$, $h = 0.05$, and vary $p_y = 2, \dots, 6$, by extending the idea

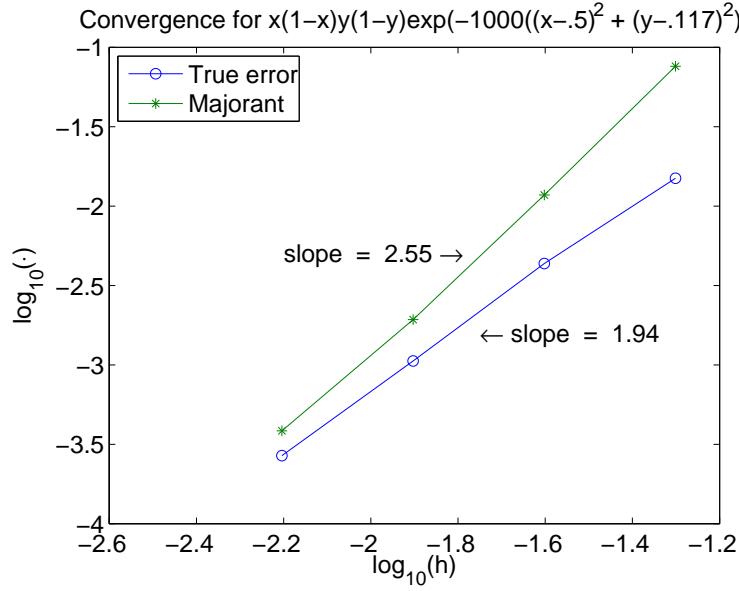


Fig. 4. Convergence of the true error and the majorant for Example 4 with $p_u = 2, p_y = 3$.

discussed in the previous example. Interestingly, we see that even for this admittedly difficult problem, with f having very large gradients, one obtains very promising results and the effectivity index can be brought as close to 1 as one pleases, though with associated high cost. Thus, this is a trade-off between the efficiency and the associated cost. As the effectivity index between 2 to 5 is considered very good, the upper bound of the true error can be *computed with guarantee at a reasonable cost*.

Table 8

Effect of p_y on the majorant, $\|\nabla \tilde{u}_h - \nabla_h u_h\|_a^2 = 2.826e-08$

p_y	β	$\ \underline{y} - \underline{A}\nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}	M_{\oplus}^{DG}	Time
2	3.655	2.524e-03	6.653e-01	15.630	4.267	
3	2.047	6.177e-04	5.110e-02	5.070	12.775	
4	0.664	3.700e-04	3.217e-03	2.149	27.626	
5	0.251	2.310e-04	2.880e-04	1.282	52.478	
6	0.047	2.261e-04	9.669e-06	1.062	89.291	

Finally, we present the results for Example 5 in Table 9 with $p_u = 1, p_y = 2$. We see that the estimates are robust with respect to the jumps in the coefficients.

Now we discuss the results on lower bound (20).

Table 9
Effectivity of the majorant for Example 5

DOF	β	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$\ y - \underline{A} \nabla_h u_h\ _{\bar{a}}^2$	$\ \operatorname{div} y + f\ ^2$	I_{\oplus}
400	0.000	6.236e-07	2.632e+00	1.872e-18	1.077
1600	0.000	1.685e-07	6.588e-01	2.047e-20	1.068
6400	0.000	4.434e-08	1.656e-01	2.572e-22	1.068
25600	0.000	1.145e-08	4.258e-02	4.223e-24	1.083

5.2 Lower bounds

For brevity reasons we report the lower bounds results only for Example 2 and Example 4. Since $w \in \mathcal{V}_0$ is to be chosen from a space larger than for u_h , the polynomial degree we take for w is also one higher than for u_h . In Table 10 we present the results for Example 2. The effectivity is again excellent.

Table 10

Effectivity of the minorant for Example 2, $p_u=1, p_w=2$

DOF	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$-\ \nabla w\ _{a,\Omega}^2$	$-2 \int_{\Omega} \underline{A} \nabla \tilde{u}_h \cdot \nabla w$	$2 \int_{\Omega} f w$	I_{\ominus}
400	5.883e-05	-7.174e-01	-2.258	3.693	.989
1600	7.294e-06	-1.670e-01	-6.271e-01	9.612e-01	.993
6400	8.800e-07	-4.089e-02	-1.609e-01	2.427e-01	.995
25600	1.073e-07	-1.017e-02	-4.050e-02	6.083e-02	.997

In Table 11 we present the results for Example 4. A behavior similar to that of majorant is observed. Further, the variation in p_w , with fixed p_u and h , also exhibits a similar effect on the overall minorant efficiency as the variation in p_y has on majorant efficiency. For fixed $p_u = 2$ and $h = 0.05$ we get $I_{\ominus} = .933, .986, .988, .989$ for $p_w = 3, 4, 5, 6$, respectively.

Table 11

Effectivity of the minorant for Example 4, $p_u=2, p_w=3$

DOF	$\ \nabla \tilde{u}_h - \nabla_h u_h\ _a^2$	$-\ \nabla w\ _{a,\Omega}^2$	$-2 \int_{\Omega} \underline{A} \nabla \tilde{u}_h \cdot \nabla w$	$2 \int_{\Omega} f w$	I_{\ominus}
3600	2.826e-08	-2.127e-04	-2.439e-04	6.565e-04	.933
14400	2.984e-08	-1.819e-05	-3.280e-05	6.903e-05	.938
57600	7.623e-09	-1.101e-06	-2.182e-06	4.384e-06	.909
230400	1.148e-09	-7.057e-08	-1.430e-07	2.842e-07	.864

The convergence of the true error and the minorant is shown in Figure 5.

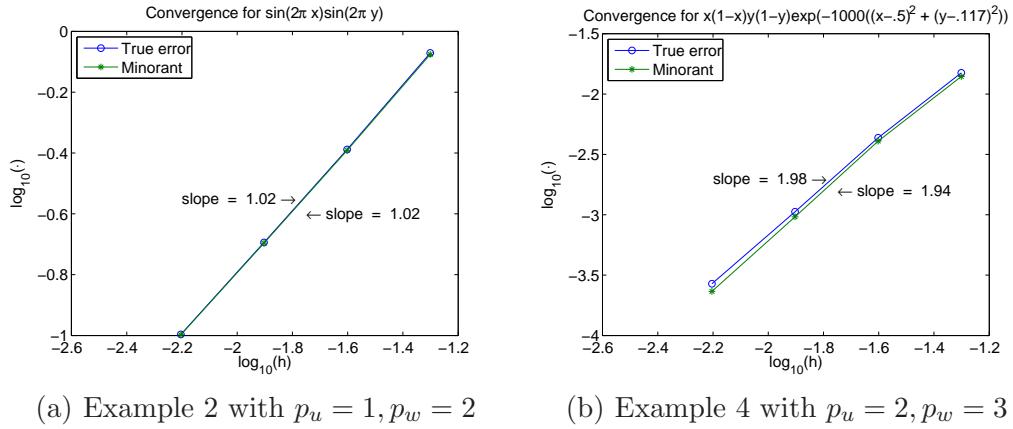


Fig. 5. Convergence of the true error and the minorant.

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