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Abstract. We consider the numerical solution of a nonlinear problem arising in in nondestructive detection of a corrosion at a hidden surface of a conductor. The problem can be naturally linearized and reduced to an elliptic Cauchy problem. In this paper we describe and test a regularized reconstruction algorithm based on a regularization by discretization, where the discretization level is chosen in data driven way.

1. Introduction

The aim of the corrosion detection is to determine a possible presence of a corrosion damage by performing current and voltage measurements on the boundary. This means to apply a current of a density g on an accessible portion of the boundary of the conductor and to measure the corresponding voltage potential u on the same portion.

This physical problem is modeled as follows. A regular bounded domain Ω represents the region occupied by the electrostatic conductor which contains no electric sources, and this is modeled by the Laplace equation, so that the voltage potential u satisfies

$$\Delta u = 0 \text{ in } \Omega. \tag{1}$$

We assume that the boundary $\partial\Omega$ of the conductor is decomposed into three open nonempty and disjoint portions Γ_c, Γ_{un}, $\Gamma_{\mathcal{D}}$. The portion Γ_c is accessible to the electrostatic measurements, whereas the portion Γ_{un}, where the corrosion may take place, is out of reach. The remaining portion $\Gamma_{\mathcal{D}}$ is assumed to be grounded, which means that the voltage potential u vanishes there, i.e.

$$u = 0 \text{ on } \Gamma_{\mathcal{D}}. \tag{2}$$

We impose a current of a density g on Γ_c and measure the corresponding potential h upon the same portion of the boundary.

By the pair of boundary measurements

$$
\begin{aligned}
u &= h &&\text{on } \Gamma_c, \\
\frac{\partial u}{\partial \nu} &= g &&\text{on } \Gamma_c
\end{aligned}
\tag{3}
$$

we want to recover a relationship between voltage u and current density $\frac{\partial u}{\partial \nu}$ on the inaccessible boundary portion Γ_{un} suspected to be corroded.

A simple model of corrosion appearance consists in finding a coefficient $\gamma = \gamma(x)$ in a linear boundary condition of the type

$$\frac{\partial u}{\partial \nu} = -\gamma u \quad \text{on } \Gamma_{un}, \tag{4}$$

where ν is the outward unit normal at the boundary, and $\gamma \geq 0$ is the so-called Robin coefficient. The study of such a problem has been developed by many authors, among them, let us mention Fasino, Inglese [13, 14], Kaup, Santosa, Vogelius [16, 17], Alessandrini, Del Piero, Rondi [1], Chaabane, Fellah, Jaoua, Leblond [7], and [20].

A more accurate treatment of corrosion requires a nonlinear relationship between voltage and current density on the corroded surface. A model of this kind is associated to the name of Butler and Volmer and postulates the boundary condition

$$\frac{\partial u}{\partial \nu} = \lambda(\exp(\alpha u) - \exp(-(1-\alpha)u)) \quad \text{on } \Gamma_{un}. \tag{5}$$

This nonlinear boundary value problem, has been recently discussed by Bryan, Kavian, Vogelius and Xu in [5, 18, 21]. The authors have examined the questions of the existence and the uniqueness of the solution of the problem with a given nonlinearity of the type (5). Namely, they have assumed to know the nonlinearity (5) by prescribing the coefficients λ and α in suitable ranges, and they have discussed the existence and the uniqueness issues for the direct problem.

Motivated by these studies, in [2, 3, 19] Alessandrini and the third author have also considered a more general choice of the nonlinear profile, namely of the form

$$\frac{\partial u}{\partial \nu} = f(u) \quad \text{on } \Gamma_{un}, \tag{6}$$

and they have dealt with the inverse problem of a reconstruction of the nonlinear profile f from boundary measurements at Γ_c, and proved a logarithmic stability estimate for this problem. See also [10] for a related work.

To the best of our knowledge there are only a few papers devoted to a numerical study of parameter identification problem in nonlinear model of the corrosion detection. In [15] only the case of $\Omega = (0,1) \times (0,a), a \ll 1$, has been discussed. In [22] the method from [6] for solving intermediate Cauchy problem has been employed. This method requires a sufficiently smooth domain boundary $\partial\Omega$. Moreover, only its convergence for vanishing noise level has been proven.

At the same time, a rather general method for solving elliptic Cauchy problem has been proposed recently in [9]. In this method a Cauchy problem is regularized by discretization, and the discretization level is chosen in data driven way. Under rather mild assumption an order-optimality of the method has been proven in [9]. In the present paper our goal is to adjust the method [9] for solving parameter identification problem in nonlinear model of the corrosion detection (1) - (3), (6). We also present numerical experiments demonstrating the performance of the adjusted method.

2. Natural linearization

It is easy to realize that the reconstruction of a nonlinearity f in (6) can be achieved in two steps. First, an elliptic Cauchy problem (1)-(3) can be reformulated as a linear operator equation with a compact operator. Then, using an approximate solution of the problem (1)-(3) one is able to find its Dirichlet and Neumann traces at Γ_{un}, and reconstruct f from (6) pointwise.

There are several ways to reduce the Cauchy problem to a linear operator equation. Here we follow the way leading to an equation with a compact operator for unknown Neumann trace $s(x,y)$ of the solution $u(x,y)$ at the part Γ_{un} of the boundary where no data was prescribed. Namely, we introduce a linear continuous operator $A : L_2(\Gamma_{un}) \to L_2(\Gamma_c)$ that assigns to $s \in L_2(\Gamma_{un})$ a Dirichlet trace of the (weak) solution u_s of the mixed boundary value problem

$$
\begin{cases}
\Delta u = 0, & \text{in } \Omega \\
\dfrac{\partial u}{\partial \nu} = s, & \text{on } \Gamma_{un} \\
\dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_c \\
u = 0 & \text{in } \Gamma_{\mathcal{D}}
\end{cases}
\tag{7}
$$

at Γ_c, i.e. $As = u_s|_{\Gamma_c}$.

Then we consider a function u_0 solving an auxiliary mixed boundary value problem

$$
\begin{cases}
\Delta u = 0, & \text{in } \Omega \\
\dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_{un} \\
\dfrac{\partial u}{\partial \nu} = g, & \text{on } \Gamma_c \\
u = 0, & \text{in } \Gamma_{\mathcal{D}}.
\end{cases}
\tag{8}
$$

If s in (7) is such that $u_s|_{\Gamma_c} + u_0|_{\Gamma_c} = h$ then in view of uniqueness results the function $u_s + u_0$ solves the Cauchy problem (1)-(3). Therefore, unknown Neumann trace s can be found from the operator equation

$$
As = r, \quad r = h - u_0|_{\Gamma_c}.
\tag{9}
$$

If only noisy measurements g^δ, h^δ are available, then one can construct a noisy version of (9)

$$
As = r^\delta, \quad r^\delta = h^\delta - u_0^\delta|_{\Gamma_c},
\tag{10}
$$

where u_0^δ is the solution of (8) with g substituted for g^δ. As in [11], we assume that we are given noisy Cauchy data h^δ, g^δ, or alternatively r^δ, such that

$$
\|r - r^\delta\|_{L_2(\Gamma_c)} \leq \delta.
\tag{11}
$$

Note, that the existence of weak solutions with L_2-traces on Γ_c can be guaranteed for auxiliary problems (7), (8) under rather mild assumptions, which are much weaker than the usual assumption about Sobolev regularity $1/2$ and $-1/2$ for Dirichlet and Neumann data that are usually adopted in the literature. We refer to [9] for further discussions.

3. Regularization by projection

The equation (10) is regularized by projection. Let $P = P_n$ be the orthogonal projector from $L_2(\Gamma_{un})$ onto n-dimensional subspace $U_n \subset L_2(\Gamma_{un})$. We seek for the minimal norm solution $s_n^\delta \in U_n$ meeting

$$
P_n A^* A P_n s = P_n A^* r^\delta,
\tag{12}
$$

where $A^* : L_2(\Gamma_c) \to L_2(\Gamma_{un})$ is the adjoint of A.

Seeking for an approximate solution s_n^δ of (10) in the subspace U_n one needs to fix a linearly independent system $\{\Phi_i^n\}_{i=1}^n$ forming a basis in U_n. Then s_n^δ can be represented in the form of a linear combination

$$
s_n^\delta = \sum_{i=1}^n \gamma_i \Phi_i^n
\tag{13}
$$

of the fixed system $\{\Phi_i^n\}$ with a coefficient vector $\gamma = \{\gamma_i\}_{i=1}^n$.

Plugging the representation (13) into (12) and keeping in mind that $P_n \Phi_i^n = \Phi_i^n$, $i = 1, 2, \ldots, n$, and $\langle \Phi_i^n, A^* A \Phi_j^n \rangle = \langle A\Phi_i^n, A\Phi_j^n \rangle$, we obtain the following system of linear algebraic equations for the vector γ

$$M\gamma = Y_\delta, \tag{14}$$

where matrix and right-hand side vector are given as

$$
\begin{aligned}
M &:= (\langle A\Phi_i^n, A\Phi_j^n \rangle)_{i,j=1,\ldots,n}, \\
Y_\delta &:= (\langle A\Phi_i^n, r^\delta \rangle)_{i=1,\ldots,n}.
\end{aligned}
\tag{15}
$$

The error $\|s - s_n^\delta\|_{L_2(\Gamma_{un})}$ is analyzed under the following assumptions:

- Convergence for noise free data: The solution s_n^0 of (12) for noise free data converges to the solution s of (9) as $n \to \infty$. It means that there exists an increasing continuous function $\psi(\lambda) = \psi(A, s; \lambda)$ such that $\psi(0) = 0$, and

$$\|s - s_n^0\|_{L_2(\Gamma_{un})} \le \psi(n^{-1}). \tag{16}$$

- Noise propagation error: In discrete approximations s_n^δ a noise propagates with an exponential rate such that

$$e^{an}\delta \le \|s_n^\delta - s_n^0\|_{L_2(\Gamma_{un})} \le e^{aqn}\delta, \quad n = 1, 2, \ldots \tag{17}$$

for some unknown $a > 0$. The quantity $q > 1$ in (17) is a priori chosen design parameter which reflects the magnitude of a gap in our knowledge of the problem. In [9] we argue that this assumption reflects a typical noise propagation for severely ill-posed problems. Note that a similar observation has been made in [15] (see the formula (2.10) there).

4. Balancing principle for the choice of n

In this section we describe an adaptive procedure for choosing a discretization level n without knowledge of a in (17) and ψ in (16). This procedure can be seen as successive testing of the hypotheses that the power aq in a robustness estimation (17) is not larger than some $b_j \in \{b_i = b_0 q^i, \ i = 0, 1, \ldots M\}$, where $q > 1$ is the same as in (17). For each of these hypotheses one can choose $n = n(b_j)$ using the strategy from [4]:

$$n(b_j) = \min\{n : \|s_n^\delta - s_m^\delta\| \le 4\delta e^{b_j m}, \ m = N, N-1, \ldots, n+1\},$$

where N is so large that

$$N > \max\left\{ \frac{4\ln 2}{b_0(q-1)}, q^2 \left(\frac{1}{b_0} \ln\left(\frac{6}{\delta} \right) + 1 \right) \right\}.$$

Then it is easy to verify that the sequence $\{n(b_j)\}$ is non-increasing, and it turns out that one can detect whether the hypothesis $aq \le b_j$ is untrue. The following statements can be proven along the lines of [9] and presented here for the sake of completeness.

Lemma 4.1 *If $b_{j+2} < aq$ then*

$$n(b_{j-1}) > \tau(N, q, b_0, \delta) := \frac{N}{q^2} - \frac{1}{b_0} \ln\left(\frac{3e^{N(q-1)b_0}}{e^{N(q-1)b_0} - 4} \right).$$

Let
$$n_\tau = \max\left\{n(b_j) : n(b_j) \leq \tau(N, q, b_0, \delta)\right\}.$$

If $n_\tau = n(b_i)$, then we take $n_+ := n(b_{i+2})$ as the final discretization level.

Theorem 4.1 *Under assumptions above, if the function ψ increases with a polynomial rate, then*
$$\|s - s_{n_+}^\delta\| \leq c\psi(n_{opt}^{-1}) \leq c\psi(\ln^{-1}\frac{1}{\delta}).$$

where c does not depend on δ, and n_{opt} is the solution of the equation $\psi(n) = \delta e^{aqn}$, which balances unknown approximation error bound (16) with unknown noise propagation rate (17).

5. Numerical examples

Using the approximate reconstruction $s_{n_+}^\delta$, we can find an approximate voltage potential in Ω as the solution $u_{n_+}^\delta$ of the mixed boundary value problem (1), (2) completed by the boundary conditions
$$\frac{\partial u}{\partial \nu} = g^\delta \text{ in } \Gamma_c, \quad \frac{\partial u}{\partial \nu} = s_{n_+}^\delta \text{ in } \Gamma_{un}.$$

Now the searched-for nonlinearity f can be reconstructed from (6), where u and $\frac{\partial u}{\partial \nu}$ are substituted for $u_{n_+}^\delta|_{\Gamma_{un}}$ and $s_{n_+}^\delta$ respectively. It gives an equation

$$f(u_{n_+}^\delta(x, y)) = s_{n_+}^\delta(x, y), \quad (x, y) \in \Gamma_{un}. \tag{18}$$

In numerical tests below a regularized approximation $s_{n_+}^\delta$ is so accurate that $u_{n_+}^\delta|_{\Gamma_{un}}$ preserves a monotony of the exact trace. Then f can be reconstructed point-wise directly from (18). If $u_{n_+}^\delta|_{\Gamma_{un}}$ is a non-monotone then (18) can be regularized in the same way as in [12, 8].

In our tests the basis $\{\Phi_i^n\}$ in the space U_n consists of the traces at Γ_{un} of two-dimensional polynomials $1, x, y, x^2, xy, y^2, \ldots, x^{k-1}, x^{k-2}y, \ldots, y^{k-1}$, $k = 1, 2, \ldots$, $n = k(k+1)/2$.

Then to calculate the matrix M in (15) one needs to find $A\Phi_i^n$, $i = 1, 2, \ldots, n$. Note that $A\Phi_i^n$ is the Dirichlet trace at Γ_c of the solution of (7), where s is substituted for Φ_i^n. To find such solutions one can use Matlab PDE toolbox with FEM method, where the domain Ω is partitioned into triangles, and an approximate solution is constructed as a corresponding sum of piece-wise linear finite elements. The auxiliary problem (8) is solved in the same way.

The inner products $\langle A\Phi_i^n, A\Phi_j^n \rangle$, $\langle A\Phi_i^n, r^\delta \rangle$ in $L_2(\Gamma_c)$ are computed approximately by trapezoidal rule using as knots on Γ_c the apexes of triangles forming Ω-triangulation. Moreover, calculating $\langle A\Phi_i^n, r^\delta \rangle$ we add to corresponding values of h and u_0 the values of independent zero mean random variables uniformly distributed on $[-\delta, \delta]$. In this way noisy data are simulated.

In the considered case it means that a mixed boundary value problem (7) should be solved n times for $s = \Phi_i^n, i = 1, 2, \ldots, n$. Usually in practice it is enough to take n much less than 100 (see, for example, numerical test presented below). At the same time, some other methods for solving elliptic Cauchy problems may require hundreds or even thousands calls of the direct solver for similar mixed boundary value problems (see, for example, numerical experiments reported in [11] for Mann-Maz'ya iterations).

To apply the balancing principle presented in the previous section one should choose design parameters N, q, b_0. One possible way to specify these parameters is the application of the balancing principle to a simple problem with a known solution. In this case one can try several parameter values, and the values corresponding to the smallest error can be used in applications of the balancing principle to problems with unknown solutions.

To illustrate this approach we consider at first a problem (1) - (3), (6) with $\Omega = [0, \pi] \times [0, 1]$ $\Gamma_c = [0, \pi] \times \{0\}$, $\Gamma_{un} = [0, \pi] \times \{1\}$, $\Gamma_D = \{0\} \times [0, 1] \cup \{\pi\} \times [0, 1]$, $g = -\sin(x)$, $f(t) = -t$. Noise level is $4 \cdot 10^{-4}$.
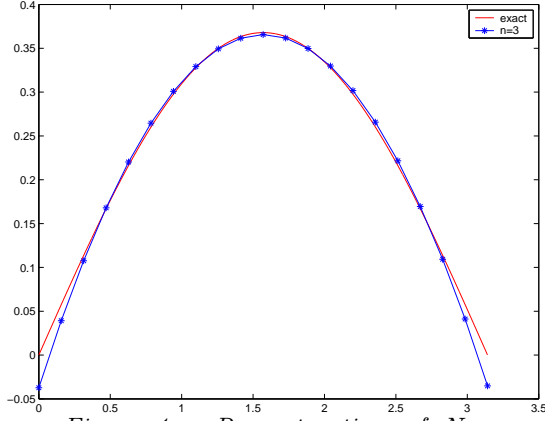
Figure 1. Reconstruction of Neumann trace on Γ_{un} for $\hat{n}_+$ selected by the balancing principle.
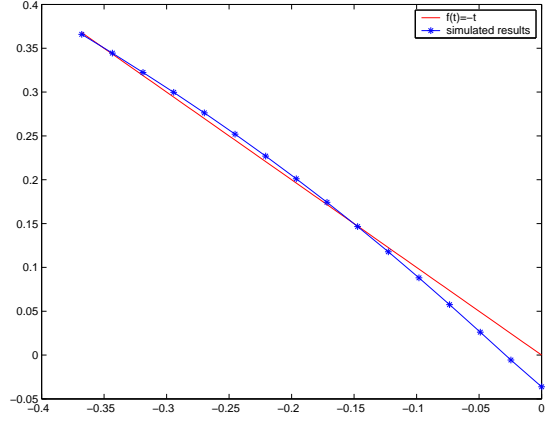

Figure 2. Approximation of $f(t)$.

Numerical experiments with this calibrating problem suggest the values $N = 66$, $q = 1.3$, $b_0 = 0.084$. For $\{b_j = 0.084 \times (1.3)^j, j = 0, 1, \cdots, 20\}$ the threshold $\tau(N, q, b_0, \delta) = 9.08$. The balancing principle selects $\{n(b_j)\}_{j=1}^{20}$ as 66, 66, 66, 66, 52, 39, 21, 11, 7, 3, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1. In this case $n_\tau := n(b_9) = 7$ and $n_+ := n(b_{11}) = 3$. The results of a reconstruction are displayed in Figure 1 and 2.

In the second example we apply the balancing principle calibrated as above to a problem (1) - (3), (6) with the domain Ω displayed in Figure 3, where

$$\begin{aligned}
\Gamma_{un} &= \{(x, y) \in \partial B(0, 1), \ x \geq 0, y \geq 0\}, \\
\Gamma_c &:= \Gamma_{c,1} \cup \Gamma_{c,2} \cup \Gamma_{c,3}
\end{aligned}$$

with

$$\begin{aligned}
\Gamma_{c,1} &= \{(x, y) \in \partial B(0, 1), \ x \leq 0, y \geq 0\}, \\
\Gamma_{c,2} &= \{(x, y), \ 3y + 4x + 4 = 0, -1 \leq x \leq 1/2\}, \\
\Gamma_{c,3} &= \{(x, y), \ x = 1, -2 \leq y \leq 0\},
\end{aligned}$$

and

$$\Gamma_D = \{(x, y), \ y = -2, \frac{1}{2} \leq x \leq 1\}.$$

In (3) synthetic data h, g are chosen in such a way that the exact solution is given as

$$u(x, y) = \frac{y + 2}{(y + 2)^2 + x^2}, \quad (x, y) \in \Omega.$$

Then it is easy to check that at Γ_{un}

$$\frac{\partial u}{\partial \nu} = -\frac{5y + 4}{(4y + 5)^2}, \quad u = \frac{y + 2}{4y + 5},$$

and these two functions are related by the equation (6), where

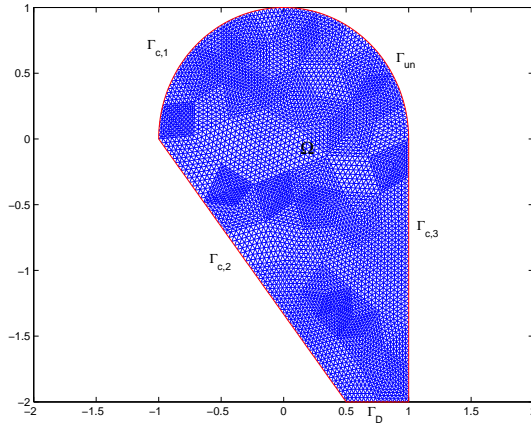$$f(t) = 4t^2 - \frac{11}{3}t + \frac{2}{3}, \quad \frac{1}{3} \leq t \leq \frac{2}{5}.$$

Figure 3. Domain from the second example.

We simulate a noise with the level $\delta = 10^{-5}$. The balancing principle selects $\{n(b_j)\}_{j=1}^{20}$ as 66, 66, 65, 59, 59, 28, 25, 14, 10, 8, 5, 3, 3, 2, 2, 1, 1, 1, 1, 1. Then $n_\tau := n(b_{10}) = 8$, and the final discretization level $n_+ := n(b_{12}) = 3$. The results of a reconstruction are displayed in Figure 4 and 5.
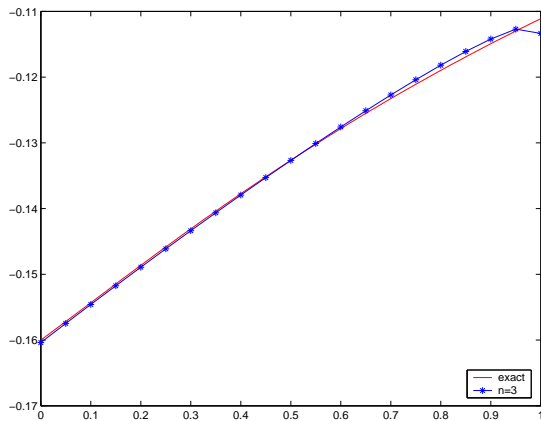

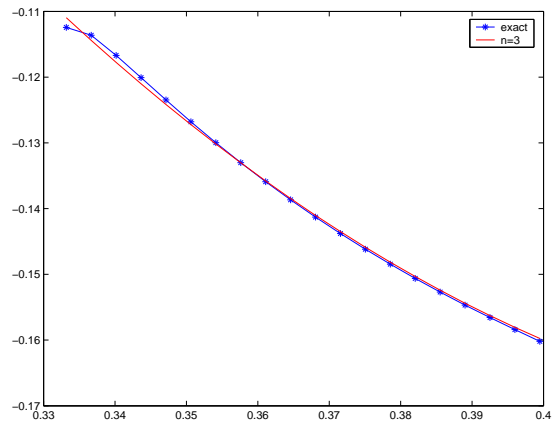*Figure 4. Reconstruction of Neumann trace on Γ_{un} for $\hat{n}_+ = 3$ selected by the balancing principle.*

Figure 5. Approximation of $f(t)$.

Acknowledgement

References

[1] Alessandrini G, Del Piero L and Rondi L 2003 Stable determination of corrosion by a single electrostatic boundary measurement I*nverse Problem* **19** no.4 973-84

[2] Alessandrini G, Sincich E 2006 Detecting nonlinear corrosion by electrostatic measurements, *Applicable Anal* **85** no.1-3 107-28.

[3] Alessandrini G, Sincich E 2007 Solving elliptic Cauchy problems and the identification of nonlinear corrosion *J. Comput. Appl. Math.* **198** no.2 307-20

[4] Bauer F, Pereverzev S V 2005 Regularization without preliminary knowledge of smoothness and error behaviour *Euro. Jnl. of Applied Math.* **16** 303-17

[5] Bryan K, Vogelius M2002 Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling *Quart. Appl. Math.* **60** 675-94

[6] Chaabane S, Jaoua M and Leblond J 2003 Parameter identification for Laplace equation and approximation in Hardy classes *J. Inv. Ill-Posed Problems* **11** 33-57

[7] Chaabane S, Fellah I, Jaoua M and Leblond J 2004 Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems *Inverse Problems* **20** no.1 47-59

[8] Cao H and Pereverzev S V 2006 Natural linearization for the identification of a diffusion coefficient in a quasi-linear parabolic system from short-time observations *Inverse Problems* **22** 2311–30

[9] Cao H and Pereverzev S V 2007 Balancing principle for the regularization of elliptic Cauchy problems *Inverse Problems* **23** 1943–61

[10] Guénichi M El 2006 Etude de la stabilité du problème inverse d'identification du coefficient de Robin dans le cas non linéare, Mémoire du Mastère en Mathématique Appliquées, ENIT

[11] Engl H W, Leitão A 2001 A Mann iterative regularization method for elliptic Cauchy problems *Numer. Funct. Anal. Optim.* **22** 861–84.

[12] Engl H W, Fusek P and Pereverzev S V 2005 Natural linearization for the identification of nonlinear heat transfer laws *J. Inverse Ill-Posed Probl.* **13** no. 3-6 567–82

[13] Fasino D, Inglese G 1999 An inverse Robin problem for Laplaces equation: theoretical results and numerical methods *Inverse Problems* **15** 41-48

[14] Fasino D, Inglese G 1999 Discrete methods in the study of an inverse problem for Laplace's equation I*MA J. Numer. Anal.* **19** no.1 105-18

[15] Fasino D, Inglese E and Mariani F 2007 Corrosion detection in conducting boundaries. II. Linearization, stability and discretization. Inverse Problems **23** no.3 1101–14

[16] Kaup P G, Santosa F 1995 Nondestructive evaluation of corrosion damage using electrostatic measurements *J. Nondestructive Eval.* **14** 127-36

[17] Kaup P G, Santosa F and Vogelius M 1996 Method for imaging corrosion damage in thin plates from electrostatic data *Inverse Problems* **12** 279-93

[18] Kavian D, Vogelius M 2003 On the existence and "blow-up" of solutions of a two dimensional nonlinear boundary value problem arising in corrosion modeling, *Proc. Roy. Soc. Edinburgh* Vol.133A 119-49

[19] Sincich E 2005 Stability and Reconstruction for the Determination of Boundary Terms by a Single Measurements *PhD Thesis* SISSA-ISAS, Trieste (down-loadable at http:// www.sissa.it/library/).

[20] Sincich E 2007 Lipschitz stability for the inverse Robin problem, Inverse Problems *23* 1311–26.

[21] Vogelius M, Xu J M 1998 A nonlinear elliptic boundary value problem related to corrosion modeling *Quart. Appl. Math.* **56** 479-505

[22] Zghal M 2006 Identification du paramètre de Robin dans un cas non linéare Mémoire du Mastère en Mathématique Appliquées ENIT