Local stability for soft obstacle by a single measurement

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E. Sincich* , M.Sini†

Abstract

We consider an inverse scattering problem arising in target identification. We prove a local stability result of logarithmic type for the determination of a sound soft obstacle from the far field measurements associated to one single incident wave.

Keywords : inverse scattering problem, soft obstacles, stability.
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1 Introduction

We consider the scattering of an acoustic incident time-harmonic plane wave, at a given wave number $\kappa > 0$ and at a given incident direction $\omega \in \mathbb{S}^2$, by a sound soft obstacle $D \subset \mathbb{R}^3$. Such a problem is modeled by the following boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta u + \kappa^2 u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ u = 0, & \text{on } \partial D, \end{cases} \tag{1.1}$$

where $u = u^s + \exp(i\kappa x \cdot \omega)$ is the total field, that is given as the sum of the scattered wave u^s and the incident plane waves $\exp(i\kappa x \cdot \omega)$. Let us recall that with obstacle we mean a bounded domain with connected complement.
Moreover, the scattered field u^s is required to satisfy the so-called *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - i\kappa u^s\right) = 0, \qquad r = \|x\|. \tag{1.2}$$

It is well-known, that the scattered field u^s has the following asymptotic behavior

$$u^s(x) = \frac{\exp(i\kappa r)}{r}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{r}\right)\right\} \tag{1.3}$$

as r tends to ∞, uniformly with respect to $\hat{x} = \frac{x}{\|x\|}$ and where u_∞ is the so-called far field pattern of the scattered wave (see for instance [11]).

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Here we are concerned with the inverse problem of detecting the obstacle D by the knowledge of the far field pattern provided some a-priori assumptions on the location of the obstacle are made.
Such kind of problem has been discussed, in a recent paper [26], by P. Stefanov and G. Uhlmann. The authors proved a uniqueness result for the obstacle provided an a-priori *closeness* condition is satisfied. Namely, there exist two obstacles D_+ and D_- such that

$$D_- \subset D \subset D_+ \tag{1.4}$$

and moreover

$$\mathrm{Vol}(D_+ \setminus D_-) < \omega_3 \kappa^{-3} \ , \tag{1.5}$$

where ω_3 is the volume of the unit sphere in $\mathbb{R}^3$.
In this paper we shall deal with the stability issue for this problem. Actually, we replace the condition (1.5) by the sharper one

$$\mathrm{Vol}(D_+ \setminus D_-) < \frac{4\pi^4}{3}\kappa^{-3} \ . \tag{1.6}$$

We will provide a stability estimate of *logarithmic* type for the obstacle by the far field measurements provided the above mentioned *closeness* condition (1.6) is met.
Recently, D. Gintides [16] presented an improvement, in two dimensions, of the local uniqueness result contained in [26]. Such an improvement has been accomplished by an optimal lower estimate for the eigenvalues of the negative Laplace operator for a domain by the Faber-Krahn inequality instead of the Poincaré one as in [26].
The condition (1.6), that we use here, derives indeed by the Faber-Krahn inequality in three dimensions.
Formerly, D. Colton and B.D. Sleeman [12] showed that a single incident wave is enough to ensure the uniqueness for *small* obstacles. More precisely, they suppose that there exists a radius $R > 0$, such that

$$D \subset B_R(0) \tag{1.7}$$

and furthermore

$$\kappa R \leqslant \pi. \tag{1.8}$$

Consequently, V. Isakov [17] solved the stability issue for the above problem, giving a *loglog* stability estimate for starshaped obstacles satisfying the *smalleness* condition (1.8). In a further paper [18], the author improved the *loglog* rate of stability to a *log* one in the case of analytic obstacles.
Moreover, J. Chen and M. Yamamoto [9], G. Alessandrini and L. Rondi [5] gave uniqueness results for 2-dimensional polygonal obstacle and for N-dimensional polyhedron, with $N \geqslant 2$, respectively, by a single incident wave. Furthermore, J. Elschner, H. Liu, M. Yamamoto and J. Zou [13, 20] extended the uniqueness result to other types of boundary condition, as sound-hard scatterers.
The stability issue for this problem has been treated by L. Rondi [22] providing a logarithmic rate of stability for polyhedral scatterers and an improved Hölder type one for polyhedral obstacles.

As already pointed out, in the present paper we are interested in the determination of the obstacles under a *closeness* type condition. However, up to some suitable adaptations, our arguments work also for the treatment of the stability issue for *small* and $C^{1,\alpha}$ smooth obstacles, leading to a *log* type stability rate as well.
For what concerns the methods used here, we shall adapt to our context several arguments introduced by G. Alessandrini, E. Beretta, E. Rosset and S. Vessella in [3] for the stable determination of unknown boundaries in the case of a scalar elliptic equation in divergence form, and then developed by A. Morassi and E. Rosset [21] for the Lamé system.
Indeed, as in [3, 21], the main tools employed arise in quantitative estimate of unique continuation as the *three spheres inequality* for elliptic systems with Laplacian principal part proved in [4], namely

$$\int_{B_{\beta_1\rho}(x)} |u|^2 \leqslant const. \left(\int_{B_\rho(x)} |u|^2\right)^\tau \cdot \left(\int_{B_{\beta_2\rho}(x)} |u|^2\right)^{1-\tau} . \tag{1.9}$$

with $1 < \beta_1 < \beta_2$ and $0 < \tau < 1$.
Beside this, we shall make use of the *doubling inequality* at the boundary discussed by L. Escauriaza and V. Adolfsson in [2] (see also [3, 24]), namely

$$\int_{B_{\beta r}(x_0)\setminus D} |u|^2 dx \leq const. \beta^K \int_{B_r(x_0)\setminus D} |u|^2 dx \tag{1.10}$$

with $\beta > 1$ and $x_0 \in \partial D$.
A main difference with respect to [3] is the lack of the maximum principle due to the fact that we are dealing with the Helmholtz equation instead of an equation in divergence form. However, by the closeness condition (1.6) and by the Faber-Krahn inequality, which gives an optimal estimate for the first Dirichlet eigenvalue for the Laplace operator with respect to the domain volume, we shall restore, up to a suitable change of variable, the maximum principle for u.
The paper is organized as follows.
In Section 2, we introduce the main hypothesis and we formulate our main result.
In Section 3, we briefly analyze the direct problem. In Lemma 3.1, we observe that using arguments arising in potential theory treated for instance in [8, 23], it can be proved that the direct problem is well posed. Moreover, in Theorem 3.2 by a classical regularity result for solutions to the Dirichlet problems, we prove that the solution and its first order derivatives are Hölder continuous up to the boundary of D. Finally, in Corollary 3.3, we state a lower bound for the total field u on sets far from the obstacle.
In Section 4, we deal with the inverse scattering problem.
In Lemma 4.1 we provide a stability estimate of the near field by the far field. Such an estimate, has been discussed by V. Isakov [17, 18], and then developed by I. Bushuyev [7]. It means that if u_1 and u_2 are two acoustic fields corresponding to obstacles D_1 and D_2 such that their scattering amplitudes, $u_{1,\infty}$ and $u_{2,\infty}$ respectively, are close

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon, \tag{1.11}$$

then u_1 and u_2 satisfy

$$\|u_1 - u_2\|_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))} \leqslant const.\varepsilon^{\alpha(\varepsilon)}, \tag{1.12}$$

where $R_1 > 0$ is a suitable radius such that $B_{R_1}(0) \supset \overline{D_i},\ i = 1,2$ and $\alpha(\varepsilon)$ is the function introduced in (4.2).
In Lemma 4.2 we prove, in a quantitative manner, that the closeness condition (1.6) still holds if we move a little bit away from $D_+ \setminus D_-$.
In Theorem 4.3 and in Theorem 4.4 we are concerned with a smallness estimate of the acoustic field u_1 on the set $D_{1,2}^{int} \setminus \overline{D_1}$ (see (2.11) for a precise definition). In Theorem 4.3, we obtain a smallness control of *loglog* type, namely

$$\|u_1\|_{L^2(D_{1,2}^{int}\setminus\overline{D_1})} \leqslant const.(\log|\log(\varepsilon)|)^{-\theta} \tag{1.13}$$

with $\theta > 0$. The proof may be summarized as follows. By means of an iterated use of the three spheres inequality we obtain a *loglog* type estimate of u_1 near the boundary of $D_{1,2}^{int}\setminus\overline{D_1}$. Then, by performing a suitable change of variable and by using a slight variation of the *closeness* condition (1.6) (see (2.14)), we observe that the conditions of the maximum principle are fulfilled. Hence we deduce the estimate (1.13). In Theorem 4.4, due to a further regularity hypothesis of the boundary $D_{1,2}^{int}$ we give an improvement of the rate of smallness found above. Indeed, the Lipschitz regularity of the boundary $D_{1,2}^{int}$ allows us to use the cone condition to approach the boundary and to achieve the following estimate

$$\|u_1\|_{L^2(D_{1,2}^{int}\setminus\overline{D_1})} \leqslant const.(|\log(\varepsilon)|)^{-\vartheta} \tag{1.14}$$

with $\vartheta > 0$.
In Lemma 4.5 we state the doubling inequality (1.10) at the boundary. Such an inequality combined with the *loglog* smallness control provided in Theorem 4.3 allows us to state a first rough estimate of *loglog* type for the obstacle contained in Theorem 4.6, namely

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leqslant const.(\log|\log(\varepsilon)|)^{-\theta} \tag{1.15}$$

with $\theta > 0$. Consequently in Proposition 4.7 we recall a result obtained in [3], which gives sufficiently conditions in order to guarantee that the boundaries of the two $C^{1,\alpha}$ domains D_1 and D_2 are locally represented as Lipschitz graphs in a common reference system. As a consequence, we notice in Proposition 4.8 that, up to choosing the threshold of the error ε in (1.11) sufficiently small, the hypothesis of Theorem 4.4 are satisfied.
Finally in the proof of the Theorem 2.3 we observe that in view of the Lipschitz regularity of the boundary of $D_{1,2}^{int}$ achieved in Proposition 4.8, the techniques developed in Theorem 4.6 can be carried over by replacing the *loglog* type estimate (1.13) by the *log* type one (1.14), leading to the desired estimate

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leqslant const.(|\log(\varepsilon)|)^{-\vartheta} \tag{1.16}$$

with $\vartheta > 0$.

2 The main result

2.1 Definitions and notations

Definition 2.1. *Let D be a bounded domain in $\mathbb{R}^3$.*
We shall say that the boundary ∂D of D is of Lipschitz class with constants $r_0,\ M>0$ if, for every $P\in\partial D$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$D\cap B_{r_0}=\{(x',x_3):x_3>\gamma(x')\} \tag{2.1}$$

where

$$\gamma: B^{'}_{r_0}\subset\mathbb{R}^2\to\mathbb{R}\ ,$$

satisfying $\gamma(0)=0$ and

$$\|\gamma\|_{C^{0,1}(B'_{r_0})}\leqslant Mr_0\ ,$$

where we denote by

$$\|\gamma\|_{C^{0,1}(B'_{r_0})}=\|\gamma\|_{L^\infty(B'_{r_0})}+\ r_0\sup_{\substack{x,y\in B^{'}_{r_0}\\ x\neq y}}\frac{|\gamma(x)-\gamma(y)|}{|x-y|}\ .$$

Definition 2.2. *Given $\alpha,\ 0<\alpha\leqslant 1$, we shall say that a domain D is of* class $C^{1,\alpha}$ with constants $r_0,\ M>0$ *if for any $P\in D$, there exists a rigid transformation of coordinates under which we have $P=0$ and*

$$D\cap B_{r_0}=\{(x',x_3):x_3>\varphi(x')\} \tag{2.2}$$

where

$$\varphi: B^{'}_{r_0}\subset\mathbb{R}^2\to\mathbb{R} \tag{2.3}$$

is a $C^{1,\alpha}$ function satisfying

$$|\varphi(0)|=|\nabla\varphi(0)|=0\ \ \textit{and}\ \ \|\varphi\|_{C^{1,\alpha}(B'_{r_0})}\leqslant Mr_0\ , \tag{2.4}$$

where we denote

$$\begin{aligned}\|\varphi\|_{C^{1,\alpha}(B'_{r_0})}\ &=\ \|\varphi\|_{L^\infty(B'_{r_0})}+r_0\|\nabla\varphi\|_{L^\infty(B'_{r_0})}+\\ &+\ \ {r_0}^{1+\alpha}\sup_{\substack{x,y\in B^{'}_{r_0}\\ x\neq y}}\frac{|\nabla\varphi(x)-\nabla\varphi(y)|}{|x-y|^\alpha}\ .\end{aligned} \tag{2.5}$$

We introduce some notation that we shall use in the sequel.
For a sake of simplicity we shall assume that $0\in D$.
Fixed $R>\text{diam}(D)$, let us define the following sets

$$D^{ext}=\mathbb{R}^3\setminus\overline{D}, \tag{2.6}$$
$$D^{ext}_R=B_R(0)\cap D^{ext}, \tag{2.7}$$
$$\tag{2.8}$$

Given D_1 and D_2 two bounded domains in $\mathbb{R}^3$ we shall denote

$$D_{1,2}^{ext} = \text{the unbounded connected component of } \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2}), \quad (2.9)$$

$$\Gamma = \partial D_{1,2}^{ext}, \quad (2.10)$$

$$D_{1,2}^{int} = \mathbb{R}^3 \setminus \overline{D_{1,2}^{ext}} \,. \quad (2.11)$$

2.2 A-priori informations on the obstacle

We shall refer as an obstacle a bounded domain with connected complement. Given $d_0, r_0, M, \alpha, h > 0$ with $0 < \alpha < 1$ and given obstacles D_+ and D_- such that

$$\text{the diameters of } D_+ \text{ and } D_- \text{ are bounded by } d_0, \quad (2.12)$$

and

$$D_+ \text{ and } D_- \text{ are of } C^{1,\alpha} \text{ class with constants } r_0, M \quad (2.13)$$

and furthermore

$$\mathrm{Vol}(D_+ \setminus D_-) < \frac{4}{3}\pi^4\kappa^{-3} - h\,, \quad (2.14)$$

we consider the family of obstacles D such that

$$D \text{ is of } C^{1,\alpha} \text{ class with constants } r_0, M, \quad (2.15)$$

and

$$D_- \subset D \subset D_+. \quad (2.16)$$

From now on we shall refer to the *a priori data* as to the following set of quantities: $\kappa, \omega, r_0, M, \alpha, d_0, h$.
In the sequel we shall denote by $\eta(t), \omega(t)$ two positive increasing functions defined on $(0, +\infty)$, that satisfy

$$\eta(t) \leqslant C|\log(t)|^{-\vartheta}, \quad \text{for every } \ 0 < t < 1 \,, \quad (2.17)$$

$$\omega(t) \leqslant C(\log|\log(t)|)^{-\theta}, \quad \text{for every } \ 0 < t < e^{-1} \,, \quad (2.18)$$

where $C > 0, \vartheta, \theta > 0$ are constants depending on the *a priori data* only.

2.3 The stability result

Theorem 2.3 (**Log stability for** D)**.** *Let u_i, $i = 1, 2$, be the weak solutions to the problem* (1.1) *with $D = D_i$ respectively and let $u_{i,\infty}$ be their respectively far field patterns. There exists $\varepsilon_0 > 0$ constant only depending on the* a priori data*, such that, if for some ε, $0 < \varepsilon < \varepsilon_0$, we have*

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon, \quad (2.19)$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leqslant \eta(\varepsilon), \quad (2.20)$$

where η is given by (2.17)*.*

3 The direct scattering problem

Let us introduce the following space

$H^1_{\text{loc}}(D^{ext}) = \{v \in D^*(D^{ext}) : \; v|_{D_R^{ext}} \in H^1(D_R^{ext}),$ for every $R > 0$ s.t. $\overline{D} \subset B_R(0)\}$

where $D^*(D^{ext})$ is the space of distributions on D^{ext}.
A weak solution to the problem (1.1) is a function $u = \exp(ik\omega \cdot x) + u^s$, where $u^s \in H^1_{\text{loc}}(D^{ext})$ is a weak solution to the problem

$$\begin{cases} \Delta u^s + \kappa^2 u^s = 0, & \text{in } D^{ext}, \\ u^s = -\exp(i\kappa\omega \cdot x), & \text{on } \partial D, \\ \lim_{r\to\infty} r\left(\dfrac{\partial u^s}{\partial r}(r\hat{x}) - i\kappa u^s(r\hat{x})\right) = 0, & \text{uniformly in } \hat{x}. \end{cases} \tag{3.1}$$

Let us recall that a weak solution of (3.1) is a function $u^s \in H^1_{\text{loc}}(D^{ext})$, with $u^s|_{\partial D} = -\exp(i\kappa\omega \cdot x)$ in the trace sense, such that, for all test functions $\eta \in H^1(D^{ext})$ with compact support in $\mathbb{R}^n$ and $\eta|_{\partial D} = 0$, the following holds

$$\int_{D^{ext}} \nabla u^s \cdot \nabla \overline{\eta} - \kappa^2 \int_{D^{ext}} .u^s \overline{\eta} = 0 \tag{3.2}$$

Furthermore, u^s satisfies the asymptotic condition (1.2).

Lemma 3.1 (**Well-posedness**). *The problem* (3.1) *has one and only one weak solution u^s. Moreover, for every $R > 0$ satisfying $\overline{D} \subset B_R(0, R)$, there exists a constant $C_R > 0$ depending on the* a priori data *and on R only, such that the following holds*

$$\|u^s\|_{H^1(D_R^+)} \leqslant C_R. \tag{3.3}$$

Proof The proof can be found for instance in [8, chap 8] or [23, chap 4]. It relies on the reformulation of the exterior Dirichlet boundary value problem (3.1) into a boundary integral equation. □

Theorem 3.2 ($C^{1,\alpha}$ **regularity at the boundary**). *Let u be the weak solution to* (1.1)-(1.2)*, then there exists a constant α, $0 < \alpha < 1$, such that for every $R > 0$ satisfying $\overline{D} \subset B_R(0, R)$, $u \in C^{1,\alpha}(D_R^{ext})$. Moreover, there exists a constant $C_R > 0$ depending on the* a priori data*, on R and on ρ only, such that*

$$\|u\|_{C^{1,\alpha}(D_R^{ext})} \leqslant C_R \; . \tag{3.4}$$

Proof The proof is based on well-known regularity bounds for solutions of Dirichlet type problems. We refer the reader to [15, Chap. 8]. □

Corollary 3.3 (**Lower bound**). *Let u be the weak solution to* (1.1)*, then there exists a radius $R_0 > 0$ depending on the* a priori data *only, such that*

$$|u(x)| > \frac{1}{2} \quad \textit{for every } x, \; |x| > R_0 \; . \tag{3.5}$$

Proof For the proof we refer to [24, Corollary 3.3]. □

4 The inverse scattering problem

Lemma 4.1 (From the far field to the near field). *Let $u_i, u_{i,\infty},\ i = 1, 2$, be as in Theorem 2.3. Suppose that, for some $\varepsilon,\ 0 < \varepsilon < 1$, (2.19) holds, then there exist a radius $R_1 > 0$ and a constant $C > 0$, depending on the* a priori data *only, such that*

$$\|u_1 - u_2\|_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))} \leqslant C\varepsilon^{\alpha(\varepsilon)}, \tag{4.1}$$

where $\alpha(\varepsilon)$ is the following function

$$\alpha(t) = \frac{1}{1 + \log(\log(t^{-1}) + e)} \ . \tag{4.2}$$

Proof Let us choose $R = 4d_0 + 4r_0$ and denote by $u_i^s,\ i = 1, 2$, the scattered wave of the problem (1.1) with $D = D_i$ respectively. By (3.4) it follows that

$$\|u_1^s - u_2^s\|_{L^2(\partial B_R(0))} \leqslant C \ , \tag{4.3}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Choosing $R_1 = 16d_0 + 16r_0$, the thesis follows by the argument in [18] (see also [17] and [7]).

□

Lemma 4.2. *There exists δ_1 depending only on the a priori data such that*

$$Vol(D_+^\delta \setminus D_{-,\delta}) \leqslant \frac{4\pi^4}{3}\kappa^{-3}, \ \textit{for every}\ \delta, 0 < \delta \leq \delta_1 \ , \tag{4.4}$$

where

$$D_{-,\delta} = \{x \in D_- : dist(x, \partial D_-) > \delta\} \ , \tag{4.5}$$
$$D_+^\delta = \{x \in \mathbb{R}^3 : dist(x, D_+) < \delta\} \ . \tag{4.6}$$

Proof We start by the inequality

$$\mathrm{Vol}(D_+^\delta \setminus D_{-,\delta}) \leq \mathrm{Vol}(D_+^\delta \setminus D_+) + \mathrm{Vol}(D_+ \setminus D_-) + \mathrm{Vol}(D_- \setminus D_{-,\delta}) \tag{4.7}$$

which we rewrite as

$$\mathrm{Vol}(D_+^\delta \setminus D_{-,\delta}) - \mathrm{Vol}(D_+ \setminus D_-) \leq \mathrm{Vol}(D_+^\delta \setminus D_+) + \mathrm{Vol}(D_- \setminus D_{-,\delta}) \tag{4.8}$$

Following [6, Lemma 2.8], we have that there exist $\overline{\delta}_1 > 0$ and $C^- > 0$ only depending only on the a priori data such that

$$\mathrm{Vol}(D_- \setminus D_{-,\delta}) \leq C^-\delta, \ \text{for every}\ \delta, 0 < \delta \leq \overline{\delta}_1. \tag{4.9}$$

Precisely, it is shown that $C^- = C'\frac{Vol(D_-)}{r_0}$ and $\overline{\delta}_1 = \min((1/2 - \sqrt{3}t)r_0, \frac{1}{2C'}r_0)$, where $t = \frac{1}{\sqrt{3}(k_0+1)}$, $k_0 = \sqrt{1 + E_0^2 r_0^2}$, $C' = \frac{\omega_2 k_0 C''}{4}$, $C'' = (\frac{2K+1}{t})^3$ and $K = [\frac{1}{t}] + 1$.
To obtain a similar estimate for $\mathrm{Vol}(D_+^\delta \setminus D_+)$, we write $\mathrm{Vol}(D_+^\delta \setminus D_+) = \mathrm{Vol}(K_+ \setminus K_{+,\delta})$ where $K_+ = B_{2d_0}(0) \setminus D_+$ and $K_{+,\delta} = B_{2d_0}(0) \setminus D_+^\delta$.

Hence

$$\mathrm{Vol}(D_+^\delta \backslash D_+) = \mathrm{Vol}(K_+ \backslash K_{+,\delta}) \leq C' \frac{\mathrm{Vol}(K_+)}{r_0}\delta, \text{ for every } \delta, 0 < \delta \leq \overline{\delta}_1. \quad (4.10)$$

We set $C = \frac{C'}{2r_0} \max(\mathrm{Vol}(D_-), \mathrm{Vol}(K_+))$. From (4.8)-(4.9)-(4.10), we have

$$\mathrm{Vol}(D_+^\delta \setminus D_{-,\delta}) - \mathrm{Vol}(D_+ \setminus D_-) \leq C\delta, \text{ for every } \delta, 0 < \delta \leq \overline{\delta}_1.$$

Using the hypothesis (2.14), we obtain:

$$\mathrm{Vol}(D_+^\delta \setminus D_{-,\delta}) \leqslant \frac{4\pi^4}{3}\kappa^{-3} - h + C\delta, \text{ for every } \delta, 0 < \delta \leq \overline{\delta}_1.$$

Hence taking δ_1 as $\delta_1 = \min(\overline{\delta}_1, \frac{h}{C})$, we get

$$\mathrm{Vol}(D_+^\delta \setminus D_{-,\delta}) \leqslant \frac{4\pi^4}{3}\kappa^{-3}, \text{ for every } \delta, 0 < \delta \leq \delta_1.$$

□

Theorem 4.3 (**Loglog stability estimate of continuation from the near field).** *Let $u_i, u_{i,\infty}$, $i = 1, 2$, be as in Theorem 2.3. We have*

$$\|u_1\|_{L^2(D_{1,2}^{int} \setminus \overline{D_1})} \leqslant \omega(\varepsilon) \quad (4.11)$$

where ω is given by (2.18)*, where $C > 0$ and θ are constants depending on the* a priori data *only.*

Theorem 4.4 (**Log stability estimate of continuation from the near field**). *Let $u_i, u_{i,\infty}$, $i = 1, 2$, be as in Theorem 2.3. In addition, let us assume that $\partial D_{1,2}^{int}$ is of Lipschitz class with constants $\tilde{r_0}, L$. We have that there exists $\varepsilon_0 > 0$ depending on the* a priori data *only, such that, if for some ε, $0 < \varepsilon < \varepsilon_0$,* (2.19) *holds, then we have*

$$\|u_1\|_{L^2(D_{1,2}^{int} \setminus \overline{D_1})} \leqslant \eta(\varepsilon) \quad (4.12)$$

where η is given by (2.17)*, with a constants $C, \vartheta > 0$ depending on $\tilde{r_0}, L$ and on the* a priori data *only.*

Proof [**Theorem 4.4**] By the Lipschitz regularity of the boundary $\partial D_{1,2}^{int}$, it follows that the cone property holds. Precisely, for every point $Q \in \partial D_{1,2}^{int} = \Gamma$, there exists a rigid transformation of coordinates under which we have $Q = 0$ and the finite cone

$$\mathcal{C} = \left\{ x : |x| < r_0, \ \frac{x \cdot \xi}{|x|} > \cos\theta \right\}$$

with axis in the direction ξ and width 2θ, where $\theta = \arctan \frac{1}{L}$, is such that $\mathcal{C} \subset \mathbb{R}^3 \setminus \overline{D_1} \cup \overline{D_2}$. Let Q be a point in Γ and let Q_0 be a point lying on the axis ξ of the cone with vertex in $Q = 0$ such that $d_0 = \mathrm{dist}(Q_0, 0) < \frac{\tilde{r_0}}{2}$.

We define $R_2 = 2R_1 + 2$, where R_1 is the radius introduced in the statement of Lemma 4.1. Let us define for every $\rho > 0$

$$D^\rho = \{x \in D_{1,2}^{ext} \cap B_{R_2}(0) : \mathrm{dist}(x, \Gamma) > \rho\} \ . \tag{4.13}$$

Let us notice that if $\rho < \frac{\tilde{r_0}}{3}$ then D^ρ is connected. Let us now define $\rho_0 = \min\{\frac{1}{16}, \frac{\tilde{r_0}}{4}\sin\theta\}$ and let P be a point in the annulus $B_{R_1+1}(0) \setminus B_{R_1}(0))$, such that $B_{4\rho_0}(P) \subset B_{R_1+1}(0)\setminus B_{R_1}(0))$. Furthermore, let γ be a path in D^{ρ_0} joining P to Q_0 and let us define $\{y_i\}$, $i = 0, \dots, s$ as follows $y_0 = Q_0$, $y_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \text{ s.t. } |\gamma(t) - y_i| = 2\rho_0\}$ if $|P - y_i| > 2\rho_0$, otherwise let $i = s$ and stop the process.
Let us introduce the function $U \in H^1_{\mathrm{loc}}(D_{1,2}^{ext})$ defined as follows

$$U(x) = u_1(x) - u_2(x). \tag{4.14}$$

We shall denote with U_1 and U_2 the real and the imaginary part of U respectively. Namely

$$U(x) = U_1(x) + iU_2(x).$$

It immediately follows that U_1 and U_2 are both real valued solutions to the Helmholtz equation in $D_{1,2}^{ext}$.
Thus, by the three spheres inequalities for elliptic system with Laplacian principal part, (see [4, Theorem 3.1]), we have that for every $\beta_1, \beta_2,\ 1 < \beta_1 < \beta_2$, there exist $\bar{r} > 0, \tau,\ 0 < \tau < 1$ and $C > 0$ depending on the *a priori data* and on β_1, β_2 only, such that for every $x \in D^{\beta_2\rho}$ the following holds

$$\int_{B_{\beta_1\rho}(x)} |U|^2 \leqslant C \left(\int_{B_\rho(x)} |U|^2 \right)^\tau \cdot \left(\int_{B_{\beta_2\rho}(x)} |U|^2 \right)^{1-\tau} \tag{4.15}$$

for every $\rho \in (0, \bar{r})$. By a possible replacement of ρ_0 with $\bar{r}$ if $\rho_0 > \bar{r}$ and choosing in (4.15) $\beta_1 = 3,\ \beta_2 = 4, \rho = \rho_0,\ x = y_0$, we infer that

$$\int_{B_{3\rho_0}(y_0)} |U|^2 \leqslant C \left(\int_{B_{\rho_0}(y_0)} |U|^2 \right)^\tau \cdot \left(\int_{B_{4\rho_0}(y_0)} |U|^2 \right)^{1-\tau} . \tag{4.16}$$

As a consequence of Lemma 3.1, we have that

$$\|U\|_{H^1(D_{1,2}^{ext} \cap B_{R_2}(0))} \leqslant C, \tag{4.17}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us observe that $B_{4\rho_0}(y_0) \subset D_{1,2}^{ext} \cap B_{R_2}(0)$ and $B_{\rho_0}(y_0) \subset B_{3\rho_0}(y_1)$. Thus by (4.16) and (4.17) we deduce that

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leqslant C \left(\int_{B_{3\rho_0}(y_1)} |U|^2 \right)^\tau \cdot C^{1-\tau} \ .$$

An iterated application of the three spheres inequality leads to

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leqslant \left(\int_{B_{\rho_0}(y_s)} |U|^2 \right)^{\tau^s} \cdot C^{1-\tau^s} \ .$$

Finally, since $B_{\rho_0}(y_s) \subset B_{R_1+1}(0) \setminus B_{R_1}(0))$, by (4.1) we obtain that

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leqslant C\{\varepsilon^{\alpha(\varepsilon)}\}^{\tau^s} .$$

We shall construct a chain of balls $B_{\rho_k}(Q_k)$ centered on the axis of the cone, pairwise tangent to each other and all contained in the cone

$$\mathcal{C}' = \left\{ x : |x| < r_0,\ \frac{x \cdot \xi}{|x|} > \cos\theta' \right\} ,$$

where $\theta' = \arcsin\left(\frac{\rho_0}{d_0}\right)$. Let $B_{\rho_0}(Q_0)$ be the first of them, the following are defined by induction in such a way

$$\begin{aligned} &Q_{k+1} = Q_k - (1+\mu)\rho_k\xi \ , \\ &\rho_{k+1} = \mu\rho_k \ , \\ &d_{k+1} = \mu d_k \ , \end{aligned}$$

with

$$\mu = \frac{1-\sin\theta'}{1+\sin\theta'} \ .$$

Hence, with this choice, we have $\rho_k = \mu^k\rho_0$ and $B_{\rho_{k+1}}(Q_{k+1}) \subset B_{3\rho_k}(Q_k)$. Considering the following estimate obtained by a repeated application of the three spheres inequality, we have that

$$\begin{aligned} \|U\|_{L^2(B_{\rho_k}(Q_k))} &\leqslant \|U\|_{L^2(B_{3\rho_{k-1}}(Q_{k-1}))} \leqslant \\ &\leqslant \|U\|^{\tau}_{L^2(B_{\rho_{k-1}}(Q_{k-1}))}\|U\|^{1-\tau}_{L^2(B_{4\rho_{l-1}}(Q_{k-1}))} \\ &\leqslant C\|U\|^{\tau^k}_{L^2(B_{\rho_0}(Q_0))} \leqslant C\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^k} . \end{aligned} \tag{4.18}$$

For every r, $0 < r < d_0$, let $k(r)$ be the smallest positive integer such that $d_k \leqslant r$ then, since $d_k = \mu^k d_0$, it follows

$$\frac{|\log(\frac{r}{d_0})|}{\log\mu} \leqslant k(r) \leqslant \frac{|\log(\frac{r}{d_0})|}{\log\mu} + 1 , \tag{4.19}$$

and by (4.18) we deduce

$$\|U\|_{L^2(B_{\rho_{k(r)}}(Q_{k(r)}))} \leqslant C\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^{k(r)}} . \tag{4.20}$$

Let $\bar{x} \in \Gamma$ with and let $x \in B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$. By Theorem 3.2, in particular, it follows that $U \in C^{1,\alpha}(D^{ext}_{1,2} \cap B_{R_2}(0)$ with

$$\|U\|_{C^{1,\alpha}(D^{ext}_{1,2} \cap B_{R_2}(0))} \leqslant C, \tag{4.21}$$

where $C > 0$ is a constant depending on the *a priori data* only. Then (4.21) yields to

$$|U(\bar{x})| \leqslant |U(x)| + C|x-\bar{x}|^{\alpha} \leqslant |U(x)| + C\left(\frac{2}{\mu}r\right)^{\alpha} .$$

Integrating this inequality over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$, we have that

$$|U(\bar{x})|^2 \leqslant \frac{2}{\omega_3\big(\frac{\rho_{k-1}}{2}\big)^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}\left(Q_{k(r)-1}\right)} |U(x)|^2 \mathrm{d}x + 2C^2\left(\frac{4r^2}{\mu^2}\right)^{\alpha}. \quad (4.22)$$

Being k the smallest integer such that $d_k \leqslant r$, then $d_{k-1} > r$ and thus (4.22) yields to

$$|U(\bar{x})|^2 \leqslant \frac{C}{\left(r\sin\theta'\right)^3}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} |U(x)|^2 \mathrm{d}x + Cr^{2\alpha} \ .$$

By (4.20) we deduce that

$$|U(\bar{x})|^2 \leqslant \frac{C}{r^3}\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^{k(r)-1}} + Cr^{2\alpha} \ . \quad (4.23)$$

The choice in (4.19) guarantees that

$$\tau^{k(r)-1} \geqslant \left(\frac{r}{d_0}\right)^{\nu},$$

where $\nu = -\log\left(\frac{1}{\mu}\right)\log\tau$. Thus, by (4.23), it follows that

$$|U(\bar{x})| \leqslant C\left\{r^{-\frac{3}{2}}\left[\left(\varepsilon^{\alpha(\varepsilon)}\right)^{\tau^s}\right]^{\frac{r^{\nu}}{2}} + r^{\alpha}\right\} \quad (4.24)$$

Minimizing the right hand sides of the above inequality with respect to r, with $r \in (0, \frac{r_0}{4})$, we deduce

$$|U(\bar{x})| \leqslant C\big(\log\left(\varepsilon^{-\alpha(\varepsilon)}\right)\big)^{-\frac{2\alpha}{\nu+2}} \ , \quad (4.25)$$

where $C > 0$ is a constant depending on the *a priori data* and on ρ only. Thus, since $\bar{x}$ is an arbitrary point in Γ, by (4.25) we have that

$$\|U(\bar{x})\|_{L^\infty(\Gamma)} \leqslant C\big(\log\left(\varepsilon^{-\alpha(\varepsilon)}\right)\big)^{-\frac{2\alpha}{\nu+2}} \ . \quad (4.26)$$

Let us denote with ${\lambda_1}^2$ and with ϕ_1 the first Dirichlet eigenvalue and the first eigenfunction of $-\Delta$ on the domain $D_+^\delta \setminus D_{-,\delta}$, namely

$$\begin{cases} \Delta\phi_1 + {\lambda_1}^2\phi_1 = 0, & \text{in } D_+^\delta \setminus \overline{D_{-,\delta}}, \\ \phi_1 \in H_0^1(D_+^\delta \setminus D_{-,\delta}) \end{cases} \quad (4.27)$$

with $\delta < \delta_1$ where δ_1 is the one defined in Lemma 4.2.
We observe that there exists a point $z_0 \in D_+^\delta \setminus D_{-,\delta}$ such that the cube $Q_{\frac{\delta}{8\sqrt{3}}}(z_0) \subset D_+^\delta \setminus \overline{D_{-,\delta}}$. Hence by the monotonicity property of the eigenvalues with respect the domains we can infer that

$${\lambda_1}^2 \leqslant \frac{9}{256}\pi^2\delta^2 =: \nu \ . \quad (4.28)$$

Let us now set $\bar{r} = \frac{\delta}{16}$. By the Harnack inequality (see for instance [15, Theorem 8.2]) we have that there exists a constant $C > 0$ depending on $\bar{r}$ and ν only such that for every $y \in D_+ \setminus \overline{D_-}$ we have

$$\sup_{B_{\bar{r}}(y)} \phi_1 \leqslant C \inf_{B_{\bar{r}}(y)} \phi_1 \ . \quad (4.29)$$

Let us prove that the set $\overline{D_+} \setminus D_-$ is connected. We consider two points x_1 and x_2 in $\overline{D_+} \setminus D_-$. We shall prove that there exists a continuous path contained in $\overline{D_+} \setminus D_-$ joining them.
We observe that being D_+ and D_- obstacles, their boundaries consist of a closed surface. Precisely, they are 2-dimensional connected and closed $C^{1,\alpha}$-manifolds. We consider two points y_1 and y_2 on ∂D_- such that

$$d(x_1, \partial D_-) = d(x_1, y_1) \text{ and } d(x_2, \partial D_-) = d(x_2, y_2).$$

We denote by j the segment $\overline{x_1 y_1}$ joining x_1 and y_1. We observe that we may have two situations:
1. The segment $\overline{x_1 y_1}$ does not cross ∂D_+ and then it gives a continuous path in $\overline{D_+} \setminus D_-$ joining x_1 and y_1.
2. The segment $\overline{x_1 y_1}$ crosses ∂D_+. In this case, we construct another continuous path joining x_1 and y_1 as follows. Le $p \in \partial D_+$ such that the segment $\overline{x_1 p}$ is contained in $\overline{D_+} \setminus D_-$ and let $q \in \partial D_+$ such that the segment $\overline{q y_1}$ is contained in $\overline{D_+} \setminus D_-$. Since ∂D_+ is a connected manifold, we can construct a continuous path j_1 on it joining p and q. Hence the union of the paths $\overline{x_1 p}, j_1$ and $\overline{q y_1}$ gives a continuous path contained in $\overline{D_+} \setminus D_-$ and joining x_1 and y_1.
In a similar manner, we construct a continuous path in $\overline{D_+} \setminus D_-$ joining x_2 and y_2. Also, being ∂D_- a connected manifold, we can construct a continuous path on ∂D_- joining y_1 and y_2. Finally, gathering all these paths, we get a continuous path contained in $\overline{D_+} \setminus D_-$ which join x_1 and x_2.
Being $\overline{D_+} \setminus D_-$ connected, we can cover it by a chain of finitely many balls $\{B_i\}_{i=1}^N$ with $N \leqslant \frac{2d_0\sqrt{3}}{\bar{r}}$, each of which has radius $\bar{r}$ and $B_i \cap B_{i-1} \neq \emptyset$. Then by an iterated use of the Harnack inequality over the chain of balls we have that

$$\sup_{D_+ \setminus D_-} \phi_1 \leqslant C^N \inf_{D_+ \setminus D_-} \phi_1 \ . \tag{4.30}$$

priori data only, such that Since D_+ and D_- are assume to be obstacles, we can deal as in [26, Theorem 1] to prove that $D_{1,2}^{int} \setminus D_1 \subset D_+ \setminus D_-$. We define the following function $v_1 = \dfrac{u_1}{\phi_1}$. It follows that the function v satisfies the following equation in $D_{1,2}^{int} \setminus \overline{D_1}$

$$\text{div}(\phi_1^2 \nabla v) + (\kappa^2 - {\lambda_1}^2)\phi_1 v = 0 \ . \tag{4.31}$$

By the Faber-Krahn inequality (see for instance [14]) we have that

$${\lambda_1}^3 > \frac{4\pi^4}{3\text{Vol}(D_+^\delta \setminus D_{-,\delta})} \tag{4.32}$$

and by hypothesis (2.14) we have $\kappa < \lambda_1$, hence the conditions of the maximum principle are fulfilled by the equation (4.31).
Then by the maximum principle for v_1 and the bound in (4.30) we have that

$$\max_{D_{1,2}^{int} \setminus \overline{D_1}} |u_1| \leqslant C^N \max_{\partial (D_{1,2}^{int} \setminus \overline{D_1})} |u_1| \ . \tag{4.33}$$

Finally, being $u_1 = 0$ on ∂D_1 and since $|u_1| = |u_2 + (u_1 - u_2)| = |u_1 - u_2|$ on $\partial D_2 \setminus \overline{D_1}$ where the estimate (4.26) holds, we obtain the thesis.

$\square$

Proof [**Theorem 4.3**] As in the proof of Theorem 4.4, we set $U = u_1 - u_2$. Let z_0 and $\bar{\rho}$ such that $B_{\bar{\rho}}(z_0) \subset B_{R_1+1}(0) \setminus B_{R_1}(0)$.
By Lemma 4.1 we have that

$$\int_{B_{\bar{\rho}}(z_0)} |U|^2 \leqslant C\varepsilon^{\alpha(\varepsilon)} \ . \tag{4.34}$$

Let us define the following set

$$V_r = \{x \in \mathbb{R}^3 : \text{dist}(x, D_{1,2}^{int}) > r\}. \tag{4.35}$$

Let x be a point in V_r. By an iterated use of the three spheres inequality as in the proof of Theorem 4.4, we obtain that there exists a $\tau,\ 0 < \tau < 1$ such that for any $r,\ 0 < r < \bar{\rho}$, we have that

$$\int_{B_{\frac{r}{4}}(x)} |U|^2 \leqslant C(\varepsilon^{\alpha(\varepsilon)})^{2\tau^s} \ . \tag{4.36}$$

where s is a positive integer depending on r and on the a priori data only. Moreover, by the local boundedness of solutions to elliptic equations (see [15, Chap. 8]), we have that

$$\|U\|_{L^\infty(B_{\frac{r}{8}}(x))} \leqslant Cr^{-\frac{3}{2}}(\varepsilon^{\alpha(\varepsilon)})^{2\tau^s} \ . \tag{4.37}$$

Let us define the set $W_r = \mathbb{R}^3 \setminus V_r$. Let $r > 0$ be sufficiently small so that $W_r \subset D_+^\delta$.
Since $D_{1,2}^{int} \setminus \overline{D_1} \subset W_r \setminus \overline{D_1}$ we have that, by dealing with the same change of variable performed in Theorem 4.4 , it follows that
∂B_R.

$$\max_{D_{1,2}^{int} \setminus \overline{D_1}} |u_1| \leqslant C^N \max_{\partial(W_r \setminus D_1)} |u_1| \ . \tag{4.38}$$

Hence since $u_1 = 0$ on ∂D_1 it is enough to estimate $\sup_{\partial W_r} |u_1|$.
Let us take $x \in \partial W_r$. Hence $\text{dist}(x, \partial D_{1,2}^{int}) = r$.
We may distinguish two cases:

i) there exists $y \in \partial D_1$ such that $|x - y| = \text{dist}(x, D_{1,2}^{int}) = r$,

ii) there exists $y \in \partial D_2$ such that $|x - y| = \text{dist}(x, D_{1,2}^{int}) = r$.

If the case i) occurs then by Theorem 3.2 we have that

$$|u_1(x)| = |u_1(x) - u_1(y)| \leqslant C|x - y| \leqslant Cr \ . \tag{4.39}$$

where $C > 0$ is a constant depending on the a priori data only. If the case ii) occurs we have

$$|u_1(x)| = |u_1(x) - u_2(x) + u_2(x) - u_2(y)| \leqslant |U(x)| + |u_2(x) - u_2(y)| \ . \tag{4.40}$$

Hence by (4.37) and by Theorem 3.2 we have that

$$|u_1(x)| \leqslant Cr^{-\frac{3}{2}}(\varepsilon^{\alpha(\varepsilon)})^{2\tau^s} + Cr. \tag{4.41}$$

where $C > 0$ is a constant depending on the a priori data only. Hence we found that in both cases i) and ii) we have that there exists a constant $C > 0$ depending on the a priori data only, such that

$$|u_1(x)| \leqslant C(r^{-\frac{3}{2}}(\varepsilon^{\alpha(\varepsilon)})^{2\tau^s} + r). \tag{4.42}$$

Minimizing the right hand side of (4.42) with respect to r we find that there exists $C > 0$ is a constant depending on the a priori data only, such that

$$\max_{D_{1,2}^{int}\setminus\overline{D_1}} |u_1| \leqslant \omega(\varepsilon) \ , \tag{4.43}$$

up to a possible replacement of the constants $C, \theta > 0$ in (2.18).

□

Lemma 4.5 (**Doubling inequality at the boundary**). . *Let u be the solution of* (1.1)*, then there exists a radius $\overline{r}$ such that for every $x_0 \in \partial D$, the following holds:*

$$\int_{B_{\beta r}(x_0)\setminus D} |u|^2 dx \leq C\beta^K \int_{B_r(x_0)\setminus D} |u|^2 dx \tag{4.44}$$

for every r, β such that $\beta > 1$ and $0 < \beta r < \overline{r}$, where $C > 0$ and $K > 0$ depend only on the a-priori data.

Proof For the proof we refer to [2, Theorem 1.1]. The only differences here rely on an adaptation to complex valued solutions (see [24, 25]) and a more explicit evaluation of the constants C and K in terms of the a priori data (see [3, 24, 25]). □

Theorem 4.6 (**Loglog stability for the obstacles**). *Let u_i, $i = 1, 2$, be the weak solutions to the problem* (1.1) *with $D = D_i$ respectively and let $u_{i,\infty}$ be their far field patterns respectively. If we have*

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon, \tag{4.45}$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leqslant \omega(\varepsilon), \tag{4.46}$$

where ω is given by (2.18) *and C and θ are constants depending on the a priori data only.*

Proof
We recall that

$$d := d_{\mathcal{H}}(\partial D_1, \partial D_2) := \max(\sup_{x\in\partial D_1} d(x, \partial D_2), \sup_{x\in\partial D_2} d(x, \partial D_1)).$$

We may assume without loss of generality, that There exists $x \in \partial D_2$ such that $d = d(x, \partial D_1)$. We distinguish two cases
i) $B_d(x) \cap D_1 = \emptyset$.
ii) $B_d(x) \subset D_1$.
Let us begin with the case i). There exists $x_1 \in \partial D_1$ such that $d = d(x, x_1)$. Suppose that $r_0 > d$ and set $\beta := \frac{r_0}{d}$. From the doubling inequality at the boundary, we have:

$$\int_{B_{r_0}(x_1)\setminus D_1} |u_1|^2 dx \leq C\beta^K \int_{B_d(x_1)\setminus D_1} |u_1|^2 dx.$$

Since $B_d(x_1) \setminus D_1 \subset D_{1,2}^{int} \setminus D_1$ then from Theorem 4.3, we deduce that

$$\int_{B_{r_0}(x_1)\setminus D_1} |u_1|^2 dx \leq C' d^{-K} \omega(\varepsilon). \tag{4.47}$$

Combining the lower bound in Corollary 3.3 with an iterated use of the three spheres inequality as in Theorem 4.4 we infer that

$$r_0^3 C \leq \int_{B_{r_0}(x_1)\setminus D_1} |u_1|^2 dx \tag{4.48}$$

where $C > 0$ is a constant depending on the a priori data only. Hence combining (4.48) with (4.47) we obtain

$$d \leq \omega(\varepsilon), \tag{4.49}$$

up to a possible replacing of the constants C, θ in (2.18). Suppose now that $r_0 \leq d$, then obviously we have:

$$\frac{d}{r_0} \leq \frac{d_0}{r_0} \leq C_1. \tag{4.50}$$

Since $B_{r_0}(x_1) \setminus D_1 \subset D_{1,2}^{int} \setminus D_1$, by the estimate of Theorem 4.3, we have

$$\int_{B_{r_0(x_1)}\setminus D_1} |u_1|^2 dx \leq \omega(\varepsilon) \tag{4.51}$$

Combining (4.50),(4.48) and (4.51) we obtain

$$d \leq \omega(\varepsilon) \tag{4.52}$$

up to a possible replacing of the constants C, θ in (2.18). Summing up (4.49) and (4.52) gives

$$d \leq \omega(\varepsilon) \tag{4.53}$$

with a possible replacement of the constant $C, \theta > 0$ in (2.18).
In the case ii), we proceed similarly by replacing u_1 with u_2.

□

Proposition 4.7 (**Graphs condition**). *Let* Ω_1 *and* Ω_2 *be bounded domains of class* $C^{1,\alpha}$ *with constants* ρ_0 *and* E*. There exist numbers* $d_0, \tilde{\rho}_0$*,* $d_0 > 0$*,* $0 \le \tilde{\rho}_0 \le \rho_0$ *for which the ratios* $\frac{d_0}{\rho_0}$ *and* $\frac{\tilde{\rho}_0}{\rho_0}$ *only depend on* α *and* E *such that if we have*

$$d_H(\overline{\Omega}_1, \overline{\Omega}_2) \le d_0 \tag{4.54}$$

then every connected component G *of* $\Omega_1 \cap \Omega_2$ *has a boundary of Lipschitz class with constants* $\tilde{\rho}_0$*,* $\tilde{E}$ *where* $\tilde{\rho}_0$ *is as above and* $\tilde{E} > 0$ *only depends on* α *and* E*.*

Proof For the proof, we refer to [3, Proposition 3.6]. □

Proposition 4.8. *Let* u_i *and* $u_{i,\infty}$*,* $i = 1, 2$*, be as in Theorem 4.6. There exists* $\epsilon_0 > 0$ *depending on the a priori data only, such that for any* $\epsilon < \epsilon_0$ *if*

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \le \epsilon$$

then $\partial D_{1,2}^{int}$ *is of Lipschitz class with constant* $\tilde{r}_0$*,* L *depending on the a priori data only.*

Proof Let us define $\Omega_i := B_R(0) \setminus \overline{D}_i$, $i = 1, 2$. We know that in general the Hausdorff distances $d_H(\overline{\Omega}_1, \overline{\Omega}_2)$ and $d_H(\partial D_1, \partial D_2)$ are not equivalent. However, with our regularity assumptions on D_i, $i = 1, 2$, the estimate (4.54) can be derived from (2.20) by arguing as in the proof of Lemma 8.1 in [3]. Hence the Lipschitz regularity of $\partial D_{1,2}^{int}$ follows from Proposition 4.7.

□

Proof [**Theorem 2.3**] In view of Proposition 4.8 we have that the hypothesis of the Theorem 4.4 are fulfilled. Hence by replacing the rate of stability ω (defined in (2.18)) with the improved one η (defined in (2.17)) in the proof of Theorem 4.6 the thesis follows. □

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