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Abstract

We consider a differential system based on the coupling of the Navier-Stokes and Darcy equations for modeling the interaction between surface and subsurface flows. We formulate the problem as an interface equation, we analyze the associated (nonlinear) Steklov-Poincaré operators, and we prove its well-posedness.

1 Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain, decomposed as the union of two non intersecting subdomains Ω_f and Ω_p separated by an interface Γ, i.e. $\bar{\Omega} = \bar{\Omega}_f \cup \bar{\Omega}_p$, $\Omega_f \cap \Omega_p = \emptyset$ and $\bar{\Omega}_f \cap \bar{\Omega}_p = \Gamma$. We suppose the boundaries $\partial\Omega_f$ and $\partial\Omega_p$ to be Lipschitz continuous. From the physical point of view, Γ is a surface separating an upper domain Ω_f filled by a fluid, from a lower domain Ω_p formed by a porous medium. We assume that the fluid contained in Ω_f has a fixed upper surface (i.e., we do not consider the case of free-surface fluid) and can filtrate through the underlying porous medium.

In order to describe the motion of the fluid in Ω_f, we introduce the Navier-Stokes equations: $\forall t > 0$,

$$\begin{array}{rcll} \partial_t \mathbf{u}_f - \boldsymbol{\nabla} \cdot \mathsf{T}(\mathbf{u}_f, p_f) + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f & = & \mathbf{f} & \text{in } \Omega_f\ , \\ \nabla \cdot \mathbf{u}_f & = & 0 & \text{in } \Omega_f\ , \end{array} \qquad (1)$$

where $\mathsf{T}(\mathbf{u}_f, p_f) = \nu(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f) - p_f \mathsf{I}$ is the Cauchy stress tensor, $\nu > 0$ is the kinematic viscosity of the fluid, while $\mathbf{u}_f$ and p_f are the fluid velocity and pressure, respectively; ∇ is the gradient operator with respect to the space coordinates.

In the lower domain Ω_p, we define the piezometric head $\varphi = z + p_p/(\rho_f g)$, where z is the elevation from a reference level, p_p is the pressure of the fluid in Ω_p, ρ_f its density and g is the gravity acceleration.

The fluid motion in Ω_p is described by the equations:

$$\begin{array}{rcll} n\mathbf{u}_p &=& -K\nabla\varphi & \text{in } \Omega_p\ , \\ \nabla\cdot\mathbf{u}_p &=& 0 & \text{in } \Omega_p\ , \end{array} \tag{2}$$

where $\mathbf{u}_p$ is the fluid velocity, n is the volumetric porosity and K is the hydraulic conductivity tensor $K = diag(K_1, \ldots, K_d)$ with $K_i \in L^\infty(\Omega_p)$, $i = 1, \ldots, d$. The first equation is Darcy's law. In the following, we shall denote $\mathsf{K} = K/n = diag(K_i/n)$ $(i = 1, \ldots, d)$.

For the sake of clarity, in our analysis we shall adopt homogenous boundary conditions. The treatment of non-homogeneous conditions involves some additional technicalities, but neither the guidelines of the theory nor the final results are affected. We refer the reader to [7]. In particular, for the Navier-Stokes problem we impose the no-slip condition $\mathbf{u}_f = \mathbf{0}$ on $\partial\Omega_f \setminus \Gamma$, while for the Darcy problem, we set the piezometric head $\varphi = 0$ on Γ_p^b and we require the normal velocity to be null on Γ_p: $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p, where $\partial\Omega_p = \Gamma \cup \Gamma_p^b \cup \Gamma_p$ (see Fig. 1). $\mathbf{n}_p$ and $\mathbf{n}_f$ denote the unit outward normal vectors to the surfaces $\partial\Omega_p$ and $\partial\Omega_f$, respectively, and we have $\mathbf{n}_f = -\mathbf{n}_p$ on Γ. We suppose $\mathbf{n}_f$ and $\mathbf{n}_p$ to be regular enough, and we indicate $\mathbf{n} = \mathbf{n}_f$ for simplicity of notation.

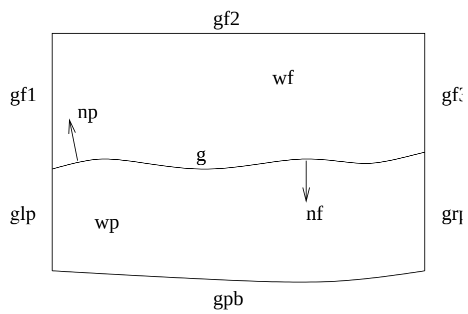

Figure 1: Schematic representation of a 2D vertical section of the computational domain

We supplement the Navier-Stokes and Darcy problems with the following conditions on Γ:

$$\mathbf{u}_p \cdot \mathbf{n} = \mathbf{u}_f \cdot \mathbf{n}\ , \tag{3}$$

$$-\varepsilon\boldsymbol{\tau}_i \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) = \nu\mathbf{u}_f \cdot \boldsymbol{\tau}_i\ , \qquad i = 1, \ldots, d-1\ , \tag{4}$$

$$-\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) = g\varphi\ , \tag{5}$$

where $\boldsymbol{\tau}_i$ $(i = 1, \ldots, d-1)$ are linear independent unit tangential vectors to the boundary Γ, and ε is the characteristic length of the pores of the porous medium.
Conditions (3)-(5) impose the continuity of the normal velocity on Γ, as well as that of the normal component of the normal stress, but they allow pressure to be discontinuous across the interface.
A rigorous mathematical justification of these interface conditions, usually denoted as Beavers-Joseph-Saffman conditions, can be found in [12, 13, 14].

The same interface conditions have been considered in [8, 9, 10, 15] for the coupling of Stokes and Darcy equations.
From Sect. 1.1 on, we focus on the steady problem obtained by dropping the time derivative in the momentum equation (1).
Then, after writing the weak form of the global problem (Sect. 1.1), we reformulate it as a nonlinear problem in one sole interface unknown. In Sect. 3, we introduce and analyze some nonlinear extension operator that will be used in Sect. 4 to write the nonlinear interface problem $\mathcal{S}\lambda = 0$, whose unknown solution λ is the common value of the normal velocity $\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_p \cdot \mathbf{n}$ across Γ. $\mathcal{S}$ is a nonlinear Steklov-Poincaré operator. Finally, in Sect. 4.1, we prove the well-posedness of this interface problem.

1.1 Weak form of the global problem

From now on, we shall consider the steady case for the Navier-Stokes equations and, instead of (2), we will use the following formulation for Darcy problem:

$$\text{find } \varphi \ : \quad -\nabla \cdot (\mathsf{K}\nabla\varphi) = 0 \quad \text{in } \Omega_p \, . \tag{6}$$

We define the functional spaces:

$$H_f = \{\mathbf{v} \in (H^1(\Omega_f))^d \, : \, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_f \setminus \Gamma\}, \tag{7}$$
$$H_p = \{\psi \in H^1(\Omega_p) \, : \, \psi = 0 \text{ on } \Gamma_p^b\}, \tag{8}$$

and let $Q = L^2(\Omega_f)$. We denote by $|\cdot|_1$ the H^1-seminorm, and by $\|\cdot\|_0$ the L^2-norm; it will always be clear form the context whether we are referring to the norms and seminorms in Ω_f or Ω_p.
Then, we introduce the bilinear forms

$$a_f(\mathbf{v},\mathbf{w}) = \int_{\Omega_f} \frac{\nu}{2}(\nabla\mathbf{v} + \nabla^T\mathbf{v}) \cdot (\nabla\mathbf{w} + \nabla^T\mathbf{w}) \qquad \forall \mathbf{v},\mathbf{w} \in (H^1(\Omega_f))^d \, , \tag{9}$$

$$b_f(\mathbf{v}, q) = -\int_{\Omega_f} q \, \nabla \cdot \mathbf{v} \qquad \forall \mathbf{v} \in (H^1(\Omega_f))^d, \quad \forall q \in Q \, , \tag{10}$$

$$a_p(\varphi, \psi) = \int_{\Omega_p} \nabla\psi \cdot \mathsf{K}\nabla\varphi \qquad \forall \varphi, \psi \in H^1(\Omega_p) \, , \tag{11}$$

and, for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in (H^1(\Omega_f))^d$, the trilinear form

$$c_f(\mathbf{w}; \mathbf{z}, \mathbf{v}) = \int_{\Omega_f} [(\mathbf{w} \cdot \nabla)\mathbf{z}] \cdot \mathbf{v} = \sum_{i,j=1}^{d} \int_{\Omega_f} w_j \, \frac{\partial z_i}{\partial x_j} \, v_i \, . \tag{12}$$

The coupling conditions (3)-(5) can be incorporated in the weak formulation of the Navier-Stokes/Darcy problem as natural conditions on Γ. In fact, the latter reads:
find $\mathbf{u}_f \in H_f$, $p_f \in Q$, $\varphi \in H_p$ such that for all $\mathbf{v} \in H_f$, $q \in Q$, $\psi \in H_p$,

$$a_f(\mathbf{u}_f, \mathbf{v}) + c_f(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}) + b_f(\mathbf{v}, p_f) + g \, a_p(\varphi, \psi)$$
$$+ \int_\Gamma g \, \varphi(\mathbf{v} \cdot \mathbf{n}) - \int_\Gamma g \, \psi(\mathbf{u}_f \cdot \mathbf{n}) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \, , \tag{13}$$
$$b_f(\mathbf{u}_f, q) = 0 \, . \tag{14}$$

In the next sections, we will consider the issue of the well-posedness of (13)-(14). In our analysis, we shall use some classical existence and uniqueness results for nonlinear problems, that we anticipate, for the sake of clarity, in the following section.

2 General existence and uniqueness results

In this section we recall some existence and uniqueness results for nonlinear saddle-point problems, referring the reader to, e.g., [3, 4, 5, 6, 11] for a rigorous study.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real Hilbert spaces. Consider a bilinear continuous form $\mathrm{b}(\cdot,\cdot) : X \times Y \to \mathbb{R}$, $(v,q) \to \mathrm{b}(v,q)$, and a trilinear form $\mathrm{a}(\cdot;\cdot,\cdot) : X \times X \times X \to \mathbb{R}$, $(w,u,v) \to \mathrm{a}(w;u,v)$, where, for $w \in X$ the mapping $(u,v) \to \mathrm{a}(w;u,v)$ is a bilinear continuous form on $X \times X$.
Then, we consider the following problem: given $l \in X'$, find a pair $(u,p) \in X \times Y$ satisfying

$$\begin{array}{rcll} \mathrm{a}(u;u,v) + \mathrm{b}(v,p) & = & \langle l, v\rangle & \forall v \in X, \\ \mathrm{b}(u,q) & = & 0 & \forall q \in Y. \end{array} \tag{15}$$

Introducing the linear operators $A(w) \in \mathcal{L}(X;X')$ for $w \in X$, and $B \in \mathcal{L}(X;Y')$:

$$\langle A(w)u, v\rangle = \mathrm{a}(w;u,v) \quad \forall u,v \in X \ ,$$
$$\langle Bv, q\rangle = \mathrm{b}(v,q) \quad \forall v \in X,\ \forall q \in Y \ ,$$

problem (15) becomes: find $(u,p) \in X \times Y$ such that

$$\begin{array}{rcll} A(u)u + B^T p & = & l & \text{in } X' \ , \\ Bu & = & 0 & \text{in } Y' \ . \end{array} \tag{16}$$

Taking $V = \mathrm{Ker}(B)$, we associate (15) with the problem:

$$\text{find } u \in V \ : \quad \mathrm{a}(u;u,v) = \langle l, v\rangle \qquad \forall v \in V \ , \tag{17}$$

or, equivalently: find $u \in V$ s.t. $\Pi A(u)u = \Pi l$ in V', where the linear operator $\Pi \in \mathcal{L}(X';V')$ is defined by $\langle \Pi l, v\rangle = \langle l, v\rangle$, $\forall v \in V$.
If (u,p) is a solution of problem (15), then u solves (17). The converse may be proven provided an inf-sup condition holds. Indeed, the following results can be proved.

Theorem 2.1 (Existence and uniqueness) *Suppose that:*

1. *the bilinear form* $\mathrm{a}(w;\cdot,\cdot)$ *is uniformly elliptic in the Hilbert space* V *with respect to* w*, i.e. there exists a constant* $\alpha > 0$ *such that*

$$\mathrm{a}(w;v,v) \geq \alpha \|v\|_X^2 \qquad \forall v,w \in V\,;$$

2. *the mapping* $w \to \Pi A(w)$ *is locally Lipschitz-continuous in* V*, i.e. there exists a continuous and monotonically increasing function* $L : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for all* $m > 0$

$$|\mathrm{a}(w_1;u,v) - \mathrm{a}(w_2;u,v)| \leq L(m)\|u\|_X\|v\|_X\|w_1 - w_2\|_X \tag{18}$$

$\forall u,v \in V$, $\forall w_1,w_2 \in S_m$ *with* $S_m = \{w \in V \ : \ \|w\|_X \leq m\}$*;*

3. it holds

$$\frac{\|\Pi l\|_{V'}}{\alpha^2}\; L\left(\frac{\|\Pi l\|_{V'}}{\alpha}\right) < 1\,. \tag{19}$$

Then, problem (17) has a unique solution $u \in V$.

We consider now problem (15).

Theorem 2.2 *Assume that the bilinear form* $\mathrm{b}(\cdot,\cdot)$ *satisfies the inf-sup condition: $\exists \beta > 0$*

$$\inf_{q\in Y}\sup_{v\in X}\frac{\mathrm{b}(v,q)}{\|v\|_X\|q\|_Y} \geq \beta\,. \tag{20}$$

Then, for each solution u of (17) there exists a unique $p \in Y$ such that the pair (u,p) is a solution of (15).

3 Some nonlinear extension operators: definition and analysis

In this section we apply domain decomposition methods at the differential level to study the Navier-Stokes/Darcy problem. We identify the subdomains with Ω_f and Ω_p, then we introduce and analyze some nonlinear extension operators that will be used in Sect. 4 to write the Steklov-Poincaré interface equation associated to the coupled problem.
In our analysis we adopt the interface condition

$$\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0 \quad \text{on } \Gamma, \qquad j = 1,\dots,d-1, \tag{21}$$

instead of (4). This simplification is acceptable from the physical viewpoint since the term in (4) involving the normal derivatives of $\mathbf{u}_f$ is multiplied by ε, and the velocity $\mathbf{u}_f$ can be supposed of order $O(\varepsilon)$ in the neighborhood of Γ (see [13]), so that the left hand side can be approximated to zero. We point out that this simplification does not alter the coupling structure, since (4) is a boundary condition for the fluid problem in Ω_f and not a coupling condition.
We consider the trace space $\Lambda = H^{1/2}_{00}(\Gamma)$ (see [16]) and the spaces

$$H_f^\tau = \{\mathbf{v} \in H_f \,:\, \mathbf{v}\cdot\boldsymbol{\tau}_j = 0 \text{ on } \Gamma,\, j = 1,\dots,d-1\}\,, \tag{22}$$

$$H_p^0 = \{\psi \in H_p \,:\, \psi = 0 \text{ on } \Gamma\}\,. \tag{23}$$

Moreover, we consider two linear continuous extension operators:

$$\boldsymbol{R}_1^\tau : \Lambda \to H_f^\tau \quad \text{such that} \quad (\boldsymbol{R}_1^\tau\mu)\cdot\mathbf{n} = \mu \text{ on } \Gamma \quad \forall \mu \in \Lambda\,, \tag{24}$$

$$R_2 : H^{1/2}(\Gamma) \to H_p \quad \text{such that} \quad R_2\mu = \mu \text{ on } \Gamma, \quad \forall \mu \in H^{1/2}(\Gamma)\,. \tag{25}$$

We can prove the following result (see [7]).

Proposition 3.1 *The coupled Navier-Stokes/Darcy problem (13)-(14) can be reformulated in the equivalent multidomain form:*

find $\mathbf{u}_f \in H_f^\tau$, $p_f \in Q$, $\varphi \in H_p$ *such that*

$$a_f(\mathbf{u}_f, \mathbf{w}) + c_f(\mathbf{u}_f; \mathbf{u}_f, \mathbf{w}) + b_f(\mathbf{w}, p_f) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{w} \quad \forall \mathbf{w} \in (H_0^1(\Omega_f))^d \ , \tag{26}$$

$$b_f(\mathbf{u}_f, q) = 0 \qquad \forall q \in Q \ , \tag{27}$$

$$a_p(\varphi, \psi) = 0 \qquad \forall \psi \in H_p^0 \ , \tag{28}$$

$$\int_\Gamma (\mathbf{u}_f \cdot \mathbf{n})\mu = a_p(\varphi, R_2\mu) \qquad \forall \mu \in \Lambda \ , \tag{29}$$

$$\begin{aligned} \int_\Gamma g\, \varphi\mu = \int_{\Omega_f} &\mathbf{f}\ (\boldsymbol{R}_1^\tau \mu) - a_f(\mathbf{u}_f, \boldsymbol{R}_1^\tau \mu) \\ &-c_f(\mathbf{u}_f; \mathbf{u}_f, \boldsymbol{R}_1^\tau \mu) - b_f(\boldsymbol{R}_1^\tau \mu, p_f) \qquad \forall \mu \in \Lambda \ . \end{aligned} \tag{30}$$

Our aim is now to reformulate the coupled problem (26)-(30) as an interface equation in a scalar unknown defined on Γ corresponding to the trace of the fluid normal velocity $\mathbf{u}_f \cdot \mathbf{n}$ on Γ. First of all, we need to introduce and analyze some further extension operators.

Let us consider the (unknown) interface variable $\lambda = (\mathbf{u}_f \cdot \mathbf{n})_{|\Gamma}$.

Due to the incompressibility constraint in Ω_f and to the boundary conditions imposed on $\partial\Omega_f \setminus \Gamma$, it must be $\lambda \in \Lambda_0$ with

$$\Lambda_0 = \left\{ \mu \in \Lambda \ : \ \int_\Gamma \mu = 0 \right\} \ . \tag{31}$$

Then, let us define the *linear* extension operator:

$$R_f \colon \Lambda_0 \to H_f^\tau \times Q_0, \quad \eta \to R_f\eta = (\boldsymbol{R}_f^1 \eta, R_f^2 \eta), \tag{32}$$

satisfying $(\boldsymbol{R}_f^1\eta) \cdot \mathbf{n} = \eta$ on Γ, and

$$\begin{aligned} a_f(\boldsymbol{R}_f^1\eta, \mathbf{w}) + b_f(\mathbf{w}, R_f^2\eta) &= 0 \qquad \forall \mathbf{w} \in (H_0^1(\Omega_f))^d \ , \\ b_f(\boldsymbol{R}_f^1\eta, q) &= 0 \qquad \forall q \in Q_0 \ , \end{aligned} \tag{33}$$

where $Q_0 = \{q \in Q \ : \ \int_{\Omega_f} q = 0\}$. Moreover, we consider the *linear* extension operator

$$R_p \colon \Lambda_0 \to H_p, \qquad \eta \to R_p\eta \tag{34}$$

such that

$$a_p(R_p\eta, \psi) = \int_\Gamma \eta\psi \qquad \forall \psi \in H_p \ . \tag{35}$$

Finally, let us introduce the following *nonlinear* extension operator:

$$\mathcal{R}_f \colon \Lambda_0 \to H_f^\tau \times Q_0, \quad \eta \to \mathcal{R}_f\eta \ = (\boldsymbol{\mathcal{R}}_f^1\eta, \mathcal{R}_f^2\eta)$$

such that $(\boldsymbol{\mathcal{R}}_f^1\eta) \cdot \mathbf{n} = \eta$ on Γ, and, for all $\mathbf{v} \in (H_0^1(\Omega_f))^d$, $q \in Q_0$,

$$\begin{aligned} a_f(\boldsymbol{\mathcal{R}}_f^1\eta, \mathbf{v}) + c_f(\mathbf{u}_* + \boldsymbol{\mathcal{R}}_f^1\eta; \mathbf{u}_* + \boldsymbol{\mathcal{R}}_f^1\eta, \mathbf{v}) + b_f(\mathbf{v}, \mathcal{R}_f^2\eta) &= 0 \ , \\ b_f(\boldsymbol{\mathcal{R}}_f^1\eta, q) &= 0 \ , \end{aligned} \tag{36}$$

where $\mathbf{u}_* \in (H_0^1(\Omega_f))^d$ satisfies the linear auxiliary Stokes problem:

find $\mathbf{u}_* \in (H^1_0(\Omega_f))^d$, $\pi_* \in Q_0$:

$$\begin{aligned} a_f(\mathbf{u}_*, \mathbf{v}) + b_f(\mathbf{v}, \pi_*) &= \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in (H^1_0(\Omega_f))^d , \\ b_f(\mathbf{u}_*, q) &= 0 \qquad\qquad \forall q \in Q_0 . \end{aligned} \tag{37}$$

Let us finally recall that the following estimate holds:

$$|\mathbf{u}_*|_1 \le \frac{2}{C_\kappa \nu} \|\mathbf{f}\|_0 . \tag{38}$$

3.1 Existence and uniqueness of the operator $\mathcal{R}_f$

We face now the issue of the existence and uniqueness of the extension operator $\mathcal{R}_f$. With this purpose, we define the auxiliary (homogeneous) nonlinear operator

$$\begin{aligned} &\mathcal{R}_0 : \Lambda_0 \to (H^1_0(\Omega_f))^d \times Q_0, \qquad \eta \to \mathcal{R}_0\eta = (\boldsymbol{\mathcal{R}}^1_0\eta, \mathcal{R}^2_0\eta), \\ &\text{with } \mathcal{R}^i_0\eta = \mathcal{R}^i_f\eta - R^i_f\eta, \quad i = 1, 2. \end{aligned} \tag{39}$$

Clearly, $\boldsymbol{\mathcal{R}}^1_0\eta \cdot \mathbf{n} = 0$ on Γ, and it satisfies:

$$\begin{aligned} a_f(\boldsymbol{\mathcal{R}}^1_0\eta, \mathbf{v}) + c_f(\mathbf{u}_* + \boldsymbol{R}^1_f\eta + \boldsymbol{\mathcal{R}}^1_0\eta; \mathbf{u}_* + \boldsymbol{R}^1_f\eta + \boldsymbol{\mathcal{R}}^1_0\eta, \mathbf{v}) \\ + b_f(\mathbf{v}, \mathcal{R}^2_0\eta) &= 0 , \\ b_f(\boldsymbol{\mathcal{R}}^1_0\eta, q) &= 0 , \end{aligned} \tag{40}$$

for all $\mathbf{v} \in (H^1_0(\Omega_f))^d$, $q \in Q_0$. Remark that problem (40) is analogous to (36), but here $\boldsymbol{\mathcal{R}}^1_0\eta \in (H^1_0(\Omega_f))^d$, while $\boldsymbol{\mathcal{R}}^1_f\eta \in H^\tau_f$.

We consider the functional space

$$V^0_f = \{\mathbf{v} \in (H^1_0(\Omega_f))^d \,:\, \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_f\} , \tag{41}$$

and, given $\eta \in \Lambda_0$, we define the form:

$$\begin{aligned} a(\mathbf{w}; \mathbf{z}, \mathbf{v}) = a_f(\mathbf{z}, \mathbf{v}) + c_f(\mathbf{w}; \mathbf{z}, \mathbf{v}) + c_f(\mathbf{u}_* + \boldsymbol{R}^1_f\eta; \mathbf{z}, \mathbf{v}) \\ + c_f(\mathbf{z}; \mathbf{u}_* + \boldsymbol{R}^1_f\eta, \mathbf{v}) \qquad \forall \mathbf{w}, \mathbf{z}, \mathbf{v} \in (H^1(\Omega_f))^d , \end{aligned} \tag{42}$$

and the functional

$$\langle \ell, \mathbf{v} \rangle = -c_f(\mathbf{u}_* + \boldsymbol{R}^1_f\eta; \mathbf{u}_* + \boldsymbol{R}^1_f\eta, \mathbf{v}) \qquad \forall \mathbf{v} \in (H^1(\Omega_f))^d . \tag{43}$$

Thus, we can rewrite (40) as: given $\eta \in \Lambda_0$,

$$\text{find } \boldsymbol{\mathcal{R}}^1_0\eta \in V^0_f \quad : \quad a(\boldsymbol{\mathcal{R}}^1_0\eta; \boldsymbol{\mathcal{R}}^1_0\eta, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in V^0_f . \tag{44}$$

Finally, let us recall some useful inequalities: the Poincaré inequality (see, e.g., [17] p. 11)

$$\exists C_{\Omega_f} > 0 \;:\quad \|\mathbf{v}\|_0 \le C_{\Omega_f} |\mathbf{v}|_1 \qquad \forall \mathbf{v} \in H_f , \tag{45}$$

the Korn inequality (see, e.g., [18] p. 149): $\forall \mathbf{v} = (v_1, \ldots, v_d) \in H_f$

$$\exists C_\kappa > 0 \;:\quad \int_{\Omega_f} \sum_{j,l=1}^{d} \left(\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j} \right)^2 \ge C_\kappa \|\mathbf{v}\|_1^2 , \tag{46}$$

and

$$\exists \hat{C} > 0 : \quad |c(\mathbf{w};\mathbf{z},\mathbf{v})| \le \hat{C}|\mathbf{w}|_1 \, |\mathbf{z}|_1 \, |\mathbf{v}|_1 \qquad \forall \mathbf{w},\mathbf{z},\mathbf{v} \in H_f, \tag{47}$$

which follows from the Poincaré inequality (45) and the inclusion $(H^1(\Omega_f))^d \subset (L^4(\Omega_f))^d$ (for $d = 2, 3$) due to the Sobolev embedding theorem (see [1]).

We can now state the following result.

Proposition 3.2 *Let $\mathbf{f} \in L^2(\Omega_f)$ such that*

$$\frac{\|\mathbf{f}\|_0}{\nu^2} < \frac{1}{8} \cdot \frac{C_\kappa^2}{\hat{C}}\,, \tag{48}$$

where C_κ and $\hat{C}$ are the constants introduced in (46) and (47), respectively. If

$$\eta \in \left\{ \mu \in \Lambda_0 \,:\, |\boldsymbol{R}_f^1 \mu|_1 < \frac{\nu C_\kappa}{4\hat{C}} - \frac{2}{C_\kappa \nu}\|\mathbf{f}\|_0 \right\}\,, \tag{49}$$

then, there exists a unique nonlinear extension $\mathcal{R}_f\eta = (\boldsymbol{\mathcal{R}}_f^1\eta, \mathcal{R}_f^2\eta) \in H_f^\tau \times Q_0$.

Remark 3.1 *Notice that (49) imposes a constraint on η. In particular, since the norms $|\boldsymbol{R}_f^1\eta|_1$ and $\|\eta\|_\Lambda$ are equivalent (see [9], Lemma 4.1), this condition implies that a unique extension $\mathcal{R}_f\eta$ exists, provided the norm of η is small enough. In our specific case, this means that we would be able to consider an extension $\mathcal{R}_f\lambda$ only if the normal velocity λ across the interface Γ is sufficiently small.*
Finally, remark that (48) guarantees that the radius of the ball in (49) is positive.

Proof. The proof is made of several steps and it is based on Theorems 2.1-2.2.
1. Let $\mathbf{v}, \mathbf{w} \in V_f^0$ and $\eta \in \Lambda_0$. Then, we have

$$\begin{aligned} a(\mathbf{w};\mathbf{v},\mathbf{v}) = a_f(\mathbf{v},\mathbf{v}) &+ c_f(\mathbf{w};\mathbf{v},\mathbf{v}) \\ &+ c_f(\mathbf{u}_* + \boldsymbol{R}_f^1\eta;\mathbf{v},\mathbf{v}) + c_f(\mathbf{v};\mathbf{u}_* + \boldsymbol{R}_f^1\eta,\mathbf{v}). \end{aligned} \tag{50}$$

Integrating by parts and recalling that $\mathbf{w} \in V_f^0$, then

$$c_f(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\partial\Omega_f} \mathbf{w}\cdot\mathbf{n}|\mathbf{v}|^2 - \frac{1}{2}\int_{\Omega_f} \nabla\cdot\mathbf{w}|\mathbf{v}|^2 = 0\,,$$

where $|\mathbf{v}|$ is the Euclidian norm of the vector $\mathbf{v}$. Moreover, denoting by n_j the components of the unit outward normal vector $\mathbf{n}_f$ to $\partial\Omega_f$, we have

$$\begin{aligned} c_f(\mathbf{v};\mathbf{u}_* + \boldsymbol{R}_f^1\eta,\mathbf{v}) &= \int_{\Omega_f} \sum_{i,j=1}^d v_j \frac{\partial(\mathbf{u}_* + \boldsymbol{R}_f^1\eta)_i}{\partial x_j}\, v_i \\ &= -\sum_{i,j=1}^d \int_{\Omega_f} \frac{\partial}{\partial x_j}(v_i v_j)(\mathbf{u}_* + \boldsymbol{R}_f^1\eta)_i + \sum_{i,j=1}^d \int_{\partial\Omega_f} (v_i v_j)(\mathbf{u}_* + \boldsymbol{R}_f^1\eta)_i n_j \\ &= -\sum_{i,j=1}^d \int_{\Omega_f} \frac{\partial v_i}{\partial x_j} v_j (\mathbf{u}_* + \boldsymbol{R}_f^1\eta)_i - \sum_{i,j=1}^d \int_{\Omega_f} \frac{\partial v_j}{\partial x_j} v_i (\mathbf{u}_* + \boldsymbol{R}_f^1\eta)_i \\ &\quad + \sum_{i,j=1}^d \int_{\partial\Omega_f} v_j\, n_j (\mathbf{u}_* + \boldsymbol{R}_f^1\eta)_i v_i \\ &= -c_f(\mathbf{v};\mathbf{v},\mathbf{u}_* + \boldsymbol{R}_f^1\eta). \end{aligned}$$

Finally, $\nabla \cdot (\mathbf{u}_* + \boldsymbol{R}_f^1 \eta) = 0$ by construction, so that $c_f(\mathbf{u}_* + \boldsymbol{R}_f^1\eta; \mathbf{v}, \mathbf{v}) = 0$. Then (50) becomes:

$$a(\mathbf{w}; \mathbf{v}, \mathbf{v}) = a_f(\mathbf{v}, \mathbf{v}) - c_f(\mathbf{v}; \mathbf{v}, \mathbf{u}_* + \boldsymbol{R}_f^1\eta)\,, \tag{51}$$

and using the inequalities (46) and (47) we obtain:

$$\begin{aligned} a(\mathbf{w}; \mathbf{v}, \mathbf{v}) &\geq \frac{C_\kappa \nu}{2}|\mathbf{v}|_1^2 - \hat{C}|\mathbf{v}|_1^2\, |\mathbf{u}_* + \boldsymbol{R}_f^1\eta|_1 \\ &\geq |\mathbf{v}|_1^2 \left(\frac{C_\kappa \nu}{2} - \hat{C}(|\mathbf{u}_*|_1 + |\boldsymbol{R}_f^1\eta|_1) \right) \\ &\geq |\mathbf{v}|_1^2 \left(\frac{C_\kappa \nu}{2} - \hat{C}\left(\frac{2\|\mathbf{f}\|_0}{\nu C_\kappa} + |\boldsymbol{R}_f^1\eta|_1 \right) \right)\,, \end{aligned}$$

the last inequality following from (38). Then, thanks to (49), the bilinear form $a(\mathbf{w}; \cdot, \cdot)$ is uniformly elliptic on V_f^0 with respect to $\mathbf{w}$, with constant α_a (independent of $\mathbf{w}$):

$$\alpha_a = \frac{C_\kappa \nu}{2} - \hat{C}\left(\frac{2\|\mathbf{f}\|_0}{\nu C_\kappa} + |\boldsymbol{R}_f^1\eta|_1 \right)\,.$$

2. Still using (47), we easily obtain:

$$|a(\mathbf{w}_1; \mathbf{z}, \mathbf{v}) - a(\mathbf{w}_2; \mathbf{z}, \mathbf{v})| = |c_f(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{z}, \mathbf{v})| \leq \hat{C}|\mathbf{w}_1 - \mathbf{w}_2|_1 |\mathbf{v}|_1 |\mathbf{z}|_1.$$

3. We have

$$\begin{aligned} \|\Pi \ell\|_{(V_f^0)'} &= \sup_{\mathbf{v} \in V_f^0, \mathbf{v} \neq 0} \frac{|-c_f(\mathbf{u}_* + \boldsymbol{R}_f^1\eta; \mathbf{u}_* + \boldsymbol{R}_f^1\eta, \mathbf{v})|}{|\mathbf{v}|_1} \\ &\leq \sup_{\mathbf{v} \in V_f^0, \mathbf{v} \neq 0} \frac{\hat{C}|\mathbf{u}_* + \boldsymbol{R}_f^1\eta|_1^2\, |\mathbf{v}|_1}{|\mathbf{v}|_1} \\ &\leq \hat{C}\left(\frac{2}{C_\kappa \nu}\|\mathbf{f}\|_0 + |\boldsymbol{R}_f^1\eta|_1 \right)^2, \end{aligned}$$

so that

$$\hat{C}\frac{\|\Pi \ell\|_{(V_f^0)'}}{\alpha_a^2} < 1$$

owing to (49).

4. Thanks to (49) and 1–3, $a(\cdot; \cdot, \cdot)$ and ℓ satisfy the hypotheses of Theorem 2.1, which allows us to conclude that there exists a unique solution $\boldsymbol{\mathcal{R}}_0^1\eta \in V_f^0$ to (44).

5. Since the inf-sup condition is satisfied, Theorem 2.2 guarantees that there exists a unique solution $(\boldsymbol{\mathcal{R}}_0^1\eta, \mathcal{R}_0^2\eta)$ to (40). The thesis follows from (39). □

4 The interface equation associated to the coupled problem

In this section we reformulate the global coupled problem (26)-(30) as an interface equation depending solely on $\lambda = (\mathbf{u}_f \cdot \mathbf{n})_{|\Gamma}$.

We formally define the *nonlinear* pseudo-differential operator $\mathcal{S} : \Lambda_0 \to \Lambda_0'$,

$$\begin{aligned}\langle \mathcal{S}\eta, \mu \rangle &= a_f(\boldsymbol{\mathcal{R}}_f^1\eta + \mathbf{u}_*, \boldsymbol{R}_1^\tau\mu) + c_f(\boldsymbol{\mathcal{R}}_f^1\eta + \mathbf{u}_*; \boldsymbol{\mathcal{R}}_f^1\eta + \mathbf{u}_*, \boldsymbol{R}_1^\tau\mu) \\ &\quad + b_f(\boldsymbol{R}_1^\tau\mu, \mathcal{R}_f^2\eta + \pi_*) - \int_{\Omega_f} \mathbf{f}\,(\boldsymbol{R}_1^\tau\mu) \\ &\quad + \int_\Gamma g(R_p\eta)\mu \qquad \forall\eta \in \Lambda_0, \forall\mu \in \Lambda\ . \end{aligned} \tag{52}$$

We have the following equivalence result, whose proof follows the guidelines of Theorem 4.1 in [9].

Theorem 4.1 *The solution to (26)-(30) can be characterized as follows:*

$$\mathbf{u}_f = \boldsymbol{\mathcal{R}}_f^1\lambda + \mathbf{u}_*, \quad p_f = \mathcal{R}_f^2\lambda + \pi_* + \hat{p}_f, \quad \varphi = R_p\lambda\ , \tag{53}$$

where $\hat{p}_f = (meas(\Omega_f))^{-1}\int_{\Omega_f} p_f$*, and* $\lambda \in \Lambda_0$ *is the solution of the nonlinear interface problem:*

$$\langle \mathcal{S}\lambda, \mu \rangle = 0 \qquad \forall\mu \in \Lambda_0\ . \tag{54}$$

Moreover, $\hat{p}_f$ *can be obtained from* λ *by solving the algebraic equation*

$$\hat{p}_f = (meas(\Gamma))^{-1}\langle \mathcal{S}\lambda, \varepsilon \rangle,$$

where $\varepsilon \in \Lambda$ *is a fixed function such that*

$$\frac{1}{meas(\Gamma)}\int_\Gamma \varepsilon = 1\ . \tag{55}$$

Notice that a more useful characterization of the operator $\mathcal{S}$ can be provided. Indeed, with the special choice $\boldsymbol{R}_1^\tau = \boldsymbol{R}_f^1$ in (52), thanks to (33), we obtain

$$b_f(\boldsymbol{R}_f^1\mu, \mathcal{R}_f^2\eta + \pi_*) = 0 \qquad \forall\eta, \mu \in \Lambda_0\ .$$

Moreover, owing to (39), for $\eta, \mu \in \Lambda_0$, we have

$$\begin{aligned}\langle \mathcal{S}\eta, \mu \rangle &= a_f(\boldsymbol{\mathcal{R}}_0^1\eta + \boldsymbol{R}_f^1\eta + \mathbf{u}_*, \boldsymbol{R}_f^1\mu) \\ &\quad + c_f(\boldsymbol{\mathcal{R}}_0^1\eta + \boldsymbol{R}_f^1\eta + \mathbf{u}_*; \boldsymbol{\mathcal{R}}_0^1\eta + \boldsymbol{R}_f^1\eta + \mathbf{u}_*, \boldsymbol{R}_f^1\mu) \\ &\quad - \int_{\Omega_f} \mathbf{f}\,(\boldsymbol{R}_f^1\mu) + \int_\Gamma g(R_p\eta)\mu\ .\end{aligned}$$

By taking $\boldsymbol{\mathcal{R}}_0^1\eta$ $(\in (H_0^1(\Omega_f))^d)$ as test function in (33), we obtain:

$$a_f(\boldsymbol{R}_f^1\mu, \boldsymbol{\mathcal{R}}_0^1\eta) + b_f(\boldsymbol{\mathcal{R}}_0^1\eta, R_f^2\mu) = 0\,.$$

Finally, since $R_f^2\mu \in Q_0$, owing to (40) it follows that $a_f(\boldsymbol{R}_f^1\mu, \boldsymbol{\mathcal{R}}_0^1\eta) = 0$, so that, for all $\eta, \mu \in \Lambda_0$, the operator $\mathcal{S}$ can be characterized as

$$\begin{aligned}\langle \mathcal{S}\eta, \mu \rangle &= a_f(\boldsymbol{R}_f^1\eta + \mathbf{u}_*, \boldsymbol{R}_f^1\mu) \\ &\quad + c_f(\boldsymbol{\mathcal{R}}_0^1\eta + \boldsymbol{R}_f^1\eta + \mathbf{u}_*; \boldsymbol{\mathcal{R}}_0^1\eta + \boldsymbol{R}_f^1\eta + \mathbf{u}_*, \boldsymbol{R}_f^1\mu) \\ &\quad + \int_\Gamma g(R_p\eta)\mu - \int_{\Omega_f} \mathbf{f}\,(\boldsymbol{R}_f^1\mu)\ .\end{aligned} \tag{56}$$

4.1 Existence and uniqueness result

We study the existence and uniqueness of the solution of the nonlinear interface problem (54).
Note that in view of (56), $\mathcal{S}\lambda$ is defined in terms of the operator $\boldsymbol{\mathcal{R}}_0^1\lambda$, which, thanks to (40), satisfies in its turn the following problem for all $\mathbf{v} \in V_f^0$:

$$a_f(\boldsymbol{\mathcal{R}}_0^1\lambda, \mathbf{v}) + c_f(\boldsymbol{\mathcal{R}}_0^1\lambda + \boldsymbol{R}_f^1\lambda + \mathbf{u}_*; \boldsymbol{\mathcal{R}}_0^1\lambda + \boldsymbol{R}_f^1\lambda + \mathbf{u}_*, \mathbf{v}) = 0 \,. \tag{57}$$

Therefore, in order to prove the existence and uniqueness of the solution of the interface problem, we have to consider (54), with the operator $\mathcal{S}$ as in (56), coupled with (57), i.e., we have to guarantee at once the existence and uniqueness of $\lambda \in \Lambda_0$ and $\boldsymbol{\mathcal{R}}_0^1\lambda \in V_f^0$. To this aim we apply Theorem 2.1 replacing the space V by the product space $W = \Lambda_0 \times V_f^0$ endowed with the norm:

$$\|\bar{v}\|_W = (|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2)^{1/2} \qquad \forall \bar{v} = (\mu, \mathbf{v}) \in W \,. \tag{58}$$

We introduce the trilinear form and the linear functional associated with our problem in the space W. For any fixed $\bar{w} = (\eta, \mathbf{w}) \in W$, we define the following operator depending on $\bar{w}$:

$$\begin{array}{llll} \mathcal{A}(\eta, \mathbf{w}) : & W & \rightarrow & \Lambda_0' \times (V_f^0)', \\ \mathcal{A}(\eta, \mathbf{w}) : & (\xi, \mathbf{u}) & \rightarrow & (\mathcal{A}_0(\eta, \mathbf{w})(\xi, \mathbf{u}), \mathcal{A}_f(\eta, \mathbf{w})(\xi, \mathbf{u})) \end{array}$$

where, for every test function $\mu \in \Lambda_0$,

$$\begin{aligned} \langle \mathcal{A}_0(\eta, \mathbf{w})(\xi, \mathbf{u}), \mu \rangle \;=\;& a_f(\boldsymbol{R}_f^1\xi, \boldsymbol{R}_f^1\mu) + c_f(\mathbf{w} + \boldsymbol{R}_f^1\eta; \mathbf{u} + \boldsymbol{R}_f^1\xi, \boldsymbol{R}_f^1\mu) \\ & + c_f(\mathbf{u} + \boldsymbol{R}_f^1\xi; \mathbf{u}_*, \boldsymbol{R}_f^1\mu) \\ & + c_f(\mathbf{u}_*; \mathbf{u} + \boldsymbol{R}_f^1\xi, \boldsymbol{R}_f^1\mu) + \int_\Gamma g(R_p\xi)\mu \,, \end{aligned}$$

whereas for any test function $\mathbf{v} \in V_f^0$,

$$\begin{aligned} \langle \mathcal{A}_f(\eta, \mathbf{w})(\xi, \mathbf{u}), \mathbf{v} \rangle \;=\;& a_f(\mathbf{u}, \mathbf{v}) + c_f(\mathbf{w} + \boldsymbol{R}_f^1\eta; \mathbf{u} + \boldsymbol{R}_f^1\xi, \mathbf{v}) \\ & + c_f(\mathbf{u}_*; \mathbf{u} + \boldsymbol{R}_f^1\xi, \mathbf{v}) + c_f(\mathbf{u} + \boldsymbol{R}_f^1\xi; \mathbf{u}_*, \mathbf{v}) \,. \end{aligned}$$

We indicate by $\tilde{a}$ the form associated to the operator $\mathcal{A}$:

$$\tilde{a}(\bar{w}; \bar{u}, \bar{v}) = \langle \mathcal{A}_0(\eta, \mathbf{w})(\xi, \mathbf{u}), \mu \rangle + \langle \mathcal{A}_f(\eta, \mathbf{w})(\xi, \mathbf{u}), \mathbf{v} \rangle \tag{59}$$

for all $\bar{w} = (\eta, \mathbf{w}), \bar{u} = (\xi, \mathbf{u}), \bar{v} = (\mu, \mathbf{v}) \in W$.
Next, we define two functionals $\ell_0 : \Lambda_0 \to \mathbb{R}$ and $\ell_f : V_f^0 \to \mathbb{R}$ as:

$$\begin{aligned} \langle \ell_0, \mu \rangle &= \int_{\Omega_f} \mathbf{f}\,(\boldsymbol{R}_f^1\mu) - a_f(\mathbf{u}_*, \boldsymbol{R}_f^1\mu) - c_f(\mathbf{u}_*; \mathbf{u}_*, \boldsymbol{R}_f^1\mu) \qquad && \forall \mu \in \Lambda_0 \,, \\ \langle \ell_f, \mathbf{v} \rangle &= -c_f(\mathbf{u}_*; \mathbf{u}_*, \mathbf{v}) && \forall \mathbf{v} \in V_f^0 \,, \end{aligned}$$

and denote

$$\langle \tilde{\ell}, \bar{v} \rangle = \langle \ell_0, \mu \rangle + \langle \ell_f, \mathbf{v} \rangle \qquad \forall \bar{v} = (\mu, \mathbf{v}) \in W \,. \tag{60}$$

Thus, the problem defined by (54) and (57) can be reformulated as:

$$\text{find } \bar{u} = (\lambda, \boldsymbol{\mathcal{R}}_0^1\lambda) \in W \;:\quad \tilde{a}(\bar{u}; \bar{u}, \bar{v}) = \langle \tilde{\ell}, \bar{v} \rangle \qquad \forall \bar{v} = (\mu, \mathbf{v}) \in W \,. \tag{61}$$

We shall prove the existence and uniqueness of the solution only in a closed convex subset of W.

Lemma 4.1 *Let* $\mathbf{f} \in L^2(\Omega_f)$ *be such that*

$$\frac{\|\mathbf{f}\|_0}{\nu^2} < C_{min} \frac{C_\kappa^2}{\hat{C}} \ , \tag{62}$$

where $0 < C_{min} < 1/16$ *is a suitable positive constant, depending only on* C_{Ω_f} *and* C_κ*, to guarantee that*

$$C_1^2 - \left(\frac{3\sqrt{2}}{4} + 1 \right) C_2 > 0 \ , \tag{63}$$

where

$$C_1 = \frac{C_\kappa \nu}{4\hat{C}} - \frac{4\|\mathbf{f}\|_0}{C_\kappa \nu} \ , \quad C_2 = \frac{\sqrt{2}}{\hat{C}} \left(C_{\Omega_f} \|\mathbf{f}\|_0 + \frac{4}{C_\kappa} \|\mathbf{f}\|_0 + \frac{4\hat{C}}{C_\kappa^2 \nu^2} \|\mathbf{f}\|_0^2 \right) \ . \tag{64}$$

Moreover, let

$$0 \leq r_m < r_M \leq C_1 \tag{65}$$

be two constants defined as

$$r_m = \frac{C_1 - \sqrt{C_1^2 - 2C_2}}{2} \quad \text{and} \quad r_M = C_1 - \sqrt{C_2/\sqrt{2}} \, . \tag{66}$$

If we consider

$$\overline{B}_r = \{ \bar{w} = (\eta, \mathbf{w}) \in W \ : \ |\boldsymbol{R}_f^1 \eta|_1 \leq r \} \ , \tag{67}$$

with

$$r_m < r < r_M \ , \tag{68}$$

then, there exists a unique solution $\bar{u} = (\lambda, \boldsymbol{\mathcal{R}}_0^1 \lambda) \in \overline{B}_r$ *to (61).*

Remark 4.1 *Since the constants* C_1 *and* C_2 *in (64) depend on* $\|\mathbf{f}\|_0$*, the condition (63) can be viewed as an inequality in* $\|\mathbf{f}\|_0$*. By simple algebraic calculations we can prove that there exists a constant,*

$$C_{min} = \frac{2 + (\frac{3}{2} + \sqrt{2})(C_{\Omega_f} + \frac{4}{C_\kappa}) - \sqrt{[2 + (\frac{3}{2} + \sqrt{2})(C_{\Omega_f} + \frac{4}{C_\kappa})]^2 - \frac{1}{4}(10 - 4\sqrt{2})}}{2(10 - 4\sqrt{2})} \tag{69}$$

such that any $\|\mathbf{f}\|_0$ *satisfying (62) is a solution of this inequality. Moreover, we get that* $0 < C_{min} < 1/16$*, so that* $C_1 > 0$*, and* r_m *and* r_M *in (66) satisfy (65).*

Proof. The proof is composed of several parts.

1. *For each* $\bar{w} = (\eta, \mathbf{w}) \in \overline{B}_r$ *the bilinear form* $\tilde{a}(\bar{w}; \cdot, \cdot)$ *is uniformly coercive on* W*.*

By definition, for all $\bar{v} = (\mu, \mathbf{v})$ we have

$$\begin{aligned} \tilde{a}(\bar{w}; \bar{v}, \bar{v}) \ = \ & a_f(\boldsymbol{R}_f^1 \mu, \boldsymbol{R}_f^1 \mu) + a_f(\mathbf{v}, \mathbf{v}) + \int_\Gamma g(R_p \mu) \mu \\ & + c_f(\mathbf{w} + \boldsymbol{R}_f^1 \eta; \mathbf{v} + \boldsymbol{R}_f^1 \mu, \mathbf{v} + \boldsymbol{R}_f^1 \mu) \\ & + c_f(\mathbf{v} + \boldsymbol{R}_f^1 \mu; \mathbf{u}_*, \mathbf{v} + \boldsymbol{R}_f^1 \mu) \\ & + c_f(\mathbf{u}_*; \mathbf{v} + \boldsymbol{R}_f^1 \mu, \mathbf{v} + \boldsymbol{R}_f^1 \mu) \ . \end{aligned}$$

Thanks to (35), we have $\int_\Gamma g(R_p\mu)\mu \geq 0$. Using the inequalities (46) and (47), the estimate (38) and the fact that $\mathbf{w} \in V_f^0$, we obtain

$$\begin{aligned}
\tilde{a}(\bar{w};\bar{v},\bar{v}) \;\; &\geq \;\; \frac{C_k\nu}{2}(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) - 2\hat{C}|\boldsymbol{R}_f^1\eta|_1(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) \\
&\quad -4\hat{C}|\mathbf{u}_*|_1(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) \\
&\geq \;\; \frac{C_k\nu}{2}(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) - 2\hat{C}|\boldsymbol{R}_f^1\eta|_1(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) \\
&\quad -4\hat{C}\frac{2}{C_k\nu}\|\mathbf{f}\|_0(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) \; .
\end{aligned}$$

Thus,

$$\tilde{a}(\bar{w};\bar{v},\bar{v}) \geq \alpha_{\tilde{a}}(|\boldsymbol{R}_f^1\mu|_1^2 + |\mathbf{v}|_1^2) \; , \tag{70}$$

having set

$$\alpha_{\tilde{a}} = \frac{C_k\nu}{2} - 2\hat{C}|\boldsymbol{R}_f^1\eta|_1 - \frac{8\hat{C}}{C_k\nu}\|\mathbf{f}\|_0 \; . \tag{71}$$

Condition $\alpha_{\tilde{a}} > 0$ is equivalent to $|\boldsymbol{R}_f^1\eta|_1 < C_1$, which is satisfied in view of (65) and (68). Thus, the bilinear form $\tilde{a}(\bar{w};\cdot,\cdot)$ is uniformly coercive with respect to any $\bar{w} \in \overline{B}_r$.
Thanks to the Lax-Milgram Lemma (see, e.g., [17] p. 133) the operator $\mathcal{A}(\bar{w}) \in \mathcal{L}(W;W')$ is invertible for each $\bar{w} \in \overline{B}_r$. Moreover, the inverse $\mathcal{T}(\bar{w}) = (\mathcal{A}(\bar{w}))^{-1}$ belongs to $\mathcal{L}(W';W)$ and it satisfies

$$\|\mathcal{T}(\bar{w})\|_{\mathcal{L}(W';W)} \leq \frac{1}{\alpha_{\tilde{a}}} \; .$$

Now, we prove that there exists a unique $\bar{u} \in \overline{B}_r$ such that $\bar{u} = \mathcal{T}(\bar{u})\tilde{\ell}$, i.e., (61) has a unique solution in B_r.

2. $\bar{v} \to \mathcal{T}(\bar{v})\tilde{\ell}$ *maps* $\overline{B}_r$ *into* $\overline{B}_r$ *and is a strict contraction in* $\overline{B}_r$.
For all $\bar{v} = (\mu, \mathbf{v}) \in \overline{B}_r$ we have

$$\|\mathcal{T}(\bar{v})\tilde{\ell}\|_W \leq \|\mathcal{T}(\bar{v})\|_{\mathcal{L}(W';W)}||\tilde{\ell}||_{W'} \leq \frac{||\tilde{\ell}||_{W'}}{\alpha_{\tilde{a}}} \; .$$

Moreover,

$$\begin{aligned}
||\tilde{\ell}||_{W'} \;\; &= \;\; \sup_{\bar{v}\in W,\bar{v}\neq 0} \frac{\left|\displaystyle\int_{\Omega_f} \mathbf{f}\,(\boldsymbol{R}_f^1\mu) - a_f(\mathbf{u}_*, \boldsymbol{R}_f^1\mu) - c_f(\mathbf{u}_*;\mathbf{u}_*, \mathbf{v} + \boldsymbol{R}_f^1\mu)\right|}{\|\bar{v}\|_W} \\
&\leq \;\; \sup_{\bar{v}\in W,\bar{v}\neq 0} \frac{C_{\Omega_f}\|\mathbf{f}\|_0|\boldsymbol{R}_f^1\mu|_1 + 2\nu|\mathbf{u}_*|_1|\boldsymbol{R}_f^1\mu|_1 + \hat{C}|\mathbf{u}_*|_1^2|\mathbf{v} + \boldsymbol{R}_f^1\mu|_1}{\|\bar{v}\|_W} \\
&\leq \;\; \sup_{\bar{v}\in W,\bar{v}\neq 0} \frac{\left(C_{\Omega_f}\|\mathbf{f}\|_0 + 2\nu|\mathbf{u}_*|_1 + \hat{C}|\mathbf{u}_*|_1^2\right)(|\boldsymbol{R}_f^1\mu|_1 + |\mathbf{v}|_1)}{\|\bar{v}\|_W} \\
&\leq \;\; \sqrt{2}\left(C_{\Omega_f}\|\mathbf{f}\|_0 + 2\nu|\mathbf{u}_*|_1 + \hat{C}|\mathbf{u}_*|_1^2\right) \\
&\leq \;\; \sqrt{2}\left(C_{\Omega_f}\|\mathbf{f}\|_0 + \frac{4}{C_\kappa}\|\mathbf{f}\|_0 + \frac{4\hat{C}}{C_\kappa^2\nu^2}\|\mathbf{f}\|_0^2\right) \; ,
\end{aligned} \tag{72}$$

the last inequality following from (38). From (72) and (71), corresponding to some $\bar{w} = (\eta, \mathbf{w}) \in \overline{B}_r$, condition

$$\frac{||\tilde{\ell}||_{W'}}{\alpha_{\tilde{a}}} \leq r$$

is equivalent to

$$2r^2 - 2C_1 r + C_2 \leq 0\,,$$

that is $r_{min} \leq r \leq r_{max}$ with

$$r_{min} = \frac{C_1 - \sqrt{C_1^2 - 2C_2}}{2} \qquad \text{and} \qquad r_{max} = \frac{C_1 + \sqrt{C_1^2 - 2C_2}}{2}\,.$$

Since $r_m = r_{min} < r_M \leq r_{max}$, it follows that for any $\bar{v} \in \overline{B}_r$ with r satisfying (68), $\mathcal{T}(\bar{v})\tilde{\ell}$ belongs to $\overline{B}_r$.
Finally, to prove that the map $\bar{v} \to \mathcal{T}(\bar{v})\tilde{\ell}$ is a strict contraction in $\overline{B}_r$, we should guarantee (see [11] p. 282) that for any $\bar{w}_1, \bar{w}_2 \in \overline{B}_r$

$$\|(\mathcal{T}(\bar{w}_1) - \mathcal{T}(\bar{w}_2))\tilde{\ell}\|_W \leq \frac{||\tilde{\ell}||_{W'}}{\alpha_{\tilde{a}}^2} L(r)\|\bar{w}_1 - \bar{w}_2\|_W < \|\bar{w}_1 - \bar{w}_2\|_W\,, \tag{73}$$

$L(r)$ being the Lipschitz continuity constant associated to $\mathcal{A}$. However,

$$\begin{aligned} |\langle(\mathcal{A}(\bar{w}_1) - \mathcal{A}(\bar{w}_2))(\bar{u}), \bar{v}\rangle| &= |\tilde{a}(\bar{w}_1; \bar{u}, \bar{v}) - \tilde{a}(\bar{w}_2; \bar{u}, \bar{v})| \\ &= |c_f(\mathbf{w}_1 + \boldsymbol{R}_f^1\eta_1 - (\mathbf{w}_2 + \boldsymbol{R}_f^1\eta_2); \mathbf{u} + \boldsymbol{R}_f^1\lambda, \mathbf{v} + \boldsymbol{R}_f^1\mu)| \\ &\leq \hat{C}|\mathbf{w}_1 + \boldsymbol{R}_f^1\eta_1 - \mathbf{w}_2 - \boldsymbol{R}_f^1\eta_2|_1\ |\mathbf{u} + \boldsymbol{R}_f^1\lambda|_1\ |\mathbf{v} + \boldsymbol{R}_f^1\mu|_1 \\ &\leq 2\sqrt{2}\hat{C}\|\bar{w}_1 - \bar{w}_2\|_W\|\bar{u}\|_W\|\bar{v}\|_W\,, \end{aligned}$$

so that $L(r) = 2\sqrt{2}\hat{C}$. Thus, condition

$$\frac{||\tilde{\ell}||_{W'}}{\alpha_{\tilde{a}}^2} L(r) < 1$$

is equivalent to

$$r^2 - 2C_1 r + C_1^2 - \frac{C_2}{\sqrt{2}} > 0$$

i.e.,

$$r < r_{MIN} = C_1 - \sqrt{C_2/\sqrt{2}} \qquad \text{or} \qquad r > r_{MAX} = C_1 + \sqrt{C_2/\sqrt{2}}\,.$$

Condition $r > r_{MAX}$ does not fit with the previous restrictions on r. However, since $r_M = r_{MIN}$, (73) is satisfied for any r in the interval (68).
3. The existence and uniqueness of the solution $\bar{u} = (\lambda, \boldsymbol{\mathcal{R}}_0^1\lambda) \in \overline{B}_r$ to (61) is now a simple consequence of the Banach contraction theorem (see, e.g., [19]).
□

The following theorem is a direct consequence of the previous lemma.

Theorem 4.2 *If (62) holds with C_{min} given in (69), and r_m and r_M defined in (66), then problem (61) has a unique solution $\bar{u} = (\lambda, \boldsymbol{\mathcal{R}}_0^1\lambda)$ in the set*

$$B_{r_M} = \{\bar{w} = (\eta, \mathbf{w}) \in W\ :\ |\boldsymbol{R}_f^1\eta|_1 < r_M\}\,,$$

and it satisfies $|\boldsymbol{R}_f^1\lambda|_1 \leq r_m$. *In particular, it follows that (54) has a unique solution* λ *in the set* $S_{r_M} \subset \Lambda_0$,

$$S_{r_M} = \{\eta \in \Lambda_0 \, : \, |\boldsymbol{R}_f^1\eta|_1 < r_M\} \, ,$$

and it satisfies $|\boldsymbol{R}_f^1\lambda|_1 \leq r_m$.

Proof. Since problem (54) has a solution λ if and only if $\bar{u} = (\lambda, \boldsymbol{\mathcal{R}}_0^1\lambda)$ is a solution of problem (61), we prove only the first part of theorem.
From the previous Lemma 4.1, if (62) holds, with C_{min} given in (69), (61) has at least a solution in B_{r_M} as it has a solution in $\overline{B}_r \subset B_{r_M}$, for any $r_m < r < r_M$. To prove the uniqueness, let us assume that (61) has two solutions $\bar{u}_1 = (\lambda_1, (\boldsymbol{\mathcal{R}}_0^1\lambda)_1) \neq \bar{u}_2 = (\lambda_2, (\boldsymbol{\mathcal{R}}_0^1\lambda)_2)$ in B_{r_M}. Then, $r_1 = |\boldsymbol{R}_f^1\lambda_1|_1 < r_M$ and $r_2 = |\boldsymbol{R}_f^1\lambda_2|_1 < r_M$. Therefore, any set $\overline{B}_r$ with $\max\{r_m, r_1, r_2\} < r < r_M$ contains two different solutions of problem (61). This contradicts the result of Lemma 4.1. Now, let $\bar{u} = (\lambda, \boldsymbol{\mathcal{R}}_0^1\lambda)$ be the unique solution of problem (61) in B_{r_M}. According to Lemma 4.1, it belongs to each $\overline{B}_r \subset B_{r_M}$ with $r_m < r < r_M$, and consequently $|\boldsymbol{R}_f^1\lambda|_1 \leq r_m$. □

Remark 4.2 *Notice that condition (62) is analogous to what is usually required to prove existence and uniqueness of the solution of the Navier-Stokes equations. Moreover, we have proved that the solution is unique in* S_{r_M}. *Thus, in view of Remark 3.1, Theorem 4.2 states that the solution is unique only for sufficiently small normal velocities* λ *across the interface* Γ. *Finally, notice that (62) implies (48) and that* S_{r_M} *is included in the set (49), so that the existence and uniqueness of the nonlinear extension* $\boldsymbol{\mathcal{R}}_0^1\lambda$ *is ensured as well.*

The reformulation of the coupled problem as an interface equation is interesting in view of setting up iterative substructuring methods to compute the solution of the global problem. In particular, we can split the operator $\mathcal{S}$ into a nonlinear part, say $\mathcal{S}_f$, associated to the Navier-Stokes problem in Ω_f, and a linear part S_p related to the Darcy equation in Ω_p, and exploit them to set up iterative methods inspired to the Dirichlet-Neumann and Neumann-Neumann schemes in domain decomposition (see, e.g., [18]). This study will be the object of a forthcoming work [2].

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