ÖAW AUSTRIAN ACADEMY OF SCIENCES

Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

RICAM JOHANN-RADON-INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

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RICAM-Report 2007-01

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January 19, 2007

Abstract

Recently, integral invariants and according signatures have been identified to be useful for shape classification, which is an important research topic in computer vision, artificial intelligence and pattern recognition. The modelling of integral invariants and signatures for shape analysis and in particular the analysis have not attracted attention in the inverse problems community so far. This paper is to point out a novel research area in inverse problems. For that purpose we provide an “inverse problems point of view” of integral invariants and signatures and highlight some fundamental mathematical perspectives.

Keywords: signatures, integral invariants

1 Introduction

For *shape matching* and *classification*, an object, given by its shape, is compared with representatives of classes within a data base. Similar problems are addressed in many areas of applied sciences such as *computer vision*, *artificial intelligence* and *pattern recognition*.

The problems are tackled by a-priori assigning each object class within a data base one or more typical representatives that capture the dominant features of the class. Then, the particular object under investigation is compared with the representative shapes using an appropriate notion of similarity. Common distance measures, such as the Hausdorff distance, are not appropriate, since they do not take into account the significance of dominant features.

We consider descriptors of shapes which emphasize on peaks, edges or ridges. These features can be expressed by differentials of the shape boundary, which are invariant under rigid motions. Differentials have been used successfully for shape matching and classification, but are difficult to handle numerically, since they are unstable with respect to noise. Alternatively, *integral invariants* descriptors have been proposed by Manay et al. [6]. In comparison with differential invariants, a significant advantage of integral invariants is that the integration kernel can be adapted to capture either global or local features. Consequently, they can be used to distinguish object features on different scales.

Although integral invariants have proven to be successful in practical applications for object classification and shape matching, from a point of view of inverse problems, important issues have not been addressed so far. This paper is to point out some of the open questions in inverse problems theory, to highlight the relation to other inverse problems, as well as to introduce this novel area to the inverse problems community.

The first question – from an inverse problems point of view probably the most important one – is the theoretical possibility of reconstructing a shape from its integral invariant. A positive answer implies that an integral invariant uniquely determines a shape, and thus can be considered as a token. This question is even more involved when only partial data of integral invariants of an object are available; such a handicap can be observed if the object is partially occluded during recording or the data of integral invariants has been damaged. The second question concerns stability and the effect of noise on the shape matching and classification process. For this purpose appropriate distance measures of integral invariants have to be determined.

The outline of this paper is as follows. In Section 2 we introduce integral invariants and further two particular examples of integral invariants in details. In Section 3 we introduce shape signatures derived from integral invariants, which are considered to be useful for object classification. In the last part (Section 4) of this paper we present mathematical formulations of inverse problems issues related to integral invariants and signatures. Some solutions and solution concepts are presented in Section 5.

2 Integral Invariants

In this section we introduce a mathematical framework of integral invariants. For the sake of simplicity of presentation, we restrict attention to integral invariants of two dimensional objects, although most of the definitions and results for integral invariants for higher dimensional objects can be derived analogously. Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain with finite perimeter. We assume that the boundary can be parameterized by a continuous and injective curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$. By $\bar{\gamma}$ we denote the barycenter of the area enclosed by γ. The two dimensional Lebesgue measure is denoted by $\mathcal{L}^2$.

We define integral invariants as follows:
Definition 2.1:
Let $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$ such that for every $r \geq 0$ the function $f(r, \cdot)$ is locally integrable. Moreover, assume that for every compact set $K \subset \mathbb{R}^2$ and $r_0 \geq 0$

$$\lim_{r \to r_0} \int_K |f(r, \mathbf{x}) - f(r_0, \mathbf{x})| \, d\mathcal{L}^2(\mathbf{x}) = 0. \tag{1}$$

We define $I[\gamma] : \mathbb{S}^1 \to \mathbb{R}$ by

$$I[\gamma](t) := \int_{R_{(\gamma(t)-\bar{\gamma})}(\Omega-\bar{\gamma})} f(\|\gamma(t)-\bar{\gamma}\|, \mathbf{x})\, d\mathcal{L}^2(\mathbf{x}), \tag{2}$$

where $R_{\mathbf{v}} \in \mathrm{SO}(2)$ denotes the rotation satisfying

$$R_{\mathbf{v}}\mathbf{v} = \|\mathbf{v}\|\mathbf{e_1}.$$

The function $I[\gamma]$ is called *integral invariant* of the curve γ, and f the *kernel function* of the invariant.

To fix the notation we have selected the reference point as the barycenter $\bar{\gamma}$, which guarantees that $I[\gamma]$ is invariant with respect to rigid motions. In what follows, for any other choice of the reference point which guarantees invariance with respect to rigid motions, the results can be derived analogously. For instance the barycenter of the convex hull of the curve is an equally suitable choice. We also note that (1) implies continuity of the function $I[\gamma](\cdot)$.

Examples:

1. Let $r > 0$ and define $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(s, \mathbf{x}) := \chi_{B_r(s\cdot\mathbf{e_1})}(\mathbf{x}). \tag{3}$$

   With this kernel function the corresponding integral invariant can be written as

   $$I^r_{\mathrm{Circle}}[\gamma](t) := \mathcal{L}^2\big(\Omega \cap B_r\big(\gamma(t)\big)\big) = \chi_\Omega * \chi_{B_r(0)}\big(\gamma(t)\big). \tag{4}$$

   The integral invariant I^r_{Circle} is called *circular area integral invariant* (see Figure 1).

   It has been shown in [4] that for a two times differentiable curve γ the circular area integral invariant $I^r_{\mathrm{Circle}}[\gamma]$ and the curvature κ of γ are related as follows:

   $$I^r_{\mathrm{Circle}}[\gamma](t) = \frac{\pi}{2}r^2 - \frac{\kappa_\gamma(t)}{3}r^3 + O\big(r^4\big). \tag{5}$$

   In general, if γ is not twice differentiable, then we have

   $$I^r_{\mathrm{Circle}}[\gamma](t) = \frac{\delta}{2}r^2 - \frac{\kappa^-_\gamma(t) + \kappa^+_\gamma(t)}{6}r^3 + O\big(r^4\big), \tag{6}$$

   where δ denotes the aperture of the circular sector centered at the nonsmooth point of the curve, and κ^-_γ, κ^+_γ denote the curvature to the right and left, respectively (see [7]).

2. Let $\varepsilon > 0$ and Ω be star-shaped with respect to its barycenter $\bar{\gamma}$. We consider $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$f(s, \mathbf{x}) := \chi_{[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]}\left(\arccos\left(\frac{\langle \mathbf{e_1}, \mathbf{x}\rangle}{\|\mathbf{x}\|}\right)\right), \tag{7}$$

   which is independent of s and equals the characteristic function of a cone with aperture ε and rotation axis $\mathbf{e_1}$. We call the corresponding integral invariant $I^\varepsilon_{\mathrm{Cone}}$ *cone area integral invariant* (see Figure 1).

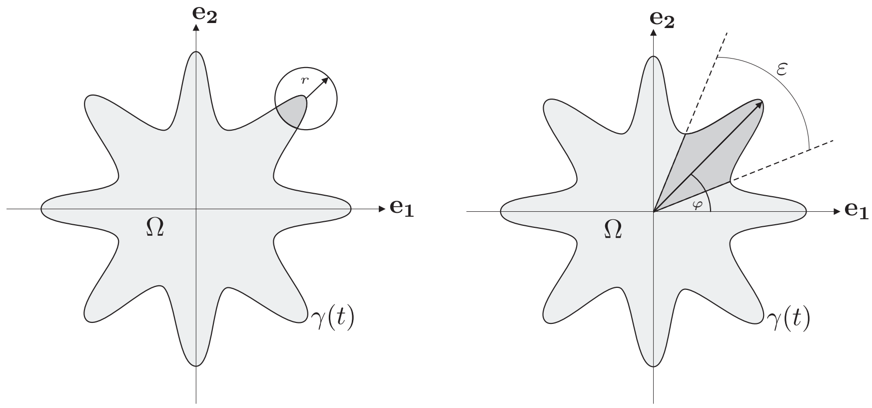

Figure 1: **(Visualization of different integral invariants)** *Left*: Circle area integral invariant. For each point located on the curve γ the area of intersection of Ω with a ball of radius r is measured. *Right*: Cone area integral invariant. For each value of the angle φ the area of intersection of a cone with aperture ε and the domain Ω is measured.

In the definitions of the circular and cone area integral invariants, the parameters r and ε, respectively, control the sensitivity of each invariant with respect to local variations in γ. Large parameters guarantee that the invariants are less sensitive to local variations (cf. Table 1). Moreover, from this table it becomes evident that $I^{\varepsilon}_{\text{Cone}}$ is less sensitive than I^{r}_{Circle} to local variations of the curve. The higher stability of $I^{\varepsilon}_{\text{Cone}}$ results from the fact that the barycenter is less affected by small changes of the curve than the center of the ball of integration, both being the reference point for the corresponding integral invariant. Thus, from a point of view of stability the cone area integral invariant is preferable; the use of $I^{\varepsilon}_{\text{Cone}}$, however, is restricted to star-shaped domains. Furthermore, in the case of incomplete data it is impossible to determine the barycenter of the domain, and then the cone area integral invariant turns out to be useless.

3 Shape Signatures

Integral invariants are invariant with respect to rotations and translations but suffer from the dependence of the parameterization of the curve γ. Without a specification of the parameterization the integral invariant is not suitable for shape classification. In particular, a different choice of the starting point of the parameterization results in a shift of the integral invariant $I[\gamma]$, which is undesirable for shape classification and matching. To get rid of the effects of reparameterization, Manay et al. [6] defined *signatures* based on integral invariants, which are curves, where $I[\gamma]$ is plotted against its derivative $I[\gamma]'$. Thus, they followed the standard approach for defining differential signatures (cf. [2])

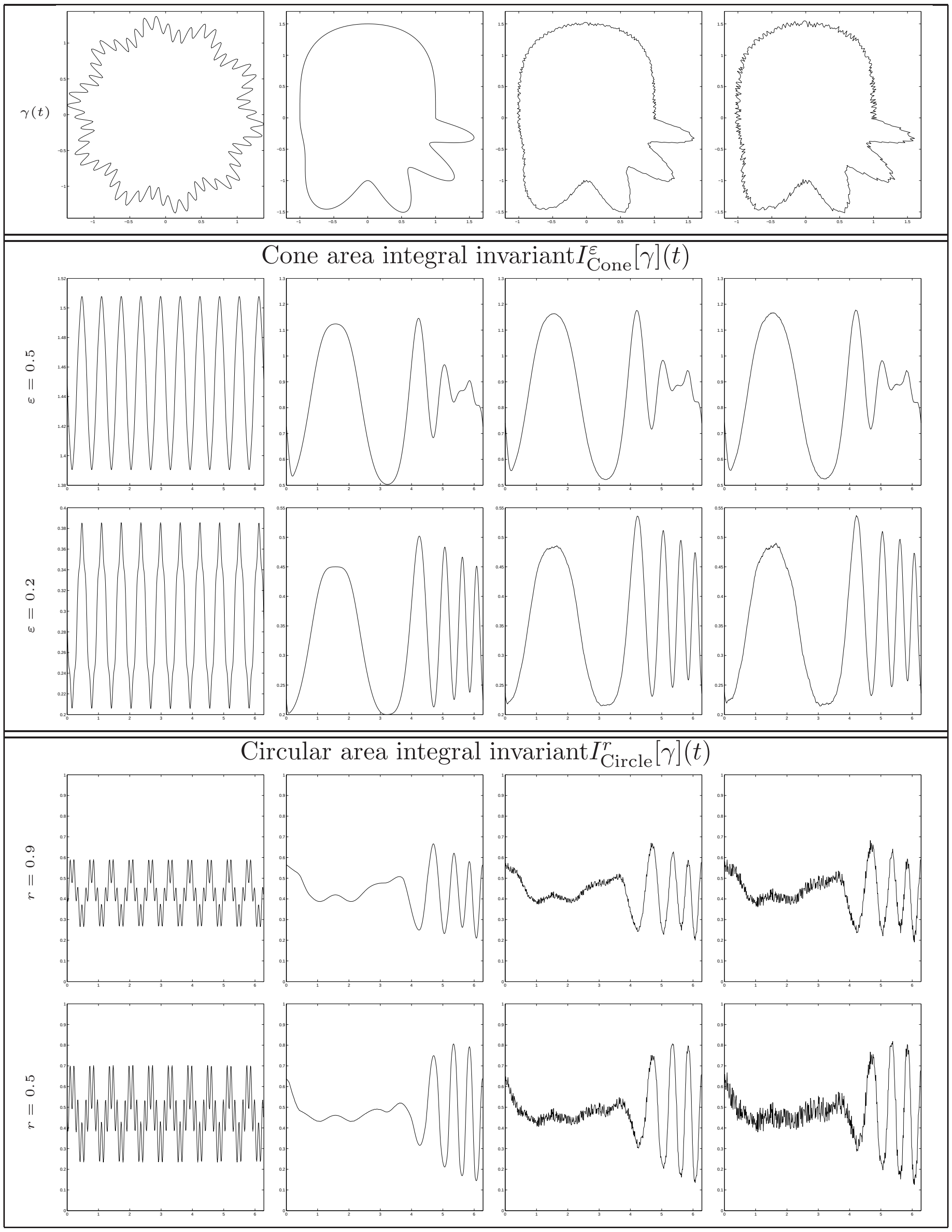

Table 1: **(Circular versus cone area integral invariant)** *Top row:* Four test curves. The curve in the second image is perturbed with $\pm 2.5\%$ (third image) and $\pm 5\%$ uniformly distributed noise (fourth image). *Middle row:* Cone area integral invariant $I^{\varepsilon}_{\text{Cone}}[\gamma]$ for the four different test curves and two values of ε. *Bottom row:* Circular area integral invariant $I^{r}_{\text{Circle}}[\gamma]$ for different values of r.

where the curvature κ is plotted against κ'. In these definitions the derivatives of integral and differential invariants, respectively are required, which have to be provided numerically. Since the stable numerical computation of derivatives is quite demanding we introduce a slightly different concept of signatures.
DEFINITION 3.1:
A *signature* of a curve γ is the set

$$\left\{ \left(I_1[\gamma](t), I_2[\gamma](t) \right) : t \in \mathbb{S}^1 \right\}, \tag{8}$$

where $I_1[\gamma]$, $I_2[\gamma]$ denote two arbitrary integral invariants.

The signature does not depend on the parameterization of the curve γ. As an advantage to previous approaches no numerical derivatives are required. Therefore, the approach is expected to be more robust with respect to curve perturbations than the original definition given by Manay.

From the invariants introduced in Section 2 we can generate two different classes of signatures of an object (cf. Table 2 and 3):

1. We may use just one class of invariants, but with different parameters, such as

$$\begin{aligned} \left\{ \left(I^{r_1}_{\text{Circle}}[\gamma](t), I^{r_2}_{\text{Circle}}[\gamma](t) \right) : t \in \mathbb{S}^1 \right\}, &\quad r_1, r_2 > 0,\ r_1 \neq r_2, \\ \left\{ \left(I^{\varepsilon_1}_{\text{Cone}}[\gamma](t), I^{\varepsilon_2}_{\text{Cone}}[\gamma](t) \right) : t \in \mathbb{S}^1 \right\}, &\quad \varepsilon_1, \varepsilon_2 > 0,\ \varepsilon_1 \neq \varepsilon_2, \end{aligned} \tag{9}$$

   respectively.

2. We may combine cone and circular invariants

$$\left\{ \left(I^{r}_{\text{Circle}}[\gamma](t), I^{\varepsilon}_{\text{Cone}}[\gamma](t) \right) : t \in \mathbb{S}^1 \right\}, \qquad r, \varepsilon > 0, \tag{10}$$

   or any two different classes of integral invariants.

Both types of signatures form closed curves that may contain self-intersections or segments that are passed through several times.

4 An Inverse Problems Point of View on Invariants and Shape Signatures

From an inverse problems point of view, in classification applications, questions to be addressed concern the identifiability of a shape or some properties of the integral invariant or signature, respectively.

The integral invariant as defined in (2) can be regarded as a mapping $I : C^0(\mathbb{S}^1) \to C^0(\mathbb{S}^1)$, where $C^0(\mathbb{S}^1)$ denotes the set of all continuous functions on $\mathbb{S}^1$ taking values in $\mathbb{R}^2$. Injectivity of this mapping can only be achieved up to some restrictions, which are highlighted in the following.

1. Since I is invariant with respect to translations and rotations of the curve, it is evident that I is only injective up to rigid motions.

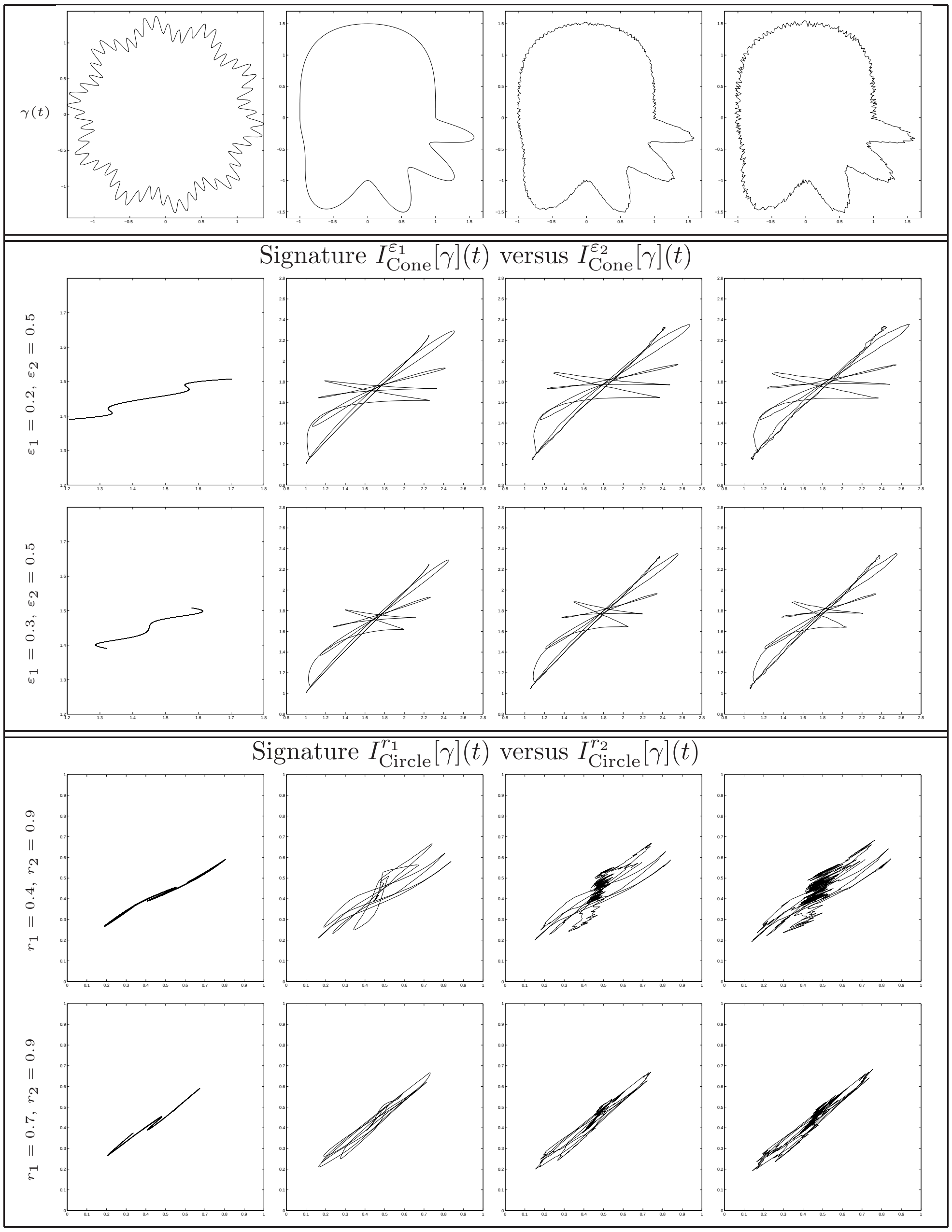

Table 2: **(First class of signatures)** *Top row:* Two different curves $\gamma(t)$ without noise, the second curve with $\pm 2.5\%$ and $\pm 5\%$ uniformly distributed noise added. *Middle row:* Corresponding signatures using the cone area integral invariant $I^{\varepsilon}_{\text{Cone}}[\gamma]$ for two different combinations of apertures ε_1 and ε_2. *Bottom row:* Corresponding signatures using the circular area integral invariant $I^{r}_{\text{Circle}}[\gamma]$ for two different combinations of radii r_1 and r_2.

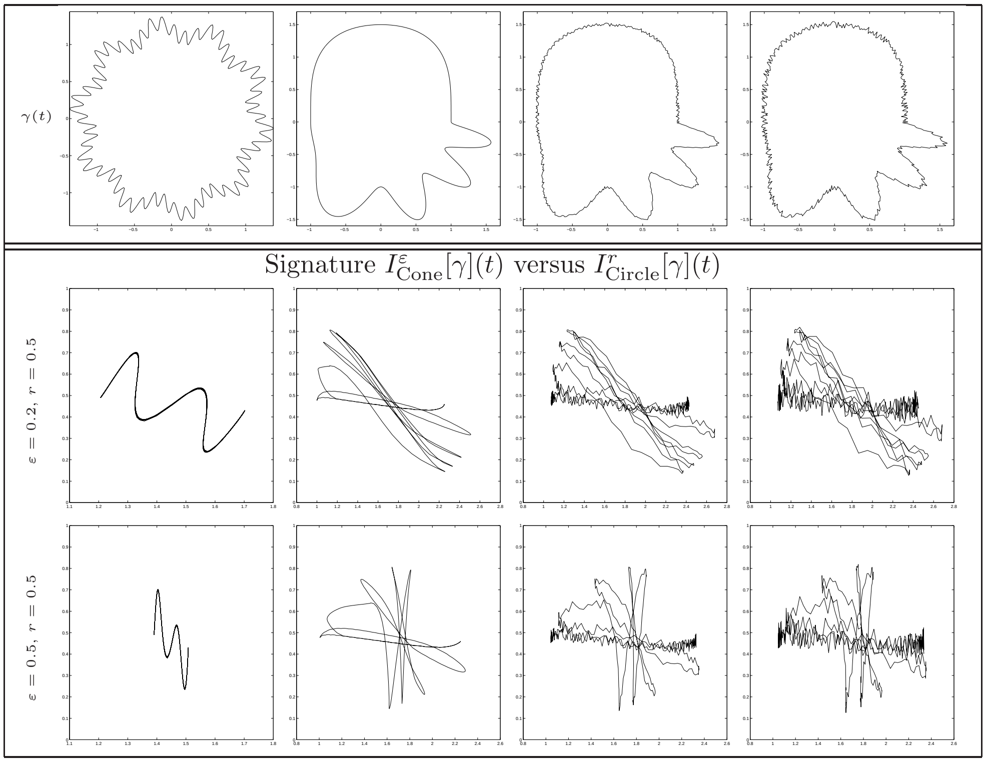

Table 3: **(Second class of signatures)** *Top row:* Two different curves $\gamma(t)$ without noise, the second curve with $\pm 2.5\%$ and $\pm 5\%$ uniformly distributed noise added. *Bottom row:* Corresponding signatures using the cone area integral invariant $I^{\varepsilon}_{\text{Cone}}[\gamma]$ and the circular area integral invariant $I^{r}_{\text{Circle}}[\gamma]$ for two different combinations of ε and r.

2. The choice of the parameters defining the integral invariants affects the possibility of differing between shapes. For instance, if $\varepsilon = 2\pi$, the function $I^{\varepsilon}_{\text{Cone}}$ is constant with function value the total area of Ω. Therefore, in this situation the functional can only be used to discriminate between objects of different size. Similarly, if $r \geq \operatorname{diam} \Omega$, then I^{r}_{Circle} equals the constant curve $t \mapsto \mathcal{L}^2(\Omega)$.

3. In applications, a difficulty for shape classification by integral invariants is the dependence on the choice of the parameterization of the curve. In fact, assume that $J : \mathbb{S}^1 \to \mathbb{R}$ is topologically equivalent to the invariant $I[\gamma]$, that is, there exists a homeomorphism η of $\mathbb{S}^1$ such that $J \circ \eta = I[\gamma]$. Then $I[\gamma] = I[\gamma \circ \eta^{-1}] \circ \eta$. Therefore, $J = I[\gamma \circ \eta^{-1}]$, is the integral invariant of the reparameterized curve $\gamma \circ \eta^{-1}$. Consequently, we may only obtain injectivity results if the parameterization of the boundary of Ω is specified, e.g. by prescribing $\gamma(0)$ and $\dot{\gamma}(0)$. A natural choice for parameterizing an arbitrary domain Ω is the arclength parameterization and for a starshaped domain an angular parameterization with respect to the barycenter.

Taking into account the considerations above, we formulate some inverse problems related to shape classification with integral invariants.

Problem 4.1:
Denote by $C^0_{\text{arc}}(\mathbb{S}^1)$ the set of all arclength parameterized closed curves γ in $\mathbb{R}^2$ satisfying $\gamma(0) = 0$ and $\dot\gamma(0) = \mathbf{e_1}$, and by

$$C^0_{\text{angle}}(\mathbb{S}^1) := \left\{\gamma \in C^0(\mathbb{S}^1) : \gamma(t) = \|\gamma(t)\|(\cos t, \sin t),\ \bar\gamma = 0\right\} \qquad (11)$$

the set of all angular parameterized closed curves with barycenter $\bar\gamma = 0$.

- Let $0 < \varepsilon < 2\pi$. Is the function $I^{\varepsilon}_{\text{Cone}}$ restricted to $C^0_{\text{angle}}(\mathbb{S}^1)$ injective?
- Let $r > 0$. Denote by $C^0_r(\mathbb{S}^1)$ the set of all curves γ in $\mathbb{R}^2$ satisfying

  $$\max\left\{\|\gamma(t) - \gamma(s)\| : t, s \in \mathbb{S}^1\right\} > r. \qquad (12)$$

  Is the function I^r_{Circle} restricted to $C^0_{\text{arc}}(\mathbb{S}^1) \cap C^0_r(\mathbb{S}^1)$ injective?
- Is it possible to reconstruct a curve from its integral invariant numerically in a stable way?
- Is it possible to identify features from its invariant? If it is possible, how can this additional information be used to tackle the problem of incomplete data? If applicable, can an algorithm be derived that partially recovers the object of interest?

Discussing the injectivity of the mapping, which assigns to each Jordan curve its signature is more complicated. Signatures have been introduced in Definition 3.1 as mere point sets to derive a shape characterization which is independent of the parameterization. In most applications, though, at least some information on the parameterization is still present, namely the direction in which this point set is passed through and the number of times one specific point in the signature is attained. This can be modelled by assuming that we only know the mapping $t \mapsto \big(I_1[\gamma](t), I_2[\gamma](t)\big)$ up to a reparameterization. Therefore, we define the quotient space

$$C^0_T(\mathbb{S}^1) := C^0(\mathbb{S}^1)/\!\sim, \qquad (13)$$

where two mappings β_1, β_2 are equivalent, if there exists a homeomorphism η of $\mathbb{S}^1$ such that $\beta_1 = \beta_2 \circ \eta$. For given integral invariants I_1 and I_2, and a curve γ we can thus define

$$\Sigma([\gamma]) := \big[\big(I_1[\gamma], I_2[\gamma]\big)\big], \qquad (14)$$

where $[\gamma]$ and $\big[\big(I_1[\gamma], I_2[\gamma]\big)\big]$ denote the equivalence classes in $C^0_T(\mathbb{S}^1)$ of the curves γ and $(I_1[\gamma], I_2[\gamma]) \in C^0(\mathbb{S}^1)$.

Problem 4.2:
Let I_1, I_2 be integral invariants.

- For which choices of I_1, I_2 and subsets $X \subset C^0_T(\mathbb{S}^1)$ of admissible curves is the mapping $\Sigma : X \to C^0_T(\mathbb{S}^1)$ injective?
- Does there exist a numerical algorithm for reconstructing a curve (up to reparameterization) from its signature?
- Similar to the case of integral invariants: Is it possible to identify features in the signature which are correlated to features of the equivalence class of a curve? If there exists a correlation, how can this be taken into account for identifying curves from incomplete signatures?

As a prerequisite step for integral invariants calculation, the shape has to be extracted from data which contains the object of interest. By this extraction perturbations and noise are introduced in the shape. To measure the influence of data errors in the shape representation of integral invariants and signatures, appropriate distance measures for objects and the integral invariants have to be introduced.

Problem 4.3:
Let γ_1, γ_2 be Jordan curves.

- What is a reasonable distance measure between two curves, for instance a noisy, perturbed curve and a representative of a class in a data base?
- Does there exist a continuous dependence of the invariants and signatures from the curves?

There is also a variety of integral invariants for surfaces and a number of applications where these have proven to be very useful (see [7]). Thus, the concepts which we have presented here only for planar curves should be extended to surfaces in spaces of dimension $n \geq 3$.

5 Relation to the Literature on Inverse Problems and Partial Solutions

It is shown in [3] that $I^{\varepsilon}_{\text{Cone}}$ is injective on $C^0_{\text{angle}}(\mathbb{S}^1)$ if and only if $\varepsilon/2\pi$ is irrational. Therefore we expect that the signatures corresponding to cone integral invariants $I^{\varepsilon_1}_{\text{Cone}}$, $I^{\varepsilon_2}_{\text{Cone}}$ are unique, if $\varepsilon_1 \neq \varepsilon_2$ and $\varepsilon_i/2\pi$ irrational for $i = 1, 2$. However, a rigorous proof is missing so far.

From the results in [4] (see also (5)) it follows that the circular invariant I^r_{Circle} approximates the curvature of γ as $r \to 0$. Since the curvature uniquely determines an arclength parameterized curve up to rigid motions, it follows that I^r_{Circle} is injective in the limit as $r \to 0$. In [3] we have used the Landweber iteration method for reconstructing the original curve $\gamma \in C^0_{\text{arc}}(\mathbb{S}^1) \cap C^0_r(\mathbb{S}^1)$ from its circular area integral invariant I^r_{Circle} for $r > 0$. Numerical experiments have shown that injectivity of that function for positive fixed radii can hold, but the proof is still missing.

There exists a relation between the circular integral invariant and the spherical Radon transform $\mathcal{R}[f] : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, which is defined by

$$\mathcal{R}[f](\mathbf{x}, r) := r^{-1} \int_{\mathbb{S}^1} f(\mathbf{x} + r\boldsymbol{\sigma})\, d\mathcal{L}^1(\boldsymbol{\sigma}) \tag{15}$$

for a function $f : \mathbb{R}^2 \to \mathbb{R}$. Indeed, we have

$$I^r_{\text{Circle}}[\gamma](t) = \int_0^r \mathcal{R}[\chi_\Omega]\big(\gamma(t), s\big)\, d\mathcal{L}^1(s). \tag{16}$$

For the spherical Radon transform there exist injectivity results, which show that for certain subsets $S \subset \mathbb{R}^2$ it is possible to reconstruct a function f with compact support from its spherical Radon transform restricted to $S \times \mathbb{R}_{\geq 0}$

(cf. [1]). In other words, it is enough to know $\mathcal{R}[f]$ for a relatively small subset S of $\mathbb{R}^2$ but all radii r, in order to be able to identify f. In [8], identifyability of f from data $\{\mathcal{R}[f](x,r) : x \in \mathbb{R}^2, r_1 < r < r_2\}$ for appropriate $r_1, r_2 \in \mathbb{R}_{\geq 0}$ has been shown.

Considering (16) the circular area integral invariant I^r_{Circle} represents the average of the spherical Radon transform over an interval $[0, r]$ evaluated at $\mathbf{x} \in \partial\Omega$. Thus, a lot less information is given than required for the two injectivity results mentioned above. However, we additionally know, that the function to be reconstructed is a characteristic function. This information is not taken into account in the standard references on injectivity of the spherical Radon transform [1].

Acknowledgement

This work has been supported by the Austrian Science Fund (FWF), Project Y123–INF, and FSP Industrial Geometry, Projects 9203-N12 and 9207-N12.

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