

# **Mathematical and numerical analysis of a steady magnetohydrodynamic problem**

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# Mathematical and numerical analysis of a steady magnetohydrodynamic problem

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**Abstract.** In this paper we consider the steady flow of a conductive incompressible fluid confined in a bounded region and subject to the Lorentz force exerted by the interaction of electric currents and magnetic fields. We study a nonlinear coupled problem where all the relevant physical unknowns appear, i.e., magnetic field, velocity and pressure of the fluid, electric current and potential. We prove the well-posedness of the coupled problem and we study its conforming finite element approximation. Finally, we propose and analyze an iterative decoupled method to compute its solution, and we present some numerical examples.

**Keywords.** Magnetohydrodynamics, Maxwell's equations, Navier-Stokes equations, Conforming finite elements.

## 1 Introduction and problem setting

Magnetohydrodynamics (MHD) concerns the interaction of electrically conductive fluids and electromagnetic fields. Applications are manifold, including astronomy and geophysics, as well as many industrial processes, especially concerning the production of metals (see, e.g., [8, 28] for a general introduction to this subject). Indeed, the interaction of magnetic fields and electric currents in a conductive fluid gives rise to a Lorentz force which permits to influence the motion of the fluid itself in a contactless way, which is clearly very useful in metallurgical industry.

The mathematical modeling of the processes taking place in such industrial plants is very involved. Indeed, it requires to take into account many phenomena, e.g., multi-phase and free-surface flows, magnetic fields, electric currents, temperature effects, chemical reactions. However, the core model describing the interaction between the fluid and magnetic fields is a nonlinear system formed by Navier-Stokes' and Maxwell's equations coupled by Ohm's law and Lorentz's force.

The literature concerning both the mathematical analysis and the finite element approximation of this coupled problem is broad (see, e.g., [11, 14, 16, 15, 17, 19, 20, 21]).

Recently, optimal control methods have also been applied to the MHD system [13] providing a mathematical tool for the optimization of magnetic fields, in order to drive the fluid flow in a desired state.

In this paper, we consider a steady MHD problem in a formulation where the physical quantities: magnetic field, velocity and pressure of the fluid, electric currents and potential appear, so that we have a nonlinear coupled system in five unknowns. After discussing the well-posedness of this problem under suitable smallness assumptions on the physical data (Sect. 2), we propose and analyze an iterative method to compute its solution (Sect. 3). In particular, we set up an operator-splitting based method which allows us to compute the global solution by independently solving the magnetic-field, the fluid-flow and the electric-currents subproblems. Concerning the finite element approximation, we consider a conforming discretization based on the classical Nédélec, Taylor-Hood and Raviart-Thomas elements (Sect. 4). Finally, we discuss the algebraic formulation of the iterative schemes that we have proposed, and we present some preliminary numerical results (Sect. 5).

The formulation of the problem that we consider is as follows.

Let  $\Omega_f \subset \mathbb{R}^3$  be a bounded Lipschitz domain, filled by an electrically conductive fluid. An external conductor  $\Omega_s$  is attached to a part of the boundary  $\Gamma_s \subset \partial\Omega_f$ , so that an electric current can be injected into the fluid domain. Finally, let  $\Omega_e$  be an additional external device, separated by the fluid domain, which possibly generates a magnetic field  $\mathbf{B}_e^\dagger$  in the whole space  $\mathbb{R}^3$ , that also influences the motion of the fluid in  $\Omega_f$ . A schematic representation of the domain is shown in fig. 1.

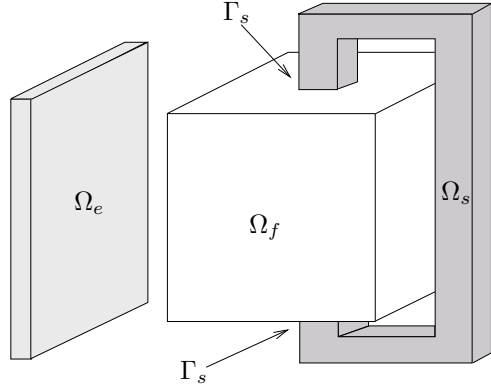


Figure 1: Schematic representation of the setting.

In the conductive device  $\Omega_s$ , we assign an electric current  $\mathbf{J}_s$  such that  $\operatorname{div} \mathbf{J}_s = 0$  in  $\Omega_s$ ,  $\mathbf{J}_s \cdot \mathbf{n} = 0$  on  $\partial\Omega_s \setminus \Gamma_s$ , and  $\mathbf{J}_s \cdot \mathbf{n} = j_s$  on  $\Gamma_s$ , where the (known) function  $j_s$  fulfills the compatibility condition  $\int_{\Gamma_s} j_s = 0$ .  $\mathbf{n}$  denotes the unit normal vector directed outward of  $\partial\Omega_f$ . The current  $\mathbf{J}_s$  originates a magnetic field  $\mathbf{B}_s^\dagger$  in  $\mathbb{R}^3$ , which may be represented by the Biot-Savart law:

$$\mathbf{B}_s^\dagger(x) = -\frac{\mu}{4\pi} \int_{\Omega_s} \frac{x-y}{|x-y|^3} \times \mathbf{J}_s(y) dy, \quad x \in \mathbb{R}^3,$$

$\mu > 0$  being the magnetic permeability, that we assume to be constant in the whole space. We suppose that the contact interface  $\Gamma_s$  between  $\Omega_f$  and  $\Omega_s$  is perfectly conductive, i.e. it holds

$$\mathbf{J}_s \cdot \mathbf{n} = j_s = \mathbf{J}_f \cdot \mathbf{n} \quad \text{on } \Gamma_s. \quad (1)$$

Thus, in the fluid domain  $\Omega_f$  we have a current  $\mathbf{J}_f$  which generates a magnetic field  $\mathbf{B}_f^\dagger$ :

$$\mathbf{B}_f^\dagger(x) = -\frac{\mu}{4\pi} \int_{\Omega_f} \frac{x-y}{|x-y|^3} \times \mathbf{J}_f(y) dy, \quad x \in \mathbb{R}^3.$$

In conclusion, we have a global magnetic field  $\mathbf{B}^\dagger$  in the space  $\mathbb{R}^3$  due to the superposition of three components:  $\mathbf{B}^\dagger(x) = \mathbf{B}_e^\dagger(x) + \mathbf{B}_s^\dagger(x) + \mathbf{B}_f^\dagger(x)$ .

The motion of the incompressible conductive fluid in  $\Omega_f$  is described by the steady Navier-Stokes' equations: find the velocity  $\mathbf{u}$  and the pressure  $p$  such that

$$\begin{aligned} -\eta \Delta \mathbf{u} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mathbf{J}_f \times \mathbf{B}_{|\Omega_f}^\dagger &= \mathbf{0} & \text{in } \Omega_f, \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega_f, \end{aligned} \quad (2)$$

with the Dirichlet boundary condition  $\mathbf{u} = \mathbf{g}$  on  $\partial\Omega_f$ ,  $\mathbf{g}$  being an assigned velocity field such that  $\int_{\partial\Omega_f} \mathbf{g} \cdot \mathbf{n} = 0$ .  $\mathbf{J}_f \times \mathbf{B}_{|\Omega_f}^\dagger$  is the Lorentz force exerted on the fluid by the interaction of the magnetic field  $\mathbf{B}^\dagger$  and the electric current  $\mathbf{J}_f$ , while  $\eta, \rho > 0$  are the fluid viscosity and density, respectively,

Finally, the electric current  $\mathbf{J}_f$  satisfies

$$\begin{aligned}\sigma^{-1}\mathbf{J}_f + \nabla\phi - \mathbf{u} \times \mathbf{B}_{|\Omega_f}^\dagger &= \mathbf{0} \quad \text{in } \Omega_f, \\ \operatorname{div} \mathbf{J}_f &= 0 \quad \text{in } \Omega_f,\end{aligned}\tag{3}$$

where  $\phi$  is the electric potential and  $\sigma > 0$  is the electric conductivity of the fluid. We impose the boundary condition (1) on  $\Gamma_s$ , while we set  $\mathbf{J}_f \cdot \mathbf{n} = 0$  on  $\partial\Omega_f \setminus \Gamma_s$ , i.e. we assume that this part of the boundary is perfectly insulated.

Remark that instead of (1), one could prescribe a difference of potential on  $\Gamma_s$  to generate the electric current  $\mathbf{J}_f$  in the fluid domain, i.e. one would impose the natural boundary condition  $\phi = \phi_s$  on  $\Gamma_s$ , for a suitable given function  $\phi_s$ .

## 2 Weak formulation and well-posedness analysis

In view of the numerical solution of the MHD problem, we restrict ourselves to a bounded domain  $\Omega \subset \mathbb{R}^3$ , which includes the fluid domain  $\Omega_f \subset \Omega$ , and is assumed to be of class  $\mathcal{C}^{1,1}$  (see, e.g., [1]).

We define the following functional spaces:

$$\mathbf{H}_0^1(\Omega_f) = \{\mathbf{v} \in \mathbf{H}^1(\Omega_f) \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_f\}, \quad \text{with } \mathbf{H}^1(\Omega_f) = \{\mathbf{v} \in (H^1(\Omega_f))^3\}, \tag{4}$$

$$\mathbf{H}_{0,\operatorname{div}}^1(\Omega_f) = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega_f) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_f\}, \tag{5}$$

$$\mathbf{H}(\operatorname{div}; \Omega_f) = \{\mathbf{v} \in (L^2(\Omega_f))^3 \mid \operatorname{div} \mathbf{v} \in L^2(\Omega_f)\}, \tag{6}$$

$$\mathbf{H}_0(\operatorname{div}; \Omega_f) = \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega_f) \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_f\}, \tag{7}$$

$$\mathbf{H}_0^0(\operatorname{div}; \Omega_f) = \{\mathbf{v} \in \mathbf{H}_0(\operatorname{div}; \Omega_f) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_f\}, \tag{8}$$

$$Q_0 = \left\{ q \in L^2(\Omega_f) \mid \int_{\Omega_f} q = 0 \right\}, \tag{9}$$

$$\mathbf{H}(\operatorname{curl}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \mid \operatorname{curl} \mathbf{v} \in (L^2(\Omega))^3\}, \tag{10}$$

$$\mathbf{H}_0(\operatorname{curl}; \Omega) = \{\mathbf{v} \in \mathbf{H}(\operatorname{curl}; \Omega) \mid \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}. \tag{11}$$

We denote by  $\|\cdot\|_{H^1(\Omega_f)}$ ,  $\|\cdot\|_{\operatorname{div}, \Omega_f}$ ,  $\|\cdot\|_{L^2(\Omega_f)}$  and  $\|\cdot\|_{\operatorname{curl}, \Omega}$  the usual norms in the spaces  $\mathbf{H}^1(\Omega_f)$ ,  $\mathbf{H}(\operatorname{div}; \Omega_f)$ ,  $L^2(\Omega_f)$  and  $\mathbf{H}(\operatorname{curl}; \Omega)$ , respectively. Remark that  $\|\mathbf{v}\|_{L^2(\Omega_f)} = \|\mathbf{v}\|_{\operatorname{div}, \Omega_f}$  for all  $\mathbf{v} \in \mathbf{H}_0^0(\operatorname{div}; \Omega_f)$ .

Finally, let us recall the Poincaré inequality (see, e.g., [18]):

$$\exists C_p > 0 : \quad \|\mathbf{v}\|_{L^2(\Omega_f)} \leq C_p \|\nabla \mathbf{v}\|_{L^2(\Omega_f)} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega_f), \tag{12}$$

and the following inequality:

$$\exists C_3 > 0 : \quad \int_{\Omega_f} (\mathbf{f} \times \mathbf{g}) \cdot \mathbf{h} \leq C_3 \|\mathbf{f}\|_{L^3(\Omega_f)} \|\mathbf{g}\|_{L^6(\Omega_f)} \|\mathbf{h}\|_{L^2(\Omega_f)}, \tag{13}$$

where the functions  $\mathbf{f}$ ,  $\mathbf{g}$ ,  $\mathbf{h}$  are assumed to be regular enough.

We introduce a continuous extension operator  $\mathbf{E}_f : (H^{1/2}(\partial\Omega_f))^3 \rightarrow \mathbf{H}^1(\Omega_f)$ , such that, taken  $\mathbf{g} \in (H^{1/2}(\partial\Omega_f))^3$  with  $\int_{\partial\Omega_f} \mathbf{g} \cdot \mathbf{n} = 0$ ,  $\mathbf{E}_f \mathbf{g} \in \mathbf{H}^1(\Omega_f)$  is the divergence-free extension of  $\mathbf{g}$ , such that  $\mathbf{E}_f \mathbf{g} = \mathbf{g}$  on  $\partial\Omega_f$ . More precisely, let  $\mathbf{E}'_f \mathbf{g} \in \mathbf{H}^1(\Omega_f)$  be a continuous extension of  $\mathbf{g}$  such that  $\mathbf{E}'_f \mathbf{g} = \mathbf{g}$  on  $\partial\Omega_f$ . Then, we can construct a function  $\mathbf{E}''_f \mathbf{g} \in \mathbf{H}_0^1(\Omega_f)$  such that

$$-\int_{\Omega_f} q \operatorname{div} (\mathbf{E}''_f \tilde{\mathbf{g}}) = \int_{\Omega_f} q \operatorname{div} (\mathbf{E}'_f \tilde{\mathbf{g}}) \quad \forall q \in L^2(\Omega_f). \tag{14}$$

The solvability of (14) is guaranteed by the inf-sup condition (see, e.g., [26] p. 157): there exists a positive constant  $\beta^* > 0$  such that

$$\forall q \in L^2(\Omega_f) \quad \exists \mathbf{v} \in \mathbf{H}_0^1(\Omega_f), \mathbf{v} \neq \mathbf{0} : \quad - \int_{\Omega_f} q \operatorname{div} \mathbf{v} \geq \beta^* \|\mathbf{v}\|_{\mathbf{H}^1(\Omega_f)} \|q\|_{L^2(\Omega_f)}.$$

Finally, we indicate by  $\mathbf{E}_f \mathbf{g} = \mathbf{E}'_f \mathbf{g} + \mathbf{E}''_f \mathbf{g}$  the divergence-free extension of the datum  $\mathbf{g}$ . Remark that by construction  $\mathbf{E}_f \mathbf{g} = \mathbf{g}$  on  $\partial\Omega_f$ , and that, thanks to (14), it holds

$$\int_{\Omega_f} q \operatorname{div} (\mathbf{E}_f \mathbf{g}) = 0 \quad \forall q \in L^2(\Omega_f).$$

Then, we split  $\mathbf{u} = \mathbf{u}_0 + \mathbf{E}_f \mathbf{g}$ , with  $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega_f)$ .

We introduce a continuous extension operator  $\mathbf{E}_s : L^2(\Gamma_s) \rightarrow \mathbf{H}(\operatorname{div}; \Omega_f)$ ,  $\mathbf{E}_s : \varphi \rightarrow \mathbf{E}_s \varphi$ , for all  $\varphi \in L^2(\Gamma_s)$  with  $\int_{\Gamma_s} \varphi = 0$ . More precisely, we consider the auxiliary Neumann problem

$$\begin{aligned} -\Delta \chi &= 0 & \text{in } \Omega_f, \\ \frac{\partial \chi}{\partial n} &= \varphi & \text{on } \Gamma_s, \\ \frac{\partial \chi}{\partial n} &= 0 & \text{on } \partial\Omega_f \setminus \Gamma_s, \end{aligned}$$

which has a unique solution in  $H^1(\Omega_f)/\mathbb{R}$ . Then, we take  $\mathbf{E}_s \varphi = \nabla \chi \in L^2(\Omega_f)$ . Remark that  $\mathbf{E}_s \varphi \cdot \mathbf{n} = \varphi$  on  $\Gamma_s$ ,  $\mathbf{E}_s \varphi \cdot \mathbf{n} = 0$  on  $\partial\Omega_f \setminus \Gamma_s$ , and  $\operatorname{div} (\mathbf{E}_s \varphi) = 0$  in  $\Omega_f$ . Thus, we can consider the divergence-free extension  $\mathbf{E}_s j_s$  of the boundary datum  $j_s$  such that  $\mathbf{E}_s j_s \cdot \mathbf{n} = 0$  on  $\partial\Omega_f \setminus \Gamma_s$ ,  $\mathbf{E}_s j_s \cdot \mathbf{n} = j_s$  on  $\Gamma_s$ ,  $\operatorname{div} (\mathbf{E}_s j_s) = 0$  in  $\Omega_f$ , and we decompose  $\mathbf{J}_f = \mathbf{J}_{f,0} + \mathbf{E}_s j_s$ , with  $\mathbf{J}_{f,0} \in \mathbf{H}_0^0(\operatorname{div}; \Omega_f)$ .

Let us now consider the electric current:

$$\bar{\mathbf{J}}_{f,0} = \begin{cases} \mathbf{J}_{f,0} & \text{in } \Omega_f, \\ \mathbf{0} & \text{in } \Omega_s, \\ \mathbf{0} & \text{in } \Omega \setminus (\bar{\Omega}_f \cup \bar{\Omega}_s), \end{cases}$$

which satisfies  $\operatorname{div} \bar{\mathbf{J}}_{f,0} = 0$  in  $\Omega$  and  $\bar{\mathbf{J}}_{f,0} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . We can represent the magnetic field, say  $\mathbf{B}_{0,f}$ , generated by  $\bar{\mathbf{J}}_{f,0}$  as the solution of the problem

$$\begin{aligned} \operatorname{curl} (\mu^{-1} \mathbf{B}_{f,0}) &= \bar{\mathbf{J}}_{f,0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{B}_{f,0} &= 0 & \text{in } \Omega, \\ \mathbf{B}_{f,0} \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{15}$$

where we have introduced the fictitious essential boundary condition  $\mathbf{B}_{f,0} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ .  $\mathbf{B}_{f,0}$  can be equivalently expressed as the extension  $\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})$  of  $\bar{\mathbf{J}}_{f,0}$  by the linear continuous operator  $\mathcal{B}_\Omega : \mathbf{X}_{\mathcal{B}_\Omega} \rightarrow \mathbf{X}_T(\Omega)$ , where

$$\begin{aligned} \mathbf{X}_{\mathcal{B}_\Omega} &= \{\mathbf{J} \in \mathbf{H}(\operatorname{div}; \Omega) \mid \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \operatorname{div} \mathbf{J} = 0 \text{ in } \Omega\}, \\ \mathbf{X}_T(\Omega) &= \{\mathbf{B} \in \mathbf{H}(\operatorname{curl}; \Omega) \cap \mathbf{H}(\operatorname{div}; \Omega) \mid \mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \operatorname{div} \mathbf{B} = 0 \text{ in } \Omega\}. \end{aligned}$$

$\mathcal{B}_\Omega$  approximates in  $\Omega$  the bounded linear Biot-Savart operator  $\mathcal{B}$  (see, e.g., [20]):

$$\mathcal{B} : (L^2(\Omega))^3 \rightarrow \mathbf{W}^1(\mathbb{R}^3), \quad \mathcal{B}(\bar{\mathbf{J}}_{f,0}) = -\frac{\mu}{4\pi} \int_{\Omega} \frac{x-y}{|x-y|^3} \times \bar{\mathbf{J}}_{f,0}(y) dy, \quad \forall x \in \mathbb{R}^3.$$

With a slight abuse in notation we will often write  $\mathcal{B}_\Omega(\mathbf{J}_{f,0})$  instead of  $\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})$ .

Under the assumption that  $\partial\Omega$  is at least of class  $\mathcal{C}^{1,1}$ , thanks to Theorem 2.9 of [2] and via Sobolev immersion, we have  $\mathbf{X}_T(\Omega) \hookrightarrow (H^1(\Omega))^3 \hookrightarrow (L^6(\Omega))^3 \hookrightarrow (L^3(\Omega))^3$ . Then, in particular, the following inequalities hold:

$$\begin{aligned} \|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{L^3(\Omega_f)} &\leq C_s^1 \|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{L^6(\Omega_f)} \\ &\leq C_s^1 \|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{L^6(\Omega)} \leq C_s^2 \|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{H^1(\Omega)} \leq C_s^3 \|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{\mathbf{X}_T}, \end{aligned} \quad (16)$$

with positive constants  $C_s^1, C_s^2, C_s^3 > 0$ , and there holds

$$\begin{aligned} \|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{\mathbf{X}_T} &\leq \bar{C} (\|\mathbf{curl} \mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{L^2(\Omega)} + \|\mathbf{div} \mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{L^2(\Omega)}) \\ &= \bar{C}\mu \|\bar{\mathbf{J}}_{f,0}\|_{L^2(\Omega)} \leq \bar{C}\mu \|\mathbf{J}_{f,0}\|_{\mathbf{div},\Omega_f}, \quad \text{for } \bar{C} > 0. \end{aligned}$$

Therefore, there exists a positive constant  $C > 0$  such that

$$\|\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})\|_{L^3(\Omega_f)} \leq C\mu \|\mathbf{J}_{f,0}\|_{\mathbf{div},\Omega_f}. \quad (17)$$

In analogous way, we can consider the current  $\bar{\mathbf{J}}_s$ :

$$\bar{\mathbf{J}}_s = \begin{cases} \mathbf{E}_s j_s & \text{in } \Omega_f, \\ \mathbf{J}_s & \text{in } \Omega_s, \\ \mathbf{0} & \text{in } \Omega \setminus (\bar{\Omega}_f \cup \bar{\Omega}_s), \end{cases}$$

which generates a magnetic field  $\mathbf{B}_s$  such that

$$\begin{aligned} \mathbf{curl} (\mu^{-1} \mathbf{B}_s) &= \bar{\mathbf{J}}_s & \text{in } \Omega, \\ \mathbf{div} \mathbf{B}_s &= 0 & \text{in } \Omega, \\ \mathbf{B}_s \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega. \end{aligned}$$

In this case, there holds:

$$\|\mathbf{B}_s\|_{L^3(\Omega_f)} \leq C\mu \|\bar{\mathbf{J}}_s\|_{\mathbf{div},\Omega} \leq C\mu (\|\mathbf{E}_s j_s\|_{\mathbf{div},\Omega_f} + \|\mathbf{J}_s\|_{\mathbf{div},\Omega_s}). \quad (18)$$

Thus, we can approximate the magnetic field  $\mathbf{B}^\dagger$  in the bounded domain  $\Omega$  by  $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})$ , where  $\mathbf{B}_e$  is the restriction of  $\mathbf{B}_e^\dagger$  to  $\Omega$ .  $\mathcal{B}_\Omega(\bar{\mathbf{J}}_{f,0})$  is the only unknown component of  $\mathbf{B}$ , since it depends on the unknown current  $\mathbf{J}_{f,0}$ .

The representation of the magnetic field  $\mathbf{B}_{f,0}$  in terms of the linear operator  $\mathcal{B}_\Omega$  will be useful for the analysis of the MHD problem, as we will see in the next section.

## 2.1 Weak form of the MHD problem

Using the notations introduced in the previous section, we can write the weak form of the MHD problem: find  $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega_f)$ ,  $p \in Q_0$ ,  $\mathbf{J}_{f,0} \in \mathbf{H}_0(\mathbf{div}; \Omega_f)$ ,  $\phi \in Q_0$  such that

$$\begin{aligned} \int_{\Omega_f} \eta \nabla \mathbf{u}_0 \cdot \nabla \mathbf{v} + \int_{\Omega_f} \rho [((\mathbf{u}_0 + \mathbf{E}_f \mathbf{g}) \cdot \nabla)(\mathbf{u}_0 + \mathbf{E}_f \mathbf{g})] \cdot \mathbf{v} - \int_{\Omega_f} p \operatorname{div} \mathbf{v} \\ - \int_{\Omega_f} [(\mathbf{J}_{f,0} + \mathbf{E}_s j_s) \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{J}_{f,0}))] \cdot \mathbf{v} = - \int_{\Omega_f} \eta \nabla(\mathbf{E}_f \mathbf{g}) \cdot \nabla \mathbf{v} \end{aligned} \quad (19)$$

$$\int_{\Omega_f} q \operatorname{div} \mathbf{u}_0 = 0 \quad (20)$$

$$\begin{aligned} \int_{\Omega_f} \sigma^{-1} \mathbf{J}_{f,0} \cdot \mathbf{K} - \int_{\Omega_f} \phi \operatorname{div} \mathbf{K} + \int_{\Omega_f} [\mathbf{K} \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{J}_{f,0}))] \cdot (\mathbf{u}_0 + \mathbf{E}_f \mathbf{g}) \\ = - \int_{\Omega_f} \sigma^{-1} \mathbf{E}_s j_s \cdot \mathbf{K} \end{aligned} \quad (21)$$

$$\int_{\Omega_f} \varphi \operatorname{div} \mathbf{J}_{f,0} = 0 \quad (22)$$

for all  $\mathbf{v} \in \mathbf{H}_0^1(\Omega_f)$ ,  $q \in Q_0$ ,  $\mathbf{K} \in \mathbf{H}_0(\text{div}; \Omega_f)$ ,  $\varphi \in Q_0$ .

Remark that the trilinear forms

$$-\int_{\Omega_f} [\mathbf{J}_f \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{J}_{f,0}))] \cdot \mathbf{v} \quad \text{and} \quad \int_{\Omega_f} [\mathbf{K} \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{J}_{f,0}))] \cdot \mathbf{u} \quad (23)$$

realize the coupling between the fluid, the electric and the magnetic-field problems in  $\Omega_f$ . Notice that they make the MHD problem nonlinear, despite the fact, e.g., to replace the Navier-Stokes' equations by the Stokes ones, that is to consider, instead of (19), the momentum equation

$$\begin{aligned} & \int_{\Omega_f} \eta \nabla \mathbf{u}_0 \cdot \nabla \mathbf{v} - \int_{\Omega_f} p \text{div } \mathbf{v} - \int_{\Omega_f} [\mathbf{J}_{f,0} \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{J}_{f,0}))] \cdot \mathbf{v} \\ & - \int_{\Omega_f} [\mathbf{E}_s j_s \times \mathcal{B}_\Omega(\mathbf{J}_{f,0})] \cdot \mathbf{v} = - \int_{\Omega_f} \eta \nabla(\mathbf{E}_f \mathbf{g}) \cdot \nabla \mathbf{v} + \int_{\Omega_f} [\mathbf{E}_s j_s \times (\mathbf{B}_e + \mathbf{B}_s)] \cdot \mathbf{v}, \end{aligned} \quad (24)$$

where the nonlinear convective term  $(\mathbf{u} \cdot \nabla) \mathbf{u}$  has been omitted.

In the next sections, we consider (24) instead of (19), and we prove the well-posedness of the MHD problem (24), (20)-(22). To this aim, we will use some classical results that, for the sake of clarity, we anticipate (in a fairly abstract form) in the following section.

### 2.1.1 General existence and uniqueness results

Let us recall some existence and uniqueness results for nonlinear saddle-point problems, referring the reader to, e.g., [4, 5, 6, 7, 12] for a rigorous study.

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two real Hilbert spaces. Consider a continuous bilinear form  $\mathfrak{b}(\cdot, \cdot) : X \times Y \rightarrow \mathbb{R}$ ,  $(v, q) \rightarrow \mathfrak{b}(v, q)$ , and a trilinear form  $\mathfrak{a}(\cdot; \cdot, \cdot) : X \times X \times X \rightarrow \mathbb{R}$ ,  $(w, u, v) \rightarrow \mathfrak{a}(w; u, v)$ , where, for  $w \in X$ , the mapping  $(u, v) \rightarrow \mathfrak{a}(w; u, v)$  is a continuous bilinear form on  $X \times X$ .

Then, consider the following problem: given  $l \in X'$ , find  $(u, p) \in X \times Y$  satisfying

$$\begin{aligned} \mathfrak{a}(u; u, v) + \mathfrak{b}(v, p) &= \langle l, v \rangle & \forall v \in X \\ \mathfrak{b}(u, q) &= 0 & \forall q \in Y. \end{aligned} \quad (25)$$

Introducing the linear operators  $A(w) \in \mathcal{L}(X; X')$  for  $w \in X$ , and  $B \in \mathcal{L}(X; Y')$ :

$$\langle A(w)u, v \rangle = \mathfrak{a}(w; u, v), \quad \langle Bv, q \rangle = \mathfrak{b}(v, q), \quad \forall u, v \in X, \forall q \in Y,$$

problem (25) becomes: find  $(u, p) \in X \times Y$  such that

$$\begin{aligned} A(u)u + B^T p &= l & \text{in } X', \\ Bu &= 0 & \text{in } Y'. \end{aligned}$$

Taking  $V = \text{Ker}(B)$ , we associate (25) to:

$$\text{find } u \in V : \quad \mathfrak{a}(u; u, v) = \langle l, v \rangle \quad \forall v \in V, \quad (26)$$

or, equivalently: find  $u \in V$  such that  $\Pi A(u)u = \Pi l$  in  $V'$ , where the linear operator  $\Pi \in \mathcal{L}(X'; V')$  is defined by  $\langle \Pi l, v \rangle = \langle l, v \rangle$ ,  $\forall v \in V$ .

If  $(u, p)$  is a solution of (25), then  $u$  solves (26). The converse may be proved as well provided an inf-sup condition holds. More generally, the following results hold.

**Theorem 2.1 (Existence and uniqueness)** *Suppose that:*

1. the bilinear form  $\mathfrak{a}(w; \cdot, \cdot)$  is uniformly elliptic in the Hilbert space  $V$  with respect to  $w$ , i.e. there exists a constant  $\alpha > 0$  such that

$$\mathfrak{a}(w; v, v) \geq \alpha \|v\|_X^2 \quad \forall v, w \in V;$$

2. the mapping  $w \rightarrow \Pi A(w)$  is locally Lipschitz-continuous in  $V$ , i.e. there exists a continuous and monotonically increasing function  $L : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that, for all  $m > 0$ ,

$$|a(w_1; u, v) - a(w_2; u, v)| \leq L(m) \|u\|_X \|v\|_X \|w_1 - w_2\|_X, \quad \forall u, v \in V, w_1, w_2 \in S_m,$$

with  $S_m = \{w \in V \mid \|w\|_X \leq m\}$ ;

3. it holds

$$\frac{\|\Pi A\|_{V'}}{\alpha^2} L \left( \frac{\|\Pi A\|_{V'}}{\alpha} \right) < 1.$$

Then (26) has a unique solution  $u \in V$ .

**Theorem 2.2** Assume that the bilinear form  $b(\cdot, \cdot)$  satisfies the inf-sup condition:  $\exists \underline{\beta} > 0$

$$\inf_{q \in Y} \sup_{v \in X} \frac{b(v, q)}{\|v\|_X \|q\|_Y} \geq \underline{\beta}.$$

Then, for each solution  $u$  of (26), there exists a unique  $p \in Y$  such that  $(u, p)$  is a solution of (25).

## 2.2 Analysis of the Stokes-MHD problem

We introduce the product spaces  $\mathbf{X} = \mathbf{H}_0^1(\Omega_f) \times \mathbf{H}_0(\text{div}; \Omega_f)$ ,  $\mathbf{Q}_0 = Q_0 \times Q_0$ , and the subspace  $\mathbf{X}_0 \subset \mathbf{X}$  with  $\mathbf{X}_0 = \mathbf{H}_{0, \text{div}}^1(\Omega_f) \times \mathbf{H}_0^0(\text{div}; \Omega_f)$ . Moreover, we define the norms

$$\begin{aligned} \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{X}} &= \left( \|\mathbf{v}\|_{H^1(\Omega_f)}^2 + \|\mathbf{K}\|_{\text{div}, \Omega_f}^2 \right)^{1/2} \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}, \\ \|(q, \varphi)\|_{\mathbf{Q}} &= \left( \|q\|_{L^2(\Omega_f)}^2 + \|\varphi\|_{L^2(\Omega_f)}^2 \right)^{1/2} \quad \forall (q, \varphi) \in \mathbf{Q}_0. \end{aligned}$$

We introduce: the trilinear form  $a(\cdot; \cdot, \cdot) : \mathbf{X} \times \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$ , such that for all  $(\mathbf{v}_i, \mathbf{K}_i) \in \mathbf{X}$  ( $i = 1, 2, 3$ ),

$$\begin{aligned} a((\mathbf{v}_1, \mathbf{K}_1); (\mathbf{v}_2, \mathbf{K}_2), (\mathbf{v}_3, \mathbf{K}_3)) &= \\ & \int_{\Omega_f} \eta \nabla \mathbf{v}_2 \cdot \nabla \mathbf{v}_3 - \int_{\Omega_f} [\mathbf{K}_2 \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{K}_1))] \cdot \mathbf{v}_3 - \int_{\Omega_f} [\mathbf{E}_s j_s \times \mathcal{B}_\Omega(\mathbf{K}_2)] \cdot \mathbf{v}_3 \\ & + \int_{\Omega_f} \sigma^{-1} \mathbf{K}_2 \cdot \mathbf{K}_3 + \int_{\Omega_f} [\mathbf{K}_3 \times (\mathbf{B}_e + \mathbf{B}_s + \mathcal{B}_\Omega(\mathbf{K}_1))] \cdot \mathbf{v}_2 + \int_{\Omega_f} [\mathbf{K}_3 \times \mathcal{B}_\Omega(\mathbf{K}_2)] \cdot \mathbf{E}_f \mathbf{g}, \end{aligned}$$

the bilinear form  $b(\cdot, \cdot) : \mathbf{X} \times \mathbf{Q}_0 \rightarrow \mathbb{R}$ :

$$b((\mathbf{v}, \mathbf{K}), (q, \varphi)) = - \int_{\Omega_f} q \text{div } \mathbf{v} - \int_{\Omega_f} \varphi \text{div } \mathbf{K} \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}, (q, \varphi) \in \mathbf{Q}_0,$$

and the linear functional  $\mathcal{F} : \mathbf{X} \rightarrow \mathbb{R}$ :

$$\begin{aligned} \mathcal{F}(\mathbf{v}, \mathbf{K}) &= - \int_{\Omega_f} \eta \nabla(\mathbf{E}_f \mathbf{g}) \cdot \nabla \mathbf{v} - \int_{\Omega_f} \sigma^{-1} \mathbf{E}_s j_s \cdot \mathbf{K} \\ & + \int_{\Omega_f} [\mathbf{E}_s j_s \times (\mathbf{B}_e + \mathbf{B}_s)] \cdot \mathbf{v} - \int_{\Omega_f} [\mathbf{K} \times (\mathbf{B}_e + \mathbf{B}_s)] \cdot \mathbf{E}_f \mathbf{g}, \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}. \end{aligned}$$

With these notations, the coupled problem (24), (20)-(22) becomes: find  $(\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}$ ,  $(p, \phi) \in \mathbf{Q}_0$  such that

$$\begin{aligned} a((\mathbf{u}_0, \mathbf{J}_{f,0}); (\mathbf{u}_0, \mathbf{J}_{f,0}), (\mathbf{v}, \mathbf{K})) + b((\mathbf{v}, \mathbf{K}), (p, \phi)) &= \mathcal{F}(\mathbf{v}, \mathbf{K}) \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}, \\ b((\mathbf{u}_0, \mathbf{J}_{f,0}), (q, \varphi)) &= 0 \quad \forall (q, \varphi) \in \mathbf{Q}_0, \end{aligned} \quad (27)$$

and it can be reformulated on the kernel of  $b$  as:

$$\text{find } (\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}_0 : \quad a((\mathbf{u}_0, \mathbf{J}_{f,0}); (\mathbf{u}_0, \mathbf{J}_{f,0}), (\mathbf{v}, \mathbf{K})) = \mathcal{F}(\mathbf{v}, \mathbf{K}) \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}_0. \quad (28)$$

Then, we can state the following result.



**Proposition 2.1** Assume that  $\mathbf{B}_{e|\Omega_f} \in (L^3(\Omega_f))^3$ . If the boundary data  $\mathbf{g} \in (H^{1/2}(\partial\Omega_f))^3$ ,  $j_s \in L^2(\Gamma_s)$  and the assigned magnetic field  $\mathbf{B}_e$  are small enough, and the physical parameters  $\mu^{-1}$ ,  $\eta$ ,  $\sigma^{-1}$  are sufficiently large, then the MHD problem (27) has a unique solution  $(\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}$ ,  $(p, \phi) \in \mathbf{Q}_0$ .

*Proof.* The proof is made of several steps and it will be based on Theorems 2.1 and 2.2.

1. Let  $(\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K}) \in \mathbf{X}_0$ . Then, using the inequalities (16), (17), (12) and (13), we have

$$\begin{aligned}
& a((\mathbf{u}, \mathbf{J}); (\mathbf{v}, \mathbf{K}), (\mathbf{v}, \mathbf{K})) \\
&= \eta \|\nabla \mathbf{v}\|_{L^2(\Omega_f)}^2 + \sigma^{-1} \|\mathbf{K}\|_{L^2(\Omega_f)}^2 \\
&\quad - \int_{\Omega_f} [\mathbf{E}_s j_s \times \mathcal{B}_\Omega(\mathbf{K})] \cdot \mathbf{v} + \int_{\Omega_f} [\mathbf{K} \times \mathcal{B}_\Omega(\mathbf{K})] \cdot \mathbf{E}_f \mathbf{g} \\
&\geq \frac{\eta}{1 + C_p^2} \|\mathbf{v}\|_{H^1(\Omega_f)}^2 + \sigma^{-1} \|\mathbf{K}\|_{L^2(\Omega_f)}^2 \\
&\quad - \tilde{C}\mu \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} \|\mathbf{K}\|_{\text{div}, \Omega_f} \|\mathbf{v}\|_{H^1(\Omega_f)} - \tilde{C}\mu \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \|\mathbf{K}\|_{\text{div}, \Omega_f}^2 \\
&\geq \frac{\eta}{1 + C_p^2} \|\mathbf{v}\|_{H^1(\Omega_f)}^2 + \sigma^{-1} \|\mathbf{K}\|_{L^2(\Omega_f)}^2 \\
&\quad - \frac{\tilde{C}\mu}{2} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} \left( \|\mathbf{K}\|_{\text{div}, \Omega_f}^2 + \|\mathbf{v}\|_{H^1(\Omega_f)}^2 \right) - \tilde{C}\mu \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \|\mathbf{K}\|_{\text{div}, \Omega_f}^2 \\
&\geq \left( \frac{\eta}{1 + C_p^2} - \frac{\tilde{C}\mu}{2} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} \right) \|\mathbf{v}\|_{H^1(\Omega_f)}^2 \\
&\quad + \left( \sigma^{-1} - \frac{\tilde{C}\mu}{2} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} - \tilde{C}\mu \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \right) \|\mathbf{K}\|_{\text{div}, \Omega_f}^2 \geq \alpha \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{X}}^2,
\end{aligned}$$

where we have denoted  $\tilde{C} = CC_3C_s^2/C_s^1$ , and

$$\alpha = \min \left( \frac{\eta}{1 + C_p^2} - \frac{\tilde{C}\mu}{2} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f}, \sigma^{-1} - \frac{\tilde{C}\mu}{2} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} - \tilde{C}\mu \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \right). \quad (29)$$

If

$$\eta > (1 + C_p^2) \frac{\tilde{C}\mu}{2} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} \quad \text{and} \quad \sigma^{-1} > \tilde{C}\mu \left( \frac{\|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f}}{2} + \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \right), \quad (30)$$

then, the bilinear form  $a((\mathbf{u}, \mathbf{J}); \cdot, \cdot)$  is uniformly elliptic in  $\mathbf{X}_0$  with constant  $\alpha$  given in (29).

2. Let  $(\mathbf{w}_i, \mathbf{J}_i), (\mathbf{u}, \mathbf{Y}), (\mathbf{v}, \mathbf{K}) \in \mathbf{X}_0$ ,  $i = 1, 2$ . Then,

$$\begin{aligned}
& |a((\mathbf{w}_1, \mathbf{J}_1); (\mathbf{u}, \mathbf{Y}), (\mathbf{v}, \mathbf{K})) - a((\mathbf{w}_2, \mathbf{J}_2); (\mathbf{u}, \mathbf{Y}), (\mathbf{v}, \mathbf{K}))| \\
&= \left| \int_{\Omega_f} [\mathbf{Y} \times \mathcal{B}_\Omega(\mathbf{J}_2 - \mathbf{J}_1)] \cdot \mathbf{v} - \int_{\Omega_f} [\mathbf{K} \times \mathcal{B}_\Omega(\mathbf{J}_2 - \mathbf{J}_1)] \cdot \mathbf{u} \right| \\
&\leq \tilde{C}\mu \|\mathbf{J}_2 - \mathbf{J}_1\|_{\text{div}, \Omega_f} \left( \|\mathbf{Y}\|_{\text{div}, \Omega_f} \|\mathbf{v}\|_{H^1(\Omega_f)} + \|\mathbf{K}\|_{\text{div}, \Omega_f} \|\mathbf{u}\|_{H^1(\Omega_f)} \right) \\
&\leq 2\sqrt{2}\tilde{C}\mu \|(\mathbf{w}_1, \mathbf{J}_1) - (\mathbf{w}_2, \mathbf{J}_2)\|_{\mathbf{X}} \|(\mathbf{u}, \mathbf{Y})\|_{\mathbf{X}} \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{X}}.
\end{aligned}$$

Thus, the bilinear form  $a$  is Lipschitz continuous with constant  $2\sqrt{2}\tilde{C}\mu$ .

3. Let  $(\mathbf{v}, \mathbf{K}) \in \mathbf{X}_0$  and let  $\|\cdot\|_{\mathbf{X}'}$  denote the norm of the dual space of  $\mathbf{X}$ . Then, we have:

$$\begin{aligned}
\mathcal{F}(\mathbf{v}, \mathbf{K}) &\leq (\eta \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} + C' \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} (\|\mathbf{B}_e\|_{L^3(\Omega_f)} + \|\mathbf{B}_s\|_{L^3(\Omega_f)})) \|\mathbf{v}\|_{H^1(\Omega_f)} \\
&\quad + (\sigma^{-1} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} + C' \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} (\|\mathbf{B}_e\|_{L^3(\Omega_f)} + \|\mathbf{B}_s\|_{L^3(\Omega_f)})) \|\mathbf{K}\|_{\text{div}, \Omega_f},
\end{aligned}$$

with  $C' = C_3 C_s^2 / C_s^1$ . Thus,  $\|\mathcal{F}\|_{\mathbf{X}'} \leq C_{\mathcal{F}}$ , where, owing to (18),

$$C_{\mathcal{F}} = \sqrt{2} \max \left( \eta \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} + C' \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} (\|\mathbf{B}_e\|_{L^3(\Omega_f)} + C\mu \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} + C\mu \|\mathbf{J}_s\|_{\text{div}, \Omega_s}), \right. \\ \left. \sigma^{-1} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} + C' \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} (\|\mathbf{B}_e\|_{L^3(\Omega_f)} + C\mu \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} + C\mu \|\mathbf{J}_s\|_{\text{div}, \Omega_s}) \right).$$

4. If the data of the problem satisfy (30), and there holds  $2\sqrt{2}\tilde{C}C_{\mathcal{F}}\mu < \alpha^2$ , Theorem 2.1 guarantees that there exists a unique solution  $(\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}_0$  to (28).

5. The bilinear form  $b$  is continuous: indeed, for all  $(\mathbf{u}, \mathbf{J}) \in \mathbf{X}$ ,  $(q, \varphi) \in \mathbf{Q}_0$ ,

$$|b((\mathbf{u}, \mathbf{J}), (q, \varphi))| \leq \|q\|_{L^2(\Omega_f)} \|\text{div } \mathbf{u}\|_{L^2(\Omega_f)} + \|\varphi\|_{L^2(\Omega_f)} \|\text{div } \mathbf{J}\|_{L^2(\Omega_f)} \leq \|(\mathbf{u}, \mathbf{J})\|_{\mathbf{X}} \|(q, \varphi)\|_{\mathbf{Q}}.$$

Moreover, it is well-known that inf-sup conditions hold between the spaces  $\mathbf{H}_0^1(\Omega_f)$  and  $Q_0$  (see, e.g., [26] p. 157), and between  $\mathbf{H}_0(\text{div}; \Omega_f)$  and  $Q_0$  (see, e.g., [25] p. 238), with positive constants, say,  $\beta_1 > 0$  and  $\beta_2 > 0$ , respectively:

$$\forall q \in Q_0 \quad \exists \mathbf{u} \in \mathbf{H}_0^1(\Omega_f), \mathbf{u} \neq \mathbf{0} : \quad - \int_{\Omega_f} q \text{div } \mathbf{u} \geq \beta_1 \|q\|_{L^2(\Omega_f)} \|\mathbf{u}\|_{H^1(\Omega_f)}, \quad (31)$$

$$\forall \varphi \in Q_0 \quad \exists \mathbf{J} \in \mathbf{H}_0(\text{div}; \Omega_f), \mathbf{J} \neq \mathbf{0} : \quad - \int_{\Omega_f} \varphi \text{div } \mathbf{J} \geq \beta_2 \|\varphi\|_{L^2(\Omega_f)} \|\mathbf{J}\|_{\text{div}, \Omega_f}. \quad (32)$$

Then, for all  $(q, \varphi) \in \mathbf{Q}_0$ , there exists  $(\mathbf{u}, \mathbf{J}) \in \mathbf{X}$ ,  $(\mathbf{u}, \mathbf{J}) \neq (\mathbf{0}, \mathbf{0})$ , such that:

$$\begin{aligned} b((\mathbf{u}, \mathbf{J}), (q, \varphi)) &\geq \beta_1 \|q\|_{L^2(\Omega_f)} \|\mathbf{u}\|_{H^1(\Omega_f)} + \beta_2 \|\varphi\|_{L^2(\Omega_f)} \|\mathbf{J}\|_{\text{div}, \Omega_f} \\ &\geq \min(\beta_1, \beta_2) (\|q\|_{L^2(\Omega_f)} \|\mathbf{u}\|_{H^1(\Omega_f)} + \|\varphi\|_{L^2(\Omega_f)} \|\mathbf{J}\|_{\text{div}, \Omega_f}) \\ &\geq \min(\beta_1, \beta_2) \min(\|\mathbf{u}\|_{H^1(\Omega_f)}, \|\mathbf{J}\|_{\text{div}, \Omega_f}) (\|q\|_{L^2(\Omega_f)} + \|\varphi\|_{L^2(\Omega_f)}) \\ &\geq \min(\beta_1, \beta_2) \|(q, \varphi)\|_{\mathbf{Q}} \min(\|\mathbf{u}\|_{H^1(\Omega_f)}, \|\mathbf{J}\|_{\text{div}, \Omega_f}) \\ &\quad \cdot \frac{\|\mathbf{u}\|_{H^1(\Omega_f)} + \|\mathbf{J}\|_{\text{div}, \Omega_f}}{\|\mathbf{u}\|_{H^1(\Omega_f)} + \|\mathbf{J}\|_{\text{div}, \Omega_f}} \\ &\geq \min(\beta_1, \beta_2) \|(q, \varphi)\|_{\mathbf{Q}} \frac{\min(\|\mathbf{u}\|_{H^1(\Omega_f)}, \|\mathbf{J}\|_{\text{div}, \Omega_f})}{\|\mathbf{u}\|_{H^1(\Omega_f)} + \|\mathbf{J}\|_{\text{div}, \Omega_f}} \|(\mathbf{u}, \mathbf{J})\|_{\mathbf{X}} \\ &\geq \beta \|(q, \varphi)\|_{\mathbf{Q}} \|(\mathbf{u}, \mathbf{J})\|_{\mathbf{X}}. \end{aligned} \quad (33)$$

Thus, the bilinear form  $b$  satisfies an inf-sup condition with constant  $\beta = C_{\beta} \min(\beta_1, \beta_2)$ , where  $0 < C_{\beta} < 1/3$ . Remark that in order to guarantee the inf-sup condition (33), we need only to ensure that the ‘‘local’’ inf-sup conditions (31) and (32) hold for the velocity–pressure spaces, and for the electric currents–potential spaces, respectively. No additional compatibility condition is required.

6. Thanks to (33), Theorem 2.2 guarantees that there exists a unique solution  $(\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}$ ,  $(p, \phi) \in \mathbf{Q}_0$  of (27).  $\square$

### 2.3 A simplified linear problem

In some applications like in the case of weakly conductive fluids (e.g., salty water), or, more generally, for low magnetic Reynolds numbers, the magnetic field  $\mathcal{B}_{\Omega}(\mathbf{J}_{f,0})$  can be neglected in comparison to the assigned external magnetic fields, that we generically indicate by  $\mathbf{B}^*$  (see, e.g., [8]).

In such cases the MHD system reduces to a linear saddle-point problem in  $\Omega_f$ . Indeed, the nonlinear coupling terms (23) become:

$$- \int_{\Omega_f} [\mathbf{J}_f \times \mathbf{B}^*] \cdot \mathbf{v} \quad \text{and} \quad \int_{\Omega_f} [\mathbf{K} \times \mathbf{B}^*] \cdot \mathbf{u}.$$

This saddle-point problem in  $\Omega_f$  may also be encountered as a step of an iterative scheme to solve the MHD system (27), when one linearizes the problem in  $\Omega_f$  considering the magnetic field, say  $\mathbf{B}^{(k)}$ , computed at the previous iterate. We will illustrate this issue in Sect. 3.

In this simplified case, the well-posedness of the linear MHD problem can be proved using the classical theory by Brezzi [3] without any assumption on the physical data. Let us briefly show it.

For all  $(\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K}) \in \mathbf{X}$ , we define the bilinear form  $a_{lin}(\cdot, \cdot) : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$ ,

$$a_{lin}((\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K})) = \int_{\Omega_f} \eta \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \int_{\Omega_f} \sigma^{-1} \mathbf{J} \cdot \mathbf{K} - \int_{\Omega_f} [\mathbf{J} \times \mathbf{B}^*] \cdot \mathbf{v} + \int_{\Omega_f} [\mathbf{K} \times \mathbf{B}^*] \cdot \mathbf{u},$$

and the linear functional:  $\mathcal{F}_{lin} : \mathbf{X} \rightarrow \mathbb{R}$

$$\mathcal{F}_{lin}(\mathbf{v}, \mathbf{K}) = - \int_{\Omega_f} \eta \nabla(\mathbf{E}_f \mathbf{g}) \cdot \nabla \mathbf{v} - \int_{\Omega_f} \sigma^{-1} \mathbf{E}_s j_s \cdot \mathbf{K} + \int_{\Omega_f} [\mathbf{E}_s j_s \times \mathbf{B}^*] \cdot \mathbf{v} - \int_{\Omega_f} [\mathbf{K} \times \mathbf{B}^*] \cdot \mathbf{E}_f \mathbf{g}.$$

The weak form of the simplified linear MHD problem reads: find  $(\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}$ ,  $(p, \phi) \in \mathbf{Q}_0$  such that

$$\begin{aligned} a_{lin}((\mathbf{u}_0, \mathbf{J}_{f,0}), (\mathbf{v}, \mathbf{K})) + b((\mathbf{v}, \mathbf{K}), (p, \phi)) &= \mathcal{F}_{lin}(\mathbf{v}, \mathbf{K}) \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}, \\ b((\mathbf{u}_0, \mathbf{J}_{f,0}), (q, \varphi)) &= 0 \quad \forall (q, \varphi) \in \mathbf{Q}_0. \end{aligned} \quad (34)$$

We can prove the following result.

**Lemma 2.1** *Assuming that  $\mathbf{B}^* \in (L^3(\Omega_f))^3$ ,  $a_{lin}(\cdot, \cdot)$  is continuous on  $\mathbf{X} \times \mathbf{X}$  and coercive on  $\mathbf{X}_0 \times \mathbf{X}_0$ . Moreover,  $\mathcal{F}_{lin}$  is a continuous linear functional on  $\mathbf{X}$ .*

*Proof.* Concerning the continuity of  $a_{lin}$ , we have:

$$\begin{aligned} |a_{lin}((\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K}))| &\leq \eta \|\mathbf{u}\|_{H^1(\Omega_f)} \|\mathbf{v}\|_{H^1(\Omega_f)} + \sigma^{-1} \|\mathbf{J}\|_{\text{div}, \Omega_f} \|\mathbf{K}\|_{\text{div}, \Omega_f} \\ &\quad + C' \|\mathbf{B}^*\|_{L^3(\Omega_f)} (\|\mathbf{J}\|_{\text{div}, \Omega_f} \|\mathbf{v}\|_{H^1(\Omega_f)} + \|\mathbf{K}\|_{\text{div}, \Omega_f} \|\mathbf{u}\|_{H^1(\Omega_f)}) \\ &\leq \gamma_{lin} \|(\mathbf{u}, \mathbf{J})\|_{\mathbf{X}} \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{X}}, \end{aligned}$$

where  $\gamma_{lin} = 2 \max(\eta, \sigma^{-1}, C' \|\mathbf{B}^*\|_{L^3(\Omega_f)})$ . On the other hand,

$$a_{lin}((\mathbf{v}, \mathbf{K}), (\mathbf{v}, \mathbf{K})) \geq \frac{\eta}{1 + C_p^2} \|\mathbf{v}\|_{H^1(\Omega_f)}^2 + \sigma^{-1} \|\mathbf{K}\|_{\text{div}, \Omega_f}^2 \geq \alpha_{lin} \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{X}}^2, \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}_0,$$

with  $\alpha_{lin} = \min(\eta/(1 + C_p^2), \sigma^{-1})$ .

Finally, the functional  $\mathcal{F}_{lin}$  is continuous on  $\mathbf{X}$ :

$$\begin{aligned} |\mathcal{F}_{lin}(\mathbf{v}, \mathbf{K})| &\leq \eta \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \|\mathbf{v}\|_{H^1(\Omega_f)} + \sigma^{-1} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} \|\mathbf{K}\|_{\text{div}, \Omega_f} \\ &\quad + C' \|\mathbf{B}^*\|_{L^3(\Omega_f)} (\|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} \|\mathbf{v}\|_{H^1(\Omega_f)} + \|\mathbf{K}\|_{\text{div}, \Omega_f} \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)}) \\ &\leq C_{\mathcal{F}_{lin}} \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{X}}, \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X}, \end{aligned}$$

with constant

$$C_{\mathcal{F}_{lin}} = \sqrt{2} \max \left( \eta \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} + C' \|\mathbf{B}^*\|_{L^3(\Omega_f)} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f}, \right. \\ \left. \sigma^{-1} \|\mathbf{E}_s j_s\|_{\text{div}, \Omega_f} + C' \|\mathbf{B}^*\|_{L^3(\Omega_f)} \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \right).$$

□

Then, we can prove the well-posedness of (34).

**Proposition 2.1** *The linear coupled problem (34) admits a unique solution  $(\mathbf{u}_0, \mathbf{J}_{f,0}) \in \mathbf{X}$ ,  $(p, \phi) \in \mathbf{Q}_0$  which satisfies the a-priori estimates:*

$$\|(\mathbf{u}_0, \mathbf{J}_{f,0})\|_{\mathbf{X}} \leq \frac{C_{\mathcal{F}_{lin}}}{\alpha_{lin}}, \quad \|(p, \phi)\|_{\mathbf{Q}} \leq \frac{1}{\beta} \left( 1 + \frac{\gamma_{lin}}{\alpha_{lin}} \right) C_{\mathcal{F}_{lin}},$$

where  $\alpha_{lin}, \gamma_{lin}, C_{\mathcal{F}_{lin}}$  are defined in Lemma 2.1, and  $\beta$  is the inf-sup constant given in (33).

*Proof.* It is a straightforward consequence of the theorem by Brezzi [3], whose hypotheses are satisfied thanks to Lemma 2.1, and to the fact that the bilinear form  $b$  is continuous and fulfills an inf-sup condition as shown in Proposition 2.1.  $\square$

### 3 Iterative solution methods

The proof of the well-posedness of (27), which is essentially based on Banach's contraction theorem (see, e.g., [29]), suggests that the solution of the coupled problem might be computed by successive approximations. More precisely, starting from any  $(\mathbf{u}_0^{(0)}, \mathbf{J}_{f,0}^{(0)}) \in \mathbf{X}$ ,  $(p^{(0)}, \phi^{(0)}) \in \mathbf{Q}_0$ , one can construct the sequences  $(\mathbf{u}_0^{(k)}, \mathbf{J}_{f,0}^{(k)}) \in \mathbf{X}$ ,  $(p^{(k)}, \phi^{(k)}) \in \mathbf{Q}_0$  solving at each step  $k \geq 0$  the linearized problem

$$\begin{aligned} a((\mathbf{u}_0^{(k-1)}, \mathbf{J}_{f,0}^{(k-1)}); (\mathbf{u}_0^{(k)}, \mathbf{J}_{f,0}^{(k)}), (\mathbf{v}, \mathbf{K})) + b((\mathbf{v}, \mathbf{K}), (p^{(k)}, \phi^{(k)})) &= \mathcal{F}(\mathbf{v}, \mathbf{K}) \quad \forall (\mathbf{v}, \mathbf{K}) \in \mathbf{X} \\ b((\mathbf{u}_0^{(k)}, \mathbf{J}_{f,0}^{(k)}), (q, \varphi)) &= 0 \quad \forall (q, \varphi) \in \mathbf{Q}_0. \end{aligned}$$

This fixed-point scheme converges under the same hypotheses of Proposition 2.1.

Each iteration of this method requires to compute the magnetic field  $\mathcal{B}_\Omega(\mathbf{J}_{f,0})$ , i.e. to solve (15). This is usually done by introducing a vector potential  $\mathbf{A}$  such that  $\mathbf{curl} \mathbf{A} = \mathcal{B}_\Omega(\mathbf{J}_{f,0})$ , and to rewrite (15) as:

$$\begin{aligned} \mathbf{curl} (\mu^{-1} \mathbf{curl} \mathbf{A}) &= \bar{\mathbf{J}}_{f,0} \quad \text{in } \Omega, \\ \mathbf{A} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \quad (35)$$

Remark that the boundary condition  $\mathbf{A} \times \mathbf{n} = \mathbf{0}$  implies  $\mathcal{B}_\Omega(\mathbf{J}_{f,0}) \cdot \mathbf{n} = 0$  on  $\partial\Omega$ .

The problem (35) has not a unique solution, since adding any arbitrary gradient to  $\mathbf{A}$  gives another solution. Notice that in (21) and (24) we always consider  $\mathbf{curl} \mathbf{A}$ , so that the presence of an arbitrary gradient does not influence the fluid/electric-currents problem in  $\Omega_f$ . However, uniqueness may be recovered either looking for  $\mathbf{A}$  in the quotient space, say,  $\mathbf{H}_0(\mathbf{curl}; \Omega)/\nabla\varphi$ , or adding a consistent penalization divergence term exploiting the fact that  $\text{div} \mathbf{A} = 0$ .

Another possible approach is to regularize (35) by a perturbation term of order  $O(\varepsilon)$ :

$$\begin{aligned} \mathbf{curl} (\mu^{-1} \mathbf{curl} \mathbf{A}) + \varepsilon \mathbf{A} &= \bar{\mathbf{J}}_{f,0} \quad \text{in } \Omega, \\ \mathbf{A} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \quad (36)$$

$0 < \varepsilon \ll 1$  being a suitably chosen regularization parameter.

The weak form of (36) reads: find  $\mathbf{A} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$  such that

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{A} \cdot \mathbf{curl} \mathbf{W} + \varepsilon \int_{\Omega} \mathbf{A} \cdot \mathbf{W} = \int_{\Omega} \bar{\mathbf{J}}_{f,0} \cdot \mathbf{W} \quad \forall \mathbf{W} \in \mathbf{H}_0(\mathbf{curl}; \Omega). \quad (37)$$

Following the idea of successive approximations, we would like to set up an iterative method where the magnetic-field problem defined in  $\Omega$  can be solved separately from the fluid/electric-currents problem in the fluid domain  $\Omega_f$ . Possibly, also a splitting of the fluid and the electric-currents subproblems might be envisaged, in order to set up a completely decoupled scheme. Thus, we propose the following algorithm.

Given  $\mathbf{J}_{f,0}^{(0)} \in \mathbf{H}_0^0(\text{div}; \Omega_f)$ , let

$$\bar{\mathbf{J}}_{f,0}^{(0)} = \begin{cases} \mathbf{J}_{f,0}^{(0)} & \text{in } \Omega_f, \\ \mathbf{0} & \text{in } \Omega_s, \\ \mathbf{0} & \text{in } \Omega \setminus (\bar{\Omega}_f \cup \bar{\Omega}_s). \end{cases}$$

Then, for  $k \geq 0$ ,

1. find  $\mathbf{A}^{(k)} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$  such that

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{A}^{(k)} \cdot \mathbf{curl} \mathbf{W} + \varepsilon \int_{\Omega} \mathbf{A}^{(k)} \cdot \mathbf{W} = \int_{\Omega} \bar{\mathbf{J}}_{f,0}^{(k)} \cdot \mathbf{W} \quad \forall \mathbf{W} \in \mathbf{H}_0(\mathbf{curl}; \Omega); \quad (38)$$

2. find  $(\mathbf{u}_0^{(k+1)}, p^{(k+1)}) \in \mathbf{H}_0^1(\Omega_f) \times Q_0$ ,  $(\mathbf{J}_{f,0}^{(k+1)}, \phi^{(k+1)}) \in \mathbf{H}_0(\text{div}; \Omega_f) \times Q_0$  such that

$$\begin{aligned} & \int_{\Omega_f} \eta \nabla \mathbf{u}_0^{(k+1)} \cdot \nabla \mathbf{v} - \int_{\Omega_f} p^{(k+1)} \text{div } \mathbf{v} - \int_{\Omega_f} [\mathbf{J}_{f,0}^{(k+k_c)} \times (\mathbf{B}_e + \mathbf{B}_s + \mathbf{curl } \mathbf{A}^{(k)})] \cdot \mathbf{v} \\ & - \int_{\Omega_f} [\mathbf{E}_s j_s \times \mathbf{curl } \mathbf{A}^{(k)}] \cdot \mathbf{v} = - \int_{\Omega_f} \eta \nabla (\mathbf{E}_f \mathbf{g}) \cdot \nabla \mathbf{v} + \int_{\Omega_f} [\mathbf{E}_s j_s \times (\mathbf{B}_e + \mathbf{B}_s)] \cdot \mathbf{v} \end{aligned} \quad (39)$$

$$\int_{\Omega_f} q \text{div } \mathbf{u}_0^{(k+1)} = 0 \quad (40)$$

$$\begin{aligned} & \int_{\Omega_f} \sigma^{-1} \mathbf{J}_{f,0}^{(k+1)} \cdot \mathbf{K} - \int_{\Omega_f} \phi^{(k+1)} \text{div } \mathbf{K} + \int_{\Omega_f} [\mathbf{K} \times (\mathbf{B}_e + \mathbf{B}_s + \mathbf{curl } \mathbf{A}^{(k)})] \cdot \mathbf{u}_0^{(k+k_f)} \\ & + \int_{\Omega_f} [\mathbf{K} \times \mathbf{curl } \mathbf{A}^{(k)}] \cdot \mathbf{E}_f \mathbf{g} = - \int_{\Omega_f} \sigma^{-1} \mathbf{E}_s j_s \cdot \mathbf{K} - \int_{\Omega_f} [\mathbf{K} \times (\mathbf{B}_e + \mathbf{B}_s)] \cdot \mathbf{E}_f \mathbf{g} \end{aligned} \quad (41)$$

$$\int_{\Omega_f} \varphi \text{div } \mathbf{J}_{f,0}^{(k+1)} = 0 \quad (42)$$

for all  $\mathbf{v} \in \mathbf{H}_0^1(\Omega_f)$ ,  $q \in Q_0$ ,  $\mathbf{K} \in \mathbf{H}_0(\text{div}; \Omega_f)$ ,  $\varphi \in Q_0$ .  $k_c$  and  $k_f$  are suitably chosen iteration indices.

Algorithm (38)-(42) solves the magnetic-field problem in  $\Omega$  separately from the fluid/electric-currents problem in  $\Omega_f$ . Moreover, the latter can be dealt with in two ways according to the choice of the indices  $k_c$  and  $k_f$ . Indeed, taking  $k_c = 0$  and  $k_f = 1$ , the problem in  $\Omega_f$  is decoupled and we have to solve (39)-(40) before (41)-(42). On the other hand, if  $k_c = k_f = 1$ , the problem in  $\Omega_f$  is coupled, and at each iteration we have to solve a linear saddle-point problem like (34), where now  $\mathbf{B}^* = \mathbf{B}_e + \mathbf{B}_s + \mathbf{curl } \mathbf{A}^{(k)}$ .

Finally, an acceleration step might be considered:

$$\mathbf{J}_{f,0}^{(k+1)} \leftarrow \theta \mathbf{J}_{f,0}^{(k+1)} + (1 - \theta) \mathbf{J}_{f,0}^{(k)}, \quad \text{with } 0 < \theta \leq 1. \quad (43)$$

### 3.1 Convergence of the iterative method

We consider now the issue of convergence of (38)-(42). We take the case  $k_c = 0$  and  $k_f = 1$  (completely decoupled algorithm) and we assume  $\varepsilon = 0$ . We can prove the following convergence result, which also proves the existence and uniqueness of the solution of the MHD problem in its formulation with the five unknowns:  $\mathbf{A}$ ,  $\mathbf{u}$ ,  $p$ ,  $\mathbf{J}$ , and  $\phi$ .

**Proposition 3.1** *Under the same hypotheses of Proposition 2.1, there exists a positive constant  $\rho_J > 0$  such that if  $\mathbf{J}_{f,0}^{(k)} \in B_J$ , with  $B_J = \{\mathbf{J} \in \mathbf{H}_0(\text{div}; \Omega_f) \mid \|\mathbf{J}\|_{L^2(\Omega_f)} \leq \rho_J\}$ , then the iterations (38)-(42) converge to the unique solution of (27).*

*Proof.* The proof is based on Banach's contraction theorem and it consists essentially of two parts.

1. *There exists a positive value  $\rho_J > 0$  such that if  $\|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)} \leq \rho_J$ , then  $\|\mathbf{J}_{f,0}^{(k+1)}\|_{L^2(\Omega_f)} \leq \rho_J$ .*

Indeed, thanks to (17), there holds

$$\|\mathbf{curl } \mathbf{A}^{(k)}\|_{L^3(\Omega_f)} \leq C \mu \|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)}. \quad (44)$$

Setting  $\mathbf{v} = \mathbf{u}_0^{(k+1)}$  in (39) and using (44), we have:

$$\begin{aligned} \|\mathbf{u}_0^{(k+1)}\|_{H^1(\Omega_f)} & \leq \frac{C_3(1 + C_p^2)}{\eta} (\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} + C \mu \|\mathbf{E}_s j_s\|_{L^2(\Omega_f)}) \|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)} \\ & + C_3 C (1 + C_p^2) \frac{\mu}{\eta} \|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)}^2 + (1 + C_p^2) \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)}. \end{aligned} \quad (45)$$

Then, taking  $\mathbf{K} = \mathbf{J}_{f,0}^{(k+1)}$  in (41), we have

$$\begin{aligned} \sigma^{-1} \|\mathbf{J}_{f,0}^{(k+1)}\|_{L^2(\Omega_f)} &\leq \left( \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} + \|\mathbf{curl} \mathbf{A}^{(k)}\|_{L^3(\Omega_f)} \right) \\ &\quad \cdot \left( \|\mathbf{u}_0^{(k+1)}\|_{H^1(\Omega_f)} + \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \right) + \sigma^{-1} \|\mathbf{E}_s j_s\|_{L^2(\Omega_f)}. \end{aligned}$$

Finally, (44) and (45) give

$$\begin{aligned} \|\mathbf{J}_{f,0}^{(k+1)}\|_{L^2(\Omega_f)} &\leq \|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)}^3 C_3 C^2 (1 + C_p^2) \frac{\mu^2 \sigma}{\eta} \\ &\quad + \|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)}^2 C_3 C (1 + C_p^2) \frac{\mu \sigma}{\eta} (2\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} + C\mu \|\mathbf{E}_s j_s\|_{L^2(\Omega_f)}) \\ &\quad + \|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)} \left( C_3 (1 + C_p^2) \frac{\sigma}{\eta} \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} (\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} \right. \\ &\quad \left. + C\mu \|\mathbf{E}_s j_s\|_{L^2(\Omega_f)}) + C(2 + C_p^2) \mu \sigma \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \right) \\ &\quad + (2 + C_p^2) \sigma \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)}. \end{aligned} \quad (46)$$

Choosing  $\rho_J = 2(2 + C_p^2) \sigma \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)}$ , and assuming that  $\|\mathbf{J}_{f,0}^{(k)}\|_{L^2(\Omega_f)} \leq \rho_J$  (we may always take  $\|\mathbf{J}_{f,0}^{(0)}\|_{L^2(\Omega_f)} \leq \rho_J$ ), then  $\|\mathbf{J}_{f,0}^{(k+1)}\|_{L^2(\Omega_f)} \leq \rho_J$ , provided that, for example,  $\sigma$  is sufficiently small, or  $\eta$  is sufficiently large, or  $\mu$  and  $\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)}$  are sufficiently small.

- Let  $\mathbf{J}_i^{(k)} \in \mathbf{H}_0(\text{div}; \Omega_f)$  and let  $\mathbf{A}_i^{(k)}$ ,  $\mathbf{u}_i^{(k+1)}$  ( $i = 1, 2$ ) be the corresponding magnetic fields and fluid velocities given by (38) and (39), respectively. Then, it holds

$$\|\mathbf{curl} (\mathbf{A}_1^{(k)} - \mathbf{A}_2^{(k)})\|_{L^3(\Omega_f)} \leq C\mu \|\mathbf{J}_1^{(k)} - \mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)}. \quad (47)$$

Moreover,  $\mathbf{u}_1^{(k+1)} - \mathbf{u}_2^{(k+1)}$  satisfies

$$\begin{aligned} &\int_{\Omega_f} \eta \nabla (\mathbf{u}_1^{(k+1)} - \mathbf{u}_2^{(k+1)}) \cdot \nabla \mathbf{v} - \int_{\Omega_f} (p_1^{(k+1)} - p_2^{(k+1)}) \text{div} \mathbf{v} \\ &\quad - \int_{\Omega_f} [(\mathbf{J}_1^{(k)} - \mathbf{J}_2^{(k)}) \times (\mathbf{B}_e + \mathbf{B}_s)] \cdot \mathbf{v} - \int_{\Omega_f} [\mathbf{E}_s j_s \times \mathbf{curl} (\mathbf{A}_1^{(k)} - \mathbf{A}_2^{(k)})] \cdot \mathbf{v} \\ &\quad - \int_{\Omega_f} [\mathbf{J}_1^{(k)} \times \mathbf{curl} \mathbf{A}_1^{(k)}] \cdot \mathbf{v} + \int_{\Omega_f} [\mathbf{J}_2^{(k)} \times \mathbf{curl} \mathbf{A}_2^{(k)}] \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega_f). \end{aligned}$$

Using (47), we have

$$\begin{aligned} \|\mathbf{u}_1^{(k+1)} - \mathbf{u}_2^{(k+1)}\|_{H^1(\Omega_f)} &\leq \frac{C_3(1 + C_p^2)}{\eta} \left( \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} + C\mu \|\mathbf{E}_s j_s\|_{L^2(\Omega_f)} \right. \\ &\quad \left. + C\mu (\|\mathbf{J}_1^{(k)}\|_{L^2(\Omega_f)} + \|\mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)}) \right) \|\mathbf{J}_1^{(k)} - \mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)}. \end{aligned} \quad (48)$$

Finally, from (41) it follows

$$\begin{aligned} \|\mathbf{J}_1^{(k+1)} - \mathbf{J}_2^{(k+1)}\|_{L^2(\Omega_f)} &\leq C_3 \sigma \left( \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} \|\mathbf{u}_1^{(k+1)} - \mathbf{u}_2^{(k+1)}\|_{H^1(\Omega_f)} \right. \\ &\quad + \|\mathbf{curl} (\mathbf{A}_1^{(k)} - \mathbf{A}_2^{(k)})\|_{L^3(\Omega_f)} \|\mathbf{E}_f \mathbf{g}\|_{H^1(\Omega_f)} \\ &\quad + \|\mathbf{curl} \mathbf{A}_1^{(k)}\|_{L^3(\Omega_f)} \|\mathbf{u}_1^{(k+1)} - \mathbf{u}_2^{(k+1)}\|_{H^1(\Omega_f)} \\ &\quad \left. + \|\mathbf{curl} (\mathbf{A}_1^{(k)} - \mathbf{A}_2^{(k)})\|_{L^3(\Omega_f)} \|\mathbf{u}_2^{(k+1)}\|_{H^1(\Omega_f)} \right). \end{aligned}$$

Thus, thanks to (47) and (48), we obtain

$$\begin{aligned}
\|\mathbf{J}_1^{(k+1)} - \mathbf{J}_2^{(k+1)}\|_{L^2(\Omega_f)} &\leq C_3\sigma\|\mathbf{J}_1^{(k)} - \mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)} \\
&\quad \cdot \left( C_3(1 + C_p^2) \frac{\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)}^2}{\eta} + (2 + C_p^2)C\mu\|\mathbf{E}_f\mathbf{g}\|_{H^1(\Omega_f)} \right. \\
&\quad + 2CC_3(1 + C_p^2) \frac{\mu}{\eta} \|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} \|\mathbf{E}_s j_s\|_{L^2(\Omega_f)} \\
&\quad + C_3C(1 + C_p^2) \frac{\mu}{\eta} \left( 2\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)} + C\mu\|\mathbf{E}_s j_s\|_{L^2(\Omega_f)} \right) \\
&\quad \cdot \left( \|\mathbf{J}_1^{(k)}\|_{L^2(\Omega_f)} + \|\mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)} \right) \\
&\quad + C_3C^2(1 + C_p^2) \frac{\mu^2}{\eta} \left( \|\mathbf{J}_1^{(k)}\|_{L^2(\Omega_f)}^2 \right. \\
&\quad \left. + \|\mathbf{J}_1^{(k)}\|_{L^2(\Omega_f)} \|\mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)} + \|\mathbf{J}_2^{(k)}\|_{L^2(\Omega_f)}^2 \right).
\end{aligned}$$

We can conclude that if  $\sigma$  is small enough, or  $\mu$  and  $\|\mathbf{B}_e + \mathbf{B}_s\|_{L^3(\Omega_f)}$  are small enough, or  $\eta$  is large enough, then the mapping  $\mathbf{J}_{f,0}^{(k)} \rightarrow \mathbf{J}_{f,0}^{(k+1)}$  is a contraction in  $B_J$ .  $\square$

## 4 A conforming finite element approximation

In this section we introduce a conforming finite element approximation of the MHD problem (24), (20)-(22), where  $\mathcal{B}_\Omega(\mathbf{J}_{f,0})$  is expressed as the curl of the auxiliary field  $\mathbf{A}$  (37).

We consider a triangulation  $\mathcal{T}_h$  of the computational domain  $\overline{\Omega}$ , depending on a positive parameter  $h > 0$ , made up of tetrahedra. We assume that  $\mathcal{T}_h$  is regular and that the triangulations induced on the subdomains  $\Omega \setminus \overline{\Omega}_f$  and  $\Omega_f$  are compatible on  $\partial\Omega_f$ , i.e. they share the same faces therein. We denote by  $\mathbb{P}_n$ , with  $n$  a non-negative integer, the space of algebraic polynomials of degree less or equal to  $n$  in the variable  $\mathbf{x} \in \mathbb{R}^3$ , and by  $\tilde{\mathbb{P}}_n$  the homogenous polynomials of total degree equal to  $n$  in  $\mathbf{x}$ . Moreover, let  $\mathbb{D}_n = (\mathbb{P}_{n-1})^3 \oplus \tilde{\mathbb{P}}_{n-1}\mathbf{x}$  ( $n > 1$ ), and  $R_n = (\mathbb{P}_{n-1})^3 \oplus S_n$ , with  $S_n = \{\mathbf{p} \in (\tilde{\mathbb{P}}_n)^3 \mid \mathbf{x} \cdot \mathbf{p}(\mathbf{x}) = 0\}$  (see, e.g., [22]).

Then, we define the  $\mathbf{H}(\mathbf{curl}; \Omega)$ -conforming discrete space (Nédélec elements):

$$\mathbf{Y}_l^A = \{\mathbf{a}_h \in \mathbf{H}_0(\mathbf{curl}; \Omega) \mid \mathbf{a}_{h|K} \in R_l \quad \forall K \in \mathcal{T}_h\}, \quad l > 1; \quad (49)$$

the Raviart-Thomas elements:

$$\mathbf{Y}_m^J = \{\mathbf{y}_h \in \mathbf{H}_0(\text{div}; \Omega_f) \mid \mathbf{y}_{h|K} \in \mathbb{D}_m \quad \forall K \in \mathcal{T}_h\}, \quad m \geq 1, \quad (50)$$

$$W_{m-1}^\phi = \{\psi_h \in Q_0 \mid \psi_{h|K} \in \mathbb{P}_{m-1} \quad \forall K \in \mathcal{T}_h\}; \quad (51)$$

the Taylor-Hood elements:

$$\mathbf{W}_r^u = \{\mathbf{v}_h \in (C^0(\overline{\Omega}_f))^3 \mid \mathbf{v}_{h|K} \in \mathbb{P}_r \quad \forall K \in \mathcal{T}_h\} \cap \mathbf{H}_0^1(\Omega_f), \quad r \geq 2, \quad (52)$$

$$W_{r-1}^p = \{q_h \in (C^0(\overline{\Omega}_f))^3 \mid q_{h|K} \in \mathbb{P}_{r-1} \quad \forall K \in \mathcal{T}_h\} \cap Q_0. \quad (53)$$

Let us consider the approximation of the boundary data. We assume that  $\mathbf{g} \in (H^{1/2}(\partial\Omega_f))^3 \cap (C^0(\partial\Omega_f))^3$ , and we consider a suitable interpolant, say  $\mathbf{g}_h$ . Then, following the guidelines of the continuous case, we can construct a discrete divergence-free extension  $\mathbf{E}_f^h \mathbf{g}_h \in \widetilde{\mathbf{W}}_r^u$ , with  $\widetilde{\mathbf{W}}_r^u = \{\mathbf{v}_h \in (C^0(\overline{\Omega}_f))^3 \mid \mathbf{v}_{h|K} \in \mathbb{P}_r \quad \forall K \in \mathcal{T}_h\}$ . Let us point out that to define  $\mathbf{E}_f^h \mathbf{g}_h$  we should consider the discrete counterpart of (14), whose solvability is now guaranteed since the Taylor-Hood elements satisfy the inf-sup condition: there exists a positive constant  $\beta_{TH} > 0$ , independent

of  $h$ , such that

$$\forall q_h \in W_{r-1}^p, \exists \mathbf{v}_h \in \mathbf{W}_r^u, \mathbf{v}_h \neq \mathbf{0} : \int_{\Omega_f} q_h \operatorname{div} \mathbf{v}_h \geq \beta_{TH} \|\mathbf{v}_h\|_{H^1(\Omega_f)} \|q_h\|_{L^2(\Omega_f)}. \quad (54)$$

Moreover, supposing that  $j_s \in L^2(\Gamma_s) \cap C^0(\Gamma_s)$ , we can proceed in an analogous way to construct the divergence-free extension  $\mathbf{E}_s^h j_s^h \in \tilde{\mathbf{Y}}_m^J$  of a suitable approximation  $j_s^h$  of  $j_s$ , where  $\tilde{\mathbf{Y}}_m^J = \{\mathbf{y}_h \in \mathbf{H}(\operatorname{div}; \Omega_f) \mid \mathbf{y}_h|_K \in \mathbb{D}_m \forall K \in \mathcal{T}_h\}$ , and  $\mathbf{E}_s^h j_s^h \cdot \mathbf{n} = j_s^h$  on  $\Gamma_s$ ,  $\mathbf{E}_s^h j_s^h \cdot \mathbf{n} = 0$  on  $\partial\Omega_f \setminus \Gamma_s$ . More precisely, taken  $\tilde{\mathbf{E}}_s^h j_s^h \in \tilde{\mathbf{Y}}_m^J$  such that  $\tilde{\mathbf{E}}_s^h j_s^h \cdot \mathbf{n} = j_s^h$  on  $\Gamma_s$  and  $\tilde{\mathbf{E}}_s^h j_s^h \cdot \mathbf{n} = 0$  on  $\partial\Omega_f \setminus \Gamma_s$ , we construct an auxiliary function  $\hat{\mathbf{E}}_s^h j_s^h \in \mathbf{Y}_m^J$  such that

$$- \int_{\Omega_f} q_h \operatorname{div} (\hat{\mathbf{E}}_s^h j_s^h) = \int_{\Omega_f} q_h \operatorname{div} (\tilde{\mathbf{E}}_s^h j_s^h) \quad \forall q_h \in W_{m-1}^\phi.$$

The solvability of this problem is guaranteed by the inf-sup condition (see [27]): there exists a positive constant  $\beta_{RT} > 0$ , independent of  $h$ , such that

$$\forall q_h \in W_{m-1}^\phi, \exists \mathbf{v}_h \in \mathbf{Y}_m^J, \mathbf{v}_h \neq \mathbf{0} : \int_{\Omega_f} q_h \operatorname{div} \mathbf{v}_h \geq \beta_{RT} \|\mathbf{v}_h\|_{\operatorname{div}, \Omega_f} \|q_h\|_{L^2(\Omega_f)}. \quad (55)$$

Finally, let  $\mathbf{E}_s^h j_s^h = \tilde{\mathbf{E}}_s^h j_s^h + \hat{\mathbf{E}}_s^h j_s^h$ , and let us split  $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{E}_f^h \mathbf{g}_h$ , with  $\mathbf{u}_{0,h} \in \mathbf{W}_r^u$ , and  $\mathbf{J}_{f,h} = \mathbf{J}_{0,h} + \mathbf{E}_s^h j_s^h$ , with  $\mathbf{J}_{0,h} \in \mathbf{Y}_m^J$ .

Then, the conforming finite element approximation of the MHD problem reads: find  $\mathbf{A}_h \in \mathbf{Y}_l^A$ ,  $\mathbf{u}_{0,h} \in \mathbf{W}_r^u$ ,  $p_h \in W_{r-1}^p$ ,  $\mathbf{J}_{0,h} \in \mathbf{Y}_m^J$ ,  $\phi_h \in W_{m-1}^\phi$  such that

$$\int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{A}_h \cdot \operatorname{curl} \mathbf{W}_h + \varepsilon \int_{\Omega} \mathbf{A}_h \cdot \mathbf{W}_h - \int_{\Omega} \bar{\mathbf{J}}_{0,h} \cdot \mathbf{W}_h = 0 \quad (56)$$

$$\begin{aligned} \int_{\Omega_f} \eta \nabla \mathbf{u}_{0,h} \cdot \nabla \mathbf{v}_h - \int_{\Omega_f} p_h \operatorname{div} \mathbf{v}_h - \int_{\Omega_f} [\mathbf{J}_{0,h} \times (\mathbf{B}_e^h + \mathbf{B}_s^h + \operatorname{curl} \mathbf{A}_h)] \cdot \mathbf{v}_h \\ - \int_{\Omega_f} [\mathbf{E}_s^h j_s^h \times \operatorname{curl} \mathbf{A}_h] \cdot \mathbf{v}_h = - \int_{\Omega_f} \eta \nabla (\mathbf{E}_f^h \mathbf{g}_h) \cdot \nabla \mathbf{v}_h + \int_{\Omega_f} [\mathbf{E}_s^h j_s^h \times (\mathbf{B}_e^h + \mathbf{B}_s^h)] \cdot \mathbf{v}_h \end{aligned} \quad (57)$$

$$\int_{\Omega_f} q_h \operatorname{div} \mathbf{u}_{0,h} = 0 \quad (58)$$

$$\begin{aligned} \int_{\Omega_f} \sigma^{-1} \mathbf{J}_{0,h} \cdot \mathbf{K}_h - \int_{\Omega_f} \phi_h \operatorname{div} \mathbf{K}_h + \int_{\Omega_f} [\mathbf{K}_h \times (\mathbf{B}_e^h + \mathbf{B}_s^h + \operatorname{curl} \mathbf{A}_h)] \cdot \mathbf{u}_{0,h} \\ + \int_{\Omega_f} [\mathbf{K}_h \times \operatorname{curl} \mathbf{A}_h] \cdot \mathbf{E}_f^h \mathbf{g}_h = - \int_{\Omega_f} \sigma^{-1} \mathbf{E}_s^h j_s^h \cdot \mathbf{K}_h - \int_{\Omega_f} [\mathbf{K}_h \times (\mathbf{B}_e^h + \mathbf{B}_s^h)] \cdot \mathbf{E}_f^h \mathbf{g}_h \end{aligned} \quad (59)$$

$$\int_{\Omega_f} \varphi_h \operatorname{div} \mathbf{J}_{0,h} = 0 \quad (60)$$

for all  $\mathbf{W}_h \in \mathbf{Y}_l^A$ ,  $\mathbf{v}_h \in \mathbf{W}_r^u$ ,  $q_h \in W_{r-1}^p$ ,  $\mathbf{K}_h \in \mathbf{Y}_m^J$ ,  $\varphi_h \in W_{m-1}^\phi$ .

We have denoted by  $\mathbf{B}_e^h$  and  $\mathbf{B}_s^h$  two suitable approximations of the magnetic fields  $\mathbf{B}_e$  and  $\mathbf{B}_s$ , respectively, on the space  $\tilde{\mathbf{Y}}_l^A = \{\mathbf{a}_h \in \mathbf{H}(\operatorname{curl}; \Omega) \mid \mathbf{a}_h|_K \in R_l \forall K \in \mathcal{T}_h\}$ . Notice that, as a consequence of the discrete de Rham diagram (see, e.g., [22]),  $\operatorname{curl} \mathbf{A}_h \in \tilde{\mathbf{Y}}_{l-1}^J$  provided that  $\mathbf{A}_h \in \mathbf{Y}_l^A$ .

The inf-sup conditions (54), (55) suffice to ensure the stability of the coupled problem: no additional compatibility condition is required between the discrete spaces. This can be seen considering the discrete counterpart of the linear coupled problem (34): find  $(\mathbf{u}_{0,h}, \mathbf{J}_{0,h}) \in \mathbf{X}_h$ ,  $(p_h, \phi_h) \in \mathbf{Q}_0^h$  such that

$$\begin{aligned} a_{lin}^h((\mathbf{u}_{0,h}, \mathbf{J}_{0,h}), (\mathbf{v}_h, \mathbf{K}_h)) + b((\mathbf{v}_h, \mathbf{K}_h), (p_h, \phi_h)) &= \mathcal{F}_{lin}^h(\mathbf{v}_h, \mathbf{K}_h) \quad \forall (\mathbf{v}_h, \mathbf{K}_h) \in \mathbf{X}_h, \\ b((\mathbf{u}_{0,h}, \mathbf{J}_{0,h}), (q_h, \varphi_h)) &= 0 \quad \forall (q_h, \varphi_h) \in \mathbf{Q}_0^h, \end{aligned} \quad (61)$$



where  $\mathbf{X}_h = \mathbf{W}_r^u \times \mathbf{Y}_m^J$ ,  $\mathbf{Q}_0^h = W_{r-1}^p \times W_{m-1}^\phi$ , and  $a_{lin}^h(\cdot, \cdot) : \mathbf{X}_h \times \mathbf{X}_h \rightarrow \mathbb{R}$ ,

$$a_{lin}^h((\mathbf{u}_h, \mathbf{J}_h), (\mathbf{v}_h, \mathbf{K}_h)) = \int_{\Omega_f} \eta \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h + \int_{\Omega_f} \sigma^{-1} \mathbf{J}_h \cdot \mathbf{K}_h - \int_{\Omega_f} [\mathbf{J}_h \times \mathbf{B}_h^*] \cdot \mathbf{v}_h + \int_{\Omega_f} [\mathbf{K}_h \times \mathbf{B}_h^*] \cdot \mathbf{u}_h.$$

The definition of the linear functional  $\mathcal{F}_{lin}^h$  is straightforward.

Thanks to (54), (55), the bilinear form  $b(\cdot, \cdot)$  satisfies an inf-sup condition with constant  $\beta^* > 0$ ,  $\beta^* = C_\beta \min(\beta_{TH}, \beta_{RT})$ , independent of  $h$ . Remark that, the mixed coupling terms

$$- \int_{\Omega_f} [\mathbf{J}_h \times \mathbf{B}_h^*] \cdot \mathbf{v}_h + \int_{\Omega_f} [\mathbf{K}_h \times \mathbf{B}_h^*] \cdot \mathbf{u}_h \quad (62)$$

vanish for  $(\mathbf{v}_h, \mathbf{K}_h) = (\mathbf{u}_h, \mathbf{J}_h)$ .

The well-posedness of (61) can be easily proved following the same ideas of Proposition 2.1. For the sake of completeness, let us point out that its solution satisfy the error estimates:

$$\begin{aligned} \|(\mathbf{u}, \mathbf{J}_f) - (\mathbf{u}_h, \mathbf{J}_{f,h})\|_{\mathbf{X}} &\leq \left(1 + \frac{\tilde{\gamma}_{lin}}{\tilde{\alpha}_{lin}}\right) \inf_{(\mathbf{v}_h, \mathbf{K}_h) \in \mathbf{X}_h^0} \|(\mathbf{u}, \mathbf{J}_f) - (\mathbf{v}_h, \mathbf{K}_h)\|_{\mathbf{X}} \\ &\quad + \frac{1}{\tilde{\alpha}_{lin}} \inf_{(q_h, \varphi_h) \in \mathbf{Q}_0^h} \|(p, \phi) - (q_h, \varphi_h)\|_{\mathbf{Q}}, \\ \|(p, \phi) - (p_h, \varphi_h)\|_{\mathbf{Q}} &\leq \frac{\tilde{\gamma}_{lin}}{\beta^*} \left(1 + \frac{\tilde{\gamma}_{lin}}{\tilde{\alpha}_{lin}}\right) \inf_{(\mathbf{v}_h, \mathbf{K}_h) \in \mathbf{X}_h^0} \|(\mathbf{u}, \mathbf{J}_f) - (\mathbf{v}_h, \mathbf{K}_h)\|_{\mathbf{X}} \\ &\quad + \left(1 + \frac{1}{\beta^*} + \frac{\tilde{\gamma}_{lin}}{\tilde{\alpha}_{lin} \beta^*}\right) \inf_{(q_h, \varphi_h) \in \mathbf{Q}_0^h} \|(p, \phi) - (q_h, \varphi_h)\|_{\mathbf{Q}}, \\ \inf_{(\mathbf{v}_h, \mathbf{K}_h) \in \mathbf{X}_h^0} \|(\mathbf{u}, \mathbf{J}_f) - (\mathbf{v}_h, \mathbf{K}_h)\|_{\mathbf{X}} &\leq \left(1 + \frac{1}{\beta^*}\right) \inf_{(\mathbf{v}_h, \mathbf{K}_h) \in \mathbf{X}_h} \|(\mathbf{u}, \mathbf{J}_f) - (\mathbf{v}_h, \mathbf{K}_h)\|_{\mathbf{X}}, \end{aligned}$$

where  $\mathbf{X}_h^0 \subset \mathbf{X}_h$  is the kernel of  $b(\cdot, \cdot)$ .  $\tilde{\alpha}_{lin} > 0$  and  $\tilde{\gamma}_{lin} > 0$  are the  $h$ -independent coercivity and continuity constants of  $a_{lin}^h(\cdot, \cdot)$ , respectively.

Finally, we point out that the iterative scheme (38)-(42) can be replicated at the discrete stage, and a convergence result analogous to Proposition 3.1 can be proved. This also ensures the existence and uniqueness of the solution of the finite element approximation (56)-(60) of the MHD problem.

## 4.1 Algebraic formulation of the iterative method

In this section we consider the algebraic formulation of the iterative method (38)-(42) to compute the solution of the MHD problem.

Let us denote by  $\{\mathbf{a}_h^j\}_{j=1, \dots, N_l^A}$ ,  $\{\mathbf{y}_h^j\}_{j=1, \dots, N_m^J}$ ,  $\{\psi_h^j\}_{j=1, \dots, N_{m-1}^\phi}$ ,  $\{\mathbf{v}_h^j\}_{j=1, \dots, N_r^u}$ ,  $\{q_h^j\}_{j=1, \dots, N_{r-1}^p}$ , the bases of the finite element spaces  $\mathbf{Y}_l^A$ ,  $\mathbf{Y}_m^J$ ,  $W_{m-1}^\phi$ ,  $\mathbf{W}_r^u$ ,  $W_{r-1}^p$ , respectively. We define the following matrices:

$$(\mathbf{M})_{ij} = \int_{\Omega} \mu^{-1} (\mathbf{curl} \mathbf{a}_h^j) \cdot (\mathbf{curl} \mathbf{a}_h^i) + \varepsilon \int_{\Omega} \mathbf{a}_h^j \cdot \mathbf{a}_h^i, \quad i, j = 1, \dots, N_l^A, \quad (63)$$

$$(\mathbf{F})_{ij} = \int_{\Omega} \mathbf{y}_h^j \cdot \mathbf{a}_h^i, \quad i = 1, \dots, N_l^A, \quad j = 1, \dots, N_m^J, \quad (64)$$

$$(\mathbf{A})_{ij} = \int_{\Omega_f} \eta \nabla \mathbf{v}_h^j \cdot \nabla \mathbf{v}_h^i, \quad i, j = 1, \dots, N_r^u, \quad (65)$$

$$(\mathbf{B})_{ij} = - \int_{\Omega_f} q_h^i \operatorname{div} \mathbf{v}_h^j, \quad i = 1, \dots, N_{r-1}^p, \quad j = 1, \dots, N_r^u, \quad (66)$$

$$(\mathbf{D})_{ij} = \int_{\Omega_f} \sigma^{-1} \mathbf{y}_h^j \cdot \mathbf{y}_h^i, \quad i, j = 1, \dots, N_m^J, \quad (67)$$

$$(\mathbf{E})_{ij} = - \int_{\Omega_f} \psi_h^i \operatorname{div} \mathbf{y}_h^j, \quad i = 1, \dots, N_{m-1}^\phi, j = 1, \dots, N_m^J, \quad (68)$$

$$(\mathbf{C}(\mathbf{B}_h))_{ij} = - \int_{\Omega_f} (\mathbf{y}_h^i \times \mathbf{B}_h) \cdot \mathbf{v}_h^j, \quad i = 1, \dots, N_m^J, j = 1, \dots, N_r^u. \quad (69)$$

In the definition of matrix  $\mathbf{C}$  we have denoted by  $\mathbf{B}_h$  a generic assigned vector function in  $\widetilde{\mathbf{Y}}_m^J$ . This matrix provides the algebraic counterpart of the terms in (62), and it realizes the coupling between the fluid and electric-currents problems in  $\Omega_f$ .

Finally, let us point out that, since the functions  $\mathbf{y}_h^j \in \mathbf{Y}_m^J$  are only defined in  $\Omega_f$ , they are to be intended as extended by zero in  $\Omega \setminus \overline{\Omega}_f$  in (64).

Let us now consider the algebraic formulation of (38)-(42). With a slight abuse in notation, we will use the same symbols to denote the discrete functions and the vectors of the coefficients of their linear expansions on the bases of the finite element spaces.

Given  $\mathbf{J}_{0,h}^{(0)} \in \mathbb{R}^{N_m^J}$ , let  $\overline{\mathbf{J}}_{0,h}^{(0)}$  be its extension by zero in the nodes in  $\Omega \setminus \overline{\Omega}_f$ . Then, for  $k \geq 0$ ,

1. compute the right-hand side  $(\mathbf{f}_A^{(k)})_i = \int_{\Omega} \overline{\mathbf{J}}_{0,h}^{(k)} \cdot \mathbf{a}_h^i$  ( $i = 1, \dots, N_l^A$ ), for example performing the matrix-vector product  $\mathbf{f}_A^{(k)} = \mathbf{F} \cdot \overline{\mathbf{J}}_{0,h}^{(k)}$ .

Then, solve the linear system

$$\mathbf{M}\mathbf{A}_h^{(k)} = \mathbf{f}_A^{(k)}. \quad (70)$$

2. Compute  $\mathbf{B}_h^{(k)} = \mathbf{curl} \mathbf{A}_h^{(k)}$ . Then, update the matrix  $\mathbf{C}^{(k)}$ :

$$\mathbf{C}^{(k)} = \mathbf{C}(\mathbf{B}_e^h + \mathbf{B}_s^h + \mathbf{B}_h^{(k)}),$$

and the vectors

$$\begin{aligned} (\mathbf{f}_{u,1}^{(k)})_i &= \int_{\Omega_f} [\mathbf{E}_s^h \mathbf{j}_s^h \times \mathbf{B}_h^{(k)}] \cdot \mathbf{v}_h^i, \quad i = 1, \dots, N_r^u, \\ (\mathbf{f}_{J,1}^{(k)})_i &= - \int_{\Omega_f} [\mathbf{y}_h^i \times \mathbf{B}_h^{(k)}] \cdot \mathbf{E}_f^h \mathbf{g}_h, \quad i = 1, \dots, N_m^J. \end{aligned}$$

3. At this stage we consider separately the two cases:  $k_c = 0, k_f = 1$  (decoupled strategy), and  $k_c = k_f = 1$  (coupled approach).

3.1. In the first case, i.e. when  $k_c = 0$  and  $k_f = 1$ , we perform the following steps.

- Compute the vector  $\mathbf{f}_{u,2}^{(k)} = -(\mathbf{C}^{(k)})^T \mathbf{J}_{0,h}^{(k)}$ , and solve the linear system

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0,h}^{(k+1)} \\ \mathbf{p}_h^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_u^{(k)} \\ \mathbf{0} \end{pmatrix}, \quad (71)$$

where  $\mathbf{f}_u^{(k)} = \mathbf{f}_{u,1}^{(k)} + \mathbf{f}_{u,2}^{(k)} + \mathbf{f}_{u,bc}$ , and  $\mathbf{f}_{u,bc}$  accounts for the boundary conditions of the Stokes problem.

- Then, compute  $\mathbf{f}_{J,2}^{(k+1)} = \mathbf{C}^{(k)} \mathbf{u}_{0,h}^{(k+1)}$ , and solve the linear system

$$\begin{pmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{J}_{0,h}^{(k+1)} \\ \phi_h^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_J^{(k+1)} \\ \mathbf{0} \end{pmatrix}, \quad (72)$$

where  $\mathbf{f}_J^{(k+1)} = \mathbf{f}_{J,1}^{(k)} + \mathbf{f}_{J,2}^{(k+1)} + \mathbf{f}_{J,bc}$ , and  $\mathbf{f}_{J,bc}$  accounts for the boundary conditions of the electric-currents problem.

3.2. On the other hand, the coupled approach, which corresponds to the choice of the indices  $k_c = k_f = 1$ , requires to solve the following  $(4 \times 4)$ -block linear system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T & \mathbf{C}^T & 0 \\ \mathbf{B} & 0 & 0 & 0 \\ -\mathbf{C} & 0 & \mathbf{D} & \mathbf{E}^T \\ 0 & 0 & \mathbf{E} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_{0,h}^{(k+1)} \\ p_h^{(k+1)} \\ \mathbf{J}_{0,h}^{(k+1)} \\ \phi_h^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_u^{(k)} \\ \mathbf{0} \\ \mathbf{f}_J^{(k)} \\ \mathbf{0} \end{pmatrix}, \quad (73)$$

where now  $\mathbf{f}_u^{(k)} = \mathbf{f}_{u,1}^{(k)} + \mathbf{f}_{u,bc}$ , and  $\mathbf{f}_J^{(k)} = \mathbf{f}_{J,1}^{(k)} + \mathbf{f}_{J,bc}$ .

At the algebraic level, the decoupled approach (71)-(72) corresponds to perform a Gauss-Seidel step on the linear system (73). This strategy is quite interesting since it allows us to deal independently with the three linear systems (70), (71) and (72) of reduced size, for which effective preconditioning techniques known in the literature may be applied.

## 5 Numerical tests

In this section we present some numerical tests obtained considering the finite element discretization proposed in Sect. 4 and the iterative method that we have analyzed in Sects. 3 and 4.1. The implementation has been done within the 3D finite element solver NGSolve [23] and the linear systems have been solved using the direct solver PARDISO [24].

First, we consider the simplified problem discussed in Sect. 2.3. Let  $\Omega_f = (0, 1)^3$  and assume that the imposed magnetic field  $\mathbf{B}^*$  grows linearly in the  $z$ -direction:  $\mathbf{B}^* = (0, 0, 10^{-4}x)^T$  T. We consider a no-slip boundary condition  $\mathbf{u} = \mathbf{0}$  on  $\partial\Omega_f$ , and we impose a difference of electric potential on two opposite faces:  $\phi = -10^{-5}$  V on  $\partial\Omega_f \cap \{x = 0\}$ , and  $\phi = 10^{-5}$  V on  $\partial\Omega_f \cap \{x = 1\}$ . The remaining part of the boundary is supposed to be non-conductive, i.e.  $\mathbf{J}_f \cdot \mathbf{n} = 0$  therein. Finally, the physical parameters are set to represent a melted metal (e.g., aluminum at  $700^\circ\text{C}$ ): in particular, the kinematic viscosity of the fluid is  $6 \cdot 10^{-7}$  m<sup>2</sup>/s, its density  $2.38 \cdot 10^3$  kg/m<sup>3</sup> and its electric conductivity  $\sigma = 5.1 \cdot 10^6$   $\Omega^{-1}\text{m}^{-1}$ .

Concerning the finite element discretization, we consider Taylor-Hood elements of order  $r = 2$ , Raviart-Thomas elements of order  $m = 2$ , and we approximate the magnetic field  $\mathbf{B}^*$  using Nédélec elements of order  $l = 1$ . The computational grid is composed of 768 tetrahedra with 23488 degrees of freedom.

In fig. 2 we show the computed velocity field  $\mathbf{u}$  and the electric current  $\mathbf{J}_f$ . We can see that the Lorentz force exerted on the fluid sets it in rotational motion.

We turn now to the coupled problem (27). We consider a rectangular domain  $\Omega_f$  as represented in fig. 3 (left), placed between two parallel conductive wires at a distance of 0.25 m from its lateral walls, and 0.25 m lower than its bottom surface. In both wires an electric current is assigned to originate the magnetic field

$$\mathbf{B}_e = \frac{10^{-4}}{(y + 0.25)^2 + (z - 0.25)^2} \begin{pmatrix} 0 \\ -z + 0.25 \\ y + 0.25 \end{pmatrix} + \frac{10^{-4}}{(y - 1.25)^2 + (z - 0.25)^2} \begin{pmatrix} 0 \\ -z + 0.25 \\ y - 1.25 \end{pmatrix} \text{ T.}$$

In fig. 3 (right) we represent  $\mathbf{B}_{e|\Omega_f}$  (see also [9, 10]).

The no-slip condition  $\mathbf{u} = \mathbf{0}$  is imposed on  $\partial\Omega_f$ . Concerning the electric currents, the lateral boundary is assumed to be insulated, i.e.  $\mathbf{J}_f \cdot \mathbf{n} = 0$  therein, while we impose  $\mathbf{J}_f \cdot \mathbf{n} = 20$  A on the top and bottom surfaces. The physical parameters are taken as in the previous test case.

We adopt two uniform computational meshes with increasing number of tetrahedra, and we consider a few possible choices for the degrees of the polynomials used for the finite element approximation.

The coupled problem is solved considering the iterative algorithm (70)-(72), that is using the fully decoupled approach corresponding to (38)-(42) with  $k_c = 0$  and  $k_f = 1$ . We adopt a relaxation

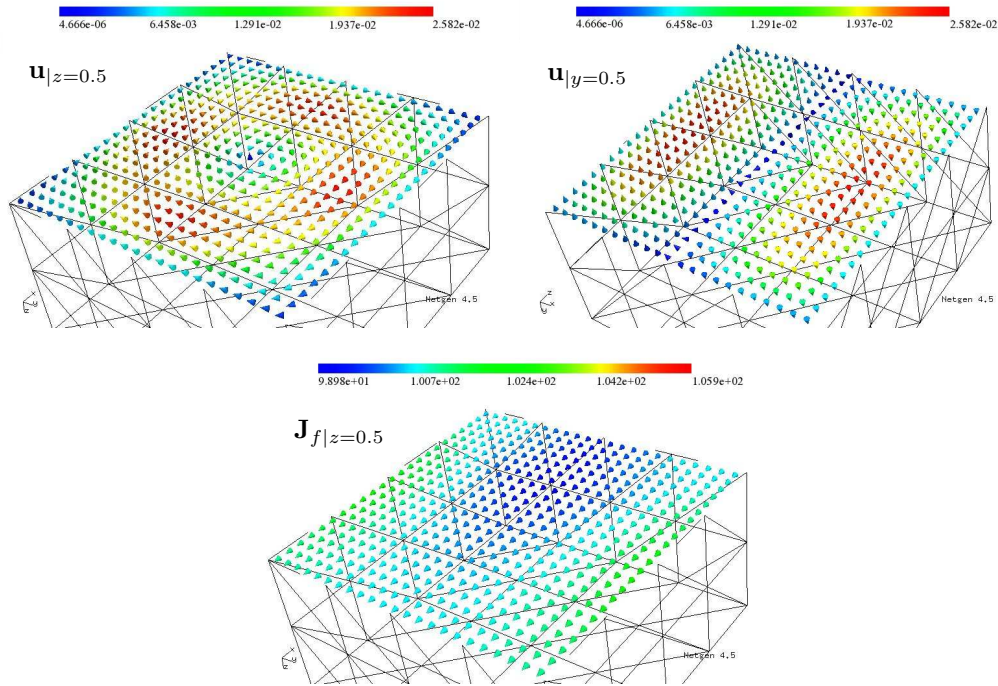


Figure 2: Computed solution for the test case with fixed magnetic field  $\mathbf{B}^*$ : velocity field across the plane  $z = 0.5$  (top right) and  $y = 0.5$  (top left), and electric current  $\mathbf{J}_f$  across the plane  $z = 0.5$  (bottom).

like (43) where we set  $\theta = 0.4$ . We use a stopping criterion based on a relative-increment test with respect to the variable  $\mathbf{J}_f$  with tolerance  $10^{-5}$ .

The number of iterations required for the different discretizations is reported in table 1, while fig. 4 shows the computed solutions. In this case, the interaction between the magnetic field  $\mathbf{B}_e + \text{curl } \mathbf{A}$  and the electric current  $\mathbf{J}_f$  gives rise to a double symmetric rotational movement of the fluid inside  $\Omega_f$ , with larger velocity towards the bottom of the domain. As we can see, the number of iterations remains bounded and essentially independent of the number of unknowns.

A more thorough investigation of the convergence rate of the iterative method is in order, especially concerning the dependence on  $h$  and on the degree of the polynomials used. Moreover, an optimal strategy to choose the acceleration parameter  $\theta$  is still not available and will be the object of future investigation.

Table 1: Convergence results for two computational grids and different choices of the degrees of the finite elements.

order $l$ of the Nédélec FE		1	2	1
order $r$ of the Taylor-Hood FE		2	2	2
order $m$ of the Raviart-Thomas FE		1	1	2
Grid 1	dofs	6071	9018	11543
(344 elements)	iterates	22	22	22
Grid 2	dofs	42602	64224	85130
(2752 elements)	iterates	22	22	22

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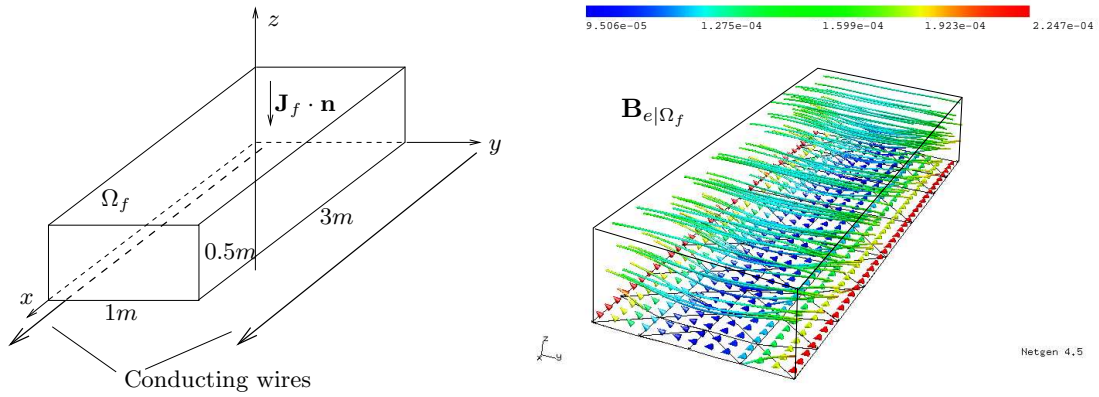


Figure 3: Schematic representation of the setting of the problem (left), and field lines of the assigned magnetic field  $\mathbf{B}_{e|\Omega_f}$  (right).

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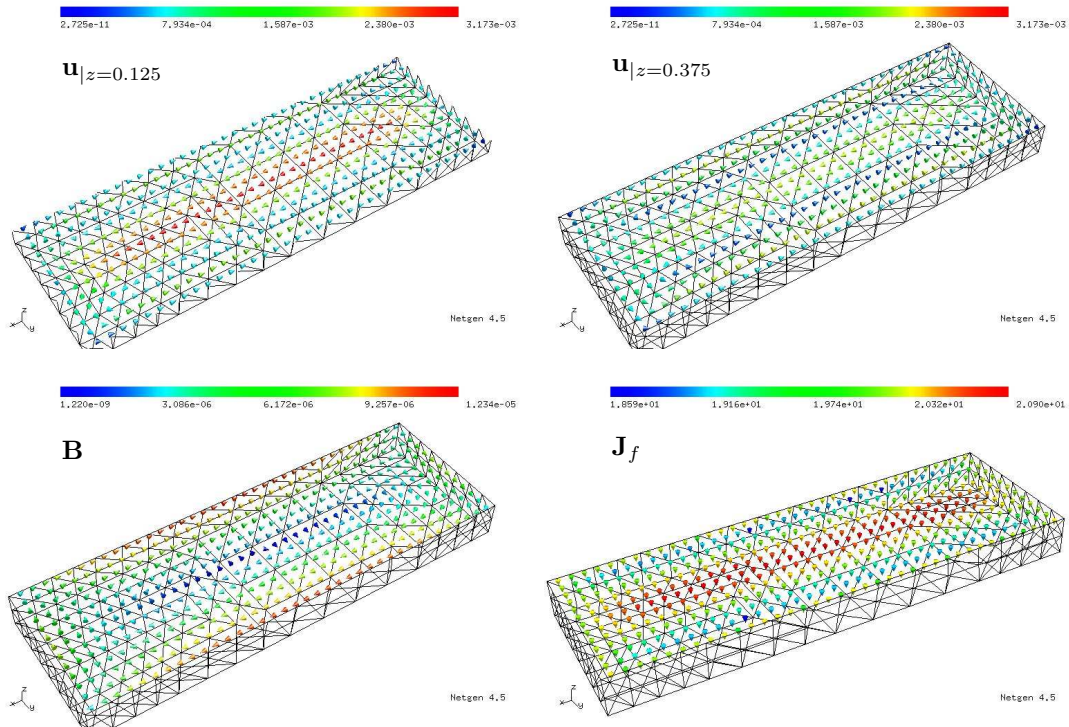


Figure 4: Computed velocity field across the plane  $z = 0.125$  (top left) and  $z = 0.375$  (top right), the magnetic field  $\mathbf{B}$  (bottom left), and the electric current  $\mathbf{J}_f$  (bottom right) in  $\Omega_f$ .

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