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Abstract

The aim of this paper is to present a DD (domain decomposition) algorithm almost optimal in the total computational work for a piece wise orthotropic discretizations on a domain composed of rectangles with arbitrary aspect ratios. The two nonzero coefficients in the diagonal matrix of coefficients before products of first order derivatives in the energy integral of the problem are assumed to be arbitrary positive numbers different for each subdomain. The rectangular mesh of the finite element discretization is uniform on each subdomain and otherwise arbitrary. The main problem in designing the algorithm is the interface Schur complement preconditioning, which is closely related to obtaining boundary norms for discrete harmonic functions on the shape irregular domains. The computational cost of the presented Schur complement and DD algorithms is $\mathcal{O}(\mathcal{N}(\log \mathcal{N})^{1/2})$ arithmetic operations, where $\mathcal{N}$ is the number of unknowns.

1 Introduction

The problem of construction of boundary norms for discrete harmonic FE (finite element) functions and the problem of the Schur complement preconditioning are closely interrelated. This is for the reason that under definite conditions the matrix of the quadratic form, induced by the square of such a norm, is spectrally equivalent to the corresponding Schur complement of the FE stiffness matrix. Apart from being close in the spectrum to the Schur complement, a good

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preconditioner must possess another property: the solution of the system of algebraic equations with the preconditioner for the matrix must be cheap. If the equation is uniformly elliptic in Ω, the domain Ω is sufficiently good, and the FE mesh is quasiuniform, then the situation is well understood. In this case, the energy space is equivalent to $H^1(\Omega)$ and a proper boundary norm for discrete (FE) harmonic functions is the norm in space $H^{1/2}(\partial\Omega)$. This fact together with the relatively simple discrete representations for the norm in $H^{1/2}(\partial\Omega)$ for the traces of FE functions fetched a good service in designing computationally cheap Schur complement preconditioners for DD (domain decomposition) algorithms. The marking point in the development of such techniques was the paper of Dryja [16], although independently of DD methods finite-difference analogues of $H^{1/2}$-norms were studied earlier, *e.g.,* by Andreev [2, 3]. Papers of Golub/Mayers [19], Bjorstad/Widlund [8], Chan [13], Nepomnyaschikh [36, 37] and others may be referred for later contributions. The situation is more difficult when some or all of the above mentioned conditions are violated. For instance, the domain may have different sizes in different directions, the elliptic equation – orthotropic, as well as the finite element mesh. The boundary norm for harmonic functions on rectangular domain $\Omega = (0, 1) \times (0, \epsilon)$, which is equivalent to the H^1-norm uniformly in $\epsilon \in (0, 1]$, can be derived along the lines, followed by Maz'ya/Poborchi [35] at the study of boundary norms for bad domains. We term it the shape dependent norm, since it is strongly influenced by the value of ϵ. If the triangulation is quasiuniform, then, by means of the results on some special interpolation operator $\Pi_h v : H^1(\Omega) \mapsto \mathcal{V}(\Omega)$, where $\mathcal{V}(\Omega)$ denotes the FE space, it can be shown that the shape dependent norm retains for FE discrete harmonic functions. One of the important for computations facts, reflected in this norm, is that in the Schur complement for a slim domain the longer edges are strongly coupled. The orthotropy of the elliptic operator is easily reduced to the orthotropy of the mesh by the transform of variables. Therefore, an important problem is Poisson equation in a slim domain discretized on the orthotropic mesh of different sizes h_1, h_2. Exactly this situation was studied, *e.g.,* by Griebel/Oswald [22], Grauschopf/Griebel/Regler [20] and Oswald [34], who developed BPX, MDS and other types preconditioners for orthotropic FE discretizations. These preconditioners stem from stable multilevel finite element and wavelet tensor product decompositions and allow to derive spectrally equivalent Schur complement preconditioners and low energy prolongation operators. However, if we deal with a collection of rectangular subdomains, having different edge aspect ratios, accompanied by changing from domain to domain orthotropy of the differential equation and rectangular fine mesh, the approach of the cited works is not directly applicable. The reason is that the preconditioners-solvers, designed in this way for different subdomains, in a definite sense are not compatible, and, therefore, the use of them with the purpose of assembling an efficient Schur complement preconditioner for the collection of subdomains meets difficulties. The model discrete problem of this sort, which is the aim of this paper is the following. Suppose, the domain $\Omega = (0, 1) \times (0, 1)$ is decomposed into subdomains

$$\Omega_j = (z_{1,j_1-1}, z_{1,j_1}) \times (z_{2,j_2-1}, z_{2,j_2})\,, \qquad j = (j_1, j_2)\,, \tag{1.1}$$

by the rectangular *decomposition grid*

$$x_k = z_{k,j_k}, \qquad j_k = 0, 1, .., J_k, \qquad z_{k,j_k} - z_{k,j_k-1} = H_{k,j_k} > 0, \qquad z_{k,0} = 0, \qquad z_{k,J_k} = 1\,. \tag{1.2}$$

The decomposition grid is imbedded in the nonuniform rerctangular *source finer grid*

$$x_k = x_{k,i_k}, \qquad i_k = 0, 1, .., N_k, \qquad x_{k,0} = 0, \qquad x_{k,N_k} = 1, \tag{1.3}$$

i.e. $x_{k,\gamma_k} = z_{k,j_k}$ for some numbers $\gamma_k = \varkappa_k(j_k)$. For simplicity, this grid may be assumed uniform on each subdomain and having the sizes $h_{k,j_k} = H_{k,j_k}/n_{k,j_k}$, where n_{k,j_k} is the number of the source mesh intervals on the decomposition grid interval $(z_{k,j_k} - z_{k,j_k-1})$. The fine source grid induces the FE (finite element) space $\overset{\circ}{\mathcal{V}}(\Omega)$ of continuous on $\overline{\Omega}$ and bilinear on each nest of the grid functions, vanishing on the boundary $\partial\Omega$. By means of this FE space we discretize the problem

$$\alpha_\Omega(u,v) = \langle f, v\rangle\,, \qquad \forall\; v \in \overset{\circ}{H}{}^1(\Omega)\,, \qquad \text{where} \qquad \alpha_\Omega(u,v) = \int_\Omega \nabla u(x)\cdot \boldsymbol{\rho}(x)\,\nabla v(x)\,dx\,, \quad (1.4)$$

with $\boldsymbol{\rho} = \operatorname{diag}[\rho_1, \rho_2]$ being a 2×2 diagonal matrix with positive piece wise constant functions $\rho_k(x)$ such that $\rho_k(x) = \rho_{k,j} = \text{const}$ for $x \in \Omega_j$. The answer, we are looking for, is whether there exists a DD algorithm, which is robust and fast uniformly for arbitrary positive H_{k,j_k}, $\rho_{k,j}$ and h_{k,j_k}. Indeed, in this paper we present the Schur complement solver, the total arithmetical complexity of which under the assumption, *e.g.,* of a constant number of subdomains is estimated by $\mathcal{O}(\mathcal{N}(\log\mathcal{N})^{1/2})$, where $\mathcal{N}$ is the number of FE unknowns. The total arithmetical complexity of our DD preconditioner-solver $\boldsymbol{\mathcal{K}}_{\rm DD}$ of the Dirichlet-Dirichlet type is $\mathcal{O}(\mathcal{N}(\log\mathcal{N}))$. In other words,

$$\sqrt{\operatorname{cond}[\boldsymbol{\mathcal{K}}_{\rm DD}^{-1}\mathbf{K}]} \times \text{ops}\ \left[\boldsymbol{\mathcal{K}}_{\rm DD}^{-1}\mathbf{f}\,\right] = \mathcal{O}(\mathcal{N}(\log\mathcal{N})) \qquad (1.5)$$

for any vector $\mathbf{f}$, where $\mathbf{K}$ is the FE stiffness matrix for the problem (1.1)–(1.4). These estimates are retained, if the number of subdomains grows along with the number of FE unknowns, but not too fast, and, in particular, if the conditions $J_k \le N_k^{1/2}$ hold. At the end of the paper, we point out the ways of the improvement of the Schur complement and DD solvers, which allow to remove the multipliers containing log's from the above estimates of arithmetical work.

Dirichlet problems on subdomains of decomposition can be efficiently solved, *e.g.,* by FDFT (Fast Discrete Fourier Transform) and by several more efficient optimal multilevel algorithms, *e.g.,* presented in Griebel/Oswald [22]. Therefore, the key problem at designing DD algorithm, satisfying the above estimate of arithmetical work, becomes obtaining a Schur complement preconditioner-solver, the complexity of which would not violate almost optimality of the DD algorithm.

Sections 2, 3 and 4 of the paper deal with a single rectangular subdomain of the arbitrary edge aspect ratio. First of all, we establish that the shape dependent boundary norm, which is equivalent to the $H^1(\Omega)$-norm for harmonic functions and which is derived in this paper, remains a proper norm for discrete harmonic functions on a shape regular meshes. This is done by means of Scott/Zhang [41] result on a special interpolation operator for functions from $H^1(\Omega)$. Then the shape dependent norm is simplified by means of finite-difference norms, equivalent to the $H^{1/2}$-norms on some 1-d sets. It is worth noting that the result reflects the fact, known from applications of boundary FE methods and referred sometimes as *absorbtion of singularities.* Suppose, we have a spectrally equivalent preconditioner for the boundary Schur complement for a FE discretization on a quasiuniform mesh. Then, this preconditioner retains the spectral equivalence to the Schur complement generated on any shape regular mesh, which is imbedded in the quasiuniform mesh and coincides with it on the boundary. Even more general meshes can be considered.

For applications it is useful to have preconditioners admitting parallelization of computations in accordance with the geometry. Schur complement preconditioners, in which vertices are split from the edges and each edge is split at least from a part of others, meet these requirements. Such preconditioners can be obtained in several ways, apart from ones based on the shape dependent type boundary norms. We discuss (secondary) DD techniques with overlapping and nonoverlapping subdomains. In particular, for FE discretizations on shape irregular rectangular domains, we use DD with shape regular subdomains of decomposition. This allows to estimate the influence of the splitting on the relative condition number in part by means of well known techniques, developed for quasiuniform triangulations in Bramble/Pasciak/Schatz [9]– [12]. One of the conclusions is that splitting pairs of adjacent edges cause insignificant losses, which are not influenced by the aspect ratio of a subdomain. However, splitting the vertices from the rest of unknowns results in an essential increase of relative condition, depending on the orthotropy of discretization. In the case of Laplace operator and the square fine mesh, this increase is proportional to $\max\left[\max_k H_k/H_{3-k}, \log(1+\min_k n_k)\right]$, where indices j are omitted.

In the case of the orthotropic differential operator and the mesh, the differential equation can be transformed into an isotropic one. In general after such a transform, the subdomain remains rectangular and the mesh remains orthotropic. The transformed FE space can be represented by the direct sum of two subspaces, one of which is FE space induced by the finest quasiuniform mesh imbedded in the transformed orthotropic mesh. Formal splitting of these subspaces worsen the relative condition not significantly, and after the splitting the problem of obtaining the Schur complement preconditioner is reduced to the case of an isotropic differential operator on a shape irregular domain and a quasiuniform mesh.

Schur complement preconditioners, expressed via finite-difference operators, are not well suitable for the FE model problem (1.1)–(1.4). For a single subdomain they are based on the hierarchy of four imbedded FE subspaces, not all of which are compatible with the corresponding subspaces for adjacent subdomains. The consequence of this is that the assembled Schur complement preconditioner for the whole interface is not easily invertible. Hence, we discuss also the compatible preconditioner, suggested by Korneev [26, 27]. For a single subdomain, it is easily defined by replacing in the Schur complement the blocks, coupling vertices with the rest of unknowns, and the blocks, coupling adjacent edges, by zero blocks. The analysis of its relative condition number is produced with the use of the results for the preconditioners, expressed via finite-difference operators. The DD algorithm with this Schur complement preconditioner for the described above piece wise orthotropic discretization of the problem (1.1)–(1.4) is discussed in Section 5. The pointed out estimate $\mathcal{N}\log\mathcal{N}$ of computational work, testifying to the almost optimality of such DD algorithm, is proved in the same section.

We do not study solvers for the subproblem of DD algorithm, which is related to the unknowns at subdomain vertices and is completely split from other unknowns in the preconditioner. Since its dimension is much smaller than $\mathcal{N}$, in general situation of varying J_k, we assume that there is a solver, which do not compromise (1.5). We add that if J_k/N_k grow not two fast, then even the direct elimination procedure will satisfy this assumption.

Let us list some notations used in the paper.

Capital letters of the styles $\mathbf{A}, \mathbb{A}, \boldsymbol{\mathcal{A}}$ are used for matrices, small boldface letters – for vectors, whereas $\mathbf{I}$ denotes identity matrices.

$(\cdot,\cdot)_\Omega$, and $\|\cdot\|_{0,\Omega}$ are the scalar product and the norm in $L^2(\Omega)$.

$|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ are the semi-norm and the norm in the Sobolev space $H^k(\Omega)$, *i.e.*,

$$|v|^2_{k,\Omega} = \sum_{|q|=k} \int_\Omega (D^q_x v)^2 dx, \qquad \|v\|^2_{k,\Omega} = \|v\|^2_{0,\Omega} + \sum_{l=1}^{k} |v|^2_{l,\Omega}\,,$$

where

$$D^q_x v := \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2}, \qquad q = (q_1, q_2), \quad q_1, q_2 \geq 0, \quad |q| = q_1 + q_2\,.$$

$\overset{\circ}{H}{}^1(\Omega)$ is the subspace of $H^1(\Omega)$ of functions having zero traces on $\partial\Omega$.

For $I = (a, b)$, $\|\cdot\|_{1/2,I}$ and ${}_{00}\|\cdot\|_{1/2,I}$ are the norms in the space $H^{1/2}(I)$ and the subspace ${}_{00}H^{1/2}(I) \subset H^{1/2}(I)$ of functions having zero values at $x = a, b$, see, *e.g.* [1]. Expressions for these norms are

$$\begin{aligned} \|v\|^2_{1/2,I} &= \|v\|^2_{0,I} + |v|^2_{1/2,I}, \\ |v|^2_{1/2,I} &= \int_a^b \int_a^b \left(\frac{v(x) - v(y)}{x - y} \right)^2 dxdy, \\ {}_{00}\|v\|^2_{1/2,I} &= \|v\|^2_{1/2,I} + \int_a^b \frac{v^2(x)}{x-a} dx + \int_a^b \frac{v^2(x)}{b-x} dx\,. \end{aligned}$$

The norm $\|\cdot\|_{1/2,\gamma_i}$, where γ_i is an edge of $\sqcap = (0, 1) \times (0, 1)$ is defined for the traces on this edge analogously with $\|\cdot\|_{1/2,I}$. For instance, for the edge γ_i, which is on the line $x_1 = c$, $c = 0, 1$, we have

$$\|v\|^2_{1/2,\gamma_i} = \|v\|^2_{0,\gamma_i} + |v|^2_{1/2,\gamma_i}\,, \qquad |v|^2_{1/2,\gamma_i} = \int_0^1 \int_0^1 \left(\frac{v(c,t) - v(c,\tau)}{t - \tau} \right)^2 dtd\tau.$$

For a sufficiently smooth and shape regular domain Ω, the norm $\|v\|_{1/2,\partial\Omega}$ is given by the formulas

$$\|v\|^2_{1/2,\partial\Omega} = \|v\|^2_{0,\partial\Omega} + |\,v\,|^2_{1/2,\partial\Omega}\,, \qquad |\,v\,|^2_{1/2,\partial\Omega} = \int_{\partial\Omega} \int_{\partial\Omega} \left(\frac{v(x) - v(y)}{x - y} \right)^2 ds_x ds_y\,,$$

in which ds_x, ds_y are the length elements at the points $x, y \in \partial\Omega$. In the case, *e.g.*, $\Omega = \sqcap$ this norm is equivalent to the norm

$$\|v\|^2_{1/2,\partial\sqcap} = \|v\|^2_{0,\partial\sqcap} + |v|^2_{1/2,\partial\sqcap}\,, \qquad |v|^2_{1/2,\partial\sqcap} = \sum_{i=1}^{4} |v|^2_{1/2,\gamma_i} + \sum_{i=1}^{4} \int_0^1 \frac{(v_{j(i)}(t) - v_{l(i)}(t))^2}{|t|} dt, \quad (1.6)$$

where $u_{j(i)}$ denotes the restriction of u on edge $\gamma_{j(i)}$, and t is the distance to the vertex V_i of $\sqcap$, which is common for $\gamma_{j(i)}$ and $\gamma_{l(i)}$. To each V_i, we associate a preceding edge $\gamma_{j(i)}$ and a succeeding edge $\gamma_{l(i)}$, *e.g.* according to a counter-clockwise orientation of the boundary. The norm and semi-norm defined in this way for the space $H^{1/2}(\partial\sqcap)$ are equivalent to $\|v\|_{1/2,\partial\sqcap} := \inf \|w\|_{1,\sqcap}$ and $|v|_{1/2,\partial\sqcap} := \inf |w|_{1,\sqcap}$ with infima taken over $w \in H^1(\sqcap)$ for which $w = v$ on $\partial\sqcap$. We refer to Grisvard [23], and Ben Belgacem [6] for additional details on the introduced boundary norms.

$\mathbf{A}^+$ is the pseudo-inverse to a matrix $\mathbf{A}$, $||\mathbf{v}||_{\mathbf{A}} = (\mathbf{v}^\top \mathbf{A}\mathbf{v})^{1/2}$ is the norm or the seminorm, induced by a nonnegative symmetric matrix $\mathbf{A}$. If $\mathbf{A}$ is a nonnegative symmetric matrix, the notation $\mathbf{A}^{1/2}$ stands for the nonnegative symmetric matrix $\mathbf{B}$ satisfying $\mathbf{A} = \mathbf{B}\mathbf{B}$, $\ker[\mathbf{A}] = \ker[\mathbf{B}]$. ops $[\cdot]$ is the number of a.o. (arithmetic operations) needed for the operation in the square brackets. The symbols $\prec$, $\succ$ denote onesided and $\asymp$ – twosided inequalities, which hold for some, mostly absolute, constants omitted, whereas $\mathbf{A} \prec \mathbf{B}$ with nonnegative matrices $\mathbf{A}, \mathbf{B}$ implies $\mathbf{v}^\top \mathbf{A}\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}\mathbf{v}$ for any vector $\mathbf{v}$, and similarly for signs $\succ$, $\asymp$. We write $\mathbf{v} \Leftrightarrow v$, if vector $\mathbf{v}$ represents the FE function v in a chosen basis.

2 Single slim domain

2.1 Discrete analogues of boundary norms for harmonic functions in slim domains

Let $\Omega = (0,1) \times (0,\epsilon)$ and ϵ, δ satisfy $0 < \epsilon, \delta \leq 1$. For the traces of functions $v \in H^1(\Omega)$ on $\partial\Omega$, we consider two norms and two seminorms. One pair of the norm and the seminorm minimizes the H^1-norm and H^1-seminorm, respectively, among all functions coinciding with a given function on the boundary:

$$\| v \|^2_{\partial\Omega} = \inf_{\phi|_{\partial\Omega}=v} \left((\delta\epsilon^{-1})^2 \|\phi\|^2_{0,\Omega} + \|\nabla\phi\|^2_{0,\Omega} \right) , \qquad | v |^2_{\partial\Omega} = \inf_{\phi|_{\partial\Omega}=v} \|\nabla\phi\|^2_{0,\Omega} , \qquad v, \phi \in H^1(\Omega) . \tag{2.1}$$

Another pair is denoted $]|\cdot|[_{\partial\Omega}$ and $]\cdot[^2_{\partial\Omega}$ and, following the approaches of Maz'ya/Poborchi [35], is introduced by the expressions

$$]|v|[^2_{\partial\Omega} = \delta^2\epsilon^{-1} \|v\|^2_{0,\partial\Omega} +]v[^2_{\partial\Omega} ,$$

$$\begin{aligned}]v[^2_{\partial\Omega} = \epsilon^{-1} \int_0^1 (v(x_1,\epsilon) - v(x_1,0))^2 dx_1 + \int_0^1 \int_{|x_1-y_1|\leq\epsilon} \frac{(v(x_1,0) - v(y_1,0))^2}{(x_1-y_1)^2} dx_1 dy_1 + \\ + \int_0^1 \int_{|x_1-y_1|\leq\epsilon} \frac{(v(x_1,\epsilon) - v(y_1,\epsilon))^2}{(x_1-y_1)^2} dx_1 dy_1 + \int_{\Gamma_0}\int_{\Gamma_0} \frac{(v(s) - v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} + \\ + \int_{\Gamma_1}\int_{\Gamma_1} \frac{(v(s) - v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} . \end{aligned} \tag{2.2}$$

Here $\Gamma_0 = (x \in \partial\Omega : x_1 < \epsilon)$, $\Gamma_1 = (x \in \partial\Omega_\epsilon : x_1 > 1-\epsilon)$ and ds, $d\overline{s}$ are the length elements of $\partial\Omega$. The set Γ_1 is symmetric to Γ_0 with respect to the line $x_1 \equiv 1/2$, see Fig. 1. We will term $]|\cdot|[_{\partial\Omega}$ and $]\cdot[^2_{\partial\Omega}$ the *shape dependent norm and seminorm* for boundary functions.

In the paper, discretizations on rectangular meshes are taken for model discretizations. Accordingly, by the FE space $\mathcal{V}(\Omega)$ is assumed the space of the piece wise bilinear functions on the rectangular mesh $x_k \equiv x_{k,l}$ with the steps $h_{k,l} = x_{k,l} - x_{k,l-1}$, $l = 1, 2, .., n_k$, satisfying the quasiuniformity conditions

$$\underline{c}h \leq h_{k,l} \leq \overline{c}h , \qquad 0 < \underline{c}, \overline{c} = \text{const} , \tag{2.3}$$

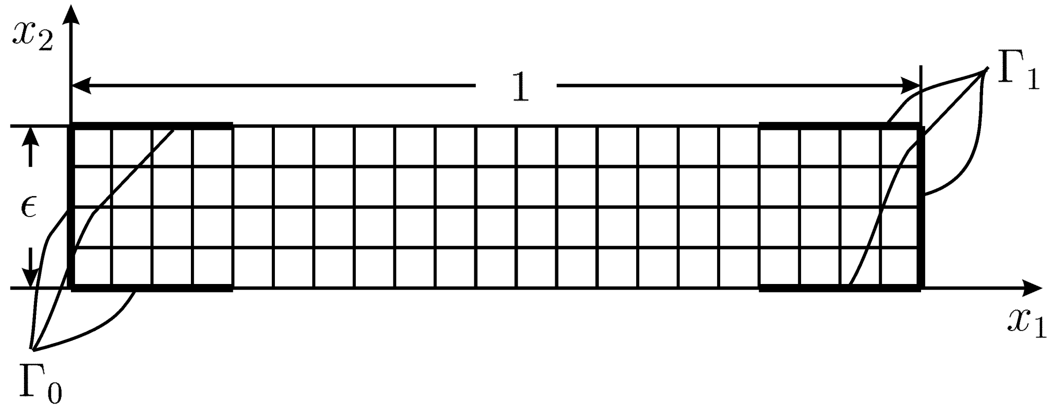

Figure 1: High aspect ratio rectangular domain triangulated by the square mesh.

and $x_{k,0} = 0,\ x_{1,n_1} = 1,\ x_{2,n_2} = \epsilon$. By $\mathcal{V}_{\rm tr}(\partial\Omega)$ is denoted the space of traces of functions from $\mathcal{V}(\Omega)$ on $\partial\Omega$. However, most of the results hold for much more general discretizations. The described above mesh may represent a *skeleton mesh*, while calculations are performed on a mesh, termed the *source mesh*, which

α) *is finer only in the interior of the domain,*

β) *has the same trace with the skeleton mesh on the boundary and*

γ) *covers the skeleton mesh, whereas the skeleton mesh itself may be a general quasiuniform quadrangular unstructured mesh with the mesh parameter h.*

The skeleton mesh can be a triangular quasiuniform mesh, as well as the source mesh. In order to simplify explicit representations of the boundary norms for FE functions and solution procedures, it is worth to assume that there are mesh nodes at the ends of the sets $\Gamma_k,\ k = 0, 1$, and that the number of nodes on the opposite edges of Ω are equal.

For the traces of FE functions $v \in \mathcal{V}_{\rm tr}(\partial\Omega)$, the simpler norm than (2.1)

$$\boldsymbol{|}\, v\, \boldsymbol{|}^2_{h,\partial\Omega} = \inf_{\phi\in\mathcal{V}(\Omega)\,;\,\phi|_{\partial\Omega}=v} \left((\delta\epsilon^{-1})^2\|\phi\|^2_{0,\Omega} + \|\nabla\phi\|^2_{0,\Omega}\right) , \quad |\, v\, |^2_{h,\partial\Omega} = \inf_{\phi\in\mathcal{V}(\Omega)\,;\,\phi|_{\partial\Omega}=v} \|\nabla\phi\|^2_{0,\Omega} , \tag{2.4}$$

can be justified, in which inf is taken only over the subspace of FE functions.

Theorem 2.1. *Let the FE space $\mathcal{V}(\Omega)$ be induced by the quasiuniform triangulation with the mesh parameter h or by its refinement satisfying $\alpha) - \gamma)$. Then for any $h > 0$ and $v \in \mathcal{V}_{\rm tr}(\partial\Omega)$, the norms and seminorms (2.4), (2.2), respectively, are equivalent uniformly in $\epsilon, \delta \in (0, 1]$.*

Proof. The proof is based, firstly, on the similar result on the equivalence of the norms (2.1),(2.2) for the traces of functions from the space $H^1(\Omega)$ and, secondly, on the interpolation results. These results are formulated in Theorem 2.2 and Lemma 2.1, respectively, at the end of this subsection after their use in the proof of Theorem 2.1.

Since $\mathcal{V}(\Omega) \subset H^1(\Omega)$, for any $v \in \mathcal{V}_{\rm tr}(\partial\Omega)$ one has the inequalities

$$]|v|[_{\partial\Omega} \prec \boldsymbol{|}\, v\, \boldsymbol{|}_{\partial\Omega} \le \boldsymbol{|}\, v\, \boldsymbol{|}_{h,\partial\Omega} , \qquad \forall\ v \in \mathcal{V}_{\rm tr}(\partial\Omega) , \tag{2.5}$$

with the first following from Theorem 2.2. For the proof of the opposite inequality

$$\boldsymbol{|}\, v\, \boldsymbol{|}_{h,\partial\Omega} \prec]|v|[_{\partial\Omega} , \qquad \forall\ v \in \mathcal{V}_{\rm tr}(\partial\Omega) , \tag{2.6}$$

we additionally use the following proposition.

Proposition 2.1. *For any $w \in H^1(\Omega)$ with the trace in $\mathcal{V}_{\rm tr}(\partial\Omega)$, there exists the interpolation $\widetilde{w} \in \mathcal{V}(\Omega)$ such that*

i) the traces of $\widetilde{w}$ and w on the boundary $\partial\Omega$ coincide, and

ii) the interpolation satisfies the estimates

$$\begin{aligned} |\, \widetilde{w}\,|_{1,\Omega} \prec |\, w\,|_{1,\Omega}\,, \qquad & \|\, \widetilde{w}\,\|_{1,\Omega} \prec \|\, w\,\|_{1,\Omega}\,, \\ \|\, w - \widetilde{w}\,\|_{k,\Omega} \prec h^{l-k} \|\, w\,\|_{l,\Omega} \qquad & k = 0,1, \;\; l = 1,2\,. \end{aligned} \tag{2.7}$$

Proof. The most direct way to prove this Proposition is to use the result of Scott/Zhang [41] for special piece wise linear interpolation operators, which figure in Lemma 2.1. It is easy to adjust it to the FE spaces of the continuous piece wise bilinear functions or to a more general case of FE spaces induced by the first order quadrangular elements. We define the triangulation of Ω subdividing each rectangular nest of the mesh in two triangles by one of the diagonals of the nest and denote by $\mathcal{V}_{\vartriangle}(\Omega)$ the space of continuous piece wise linear functions. Let $u \in \mathcal{V}(\Omega)$, $v \in \mathcal{V}_{\vartriangle}(\Omega)$ and coincide at the nodes. Then, as it is well known, see, *e.g.,* Korneev [25],

$$|\, u\,|_{1,\Omega} \prec |\, v\,|_{1,\Omega} \prec |\, u\,|_{1,\Omega}\,, \qquad \|\, u\,\|_{1,\Omega} \prec \|\, v\,\|_{1,\Omega} \prec \|\, u\,\|_{1,\Omega}\,, \tag{2.8}$$

which together with Lemma 2.1 below completes the proof. □

Let $\mathcal{H}(\Omega)$ be the subspace of $\mathcal{V}(\Omega)$ induced by the skeleton quasiuniform triangulation and $v \in \mathcal{V}_{\rm tr}(\partial\Omega)$. Suppose also that $v_{\rm harm} \in H^1(\Omega)$ and $v_{d/{\rm harm}} \in \mathcal{V}(\Omega)$ are the functions on which inf's in the first relationships of (2.1) and (2.4), respectively, are reached and $\widetilde{v}$ is the interpolation of $v_{\rm harm}$ from the space $\mathcal{H}(\Omega)$, satisfying i) and ii). First of all we note that according to Proposition 2.1

$$(\delta\epsilon^{-1})^2 \|\widetilde{v}\|^2_{0,\Omega} + \|\nabla\widetilde{v}\|^2_{0,\Omega} \prec (\delta\epsilon^{-1})^2 \|v_{\rm harm}\|^2_{0,\Omega} + \|\nabla v_{\rm harm}\|^2_{0,\Omega}\,. \tag{2.9}$$

Now one can write

$$\begin{aligned} \rfloor v \lfloor^2_{h,\partial\Omega} := (\delta\epsilon^{-1})^2 \|v_{d/{\rm harm}}\|^2_{0,\Omega} + \|\nabla v_{d/{\rm harm}}\|^2_{0,\Omega} \prec \\ \prec (\delta\epsilon^{-1})^2 \|\widetilde{v}\|^2_{0,\Omega} + \|\nabla\widetilde{v}\|^2_{0,\Omega} \prec (\delta\epsilon^{-1})^2 \|v_{\rm harm}\|^2_{0,\Omega} + \|\nabla v_{\rm harm}\|^2_{0,\Omega} \prec]|v|[_{\,\partial\Omega}\,. \end{aligned} \tag{2.10}$$

where first inequality follows from the definition of $\rfloor v \lfloor^2_{h,\partial\Omega}$, second from the definition of the same norm and the inclusion $\mathcal{H}(\Omega) \subset \mathcal{V}(\Omega)$, third is simply (2.9), and the last is the consequence of Theorem 2.2.

For the seminorms the proof is similar. □

Now we will formulate the results on the equivalence of norms and seminorms (2.1) and (2.2) and on the interpolation operator used above.

Theorem 2.2. *For the traces of functions from $H^1(\Omega)$, the norms (2.1), (2.2) are equivalent uniformly in $\epsilon, \delta \in (0,1]$.*

Proof. The proof is lengthy and given in the Appendix. □

Let $\Omega \subset \mathbb{R}^n$ be the n-dimensional domain of the arbitrary quasiuniform triangulation $\mathcal{S}_h$ with the nodal points $x_i,\ i = 1, 2, ..N$. To each node x_i we relate the $(n-1)$-dimensional simplex τ_i, which is the face of some n-dimesional simplex from the triangulation $\mathcal{S}_h$ and has n vertices $z_l,\ l = 1, 2, .., n,$ such that $z_1 = x_i$. The choice of τ_i is not unique, but it is assumed that for $x_i \in \partial\Omega$ we take $\tau_i \subset \partial\Omega$. By $\mathcal{V}_\vartriangle(\Omega)$ and $\mathcal{V}_{\rm tr}(\partial\Omega)$ are denoted the space of functions continuous on $\overline{\Omega}$ and linear on each simplex of triangulation and the space of their traces on $\partial\Omega$. Let $\theta_i \in \mathcal{P}(\tau_i)$ be the function, satisfying

$$\int_{\tau_i} \theta_i \lambda_l \, dx = \delta_{i,l} \,, \qquad\qquad l = 1, 2, .., n \,,$$

where λ_l are the barycentric coordinates in τ_i related to its vertices z_l and $\delta_{i,l}$ is Kronecker's symbol. If $\phi_i \in \mathcal{V}_\vartriangle(\Omega)$ are the Galerkin FE basis functions such that $\phi_i(x_j) = \delta_{i,j},\ i, j = 1, 2, .., N,$ then for each $v \in H^1(\Omega)$ the interpolation $\Pi_h v \in \mathcal{V}_\vartriangle(\Omega)$ is defined as

$$\Pi_h v = \sum_{i=1}^{N} \left(\int_{\tau_i} \theta_i v \, dx \right) \phi_i(x)$$

Lemma 2.1. *The interpolation operator* Π_h *satisfies*

a) $\Pi_h v : H^1(\Omega) \mapsto \mathcal{V}_\vartriangle(\Omega)$, *and, if* $v \in \mathcal{V}_\vartriangle(\Omega)$, *then* $\Pi_h v = v$,

b) $(v - \Pi_h v) \in \overset{\circ}{H}{}^1(\Omega)$, *if* $v|_{\partial\Omega} \in \mathcal{V}_{\rm tr}(\partial\Omega)$,

c) $\|v - \Pi_h v\|_{t,\Omega} \prec h^{s-t} \|v\|_{s,\Omega}$, *for* $t = 0, 1,\ s = 1, 2,$

d) $|\Pi_h v|_{1,\Omega} \prec |\, v \,|_{1,\Omega}$ *and* $\|\Pi_h v\|_{1,\Omega} \prec \|\, v \,\|_{1,\Omega}$ *for all* $v \in H^1(\Omega)$.

Proof. The proof was given by Scott/Zhang [41], in the above form Lemma is found in Xu/Zou [44]. □

Let the FE stiffness matrix $\mathbf{A}$ be induced by the Dirichlet integral over Ω and the FE space $\mathcal{V}(\Omega)$ with the nodal basis functions. We represent it in the block form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_I & \mathbf{A}_{I,B} \\ \mathbf{A}_{B,I} & \mathbf{A}_B \end{pmatrix} \tag{2.11}$$

and denote by $\mathbf{B}$ the Schur complement

$$\mathbf{B} = \mathbf{A}_B - \mathbf{A}_{B,I} \mathbf{A}_I^{-1} \mathbf{A}_{I,B} \,, \tag{2.12}$$

where lower indices I and B are related to the degrees of freedom at the nodes, living on the interior of the domain Ω and on its boundary, respectively. Obviously, the norm (2.2) defines the matrix $\mathbb{B}_{\rm MP}$ such that $\|\mathbf{v}\|_{\mathbb{B}_{\rm MP}} =]v[_{\partial\Omega}$ for all $\mathbf{v} \Leftrightarrow v \in \mathcal{V}_{\rm tr}(\partial\Omega)$. As it follows from Theorem 2.1 and the definition of the norm (2.4), this matrix can be used as a spectrally equivalent preconditioner for the Schur complement. However, in order the norm (2.2) would be more suitable for computations, it needs simplification. Indeed, the multiplication by $\mathbb{B}_{\rm MP}$ requires $\mathcal{O}(n_1 n_2)$ arithmetic operations, where $n_1 n_2$ are the numbers of the mesh intervals on the horizontal and vertical edges of Ω, respectively. It is unclear also if there is a fast solving procedure for the systems with this matrix. Below we consider a simpler boundary norm which may be represented in a more explicit form and saves the computational work at least at matrix-vector multiplications.

2.2 Finite-difference boundary norm

Let us turn to the simpler case of the skeleton mesh, the trace of which on $\partial\Omega$ coincides with the trace of the auxiliary orthogonal quasiuniform grid satisfying (2.3). We arrange the quasiuniform rectangular auxiliary coarse grid, which we term the *coarsest grid* and which is embedded in the skeleton grid. It is obtained by subdividing the domain Ω by the vertical lines $x_1 = t_{1,i}$ in such a way that $t_{1,0} = 0$, $t_{1,n_\epsilon} = 1$ for some integer $n_\epsilon \geq 1$, and the sizes $\eta_{1,i} := t_{1,i} - t_{1,i-1}$ satisfy, *e. g.,* the bounds

$$\epsilon \leq \eta_{1,i} \leq \overline{c}_\circ \epsilon \,, \qquad i = 0, 1, .., n_\epsilon \,, \tag{2.13}$$

where according to (2.3) one can set $\overline{c}_\circ = 1+\overline{c}h$. Simultaneously, we have defined the overlapping intervals

$$\tau_1 = (0, t_{1,2}) \,, \qquad \tau_i = (t_{1,i-2}, t_{1,i+1}) \,, \qquad i = 2, 3, .., n_\epsilon - 1 \,, \qquad \tau_{n_\epsilon} = (t_{1,n_\epsilon-2}, 1) \,,$$

the intersections of which with the sides γ_k, $k = 2, 3$, are denoted by $\gamma_{k,i}$. Here, it is convenient to imply by γ_k for $k = 0, 1$ the left and the right vertical edges of Ω, and by γ_k, $k = 2, 3$, the horizontal lower and upper edges, respectively. Let ν be the number of nodes on the sets $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$, which is assumed to be the same for both sets, and

$$\boldsymbol{\Delta}^2_{1/2,k} = \frac{1}{2} \begin{pmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & 0 & & \\ & & \cdot & \cdot & \cdot & & \\ & & & -1 & 2 & -1 & \\ & & & & \cdot & \cdot & \cdot \\ & & 0 & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}, \qquad k = 0, 1 \,, \tag{2.14}$$

be $\nu \times \nu$ matrices. If ν_i is the number of the nodes on $\overline{\tau}_i$, then $\boldsymbol{\Delta}_{1/2,k,i}$ is the similar to (2.14) $\nu_i \times \nu_i$ matrix related to the segment $\overline{\gamma}_{k,i}$ $k = 2, 3$. For convenience, we consider these matrices as defined on the nodes of the sets $\overline{\Gamma}_0, \overline{\Gamma}_1$ and $\overline{\gamma}_{2,i}, \overline{\gamma}_{3,i}$ and continued by zeroes on the rest nodes of the boundary.

We introduce also the matrix

$$\boldsymbol{\nabla} = \frac{h}{\epsilon} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix},$$

where $\mathbf{I}$ is the unity matrix of the dimension $n_2 + 1$ and the blocks on diagonal correspond to the nodes on the edges $\overline{\gamma}_2, \overline{\gamma}_3$, respectively. Again we consider $\boldsymbol{\nabla}$ continued by zeroes on the rest nodes.

Lemma 2.2. *The matrix*

$$\mathbf{C} = \boldsymbol{\nabla} + \boldsymbol{\Delta}_{1/2,0} + \boldsymbol{\Delta}_{1/2,1} + \sum_{k=2,3} \sum_{i=2}^{n_\epsilon - 1} \boldsymbol{\Delta}_{1/2,k,i} \tag{2.15}$$

is spectrally equivalent to the matrix $\mathbb{B}_{\rm MP}$ *uniformly with respect to* h *and* $\epsilon \in (0, 1]$*. Besides,* ops $[\mathbf{C}\mathbf{v}_B] = \mathcal{O}((n_1 + n_2)(1 + \log n_2))$.

Let us emphasize that as a rule matrices defined on different subspaces of vectors are considered as continued by zero entries in order to be defined on a whole space of vectors under consideration. Thus, sums like (2.15) should be understood as topological sums.

Proof. For the proof we introduce the seminorm $\rfloor \cdot \lfloor_{\partial\Omega}$,

$$\rfloor v \lfloor^2_{\partial\Omega} = \epsilon^{-1} \int_0^1 (v(x_1,\epsilon) - v(x_1,0))^2 dx_1 + \sum_{i=2}^{n_\epsilon - 1} \int_{\tau_i} \int_{\tau_i} [\frac{(v(x_1,0)-v(y_1,0))^2}{(x_1-y_1)^2} + \frac{(v(x_1,\epsilon)-v(y_1,\epsilon))^2}{(x_1-y_1)^2})] dx_1 dy_1 + \int_{\Gamma_0}\int_{\Gamma_0} \frac{(v(s)-v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} + \int_{\Gamma_1}\int_{\Gamma_1} \frac{(v(s)-v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s}\,, \tag{2.16}$$

and show that it is equivalent to the seminorm $]\, v\, [_{\partial\Omega}$, whereas its matrix is equivalent to $\mathbf{C}$. Indeed, if $n_\epsilon \geq 3$, then for any $f(x_1, y_1) \in L_2(\sqcap)$

$$\int_0^1 \int_{|x_1-y_1|\leq\epsilon} f^2(x_1,y_1) dy_1 dx_1 = \sum_{i=2}^{n_\epsilon-1} \int_{t_{i-1}}^{t_i} \int_{|x_1-y_1|\leq\epsilon} f^2(x_1,y_1) dx_1 dy_1 \leq \sum_{i=2}^{n_\epsilon-1} \int_{\tau_i}\int_{\tau_i} f^2(x_1,y_1) dy_1 dx_1\,,$$

and, therefore,

$$]\, v\, [_{\partial\Omega} \leq \rfloor v \lfloor_{\partial\Omega}\,, \qquad \forall\ v \in H^1(\Omega)\,. \tag{2.17}$$

Obviously this inequality holds for any $\epsilon \in (0, 1]$.

For obtaining the opposite inequality, we consider a function $v(x) \in H^{1/2}(0,1)$ of one variable and note that

$$\sum_{i=2}^{n_\epsilon-1} \int_{\tau_i}\int_{\tau_i} \frac{(v(x)-v(y))^2}{(x-y)^2} dxdy \leq \int_0^1 \int_{|x-y|\leq\varkappa} \frac{(v(x)-v(y))^2}{(x-y)^2} dxdy\,, \qquad \varkappa \leq c\epsilon\,,\ c = 3\overline{c}_\circ\,.$$

On the other hand, for any $\varkappa \leq 1$ and $z = (x+y)/2$, we have $|z-x| = |y-x|/2$, $|y-z| = |y-x|/2$, and as a consequence

$$\int_0^1 \int_{|x-y|\leq\varkappa} \frac{(v(x)-v(y))^2}{(x-y)^2} dxdy \leq 2\int_0^1 \int_{|x-y|\leq\varkappa} \frac{(v(x)-v(z))^2 + (v(z)-v(y))^2}{(x-y)^2} dydx =$$

$$4\int_0^1 \int_{|x-z|\leq\varkappa/2} \frac{(v(x)-v(z))^2}{(x-z)^2} dzdx + 4\int_0^1 \int_{|z-y|\leq\varkappa/2} \frac{(v(z)-v(y))^2}{(z-y)^2} dzdy = 8\int_0^1 \int_{|x-y|\leq\varkappa/2} \frac{(v(x)-v(y))^2}{(x-y)^2} dydx\,.$$

From the above inequality it follows

$$\sum_{i=2}^{n_\epsilon-1} \int_{\tau_i}\int_{\tau_i} \frac{(v(x)-v(y))^2}{(x-y)^2} dxdy \leq \underline{\gamma}_0^{-2} \int_0^1 \int_{|x-y|\leq\epsilon} \frac{(v(x)-v(y))^2}{(x-y)^2} dydx$$

with $\underline{\gamma}_0 \leq (1/8)^{p_0}$, $p_0 = (\log 3 + \log \overline{c}_\circ)/2\log 2$. This inequality, (2.2) and (2.16) directly lead to the opposite to (2.17) bound, so that combining with (2.17) we get

$$\underline{\gamma}_0 \rfloor v \lfloor_{\partial\Omega} \leq]\, v\, [_{\partial\Omega} \leq \rfloor v \lfloor_{\partial\Omega}\,, \qquad \forall\ v \in H^1(\Omega)\,. \tag{2.18}$$

By V is assumed the vector space corresponding to the FE space $\mathcal{V}(\Omega)$, whereas V_B denotes its restriction to the boundary. For the proof it is left to show that

$$\underline{\gamma}_C \mathbf{v}^\top \mathbf{C}\mathbf{v} \leq \rfloor v \lfloor^2_{\partial\Omega} \leq \overline{\gamma}_C \mathbf{v}^\top \mathbf{C}\mathbf{v}\,, \qquad \forall\ \mathbf{v} \Leftrightarrow v \in \mathcal{V}(\Omega)\,, \tag{2.19}$$

with some constants $\underline{\gamma}_C, \overline{\gamma}_C > 0$. These bounds follow from the equivalences

$$\begin{aligned}
\epsilon^{-1}\int_0^1 (v(x_1,\epsilon)-v(x_1,0))^2 dx_1 \asymp \mathbf{v}^\top \boldsymbol{\nabla}\mathbf{v}\,, \qquad & \forall\, \mathbf{v} \Leftrightarrow v \in \mathcal{V}_{\rm tr}(\gamma_2\cup\gamma_3)\,,\\
\int_{\tau_i}\int_{\tau_i} \tfrac{(v(x_1,0)-v(y_1,0))^2}{(x_1-y_1)^2}\, dx_1 dy_1 \asymp \mathbf{v}^\top \boldsymbol{\Delta}_{1/2,2,i}\mathbf{v}\,, \qquad & \forall\, \mathbf{v} \Leftrightarrow v \in \mathcal{V}_{\rm tr}(\gamma_2)\,,\\
\int_{\tau_i}\int_{\tau_i} \tfrac{(v(x_1,\epsilon)-v(y_1,\epsilon))^2}{(x_1-y_1)^2}\, dx_1 dy_1 \asymp \mathbf{v}^\top \boldsymbol{\Delta}_{1/2,3,i}\mathbf{v}\,, \qquad & \forall\, \mathbf{v} \Leftrightarrow v \in \mathcal{V}_{\rm tr}(\gamma_3)\,,\\
\int_{\Gamma_k}\int_{\Gamma_k} \tfrac{(v(s)-v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} \asymp \mathbf{v}^\top \boldsymbol{\Delta}_{1/2,k}\mathbf{v}\,, \qquad k=0,1\,, \qquad & \forall\, \mathbf{v} \Leftrightarrow v \in \mathcal{V}_{\rm tr}(\Gamma_k)\,,
\end{aligned} \tag{2.20}$$

where $\mathcal{V}_{\rm tr}(\gamma_2\cup\gamma_3)$, $\mathcal{V}_{\rm tr}(\gamma_2)$, $\mathcal{V}_{\rm tr}(\gamma_3)$ and $\mathcal{V}_{\rm tr}(\Gamma_k)$ are the spaces of traces of the FE space on the corresponding subsets of the boundary. The first one follows by the spectral equivalence of the FE 1-d mass matrix to its diagonal. Lines 1-3 express one well known fact. Suppose, some interval τ is subdivided by a quasiuniform grid with ν intervals and $\mathcal{H}(\tau)$ is the corresponding space of continuous piece wise linear functions. Then the matrix of the quadratic form $|\,v\,|^2_{1/2,\tau}$ on the space $\mathcal{H}(\tau)$ is spectrally equivalent to the matrix $\boldsymbol{\Delta}_{1/2}$ of the form (2.14). It seems that the the pioneering proof of this fact was given by Andreev [2, 3].

We conclude that (2.19) holds and, combining with (2.18), we get

$$\underline{\beta}_C \mathbf{C} \prec \mathbb{B}_{\rm MP} \prec \overline{\beta}_C \mathbf{C} \tag{2.21}$$

with the constants $\underline{\beta}_C, \overline{\beta}_C > 0$ depending only on $\overline{c}_\circ$ from the quasiuniformity conditions (2.3).

The multiplication $\boldsymbol{\nabla}\mathbf{v}$ requires $\mathcal{O}(n_1)$ a.o. The vector-matrix multiplications by each matrix $\boldsymbol{\Delta}_{1/2,k}$ or $\boldsymbol{\Delta}_{1/2,k,i}$ can be completed by FDFT and all these multiplications spend $\mathcal{O}((n_1+n_2)\log n_2)$ a.o. Hence, the vector-matrix multiplication by the topological sum of these matrices requires $\mathcal{O}((n_1+n_2)(1+\log n_2))$ a.o. This approves the estimate of the arithmetic work for the multiplication $\mathbf{C}\mathbf{v}$ given in Lemma. □

Corollary 2.1. *The matrix* $\mathbf{C}$ *is spectrally equivalent to the Schur complement (2.12),* i.e., $\mathbf{B} \asymp \mathbf{C}$.

Proof. The proof follows by Theorem 2.1, the definition of the matrix $\mathbb{B}_{\rm MP}$ and Lemma 2.2. □

3 Schur complement preconditioning by domain decomposition

In the case of slim domains and the quasiuniform FE mesh, preconditioners for Schur complements can be obtained by means of DD techniques employing as overlapping so nonoverlapping domain decompositions. In this way one may come to different in some aspects preconditioners, but providing essentially the same estimates of the relative condition with the preconditioners of the preceding section. At the same time, they assume almost optimal in computational complexity solvers for the systems of algebraic equations with these preconditioners for the matrices. Besides, they allow more exactly to evaluate the influence on the relative condition, caused by splitting the vertices from the rest of the unknowns in the preconditioner. These estimates are also important as a technical tool for the proof of the condition bound for the Schur complement preconditioner in DD algorithm, presented in Section 5.

3.1 Preconditioning by overlapping domain decomposition

The quasiuniform coarsest grid satisfying (2.13) introduces the decomposition of the domain Ω into the overlapping subdomains $\Omega_i,\ i=0,1,..,n_\epsilon,$ where

$$\Omega_0=(0,t_{1,1})\times(0,\epsilon),\qquad \Omega_i=(t_{1,i-1},t_{1,i+1})\times(0,\epsilon),\quad i=1,2,..,n_\epsilon-1,\qquad \Omega_{n_\epsilon}=(t_{n_\epsilon-1},1)\times(0,\epsilon),$$

and the FE space is decomposed into the direct sum

$$\mathcal{V}(\Omega)=\mathbb{V}_0(\Omega)\oplus\mathcal{V}_0(\Omega_0)\oplus\mathcal{V}_1(\Omega_1)\oplus..\oplus\mathcal{V}_{n_\epsilon}(\Omega_{n_\epsilon}) \tag{3.1}$$

of $n_\epsilon+2$ subspaces. In this decomposition, $\mathbb{V}_0(\Omega)$ is the subspace of the functions piece wise bilinear on the coarsest mesh and continuous on $\overline{\Omega}$, which we refer as the *coarsest subspace*, and

$$\mathcal{V}_0(\Omega_0)=\{v\in\mathcal{V}(\Omega):v=0\ \text{ for } x_1\ge t_{1,1}\}\,,\qquad \mathcal{V}_{n_\epsilon}(\Omega_{n_\epsilon})=\{v\in\mathcal{V}(\Omega):v=0\ \text{ for } x_1\le t_{1,n_\epsilon-1}\}\,,$$

$$\mathcal{V}_i(\Omega_i)=\{v\in\mathcal{V}(\Omega):v=0\ \text{ for } x_1\le t_{1,i-1},\ \text{ and } x_1\ge t_{1,i+1}\}\,,\qquad i=1,2,..,n_\epsilon-1\,.$$

The spaces of the traces of functions from the spaces $\mathcal{V}_i(\Omega_i)$ will be denoted by $\mathcal{V}_{i,\mathrm{tr}}(\partial\Omega_i)$, whereas for the corresponding vector spaces we use the notations $V, V_{0,\epsilon}$ and V_i respectively to $\mathcal{V},\mathbb{V}_0$ and $\mathcal{V}_i$ and the notations $V_{\mathrm{tr}},V_{i,\mathrm{tr}}$ respectively to $\mathcal{V}_{\mathrm{tr}},\mathcal{V}_{i,\mathrm{tr}}$.

Let $\mathcal{T}_i=\partial\Omega_i\cap\partial\Omega,\ k_i$ be the number of the fine mesh intervals on $\mathcal{T}_i$ for $i=0,n_\epsilon$, and for $i=1,2,..,n_\epsilon-1$ – the half of this number, $\mathbb{A}_0$ be here the stiffness matrix generated by the space $\mathbb{V}_0(\Omega)$, and

$$\mathbf{\Delta}_{1/2,i}=\mathbf{\Delta}_i^{1/2}\,,\qquad \mathbf{\Delta}_i=\mathrm{tridiag}\,[-1,2-1]_1^{k_i}\,.$$

It is assumed that for $i=0,n_\epsilon$ the matrices $\mathbf{\Delta}_{1/2,i}$ are related to the internal nodes of the corresponding part $\mathcal{T}_i$ of the boundary $\partial\Omega$. For $i=1,2,..,n_\epsilon-1$, we introduce also the matrices $\overline{\mathbf{\Delta}}_{1/2,i}=\mathrm{diag}\,[\mathbf{\Delta}_{1/2,i},\mathbf{\Delta}_{1/2,i}]$ with the two independent blocks, each related to the internal nodes of one of the two disjoint parts of $\mathcal{T}_i$ lying on the lines $x_2=0,\epsilon$, respectively. By $\boldsymbol{\mathcal{I}}_0$ is denoted the usual interpolation matrix, such that if $v_0\in\mathbb{V}_0$ and $\mathbf{v}_0$ its vector representation in the space $V_{0,\epsilon}$, then $\boldsymbol{\mathcal{I}}_0\mathbf{v}_0$ is the vector representation of v_0 in V.

Setting $\mathbb{B}_0^+=\boldsymbol{\mathcal{I}}_0\mathbb{A}_0^{-1}\boldsymbol{\mathcal{I}}_0^\top$, we define the preconditioner for the Schur complement through its inverse as

$$\mathbf{C}^{-1}=\mathbb{B}_0^++\mathbf{\Delta}_{1/2,0}^++\mathbf{\Delta}_{1/2,n_\epsilon}^++\sum_{i=1}^{n_\epsilon-1}\overline{\mathbf{\Delta}}_{1/2,i}^+\,. \tag{3.2}$$

Let us underline that the same notations $\mathbf{\Delta}_{1/2,i}$ are used in this and the preceding sections for different matrices.

Lemma 3.1. *The matrices* $\mathbf{B}$ *and* $\mathbf{C}$ *are spectrally equivalent uniformly in* h*,* i.e.,

$$\mathbf{C}\prec\mathbf{B}\prec\mathbf{C}\,. \tag{3.3}$$

Proof. As it was established by Dryja/Widlund [17], the space decomposition (3.1) is stable. This means that, if for $v \in \mathcal{V}(\Omega)$ to introduce the seminorm $|||\cdot|||_1$,

$$|||v|||_1^2 = \min_{v=w_0+v_0+v_1+..+v_{n_\epsilon}\,:\, w_0\in\mathbb{V}_0(\Omega),\, v_i\in\mathcal{V}_i(\Omega_i)} \left(|w_0|_{1,\Omega}^2 + \sum_{i=0}^{n_\epsilon} |v_i|_{1,\Omega_i}^2 \right) ,$$

then

$$|||v|||_1^2 \prec |v|_{1,\Omega}^2 \prec |||v|||_1^2 , \qquad \forall\, v \in \mathcal{V}(\Omega) . \tag{3.4}$$

Let $\mathcal{V}_{\rm harm}(\Omega) \subset \mathcal{V}(\Omega)$ and $\mathcal{V}_{i,{\rm harm}}(\Omega_i) \subset \mathcal{V}_i(\Omega_i)$ be the subspaces of the discrete harmonic functions, whereas $\mathcal{V}_{\rm tr}(\partial\Omega)$ and $\mathcal{V}_{i,{\rm tr}}(\partial\Omega_i)$ be the corresponding spaces of the traces. For $u \in \mathcal{V}_{\rm harm}(\Omega)$ we can also introduce the seminorm $]|||\cdot|||[_{\partial\Omega}$

$$]|||u|||[\,_{\partial\Omega}^2 = \min_{u=w_0+u_0+u_1+..+u_{n_\epsilon}\,:\, w_0\in\mathbb{V}_0(\Omega),\, u_i\in\mathcal{V}_{\rm tr}(\partial\Omega_i)} \left(|w_0|_{1,\Omega}^2 + \sum_{i=0}^{n_\epsilon} |u_i|_{1/2,\partial\Omega_i}^2 \right) , \tag{3.5}$$

and show that

$$]|||u|||[\,_{\partial\Omega}^2 \prec |u|_{1,\Omega}^2 \prec]|||u|||[\,_{\partial\Omega}^2 , \qquad \forall\, u \in \mathcal{V}_{\rm harm}(\Omega) . \tag{3.6}$$

In order to get the left inequality, we take the decomposition of $u = w_0 + \widetilde{u}_1 + .. + \widetilde{u}_{n_\epsilon}$ entering $|||u|||_1$, then replace $\widetilde{u}_i$ by $u_i \in \mathcal{V}_{i,{\rm harm}}(\Omega_i)$, $u_i = |_{\partial\Omega_i}\widetilde{u}_i$, and use the equivalences

$$|\phi|_{1,\Omega_i} \asymp |\phi|_{1/2,\partial\Omega_i} , \qquad \forall\, \phi \in \mathcal{V}_{\rm harm}(\Omega_i) . \tag{3.7}$$

Suppose now that for given u the functions $w_0 \in \mathbb{V}_0(\Omega)$, $u_i \in \mathcal{V}_{\rm tr}(\partial\Omega_i)$ realize the min in (3.5). Let $u_i \in \mathcal{V}_{i,{\rm harm}}(\Omega_i)$ denote also the corresponding discrete harmonic functions and $\widetilde{u} = w_0 + u_0 + u_1 + .. + u_{n_\epsilon}$. Then $|u|_{1,\Omega} \le |\widetilde{u}|_{1,\Omega}$ and, since at each point of $\overline{\Omega}$ not more than three functions among $w_0, \widetilde{u}_i$ are distinct from zero, we have by Cauchy inequality and (3.7)

$$|u|_{1,\Omega}^2 \le 3 \left(|w_0|_{1,\Omega}^2 + \sum_{i=0}^{n_\epsilon} |u_i|_{1,\Omega_i}^2 \right) \prec \left(|w_0|_{1,\Omega}^2 + \sum_{i=0}^{n_\epsilon} |u_i|_{1/2,\partial\Omega_i}^2 \right) .$$

Therefore (3.6) indeed holds.

Assume additionally that there are symmetric nonnegative matrices $\boldsymbol{\mathcal{A}}_0$ and $\boldsymbol{\mathcal{B}}_i$ for which

$$\underline{\alpha}_0\, \mathbf{w}_0^\top \boldsymbol{\mathcal{A}}_0 \mathbf{w}_0 \le |w_0|_{1,\Omega}^2 \le \overline{\alpha}_0\, \mathbf{w}_0^\top \boldsymbol{\mathcal{A}}_0 \mathbf{w}_0 , \qquad \underline{\beta}_i\, \mathbf{u}_i^\top \boldsymbol{\mathcal{B}}_i \mathbf{u}_i \le |u_i|_{1/2,\partial\Omega_i}^2 \le \overline{\beta}_i\, \mathbf{v}_i^\top \boldsymbol{\mathcal{B}}_i \mathbf{u}_i \tag{3.8}$$

with $\underline{\alpha}_0, \underline{\beta}_i > 0$, any $\mathbf{w}_0 \Leftrightarrow w_0 \in \mathbb{V}_0(\Omega)$ and any $\mathbf{u}_i \Leftrightarrow u_i \in \mathcal{V}_{\rm tr}(\partial\Omega_i)$. Then for

$$\boldsymbol{\mathcal{B}}^{-1} := \mathbb{B}_0^+ + \sum_{i=0}^{n_\epsilon} \boldsymbol{\mathcal{B}}_i^+ , \qquad \mathbb{B}_0^+ := \boldsymbol{\mathcal{I}}_0 \boldsymbol{\mathcal{A}}_0^{-1} \boldsymbol{\mathcal{I}}_0^\top ,$$

by Theorem 2.1, the definition of the Schur complement $\mathbf{S}$, the inequalities (3.6),(3.8) and Zhang's formula [45], we have

$$\underline{\gamma}\, \boldsymbol{\mathcal{B}} \prec \mathbf{S} \prec \overline{\gamma}\, \boldsymbol{\mathcal{B}} , \qquad \underline{\gamma} = \frac{1}{\max\left(\overline{\alpha}_0, \max_i \overline{\beta}_i\right)} , \qquad \overline{\gamma} = \frac{1}{\min\left(\underline{\alpha}_0, \min_i \underline{\beta}_i\right)} . \tag{3.9}$$

Now it is left to prove that, if we take

$$\boldsymbol{\mathcal{A}}_0 = \mathbb{A}_0\,, \qquad \boldsymbol{\mathcal{B}}_j = \boldsymbol{\Delta}_j^{1/2}\,, \qquad j = 0, n_\epsilon\,, \qquad \boldsymbol{\mathcal{B}}_i = \overline{\boldsymbol{\Delta}}_{1/2,i}\,, \qquad i = 1, 2, .., n_\epsilon - 1\,, \tag{3.10}$$

then $\underline{\gamma}, \overline{\gamma} \asymp 1$. We turn first to one of the subdomains $\Omega_i,\ 1 \le i \le n_\epsilon - 1$. Let $V_{i,k},\ k = 0, 1, 2, 3,$ be the vertices of Ω_i counted, *e. g.*, counter-clockwise, starting from $V_{i,0} := (t_{1,i-1}, 0)$; $\mathcal{T}_{i,k}$ be the edges situated between vertices $V_{i,k}, V_{i,k+1}$; $t \in (0,1)$ is the coordinate of a point $x \in \mathcal{T}_{i,l},\ l = k, k+1,$ equal to the distance from the vertex $V_{i,k}$ divided by the length of the edge $\mathcal{T}_{i,l}$. Taking into account that the ratio of largest/smallest edges of $\partial\Omega_i$ is less than 2, the seminorm $|\cdot|_{1/2,\partial\Omega_i}$ may be defined similarly to its definition for the unit square, *i.e.*,

$$|v|^2_{1/2,\partial\Omega_i} \simeq \sum_{k=0}^{3} |v|^2_{1/2,\mathcal{T}_{i,k}} + \sum_{k=0}^{3} \int_0^1 \frac{(v_k(t) - v_{k-1}(t))^2}{|t|}\, dt, \tag{3.11}$$

where $v_l(t)$ is the restriction of v to the edge $\mathcal{T}_{i,l}$. Due to this characterization and the fact that traces of functions from $\mathcal{V}_{i,\text{harm}}(\Omega_i)$ vanish at two edges on the lines $x_1 \equiv t_{1,i-1}, t_{1,i+1}$, we come to the representation

$$|v|^2_{1/2,\partial\Omega_i} \simeq {}_{00}|v|^2_{1/2,\mathcal{T}_{i,0}} + {}_{00}|v|^2_{1/2,\mathcal{T}_{i,2}}\,, \qquad v \in \mathcal{V}_{i,\text{harm}}(\Omega_i)\,. \tag{3.12}$$

It is known that ${}_{00}|v_{i,k}|^2_{1/2,\mathcal{T}_{i,k}} \asymp \mathbf{v}_{i,k}^\top \Delta_i^{1/2} \mathbf{v}_{i,k}$, see, *e.g.*, Andreev [2, 3] and Dryja [16], and therefore from the definition of $\overline{\boldsymbol{\Delta}}_{1/2,i}$ it follows that the matrices $\boldsymbol{\mathcal{B}}_i$ for $i = 1, 2, .., n_\epsilon - 1$ may be defined according to (3.10) for $\underline{\beta}_i, \overline{\beta}_i \asymp 1$. The choice (3.10) for $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{B}}_{n_\epsilon}$ is approved in a similar way. □

3.2 Splitting vertices and edges

Let us consider now two preconditioners: one with the vertices split from the edges and another in which additionally each vertical edge is split from all other edges. First of all, we represent the coarsest subspace $\mathbb{V}_0$ by the direct sum

$$\mathbb{V}_0 = \mathbb{V}_0^{\text{v}} \oplus \mathbb{V}_0^{\text{e}}\,,$$

where $\mathbb{V}_0^{\text{v}}$ is the subspace of the bilinear functions on Ω. By $\boldsymbol{\mathcal{I}}_{00}$ is denoted the interpolation matrix relating to each $\mathbf{v}_V \Leftrightarrow v_V \in \mathbb{V}_0^{\text{v}}$ the vector representation $\mathbf{v}_0 = \boldsymbol{\mathcal{I}}_{00}\mathbf{v}_V$ of v_V as an element of $\mathbb{V}_0$. In the basis corresponding to the above coarsest space representation, the matrix $\mathbb{A}_0$ takes the form

$$\bar{\mathbb{A}}_0 = \begin{pmatrix} \bar{\mathbb{A}}_0^{\text{e}} & \bar{\mathbb{A}}_0^{\text{ev}} \\ \bar{\mathbb{A}}_0^{\text{ve}} & \bar{\mathbb{A}}_0^{\text{v}} \end{pmatrix} \qquad \text{with} \qquad \bar{\mathbb{A}}_0^{\text{v}} = \boldsymbol{\mathcal{I}}_{00}^\top \mathbb{A}_0 \boldsymbol{\mathcal{I}}_{00}\,, \qquad \bar{\mathbb{A}}_0^{\text{e}} = \mathbb{A}_0^{\text{e}}\,,$$

assuming that $\mathbb{A}_0^{\text{e}}$, $\mathbb{A}_0^{\text{v}}$ and $\mathbb{A}_0^{\text{ve}}$ are corresponding blocks of $\mathbb{A}_0$. We define the inverse to the preconditioner $\boldsymbol{\mathcal{A}}_0$ for $\mathbb{A}_0$ by omitting in $\bar{\mathbb{A}}_0$ off diagonal blocks, inverting the block diagonal matrix and returning to the basis, induced by the coarsest grid:

$$\boldsymbol{\mathcal{A}}_0^{-1} = \boldsymbol{\mathcal{I}}_{00}(\bar{\mathbb{A}}_0^{\text{v}})^+ \boldsymbol{\mathcal{I}}_{00}^\top + (\mathbb{A}_0^{\text{e}})^+\,. \tag{3.13}$$

Let us consider the subspace $\mathcal{V}_0(\Omega_0)$ and the corresponding subspace of the traces $\mathcal{V}_{0,\mathrm{tr}}(\mathcal{T}_0)$. We have

$$\mathcal{V}_{0,\mathrm{tr}}(\mathcal{T}_0) = \mathcal{V}^{\mathrm{v}}_{0,\mathrm{tr}}(\mathcal{T}_0) \oplus \mathcal{V}^{\mathrm{e}}_{0,\mathrm{tr}}(\mathcal{T}_0)\,, \qquad V_{0,\mathrm{tr}} = V^{\mathrm{v}}_{0,\mathrm{tr}} \oplus V^{\mathrm{e}}_{0,\mathrm{tr}}$$

with $\mathcal{V}_{0,\mathrm{v},\mathrm{tr}}$ containing functions linear on each of the three parts $\gamma_{0,l},\ l=0,1,2,$ of $\mathcal{T}_0$. The matrix $\mathbf{\Delta}_{1/2,0}$, found in (3.2), may be also written in the new basis in the block form

$$\bar{\mathbf{\Delta}}_{1/2,0} = \begin{pmatrix} \bar{\mathbf{\Delta}}^{\mathrm{e}}_{1/2,0} & \bar{\mathbf{\Delta}}^{\mathrm{e,v}}_{1/2,0} \\ \bar{\mathbf{\Delta}}^{\mathrm{v,e}}_{1/2,0} & \bar{\mathbf{\Delta}}^{\mathrm{v}}_{1/2,0} \end{pmatrix},$$

where the blocks $\bar{\mathbf{\Delta}}^{\mathrm{e}}_{1/2,0} = \mathbf{\Delta}^{\mathrm{e}}_{1/2,0}$ and $\bar{\mathbf{\Delta}}^{\mathrm{v}}_{1/2,0}$ are related to the nodes on the edges and vertices, respectively, and the latter has the dimension 2×2. Let $\mathbf{T}_0$ be the interpolation matrix such that $\mathbf{v}^{\mathrm{v}}$ and $\mathbf{v} = \mathbf{T}_0\mathbf{v}^{\mathrm{v}}$ represent the same $v^{\mathrm{v}} \in \mathcal{V}^{\mathrm{v}}_{0,\mathrm{tr}}$ in the vector spaces $V^{\mathrm{v}}_{0,\mathrm{tr}}$ and $V_{0,\mathrm{tr}}$, respectively. For simplifying notations, we set $\mathbf{\Delta}_2 = \bar{\mathbf{\Delta}}^{\mathrm{v}}_{1/2,0}$, where, evidently, this matrix is defined as $\mathbf{\Delta}_2 = \mathbf{T}_0^\top \mathbf{\Delta}^{\mathrm{v}}_{1/2,0}\mathbf{T}_0$. Similarly with (3.13), we define the preconditioner $\boldsymbol{\mathcal{B}}_0 = \widetilde{\mathbf{\Delta}}_{1/2,0}$ for $\mathbf{\Delta}_{1/2,0}$ by the formula

$$\boldsymbol{\mathcal{B}}_0^{-1} = \widetilde{\mathbf{\Delta}}^{-1}_{1/2,0} = \mathbf{T}_0\mathbf{\Delta}_2^{-1}\mathbf{T}_0^\top + (\mathbf{\Delta}^{\mathrm{e}}_{1/2,0})^+ \tag{3.14}$$

and the matrix $\boldsymbol{\mathcal{B}}_0^+ = \widetilde{\mathbf{\Delta}}^{+}_{1/2,0}$ defined on V_{tr} as $\boldsymbol{\mathcal{B}}_0^{-1} = \widetilde{\mathbf{\Delta}}^{-1}_{1/2,0}$ continued by zero entries. Matrices $\boldsymbol{\mathcal{B}}_{n_\epsilon}^{-1} = \widetilde{\mathbf{\Delta}}^{-1}_{1/2,n_\epsilon}, \boldsymbol{\mathcal{B}}_{n_\epsilon}^{+} = \widetilde{\mathbf{\Delta}}^{+}_{1/2,n_\epsilon}$ are obtained exactly in the same way, if we change one variable $x_1 \Leftarrow (1-x_1)$. Let us note that for $\mathbf{\Delta}_2$ one can take the matrix

$$\mathbf{\Delta}_2 = \frac{1}{6}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

which is the block of the stiffness matrix for the bilinear reference element induced by the Dirichlet integral over the reference square.

The introduced above matrices allow us to define the Schur complement preconditioner $\mathbb{C}$ according to formula

$$\mathbb{C}^{-1} = \mathbb{B}_0^+ + \boldsymbol{\mathcal{B}}_0^+ + \boldsymbol{\mathcal{B}}_{n_\epsilon}^+ + \sum_{i=1}^{n_\epsilon-1} \overline{\mathbf{\Delta}}^{\,+}_{1/2,i}\,, \tag{3.15}$$

with $\mathbb{B}_0^+ := \boldsymbol{\mathcal{I}}_0\boldsymbol{\mathcal{A}}_0^{-1}\boldsymbol{\mathcal{I}}_0^\top$ and $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$ and $\boldsymbol{\mathcal{B}}_{n_\epsilon}$ defined by means of (3.13) and (3.14). According to this formula the subsystems relative to vertices are solved separately from the subsystems related to edges. However, the degrees of freedom for edges are coupled in subsystems to be solved. We decouple each vertical edge from the rest edges in the next step simply by replacing the corresponding off diagonal blocks in $\mathbf{\Delta}_{1/2,i},\ i=0,n_\epsilon$, by zero blocks.

For simplicity we assume that the lengths of the three parts $\gamma_{k,l},\ l=0,1,2,$ of each of the sets $\mathcal{T}_k,\ k=0,n_\epsilon,$ are equal to ϵ and contain the same number $k_0=(\nu-4)/3$ of the internal nodes. To each part $\gamma_{k,l}$ we relate the tridiagonal matrix $\hat{\mathbf{\Delta}}^{2}_{1/2} = \mathrm{tridiag}\,[-1,2-1]_1^{k_0}$, and define the blockdiagonal preconditioner

$$\hat{\mathbf{\Delta}}^{-1}_{1/2,0} = \mathrm{diag}\,[\hat{\mathbf{\Delta}}^{-1}_{1/2}, \hat{\mathbf{\Delta}}^{-1}_{1/2}, \hat{\mathbf{\Delta}}^{-1}_{1/2}, \mathbf{T}_0\mathbf{\Delta}_2^{-1}\mathbf{T}_0^\top]\,,$$

for the matrix $\mathbf{\Delta}_{1/2,0}$. The three first entries in the square brackets correspond to the three parts of $\mathcal{T}_0$ and the forth term to its vertices. The preconditioner $\hat{\mathbf{\Delta}}_{1/2,n_\epsilon}$ for $\mathbf{\Delta}_{1/2,n_\epsilon}$ is defined similarly, and under the made assumptions coincides with $\hat{\mathbf{\Delta}}_{1/2,0}$ up to the numbering of unknowns. Now a simpler than $\mathbb{C}$ preconditioner $\boldsymbol{\mathcal{C}}$ for the Schur complement may be defined:

$$\boldsymbol{\mathcal{C}}^{-1} = \mathbb{B}_0^+ + \hat{\mathbf{\Delta}}^+_{1/2,0} + \hat{\mathbf{\Delta}}^+_{1/2,n_\epsilon} + \sum_{i=1}^{n_\epsilon -1} \overline{\mathbf{\Delta}}^+_{1/2,i}\,, \tag{3.16}$$

in which only a pair of opposite longer edges are coupled.

Lemma 3.2. *For all $\epsilon \in (0,1]$ and $h \le (0,\epsilon]$, there are fulfilled the bounds*

$$\min\left(\frac{1}{n_\epsilon}, \frac{1}{1+\log \underline{n}}\right)\mathbb{C} \prec \mathbf{S} \prec \mathbb{C}\,, \qquad \min\left(\frac{1}{n_\epsilon}, \frac{1}{1+\log^2 \underline{n}}\right)\boldsymbol{\mathcal{C}} \prec \mathbf{S} \prec \boldsymbol{\mathcal{C}}\,, \qquad \underline{n} = \min(n_1, n_2)\,.$$

Proof. Different choices for $\mathbb{B}_0$ and $\boldsymbol{\mathcal{B}}_j,\ j = 0, n_\epsilon$ from those in (3.10) were used. Therefore, for obtaining the first pair of inequalities, it is necessary to take additionally into account

i) precontitioning of $\bar{\mathbb{A}}_0$ by its block diagonal part diag $[\bar{\mathbb{A}}_0^{\mathrm{e}}, \bar{\mathbb{A}}_0^{\mathrm{v}}]$ and

ii) preconditioning of $\mathbf{\Delta}_{1/2,0}$ by $\widetilde{\mathbf{\Delta}}_{1/2,0}$.

Apart from that, in order to pass to the second pair inequalities, we need to estimate

iii) replacement of the block $\mathbf{\Delta}^{(\mathrm{e})}_{1/2,0}$ in $\widetilde{\mathbf{\Delta}}_{1/2,0}$ by the block diagonal matrix

$$\hat{\mathbf{\Delta}}^{(\mathrm{e})}_{1/2,0} = \mathrm{diag}\,[\hat{\mathbf{\Delta}}_{1/2}, \hat{\mathbf{\Delta}}_{1/2}, \hat{\mathbf{\Delta}}_{1/2}]$$

and similar replacement for $\mathbf{\Delta}^{(\mathrm{e})}_{1/2,n_\epsilon}$.

Calculation of the bounds related to i) is straightforward. If we take into account results of Andreev [2, 3], Dryja [16] on the finite-difference representations of seminorms $|\cdot|_{1/2}$ and ${}_{00}|\cdot|_{1/2}$, the proof of ii) and iii) is reduced to the proof of equivalent inequalities for the corresponding boundary norms for FE functions. Since the subdomains Ω_i are shape regular, such bounds related to ii) and iii) may be found in Bramble/Pasciak/Schatz [9] - [12]. It can be added that the estimates of the lemma are sharp at least for the characteristic situations when n_2 is small, *e.g.*, $n_2 = 1$, or $n_1 = n_2$ and consequently $n_\epsilon = 1$. □

3.3 Preconditioning by nonoverlapping domain decomposition

The same coarsest grid introduces the nonoverlapping domain decomposition with the subdomains Ω_i being the nests of this grid, *e.g.*,

$$\Omega_i = (t_{1,i-1}, t_{1,i}) \times (0,\epsilon)\,, \qquad i = 1, 2, .., n_\epsilon\,.$$

For the edges of each subdomain Ω_i, we use the notations $\mathcal{T}_{i,k}$ and order them similarly to the case of the overlapping decomposition, see Subsection 3.1. We can represent the FE space as the direct sum $\mathcal{V}(\Omega) = \mathbb{V}_0(\Omega) \oplus \mathcal{W}(\Omega)$, assuming by $\mathbb{V}_0(\Omega)$ the coarsest grid space as in (3.1). Subdomains Ω_i are shape regular and according to the result of Bramble/Pasciak/Schatz [9]-[12]

$$\frac{1}{(1+\log n_2)^2}\left(|\, v_0\,|^2_{1,\Omega_i} + \sum_{k=0}^{3} {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{i,k}}\right) \prec |\, v\,|^2_{1,\Omega_i} \prec |\, v_0\,|^2_{1,\Omega_i} + \sum_{k=0}^{3} {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{i,k}} \tag{3.17}$$

for all $v = (v_0 + v_W) \in \mathcal{V}(\Omega_i)$, where $v_0 \in \mathbb{V}_0(\Omega_i)$, $v_W \in \mathcal{W}(\Omega_i)$ and $\mathbb{V}_0(\Omega_i)$ and $\mathcal{W}(\Omega_i)$ are the restrictions of the spaces $\mathbb{V}_0(\Omega)$ and $\mathcal{W}(\Omega)$ to the subdomain Ω_i. From here, we directly come to the inequalities

$$\frac{1}{(1+\log n_2)^2}\left(|\, v_0\,|^2_{1,\Omega} + {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{0,3}} + {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{n_\epsilon,1}} + \sum_{i=1}^{n_\epsilon}\sum_{k=0,2} {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{i,k}}\right) \prec$$

$$|\, v\,|^2_{1,\Omega} \prec |\, v_0\,|^2_{1,\Omega} + {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{0,3}} + {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{n_\epsilon,1}} + \sum_{i=1}^{n_\epsilon}\sum_{k=0,2} {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{i,k}}\,. \tag{3.18}$$

Let $\boldsymbol{\mathcal{B}}_W$ be the matrix spectrally equivalent to the matrix of the quadratic form (3.18) on the subspace $\mathcal{W}(\Omega)$, *i.e.,*

$$\mathbf{v}_W^\top \boldsymbol{\mathcal{B}}_W \mathbf{v}_W \prec {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{0,3}} + {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{n_\epsilon,1}} + \sum_{i=1}^{n_\epsilon}\sum_{k=0,2} {}_{00}|\, v_E\,|^2_{1/2,\mathcal{T}_{i,k}} \prec \mathbf{v}_W^\top \boldsymbol{\mathcal{B}}_W \mathbf{v}_W\,. \tag{3.19}$$

Then (3.18) is equivalent to the inequality

$$\frac{1}{(1+\log n_2)^2}\mathbf{v}^\top \mathbb{C}_{\rm hi}\,\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}^{(2)}\,\mathbf{v} \prec \mathbf{v}^\top \mathbb{C}_{\rm hi}\,\mathbf{v}\,, \tag{3.20}$$

where

$$\mathbb{C}_{\rm hi} = \begin{pmatrix} \boldsymbol{\mathcal{B}}_W & \mathbf{0} \\ \mathbf{0} & \mathbb{A}_0 \end{pmatrix}$$

and $\mathbf{B}^{(2)}$ stands for the Schur complement, transformed to the two level basis corresponding to the representation $\mathcal{V}(\Omega) = \mathbb{V}_0(\Omega) \oplus \mathcal{W}(\Omega)$ of the FE space.

Slightly changing reasoning, one also can get

$$\min\left(\frac{1}{n_\epsilon(1+\log \underline{n})}, \frac{1}{(1+\log \underline{n})^2}\right)\mathbf{v}^\top \boldsymbol{\mathcal{C}}_{\rm hi}\,\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}_{(3)}\mathbf{v} \prec \mathbf{v}^\top \boldsymbol{\mathcal{C}}_{\rm hi}\,\mathbf{v} \tag{3.21}$$

with

$$\boldsymbol{\mathcal{C}}_{\rm hi} = \begin{pmatrix} \boldsymbol{\mathcal{B}}_W & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{A}_0^{\rm e} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbb{A}}_0^{\rm v} \end{pmatrix}$$

and $\mathbf{B}_{(3)}$ standing for the Schur complement corresponding to the three level representation $\mathcal{V}(\Omega) = \mathbb{V}_0^{\rm v} \oplus \mathbb{V}_0^{\rm e}(\Omega) \oplus \mathcal{W}(\Omega)$ of the FE space.

Let us reorder the sets $\mathcal{T}_{i,k} \subset \partial\Omega$ consecutively counter clockwise starting from $\mathcal{T}_{1,0}$, introduce for them notations $\mathcal{T}^i$, $i = 1, 2, .., 2n_\epsilon + 2$, and by ν_i assume the number of the intervals of the source mesh on $\mathcal{T}^i$. Estimates (3.19) hold for

$$\boldsymbol{\mathcal{B}}_W = \mathrm{diag}\,[\boldsymbol{\Delta}_{1/2,i}]_{i=1}^{2(n_\epsilon+1)} \tag{3.22}$$

with $\boldsymbol{\Delta}_{1/2,i} = \mathrm{tridiag}\,[-1, 2, -1]_1^{\nu_i-1}$.

We have proved

Lemma 3.3. *With $\boldsymbol{\mathcal{B}}_W$ defined in (3.22), the estimates (3.21) hold for all $\epsilon \in (0,1]$ and $h \leq (0,\epsilon]$.*

The preconditioner $\boldsymbol{\mathcal{C}}_{\rm hi}$ is simpler than one figuring in (3.16). In particular, it is given explicitly in the right form and easily invertible, and, therefore, can be used for assembling Schur complement preconditioner for a domain decomposed in rectangular subdomains. The system of algebraic equations with the preconditioner $\boldsymbol{\mathcal{C}}_{\rm hi}$ of Lemma 3.3 for the matrix is solved for $\mathcal{O}(n_\epsilon \underline{n} \log \underline{n}) = \mathcal{O}(\overline{n} \log \underline{n})$ a.o., where for the case under consideration $\overline{n} = n_1$, $\underline{n} = n_2$ However, the subspaces $\mathbb{V}_0$ and $\mathcal{W}$ depend on the aspect ratio of the rectangle Ω, and, therefore, the assembled interface Schur complement preconditioner is not easy to invert.

4 Orthotropic disctetizations with arbitrary aspect ratios on thin rectangles

A more complicated situation arises when in a slim domain we have to solve a heat conduction problem with different heat conduction coefficients in different directions discretized by a uniform rectangular mesh. We consider this situation in this section under assumption that aspect ratios of the conductivity coefficients and sizes of the mesh may be arbitrary.

4.1 Reducing to isotropic discretization

The model problem, we turn to here, is

$$\alpha_\Omega(u,v) = \langle f, v\rangle\,, \qquad \forall\; v \in \mathbb{V}(\Omega) \subseteq H^1(\Omega)\,, \tag{4.1}$$

$$\alpha_\Omega(u,v) = \int_\Omega \nabla u(x) \cdot \boldsymbol{\rho}(x)\, \nabla v(x)\, dx\,,$$

in a slim domain $\Omega = (0,1) \times (0,\epsilon)$. Now $\boldsymbol{\rho} = \operatorname{diag}[\rho_1, \rho_2]$ is the 2×2 diagonal matrix, with the arbitrary positive constants ρ_k. The mesh may be quasiuniform in each direction x_k (separately) with different mesh parameters h_k or even more general, but for simplicity we restrict ourselves to the uniform rectangular mesh of arbitrary sizes $h_1, h_2 > 0$.

The FE stiffness matrix, which is denoted $\mathbf{D}$ here, is represented in the similar to (2.11) block form

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_I & \mathbf{D}_{I,B} \\ \mathbf{D}_{B,I} & \mathbf{D}_B \end{pmatrix}. \tag{4.2}$$

As previously, our purpose is to obtain a good preconditioner for the Schur complement

$$\mathbf{Y} = \mathbf{D}_B - \mathbf{D}_{B,I}\mathbf{D}_I^{-1}\mathbf{D}_{I,B}\,, \tag{4.3}$$

which will be attained in three steps. At the first step, we change variables and reduce the problem (4.1) to a transformed isotropic problem on some domain Ω_ξ. At the second step, the block diagonal preconditioner for the stiffness matrix $\mathbf{D}$ is derived, which, in turn, allows to obtain the block diagonal preconditioner for the Schur complement $\mathbf{Y}$ with two independent blocks on the diagonal. One of these blocks is the FE stiffness matrix on the finest quasiuniform mesh imbedded in the source mesh, which is called *rarefied transformed mesh*. Therefore, it is obtained by the rarefication of the transformed source mesh in one direction, corresponding to the smallest size of the transformed source mesh. For the corresponding block of the Schur complement preconditioner may be taken the matrix of the quadratic form $]\, v\, [^2_{\partial\Omega_\xi},\ \forall\ v \in \widetilde{\mathcal{V}}(\Omega_\xi)$, related to the pointed out rarefied quasiuniform mesh, which at the third step is replaced by simpler matrices, following the guidelines of the preceding sections. Another block of the Schur complement is itself a block diagonal matrix with simple explicitly written down blocks, specified on the unknowns, subjected to rarefication.

The change of variables $\xi_1 = x_1,\ \xi_2 = \sqrt{\rho_1/\rho_2}\, x_2,$ transforms the bilinear form $a_\Omega(\cdot,\cdot)$ to

$$a_\Omega(u,v) = \sqrt{\rho_1\rho_2}\, \widetilde{a}(u,v)\,, \qquad \widetilde{a}(u,v) = \int_{\Omega_\xi} \nabla_\xi u \cdot \nabla_\xi v \, d\xi\,, \tag{4.4}$$

with ∇_ξ denoting the gradient in the variables ξ and $\Omega_\xi = (0,1)\times(0,\widetilde{\epsilon})$ – the new domain with the changed thickness $\widetilde{\epsilon} = \epsilon\sqrt{\rho_1/\rho_2}$. At that the FE space $\mathcal{V}(\Omega)$ is transformed into the space $\widetilde{\mathcal{V}}(\Omega_\xi)$ of the piece wise bilinear functions on the rectangular mesh of the sizes $\hbar_1 = h_1,\ \hbar_2 = \sqrt{\rho_1/\rho_2}\, h_2$ with the mesh lines $\xi_k \equiv \xi_{k,l} = l\hbar_k$. This mesh is called the *transformed source mesh*. Respectively, $\mathbf{D} = \sqrt{\rho_1\rho_2}\,\mathbb{D},\ \mathbf{Y} = \sqrt{\rho_1\rho_2}\,\mathbb{Y}$,

$$\mathbb{Y} = \mathbb{D}_B - \mathbb{D}_{B,I}\mathbb{D}_I^{-1}\mathbb{D}_{I,B}$$

and $\mathbb{D}_I, \mathbb{D}_{I,B}, \mathbb{D}_B$ are the blocks of the stiffness matrix $\mathbb{D}$ generated by the bilinear form $\widetilde{a}(u,v)$ on the space $\widetilde{\mathcal{V}}(\Omega_\xi)$. Therefore, the preconditioning of $\mathbf{Y}$ is reduced to the preconditioning of the Schur complement $\mathbb{Y}$.

One can distinguish three cases

$$i)\ \hbar_2 \le \hbar_1 \le \widetilde{\epsilon}, \qquad ii)\ \hbar_2 \le \widetilde{\epsilon} \le \hbar_1\,, \qquad iii)\ \hbar_1 \le \hbar_2 \le \widetilde{\epsilon}, \tag{4.5}$$

to which other situations can be reduced. We start from $i)$.

Under the stated conditions, the embedded rarefied quasiuniform rectangular grid

$$\xi_k \equiv \widetilde{\xi}_{k,i}\,, \qquad k = 1,2, \qquad \text{with the steps} \qquad \eta_{k,i} = \widetilde{\xi}_{k,i} - \widetilde{\xi}_{k,i-1},$$

can be introduced, which results from the coarsening only in one direction ξ_2. In other words, it is the same uniform grid in the direction ξ_1 with $\eta_{1,i} \equiv \hbar_1 \equiv h_1$ and nonuniform in the direction ξ_2 with the sizes $\eta_{2,j}$, as much close as possible to $\hbar_1$. The mesh lines $\xi_2 \equiv \widetilde{\xi}_{2,j}$ of the rarefied mesh are defined in the following way. Firstly, we find $m_2 = \text{integer}\,[\widetilde{\epsilon}/h_1]$, secondly define the uniform mesh $\zeta_{2,j} = j\,\widetilde{\epsilon}/m_2,\ j = 0,1,..,m_2,$ and then shift the lines of this uniform nonembedded coarse mesh $\xi_2 = \zeta_{2,j}$ to the nearest lines $\xi_2 \equiv \xi_{2,l} = l\hbar_2$ of the transformed source mesh of the size $\hbar_2$ in the direction ξ_2. The number of the rarefied mesh intervals in the direction ξ_1 is $m_1 = n_1$. Obviously, the sizes of this mesh satisfy inequalities

$$\underline{c}h_1 \le \eta_{k,i} \le \bar{c}h_1\,, \qquad \underline{c} > 0\,, \qquad k = 1,2\,, \tag{4.6}$$

with the positive constants, for which, for convenience, we retained the notations as in (2.3). In a way of reducing the problem under consideration to one discussed in Sections 2 and 3, this mesh will play the role of the source mesh, see Fig. 2.

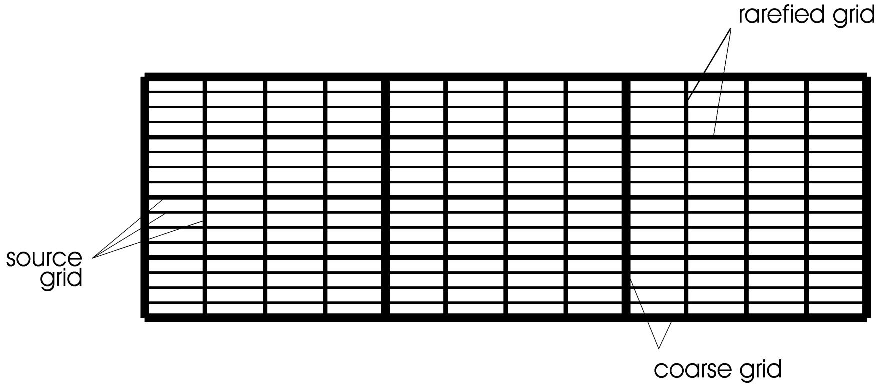

Figure 2: Transformed rectangular domain and the source, rarefied and coarse grids.

In compliance with the transformed and rarefied source grids, the space $\widetilde{\mathcal{V}}(\Omega_\xi)$ may be represented by the direct sum

$$\widetilde{\mathcal{V}}(\Omega_\xi) = \widetilde{\mathcal{V}}^{\,\mathrm{U}}(\Omega_\xi) \oplus \widetilde{\mathcal{E}}(\Omega_\xi)\,, \tag{4.7}$$

where $\widetilde{\mathcal{V}}^{\,\mathrm{U}}(\Omega_\xi)$ is the rarefied grid space of the piece wise bilinear and continuous on $\overline{\Omega}_\xi$ functions. Clearly, the space $\widetilde{\mathcal{E}}(\Omega_\xi)$ contains FE functions, which vanish on the lines $\xi_2 \equiv \widetilde{\xi}_{2,j}$ of the rarefied grid. Let $\widetilde{\mathcal{V}}_{\mathrm{tr}}(\partial\Omega_\xi)$ be the space of traces of functions from $\widetilde{\mathcal{V}}(\Omega_\xi)$ on $\partial\Omega_\xi$. For $u \in \widetilde{\mathcal{V}}(\Omega_\xi)$ and $v \in \widetilde{\mathcal{V}}_{\mathrm{tr}}(\partial\Omega_\xi)$, respectively, we introduce the norms

$$|u|_{1,\Omega_\xi} = \| u \|_{\Omega_\xi} = (\widetilde{a}(u,u))^{1/2}\,, \qquad \| v \|_{h,\partial\Omega_\xi} = \inf_{\phi\in\widetilde{\mathcal{V}}(\Omega_\xi)\,;\,\phi|_{\partial\Omega}=v} \| \phi \|_{\Omega_\xi}\,. \tag{4.8}$$

From the start, we assume that the basis in the space $\widetilde{\mathcal{V}}(\Omega_\xi)$ is chosen according to the decomposition (4.7), following which the matrix $\mathbb{D}$ can be represented also in the block form

$$\mathbb{D} = \begin{pmatrix} \mathbb{D}^E & \mathbb{D}^{EU} \\ \mathbb{D}^{UE} & \mathbb{D}^U \end{pmatrix}\,, \tag{4.9}$$

convenient for the purposes of this subsection. We underline that in (2.11) and in other block representations, like the one above, different orderings of the degrees of freedom, convenient for the specific purposes, are assumed. Let $\nu_{2,j}$ denote the number of the fine mesh intervals on the rarefied mesh interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ and

$$\boldsymbol{\Delta}_{2,j} = \mathrm{tridiag}\,[-1, 2, -1]_1^{\nu_{2,j}-1}\,. \tag{4.10}$$

An intermediate preconditioner $\widetilde{\boldsymbol{\mathcal{D}}}$ for $\mathbb{D}$ may be defined as

$$\widetilde{\boldsymbol{\mathcal{D}}} = \mathrm{diag}\,[\boldsymbol{\mathcal{D}}^E, \mathbb{D}^U]\,, \qquad \boldsymbol{\mathcal{D}}^E = \frac{\hbar_1}{\hbar_2}\,\mathrm{diag}\,[\,\underbrace{\boldsymbol{\Delta}_{2,j}, \boldsymbol{\Delta}_{2,j}, .., \boldsymbol{\Delta}_{2,j}}_{(n_1+1)\text{ times}}\,]_{j=1}^{m_2}\,, \tag{4.11}$$

Note that $\ker\widetilde{\boldsymbol{\mathcal{D}}} = \ker\mathbb{D}^U$. For a given j, the i-th block $\boldsymbol{\Delta}_{2,j}$ in the square brackets is related to the nodes on the interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ of the mesh line $\xi_1 \equiv \widetilde{\xi}_{1,i}$.

In the case ii), the rarefied quasiuniform mesh may not exist, this mesh is uniform rectangular and has the sizes h_1, $\widetilde{\epsilon}$, so that we have only one layer of the n_1 cells $h_1 \times \widetilde{\epsilon}$. Therefore $m_2 = 1$, and similarly to (4.11) we can set

$$\widetilde{\boldsymbol{\mathcal{D}}} = \text{diag}\,[\boldsymbol{\mathcal{D}}^E, \mathbb{D}^U]\,, \qquad \boldsymbol{\mathcal{D}}^E = \frac{\hbar_1}{\hbar_2}\,\text{diag}\,[\,\underbrace{\boldsymbol{\Delta}_2, \boldsymbol{\Delta}_2, .., \boldsymbol{\Delta}_2}_{(n_1+1)\text{ times}}\,]\,,$$

with the $(n_2-1)\times(n_2-1)$ blocks $\boldsymbol{\Delta}_2$. The matrix $\mathbb{D}^U$ is defined on the uniform rectangular coarsest grid, which coincides with the rarefied grid. Let us underline that in this case the matrix $\mathbb{D}^U$ is block-tridiagonal with the blocks 2×2 and does not require preconditioning.

If we have iii), the mesh parameter for the quasiuniform rectangular coarse grid is $\hbar_2$ and it satisfies

$$\underline{c}\hbar_2 \le \eta_{k,i} \le \overline{c}\hbar_2\,, \qquad \underline{c} > 0\,, \qquad k = 1, 2\,. \tag{4.12}$$

At the proper ordering of unknowns, we have again $\widetilde{\boldsymbol{\mathcal{D}}} = \text{diag}\,[\boldsymbol{\mathcal{D}}^E, \mathbb{D}^U]$, but

$$\boldsymbol{\mathcal{D}}^E = \frac{\hbar_2}{\hbar_1}\,\text{diag}\,[\,\underbrace{\boldsymbol{\Delta}_{1,i}, \boldsymbol{\Delta}_{1,i}, .., \boldsymbol{\Delta}_{1,i}}_{(n_2+1)\text{ times}}\,]_{j=1}^{m_1}\,, \qquad \boldsymbol{\Delta}_{1,i} = \text{tridiag}\,[-1, 2, -1]_1^{\nu_{1,i}-1}\,, \tag{4.13}$$

where $\nu_{1,i}$ is the number of the fine mesh intervals on the coarse mesh interval $(\widetilde{\xi}_{1,i-1}, \widetilde{\xi}_{1,i})$. For a given i, the j-th block $\boldsymbol{\Delta}_{1,i}$ in the square brackets is related to the nodes on the interval $(\widetilde{\xi}_{1,i-1}, \widetilde{\xi}_{1,i})$ of the mesh line $\xi_2 \equiv \widetilde{\xi}_{2,j}$.

Lemma 4.1. *Let positive h_1, h_2, ρ_1, ρ_2 and $\epsilon \le 1$, are such that $\hbar_k, \widetilde{\epsilon}$ satisfy i) or ii) and $\widetilde{\epsilon} \le 1$. Then*

$$\frac{1}{1 + \log\,\min(n_2, \frac{h_1\sqrt{\rho_2}}{h_2\sqrt{\rho_1}})}\,\widetilde{\boldsymbol{\mathcal{D}}} \prec \mathbb{D} \prec \widetilde{\boldsymbol{\mathcal{D}}}\,. \tag{4.14}$$

Proof. The proof follows from the inequalities

$$\min\,(\frac{1}{6}, \min_j \frac{1}{1 + \log \nu_{2,j}})\,\widetilde{\boldsymbol{\mathcal{D}}}\; \le \mathbb{D} \le c_2\,\widetilde{\boldsymbol{\mathcal{D}}}\,, \tag{4.15}$$

which are proved at the end of this subsection, see (4.30),(4.34), and from $\nu_{2,j} \le \min(n_2, \overline{c}h_1/\hbar_2)$, and $h_1/\hbar_2 = h_1\sqrt{\rho_2}\,/\,(h_2\sqrt{\rho_1})$ with $\overline{c}$ figuring in (4.6). □

Consideration of a more general situation, *i.e.,* covering all cases $i) - iii)$, allows to conclude that

$$\frac{1}{1 + \log\,\delta}\,\widetilde{\boldsymbol{\mathcal{D}}}\; \le \mathbb{D} \le c_2\,\widetilde{\boldsymbol{\mathcal{D}}}\,, \qquad \delta = \max_k \min(n_k, \frac{h_{3-k}\sqrt{\rho_k}}{h_k\sqrt{\rho_{3-k}}})\,. \tag{4.16}$$

The preconditioner $\widetilde{\boldsymbol{\mathcal{D}}}$ directly leads to some preconditioner for the Schur complement $\mathbb{Y}$. In doing this, we restrict ourselves to consideration of the case i). The counterparts of (2.11) and (2.12) for the matrices $\mathbb{D}_U$ and $\boldsymbol{\mathcal{D}}^E$, respectively, are

$$\begin{gathered} \mathbb{D}^U = \begin{pmatrix} \mathbb{D}_I^U & \mathbb{D}_{I,B}^U \\ \mathbb{D}_{B,I}^U & \mathbb{D}_B^U \end{pmatrix}, \qquad \boldsymbol{\mathcal{D}}^E = \begin{pmatrix} \boldsymbol{\mathcal{D}}_I^E & \boldsymbol{\mathcal{D}}_{I,B}^E \\ \boldsymbol{\mathcal{D}}_{B,I}^E & \boldsymbol{\mathcal{D}}_B^E \end{pmatrix}, \\ \mathbb{Y}^U = \mathbb{D}_B^U - \mathbb{D}_{B,I}^U(\mathbb{D}_I^U)^{-1}\mathbb{D}_{I,B}^U\,, \qquad \boldsymbol{\mathcal{Y}}^E = \boldsymbol{\mathcal{D}}_B^E - \boldsymbol{\mathcal{D}}_{B,I}^E(\boldsymbol{\mathcal{D}}_I^E)^{-1}\boldsymbol{\mathcal{D}}_{I,B}^E\,, \end{gathered} \tag{4.17}$$

and consequently the Schur complement $\widetilde{\boldsymbol{\mathcal{Y}}}$ for $\widetilde{\boldsymbol{\mathcal{D}}}$ has the form

$$\widetilde{\boldsymbol{\mathcal{Y}}} = \operatorname{diag}\,[\boldsymbol{\mathcal{Y}}^E, \mathbb{Y}^U]\,. \tag{4.18}$$

It is seen from (4.11) that in the matrix $\boldsymbol{\mathcal{D}}^E$ the internal degrees of freedom are not coupled with the boundary: $\boldsymbol{\mathcal{D}}^E_{I,B} = (\boldsymbol{\mathcal{D}}^E_{B,I})^\top = \mathbf{0}$. Therefore, $\boldsymbol{\mathcal{Y}}^E = \boldsymbol{\mathcal{D}}^E_B$, *i.e.*,

$$\boldsymbol{\mathcal{D}}^E_B = \frac{h_1}{\hbar_2} \operatorname{diag}\,[\, \boldsymbol{\Delta}_{2,j}, \boldsymbol{\Delta}_{2,j}\,]_{j=1}^{m_2}\,, \tag{4.19}$$

where the two matrices in the square brackets correspond to the nodes in the interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ on the left and right vertical edges of $\partial\Omega_\xi$, respectively.

Corollary 4.1. *Let* $\widetilde{\boldsymbol{\mathcal{Y}}} = \operatorname{diag}\,[\boldsymbol{\mathcal{D}}^E_B, \mathbb{Y}^U]$. *Then for all positive* h_1, h_2, ρ_1, ρ_2, *and* ϵ *we have*

$$\frac{1}{1+\log\,\delta}\,\widetilde{\boldsymbol{\mathcal{Y}}} \prec \mathbb{Y} \prec \widetilde{\boldsymbol{\mathcal{Y}}}\,. \tag{4.20}$$

Proof. The proof follows from Lemma 4.1, if to take into account that the interchange of the axes covers all possible combinations of the parameters entering the estimates, as well as the case $\epsilon \geq 1$. □

Corollary 4.1 reduces the Schur complement preconditioning to the case, considered in the preceding sections. In this way, the matrix $(\mathbb{Y}^U)^{-1}$ may be replaced by simpler matrices, but still sufficiently close to it in the spectrum.

The rest of this section contains derivation of the inequalities (4.15) of Lemma 4.1.

`The estimate from above.` We will prove the estimate (4.15),(4.14) from above for the case i). The cases $ii), iii$) are treated similarly. The matrix $\mathbb{D}$ can be obviously interpreted as assembled from the stiffness matrices of the rarefied grid nests $\delta_{i,j} := (\widetilde{\xi}_{1,i-1} < \xi_1 < \widetilde{\xi}_{1,i},\ \widetilde{\xi}_{2,j-1} < \xi_2 < \widetilde{\xi}_{2,j})$. We consider one of such nests and denote by $\mathbb{B}$ its stiffness matrix, for simplicity omitting the indices i, j in this and some other notations. This matrix can be represented as

$$\mathbb{B} = \mathbb{B}_1 \otimes \mathbb{M}_2 + \mathbb{M}_1 \otimes \mathbb{B}_2\,, \tag{4.21}$$

where $\mathbb{B}_k$ are the 1-d stiffness matrices

$$\mathbb{B}_1 = \frac{1}{h_1}\boldsymbol{\Delta}_1\,, \qquad \mathbb{B}_2 = \operatorname{diag}\,[\frac{1}{\eta_{1,i}}\boldsymbol{\Delta}_1, \frac{1}{\hbar_2}\boldsymbol{\Delta}_2]\,, \qquad \boldsymbol{\Delta}_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\,, \tag{4.22}$$

$$\boldsymbol{\Delta}_2 = \operatorname{tridiag}\,[-1, 2, -1]_1^{\nu_{2,j}-1}\,,$$

and $\mathbb{M}_k$ are the corresponding 1-d mass matrices, whereas $\nu_{2,j}$ denotes the number of the transformed source mesh intervals on the rarefied mesh interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$. If to set $\mathbb{M}_1 = h_1\mathbf{M}_1$, then another mass matrix can be represented as

$$\mathbb{M}_2 = \begin{pmatrix} \mathbb{M}_0 & \mathbb{M}_{0E} \\ \mathbb{M}_{E0} & \mathbb{M}_E \end{pmatrix} = \begin{pmatrix} \eta_{2,j}\mathbf{M}_1 & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \hbar_2\mathbf{M}_2 \end{pmatrix}$$

and as a consequence of the Cauchi inequality

$$\mathbb{M}_2 \leq 2\,\mathrm{diag}\,[\eta_{2,j}\mathbf{M}_1,\, \hbar_2\mathbf{M}_2]\,. \tag{4.23}$$

From (4.22) and (4.23), one has

$$\mathbb{B}_1\otimes\mathbb{M}_2 \leq 2\,\mathrm{diag}\,[\eta_{2,j}\mathbb{B}_1\otimes\mathbf{M}_1,\, \hbar_2\mathbb{B}_1\otimes\mathbf{M}_2] \leq 2\,\mathrm{diag}\,[\eta_{2,j}\mathbb{B}_1\otimes\mathbf{M}_1,\, \frac{2\hbar_2}{h_1}\mathbf{M}_2,\, \frac{2\hbar_2}{h_1}\mathbf{M}_2]\,, \tag{4.24}$$

where the second and third matrices in the last square brackets are related to the two vertical edges $\xi_1 \equiv \xi_{1,i-1}, \xi_{1,i}$. We have also the inequality

$$\hbar_2\mathbf{M}_2 \leq \frac{\eta_{2,j}^2}{\pi^2\hbar_2}\Delta_2\,,$$

which is the discrete form of the inequality

$$\int_{\widetilde{\xi}_{2,j-1}}^{\widetilde{\xi}_{2,j}} v^2(x)dx \leq \frac{\eta_{2,j}^2}{\pi^2}\int_{\widetilde{\xi}_{2,j-1}}^{\widetilde{\xi}_{2,j}} (v')^2(x)dx$$

for piece wise linear functions. By combining with (4.24) and (4.6), it follows

$$\frac{\hbar_2}{h_1}\mathbf{M}_2 \leq \frac{\eta_{2,j}^2}{\pi^2 h_1\hbar_2}\Delta_2 \leq \frac{\overline{c}}{\pi^2}\,\frac{h_1}{\hbar_2}\Delta_2\,. \tag{4.25}$$

Therefore,

$$\mathbb{B}_1\otimes\mathbb{M}_2 \leq 2\mathrm{diag}\,[\eta_{2,j}\mathbb{B}_1\otimes\mathbf{M}_1,\, \frac{2\overline{c}}{\pi^2}\,\frac{h_1}{\hbar_2}\Delta_2,\, \frac{2\overline{c}}{\pi^2}\,\frac{h_1}{\hbar_2}\Delta_2]\,. \tag{4.26}$$

For another summand in (4.21), one can write

$$\mathbb{M}_1\otimes\mathbb{B}_2 = \mathbb{M}_1\otimes\mathrm{diag}\,[\frac{1}{\eta_{1,i}}\boldsymbol{\Delta}_1,\, \frac{1}{\hbar_2}\boldsymbol{\Delta}_2] = \mathrm{diag}\,[\mathbb{M}_1\otimes\frac{1}{\eta_{1,i}}\boldsymbol{\Delta}_1,\, \mathbb{M}_1\otimes\frac{1}{\hbar_2}\boldsymbol{\Delta}_2] \leq$$

$$\mathrm{diag}\,[\mathbb{M}_1\otimes\frac{1}{\eta_{1,i}}\boldsymbol{\Delta}_1,\, \frac{h_1}{2\hbar_2}\boldsymbol{\Delta}_2,\, \frac{h_1}{2\hbar_2}\boldsymbol{\Delta}_2]\,, \tag{4.27}$$

where we used the right one of the inequalities

$$\frac{h_1}{6}\mathbf{I} \leq \mathbb{M}_1 \leq \frac{h_1}{2}\mathbf{I}\,, \tag{4.28}$$

in which $\mathbf{I}$ is the unity matrix.

Multiplying (4.27) by 2 and adding (4.26), we get

$$\mathbb{B} \leq 2\,\mathrm{diag}\,[\mathbb{B}_0,\, c_1\,\frac{h_1}{\hbar_2}\boldsymbol{\Delta}_2,\, c_1\,\frac{h_1}{\hbar_2}\boldsymbol{\Delta}_2] \leq c_2\,\mathrm{diag}\,[\mathbb{B}_0,\, \frac{h_1}{\hbar_2}\boldsymbol{\Delta}_2,\, \frac{h_1}{\hbar_2}\boldsymbol{\Delta}_2]\,, \qquad c_1 = 1+\frac{2\overline{c}}{\pi^2}\,, \tag{4.29}$$

where $\mathbb{B}_0$ is the 4×4 block of the matrix $\mathbb{B}$ for the vertices of the nest $\delta_{i,j}$ and $c_2 = 2c_1$. By assembling the matrices in the right part of (4.29), which are defined for each nest, one comes to the block diagonal matrix $\widetilde{\boldsymbol{\mathcal{D}}}$ of (4.11) where $\mathbb{D}^U$ is the block of the matrix $\mathbb{D}$ for the nodes of the rarefied grid and $\boldsymbol{\Delta}_{2,j}$ is the matrix for the nest $\delta_{i,j}$ figuring in (4.29) as $\boldsymbol{\Delta}_2$. Each of the matrices $\boldsymbol{\Delta}_{2,j}$ in the brackets corresponds to the interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ on the mesh line $\xi_1 = \xi_{1,i},\ i = 0, 1, .., n_1 + 1$. From (4.29) it follows that

$$\mathbb{D} \leq c_2 \widetilde{\boldsymbol{\mathcal{D}}}\,. \tag{4.30}$$

`The estimate from below.` In order to prove the estimate (4.14) from below, we note that the preconditioner $\widetilde{\mathbb{D}} := \operatorname{diag}[\mathbb{D}^E, \mathbb{D}^U,]$ satisfies the inequalities

$$\min_j \frac{1}{(1+\log \nu_{2,j})} \widetilde{\mathbb{D}} \prec \mathbb{D} \leq 2\widetilde{\mathbb{D}}\,. \tag{4.31}$$

The right one is the Cauchy inequality. For the case of the quasiuniform rectangular mesh with $(\nu_{2,j} + 1) \times (\nu_{2,j} + 1)$ nodes on each nest $\delta_{i,j}$ of the quasiuniform coarse grid, the left inequality was proved by Bramble/Pasciak/Schatz [9]-[12]. As a consequence of this result, it holds also for the space $\widetilde{\mathcal{V}}(\Omega_\xi) = \widetilde{\mathcal{V}}^{\,\mathrm{U}}(\Omega_\xi) \oplus \widetilde{\mathcal{E}}(\Omega_\xi)$.

The matrix $\mathbb{D}^E$ is assembled from the corresponding stiffness matrices for the rarefied grid nests. If $\mathbb{B}^E$ is such a matrix for some nest $\delta_{i,j}$, it has the form

$$\mathbb{B}^E = \mathbb{B}_1 \otimes \mathbb{M}^E + \hbar_2^{-1} \mathbb{M}_1 \otimes \Delta_{2,j}\,. \tag{4.32}$$

Since the first summand is nonnegative, it can be omitted, and by virtue of (4.28)

$$\mathbb{B}^E \geq \hbar_2^{-1} \mathbb{M}_1 \otimes \Delta_{2,j} \geq \frac{h_1}{6\hbar_2} \operatorname{diag}[\,\boldsymbol{\Delta}_{2,j},\, \boldsymbol{\Delta}_{2,j}]\,, \tag{4.33}$$

The inequalities (4.31) and (4.33) allow to write

$$\min\left(\frac{1}{6}, \frac{1}{1+\log \nu_{2,j}}\right) \widetilde{\boldsymbol{\mathcal{D}}} \;\leq\; \mathbb{D}\,, \tag{4.34}$$

concluding the proof.

4.2 Schur complement preconditioner-solver

In order to obtain the preconditioner for the Schur complement it is sufficient now to take into account the results of the preceding sections, which are directly applicable to Schur complement $\mathbb{Y}^U$ for the matrix $\mathbb{D}^U$. Here we assume that the transformed FE space is decomposed into the direct sum

$$\widetilde{\mathcal{V}}(\Omega_\xi) = \widetilde{\mathcal{E}}(\Omega_\xi) \oplus \widetilde{\mathcal{V}}_0^{\mathrm{e}}(\Omega_\xi) \oplus \widetilde{\mathcal{V}}_0^{\mathrm{v}}(\Omega_\xi) \oplus \widetilde{\mathcal{W}}(\Omega_\xi)\,,$$

i.e., $\widetilde{\mathcal{V}}^{\,U}(\Omega_\xi) = \widetilde{\mathcal{V}}_0^{\mathrm{e}}(\Omega_\xi) \oplus \widetilde{\mathcal{V}}_0^{\mathrm{v}}(\Omega_\xi) \oplus \widetilde{\mathcal{W}}(\Omega_\xi)$ where the subspaces $\widetilde{\mathcal{V}}_0^{\mathrm{e}}(\Omega_\xi), \widetilde{\mathcal{V}}_0^{\mathrm{v}}(\Omega_\xi), \widetilde{\mathcal{W}}(\Omega_\xi)$ are defined for $\widetilde{\mathcal{V}}^{\,U}(\Omega_\xi)$ in the same way as in Subsection 3.3 the subspaces $\mathcal{V}_0^{\mathrm{e}}(\Omega), \mathcal{V}_0^{\mathrm{v}}(\Omega), \mathcal{W}(\Omega)$ were defined for the FE space $\mathcal{V}(\Omega)$.

Without change of notations, we assume that the matrix $\mathbb{D}$ is generated in the basis corresponding to the described decomposition of the FE space $\widetilde{\mathcal{V}}(\Omega_\xi)$ and that the preconditioner

$$\boldsymbol{\mathcal{C}}_{\rm hi} = \begin{pmatrix} \boldsymbol{\mathcal{B}}_W & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{A}_0^{\rm e} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbb{A}}_0^{\rm v} \end{pmatrix}$$

for the Schur complement $\mathbb{Y}^U$ is defined according to Subsection 3.3 exactly as the preconditioner $\boldsymbol{\mathcal{C}}_{\rm hi}$ of (3.21) for the Schur complement $\mathbf{B}_{(3)}$ of the stiffness matrix, generated by means of the 3-level decomposition $\mathcal{V}(\Omega) = \mathbb{V}_0^{\rm v} \oplus \mathbb{V}_0^{\rm e}(\Omega) \oplus \mathcal{W}(\Omega)$ of the FE space. Respectively, we define the new block diagonal Schur complement preconditioner $\boldsymbol{\mathcal{Y}}$, as

$$\boldsymbol{\mathcal{Y}} = {\rm diag}\,[\boldsymbol{\mathcal{D}}^E, \boldsymbol{\mathcal{B}}_W, \mathbb{A}_0^{\rm e}, \bar{\mathbb{A}}_0^{\rm v}]\,,$$

where $\boldsymbol{\mathcal{B}}_W$ is of the same form as in Lemma 3.3.

Theorem 4.1. *Let positive $h_1,\ h_2,\ \rho_1, \rho_2$ and $\epsilon \le 1,$ are such that $\hbar_k, \widetilde{\epsilon}$ satisfy i) or ii) and $\widetilde{\epsilon} \le 1.$ Then*

$$\frac{1}{(1+\log\,\delta)(1+\log \underline{m})}\ \min\,(\frac{1}{m_\epsilon}, \frac{1}{(1+\log \underline{m})})\ \boldsymbol{\mathcal{Y}} \prec \mathbb{Y} \prec \boldsymbol{\mathcal{Y}}\,, \tag{4.35}$$

where $\underline{m} = \min_k m_k/m_{3-k}$ and $m_\epsilon = 1/\widetilde{\epsilon}$.

Proof. The proof is completed by combining the bounds of Lemma 4.1 and Lemma 3.3. □

Any other preconditioner, considered in the preceding sections, can be used for preconditioning of the matrix $\mathbb{Y}^U$.

The construction of the preconditioner and the bounds (4.35) are easily adjustable to a more general case of $\Omega = H_1 \times H_2$. In the same way we introduce the rarefied source and the coarsest meshes, while the role of ϵ is played by $\min_k\,(H_k/H_{3-k})$. The above form of the preconditioner is retained, if after the transformation to the isotropic problem the shortest edge is directed along the axis x_2. In general, with the notation

$$\theta = \max_k\ \min\,(m_k, \frac{H_k\sqrt{\rho_{3-k}}}{H_{3-k}\sqrt{\rho_k}})\,,$$

the counterpart of (4.35) for all positive $H_k,\ h_k \le H_k,\ \rho_k$ is

$$\frac{1}{(1+\log\,\delta)(1+\log \underline{m})}\ \min\,(\frac{1}{\theta}, \frac{1}{(1+\log \underline{m})})\ \boldsymbol{\mathcal{Y}} \prec \mathbb{Y} \prec \boldsymbol{\mathcal{Y}}\,. \tag{4.36}$$

Solving systems $\boldsymbol{\mathcal{Y}}\mathbf{v}_B = \mathbf{f}_B$ is cheap and namely requires not more than $\mathcal{O}(\overline{n}\log \underline{n})$ a.o., but the relative condition depends on the ratio θ of the maximal to the minimal lengths of the edges of the rectangle Ω_ξ. However, we will see that this dependence is such that it allows to obtain an almost optimal DD preconditioner-solver for piece wise orthotropic discretizations.

In what follows, the notations

$$\mathcal{U}^E(\Omega), \mathcal{U}^U(\Omega), \mathcal{U}_0^{\rm e}(\Omega), \mathcal{U}_0^{\rm v}(\Omega),\ \Psi(\Omega)\,, \tag{4.37}$$

are used for the subspaces of $\mathcal{V}(\Omega)$, corresponding to the subspaces $\widetilde{\mathcal{E}}(\Omega_\xi), \widetilde{\mathcal{V}}^U(\Omega_\xi), \widetilde{\mathcal{V}}_0^{\rm e}(\Omega_\xi)$, $\widetilde{\mathcal{V}}_0^{\rm v}(\Omega_\xi), \widetilde{\mathcal{W}}(\Omega_\xi)$ of $\widetilde{\mathcal{V}}(\Omega_\xi)$, respectively.

4.3 Compatible Schur complement preconditioner-solver

Suppose that the domain Ω is the union

$$\Omega = \cup_{j_1,j_2=1}^{J_1,J_2} \Omega_j\,, \qquad j=(j_1,j_2)\,,$$

of compatible and otherwise arbitrary rectangular subdomains Ω_j, and the boundary value problem and its discretization are described by (1.1)-(1.4). Construction of the Schur complement preconditioners $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}}_j$, figuring in (4.35)(4.36), for each subdomain involves decompositions into subspaces (4.37), which now may be denoted

$$\mathcal{U}_j^E(\Omega_j), \mathcal{U}_j^U(\Omega_j), \mathcal{U}_{j,0}^{\mathrm{e}}(\Omega_j), \mathcal{U}_{j,0}^{\mathrm{v}}(\Omega_j), \Psi_j(\Omega_j)\,. \tag{4.38}$$

Since the grids, to which these subspaces correspond, and the very subspaces for each subdomain depend on the aspect ratio ϵ_j of its edges and on $\rho_{j,k}$, in general these subspaces are not compatible. Therefore, whereas solving the system $\boldsymbol{\mathcal{Y}}_j \mathbf{v}_{B_j} = \mathbf{f}_{B_j}$ for one subdomain is easy and cheap, the existence of a fast solving procedure for the system with $\boldsymbol{\mathcal{Y}}_{\mathrm{DD}}$, assembled from the preconditioners $\boldsymbol{\mathcal{Y}}_j$ is not guaranteed. A compatible fast Schur complement preconditioner-solver, which can be used for the discrete problem (1.1)-(1.4), was suggested by Korneev [26],[27]. Since it will be used in a DD algorithm for a more general problem, than in these papers, and the estimates for the relative condition and the total computational work for the problem under consideration are also different, we briefly describe its basic properties. For more details we refer to [27].

Now we return to the problem (4.1) and derive the preconditioner, in which the unknowns at the vertices of Ω are split from the rest unknowns. Let us subdivide each rectangular nest of the source mesh into two triangles by the diagonal of a common direction for all nests. The pointed out triangulation induces the space $\hat{\mathcal{V}}(\Omega)$ of continuous functions linear on each triangle of the triangulation, which is represented by the direct sum $\hat{\mathcal{V}}(\Omega) = \hat{\mathcal{V}}_I(\Omega) \oplus \hat{\mathcal{V}}_B(\Omega)$, where $\hat{\mathcal{V}}_B(\Omega) = \hat{\mathcal{V}}_E(\Omega) \oplus \hat{\mathcal{V}}_V(\Omega)$ and $\hat{\mathcal{V}}_V(\Omega)$ is the subspace of vertex functions. Respectively the stiffness matrix generated by the space $\hat{\mathcal{V}}(\Omega)$, which is denoted $\mathbf{L}$, is represented in the block forms

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_I & \mathbf{L}_{IB} \\ \mathbf{L}_{BI} & \mathbf{L}_B \end{pmatrix} = \begin{pmatrix} \mathbf{L}_I & \mathbf{L}_{IE} & \mathbf{L}_{IV} \\ \mathbf{L}_{EI} & \mathbf{L}_E & \mathbf{L}_{EV} \\ \mathbf{L}_{VI} & \mathbf{L}_{VE} & \mathbf{L}_V \end{pmatrix}, \qquad \mathbf{L}_{IV} = \mathbf{L}_{VI} = \mathbf{0}\,. \tag{4.39}$$

Using the same indexation for the blocks of $\mathbf{D}$, one can define the tentative preconditioner $\hat{\mathbf{D}}$ for $\mathbf{D}$ by setting $\mathbf{L}_{IV} = 0, \mathbf{L}_{IV} = \mathbf{0}$. It is well known that

$$\hat{\mathbf{D}}, \mathbf{L} \prec \mathbf{D} \prec \mathbf{L}, \hat{\mathbf{D}} \tag{4.40}$$

for all $\rho_k > 0$ and all uniform rectangular source meshes with arbitrary aspect ratios. Therefore, the Schur complement $\hat{\mathbf{Y}} = \mathbf{L}_B - \mathbf{L}_{BI}\mathbf{L}_I^{-1}\mathbf{L}_{IB}$, of the matrix $\mathbf{L}$ serves as a good preconditioner for the Schur complement $\mathbf{Y}$. Due to (4.39),

$$\hat{\mathbf{Y}} = \begin{pmatrix} \hat{\mathbf{Y}}_E & \mathbf{L}_{EV} \\ \mathbf{L}_{VE} & \mathbf{L}_V \end{pmatrix}, \qquad \hat{\mathbf{Y}}_E = \mathbf{L}_E - \mathbf{L}_{EI}\mathbf{L}_I^{-1}\mathbf{L}_{IE}\,, \tag{4.41}$$

or equally one can set

$$\hat{\mathbf{Y}} = \begin{pmatrix} \hat{\mathbf{Y}}_E & \mathbf{D}_{EV} \\ \mathbf{D}_{VE} & \mathbf{D}_V \end{pmatrix}, \qquad \hat{\mathbf{Y}}_E = \mathbf{D}_E - \mathbf{D}_{EI}\mathbf{D}_I^{-1}\mathbf{D}_{IE}\,. \tag{4.42}$$

Let us represent $\hat{\mathbf{Y}}_E$ in the 4×4 block form with the blocks corresponding to the edges γ_k

$$\hat{\mathbf{Y}}_E = \left(\hat{\mathbf{Y}}_{E,k,l}\right)_{k,l=0}^{3}\,.$$

and note that all 16 blocks are nonzero, see [27]. Having replaced the blocks coupling the adjacent edges by zero blocks, one comes to the preconditioner

$$\hat{\boldsymbol{\mathcal{Y}}}_E = \begin{pmatrix} \hat{\mathbf{Y}}_{E,0,0} & \hat{\mathbf{Y}}_{E,0,1} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{Y}}_{E,1,0} & \hat{\mathbf{Y}}_{E,1,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Y}}_{E,2,2} & \hat{\mathbf{Y}}_{E,2,3} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Y}}_{E,3,2} & \hat{\mathbf{Y}}_{E,3,3} \end{pmatrix}. \tag{4.43}$$

Remark 4.1. *As it was shown in Korneev [26, 27], the computation of the matrices $\hat{\boldsymbol{\mathcal{Y}}}_E$ and $\hat{\mathbf{Y}}_E$ requires $\mathcal{O}(n_1 n_2)$ a.o. See,* e.g., *proofs of Theorem 5.1 and Lemma 5.2 in [27].*

Lemma 4.2. *Let there exist the preconditioner* $\mathbf{E}$*, satisfying the following conditions:*
1) it has the nonzero block structure similar to $\hat{\boldsymbol{\mathcal{Y}}}_E$ and
2) satisfies the inequalities

$$\underline{\gamma}_E\mathbf{E} \le \hat{\mathbf{Y}}_E \le \overline{\gamma}_E\mathbf{E}\,. \tag{4.44}$$

Then

$$\frac{\underline{\gamma}_E}{\overline{\gamma}_E}\,\hat{\boldsymbol{\mathcal{Y}}}_E \le \hat{\mathbf{Y}}_E \le 2\,\hat{\boldsymbol{\mathcal{Y}}}_E\,. \tag{4.45}$$

Proof. The right inequality (4.45) is the Cauchy inequality. In order to prove the left inequality (4.45), we rewrite the right inequality (4.44) as

$$\hat{\boldsymbol{\mathcal{Y}}}_E + (\hat{\mathbf{Y}}_E - \hat{\boldsymbol{\mathcal{Y}}}_E) \le \overline{\gamma}_E\mathbf{E}\,.$$

Let $\mathbf{v}^k$ be the subvectors of any vector $\mathbf{v}$, corresponding to edges γ_k, and $\mathbf{v}_*$ be the vector with the subvectors $\mathbf{v}_*^k = \mathbf{v}^k$ for $k = 0, 1$ and $\mathbf{v}_*^k = (-1)\mathbf{v}^k$ for $k = 2, 3$. It is easy to note that, if $\mathbf{v}^T(\hat{\mathbf{Y}}_E - \hat{\boldsymbol{\mathcal{Y}}}_E))\mathbf{v} \ne 0$, then $\mathbf{v}_*^T(\hat{\mathbf{Y}}_E - \hat{\boldsymbol{\mathcal{S}}}_E))\mathbf{v}_*$ has the opposite sign, whereas quadratic the forms $\hat{\boldsymbol{\mathcal{Y}}}_E$ and $\mathbf{E}$ are positive on the both vectors and do not change their values. Therefore,

$$\hat{\boldsymbol{\mathcal{Y}}}_E \le \overline{\gamma}_E\mathbf{E}\,, \tag{4.46}$$

and combining with the left inequality (4.44) we complete the proof. □

Corollary 4.2. *For all positive H_k, h_k, ρ_k,*

$$\underline{\beta}_E\,\hat{\boldsymbol{\mathcal{Y}}}_E \prec \hat{\mathbf{Y}}_E \prec \overline{\beta}_E\,\hat{\boldsymbol{\mathcal{Y}}}_E\,, \qquad \underline{\beta}_E = \frac{1}{(1+\log\delta)(1+\log\underline{m})^2}\,, \quad \overline{\beta}_E = 1\,. \tag{4.47}$$

Proof. Let us turn to

$$\overset{\circ}{\boldsymbol{\mathcal{Y}}}= \mathrm{diag}\,[\boldsymbol{\mathcal{D}}^E, \boldsymbol{\mathcal{B}}_W, \mathbb{A}_0^{\mathrm{e}}]\,,$$

which is the preconditioner $\boldsymbol{\mathcal{Y}}$, figuring in Theorem 4.1, restricted to the subspace of the edge unknowns. It has even a simpler structure than $\hat{\boldsymbol{\mathcal{Y}}}_E$, because in this preconditioner those edges are not coupled, which became the shortest after the transformation to the isotropic problem. If we repeat the derivation of (4.35) on the subspace of the edge unknowns, *i.e.,* omitting the steps related to the splitting of vertices, we come to the bounds

$$\frac{1}{(1+\log\,\delta)(1+\log\underline{m})^2}\;\overset{\circ}{\boldsymbol{\mathcal{Y}}}\prec \mathbb{Y}_E \prec\overset{\circ}{\boldsymbol{\mathcal{Y}}}\;. \tag{4.48}$$

If the $\mathbf{E}$ is understood as the matrix $\sqrt{\rho_1\rho_2}\,\overset{\circ}{\boldsymbol{\mathcal{Y}}}$ transformed to the nodal basis of the source mesh, then, in view of Lemma 4.2 and (4.48), Corollary 4.2 is true. □

It should be noted that the simpler preconditioner

$$\hat{\boldsymbol{\mathcal{Y}}}_E = \begin{pmatrix} \hat{\mathbf{Y}}_{E,0,0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Y}}_{E,1,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Y}}_{E,2,2} & \hat{\mathbf{Y}}_{E,2,3} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{Y}}_{E,3,2} & \hat{\mathbf{Y}}_{E,3,3} \end{pmatrix} \tag{4.49}$$

with decoupled short edges (after the transformation) may be also efficiently used in DD algorithms. The bound $\mathrm{cond}\,[\hat{\boldsymbol{\mathcal{Y}}}_E^{-1}\hat{\mathbf{Y}}_E] \prec \underline{\beta}_E^{-2}$ easily follows from the above bounds (4.44)–(4.48). The proof of a sharper bound, requires a more serious effort.

For numerical implementations of the preconditioner (4.43), it is important that FDFT, applied edge wise, makes it a block diagonal matrix with 2×2 blocks. Obviously, the matrix of FDFT, which is designated $\boldsymbol{\mathcal{F}}_E$, is the block diagonal matrix with the identical blocks for the opposite edges γ_k, γ_{k+2}:

$$\boldsymbol{\mathcal{F}}_E = [\,\boldsymbol{\mathcal{F}}_0,\ \boldsymbol{\mathcal{F}}_0,\ \boldsymbol{\mathcal{F}}_2,\ \boldsymbol{\mathcal{F}}_2\,] \tag{4.50}$$

As it was mentioned above, the matrix $\mathbf{\Lambda} := \boldsymbol{\mathcal{F}}_E^{\top}\hat{\boldsymbol{\mathcal{Y}}}_E\,\boldsymbol{\mathcal{F}}_E$ is block diagonal

$$\mathbf{\Lambda} = \mathrm{diag}\,\left[\mathrm{diag}\,[\,\mathbf{\Lambda}_{0,i}\,]_{i=1}^{n_2-1},\ \mathrm{diag}\,[\,\mathbf{\Lambda}_{2,k}\,]_{k=1}^{n_1-1}\,\right]$$

with 2×2 blocks from which $\mathbf{\Lambda}_{0,i}$ couples a pair of opposite nodes $(0,\xi_{2,i})$ and $(1,\xi_{2,i})$ on the vertical edges and $\mathbf{\Lambda}_{2,k}$ couples a pair of opposite nodes $(\xi_{1,k},0)$ and $(\xi_{1,k},\widetilde{\epsilon})$ of the horizontal edges. Due to the pointed out property the system with the matrix $\hat{\boldsymbol{\mathcal{Y}}}_E$ can be solved for $\mathcal{O}((n_1+n_2)\log\overline{n})$ a.o. For more details we refer to Section 5 in Korneev [27].

It is convenient to represent the FE space by the direct sum $\mathcal{V}(\Omega) = \mathcal{V}_E(\Omega)\oplus\mathcal{V}_V(\Omega)$, where $\mathcal{V}_E(\Omega)$ is the edge subspace with the nodal basis corresponding to the source FE space and $\mathcal{V}_V(\Omega) = \widetilde{\mathcal{V}}_0^{\mathrm{v}}(\Omega)$ is the subspace of the bilinear on Ω polynomials. Taking Lemma 4.2 into consideration, one can define the Schur complement preconditioner in the form

$$\hat{\boldsymbol{\mathcal{Y}}} = \mathrm{diag}\,[\hat{\boldsymbol{\mathcal{Y}}}_E, \sqrt{\rho_1\rho_2}\,\mathbb{A}_0^{\mathrm{v}}]\,. \tag{4.51}$$

Theorem 4.2. *For arbitrary positive H_k, $h_k \le H_k$ and ρ_k, the preconditioner $\hat{\boldsymbol{\mathcal{Y}}}$ satisfies the inequalities*

$$\frac{1}{(1+\log\,\delta)(1+\log\underline{m})}\,\min\left(\frac{1}{\theta},\frac{1}{1+\log\underline{m}}\right)\,\hat{\boldsymbol{\mathcal{Y}}} \prec \mathbf{Y} \prec \hat{\boldsymbol{\mathcal{Y}}}\,. \tag{4.52}$$

Proof. The right inequality is Cauchy's inequality. The derivation of the left inequality requires to prove two subsidiary inequalities, reflecting the block structure of $\hat{\boldsymbol{\mathcal{Y}}}$. First, we bound the block $\mathbb{A}_0^{\rm v}$. According to the decomposition

$$\widetilde{\mathcal{V}}(\Omega_\xi) = \widetilde{\mathcal{V}}_0^{\rm v}(\Omega_\xi) \oplus \widetilde{\mathcal{V}}_0^{\rm e}(\Omega_\xi) \oplus \widetilde{\mathcal{W}}(\Omega_\xi) \oplus \widetilde{\mathcal{E}}(\Omega_\xi)$$

an arbitrary transformed FE function $v \in \widetilde{\mathcal{V}}(\Omega_\xi)$ can be represented as $v = v_0^{\rm v} + v_0^{\rm e} + w + v^E$, respectively. For convenience, we introduce additionally FE functions $v_1 = v_0^{\rm v} + v_0^{\rm e}$ and $v_2 = v_1 + w$. Turning for definiteness to the case i), we can write the chain of the inequalities

$$|v_0^{\rm v}|^2_{1,\Omega_\xi} \prec m_\epsilon\,|v_1|^2_{1,\Omega_\xi} \prec m_\epsilon(1+\log\underline{m})\,|v_2|^2_{1,\Omega_\xi} \prec m_\epsilon(1+\log\underline{m})\left(1+\log\,\min(n_2\,,\frac{h_1\sqrt{\rho_2}}{h_2\sqrt{\rho_1}})\right)\,|v|^2_{1,\Omega_\xi}\,,$$

which is contained in (3.21). Its resulting inequality is equivalent to

$$\frac{1}{1+\log\,\min(n_2,\frac{h_1\sqrt{\rho_2}}{h_2\sqrt{\rho_1}})}\,\frac{1}{m_\epsilon(1+\log\underline{m})}\,\sqrt{\rho_1\rho_2}\,\mathbb{A}_0^{\rm v} \prec \mathbf{Y}\,. \tag{4.53}$$

For obtaining a bound for the block $\hat{\boldsymbol{\mathcal{Y}}}_E$ on diagonal of $\hat{\boldsymbol{\mathcal{Y}}}$, we note that, if $\mathbf{E}$ is the same as in the proof of Corollary 4.2, then according to (4.46) and (4.47)

$$\hat{\boldsymbol{\mathcal{Y}}}_E \prec \mathbf{E}\,. \tag{4.54}$$

Due to the block diagonal structure of $\boldsymbol{\mathcal{Y}}$ found in Theorem 4.1, we have $\mathring{\boldsymbol{\mathcal{Y}}} \le \boldsymbol{\mathcal{Y}}$, where, as usual, the matrix $\mathring{\boldsymbol{\mathcal{Y}}}$ is considered as continued by zeros on the whole space of boundary degrees of freedom. Therefore, applying this inequality to (4.54) and then Theorem 4.1, we come to the bound

$$\frac{1}{1+\log\,\min(n_2,\frac{h_1\sqrt{\rho_2}}{h_2\sqrt{\rho_1}})}\,\min\left(\frac{1}{m_\epsilon(1+\log\underline{m})},\frac{1}{(1+\log\underline{m})^2}\right)\,\hat{\boldsymbol{\mathcal{Y}}}_E \prec \mathbf{Y}\,, \tag{4.55}$$

which, together with (4.53), fetches the proof. □

5 Piece wise orthotropic discretizations on domain composed of rectangles with arbitrary aspect ratios

5.1 Schur complement and domain decomposition algorithms

We turn now to the piece wise orthotropic discretization (1.2)-(1.4) and first remind its definition. Note that in this section, for convenience we use the notation $\mathbf{j} = (j_1, j_2)$. We consider the domain

$\Omega = (0,1) \times (0,1)$ which is the assemblage

$$\Omega = \cup_{j_1,j_2=1}^{J_1,J_2} \Omega_{\mathbf{j}} ,$$

of rectangular subdomains

$$\Omega_{\mathbf{j}} = (z_{1,j_1-1}, z_{1,j_1}) \times (z_{2,j_2-1}, z_{2,j_2}) , \qquad \mathbf{j} = (j_1, j_2) ,$$

representing the nests of the rectangular *decomposition grid*

$$x_k = z_{k,j_k}, \qquad j_k = 0, 1, .., J_k, \qquad z_{k,j_k} - z_{k,j_k-1} = H_{k,j_k} > 0, \qquad z_{k,0} = 0, \qquad z_{k,J_k} = 1 .$$

The decomposition grid is imbedded in the nonuniform orthogonal denser grid

$$x_k = x_{k,i_k}, \qquad i_k = 0, 1, .., N_k, \qquad x_{k,0} = 0, \qquad x_{k,N_k} = 1,$$

i.e., $x_{k,\gamma_k} = z_{k,j_k}$ for some numbers $\gamma_k = \varkappa_k(j_k)$. This grid is called the *source grid*, is uniform on each subdomain and has sizes $h_{k,j_k} = H_{k,j_k}/n_{k,j_k}$, where n_{k,j_k} is the number of the fine mesh intervals on the coarse grid interval $(z_{k,j_k} - z_{k,j_k-1})$, see Fig. 3. One can also subdivide each nest of the orthogonal fine grid in two right-angled triangles and come to the triangulation, which is termed the *source triangulation.* The notations $\mathcal{V}(\Omega)$ and $\overset{\circ}{\mathcal{V}}(\Omega)$, respectively, stand for the

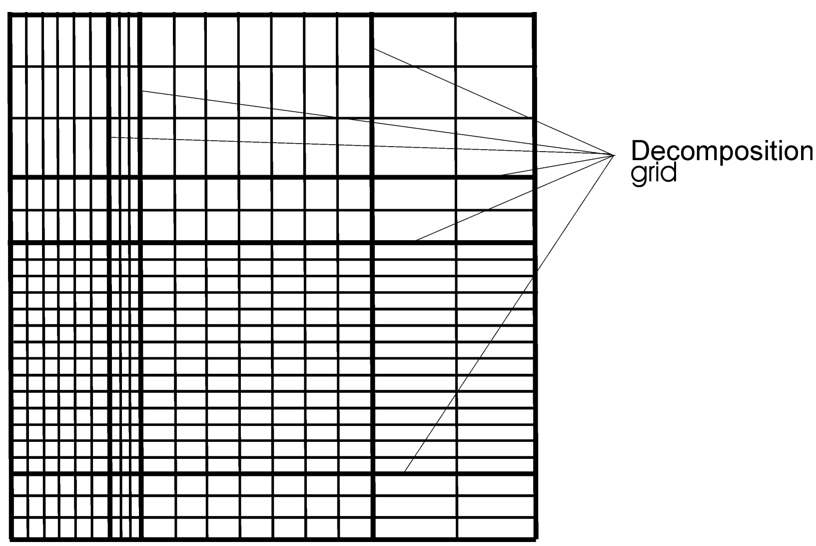

Figure 3: Decomposition grid and subdomain wise uniform rectangular source grid.

FE space of functions, continuous on $\overline{\Omega}$ and linear on each triangle of the source triangulation, and for the subspace of such functions, vanishing on $\partial\Omega$. Equally, one can understand by $\mathcal{V}(\Omega)$ and $\overset{\circ}{\mathcal{V}}(\Omega)$ the FE space of functions, which are continuous on $\overline{\Omega}$ and bilinear on each nest of the source grid, and for its subspace of the functions, vanishing on $\partial\Omega$. In the places, where the difference between these two types of FE functions should be paid attention, we make special remarks.

We consider the problem

$$\alpha_\Omega(u,v) = \langle f, v\rangle\,, \qquad \forall\ v \in \overset{\circ}{H}{}^1(\Omega)\,, \qquad\qquad \alpha_\Omega(u,v) = \int_\Omega \nabla u(x)\cdot \boldsymbol{\rho}(x)\,\nabla v(x)\,dx\,, \tag{5.1}$$

with the 2×2 diagonal matrix $\boldsymbol{\rho} = \operatorname{diag}[\rho_1,\rho_2]$ and positive piece wise constant functions $\rho_k(x) = \rho_{k,\mathbf{j}}$ for $x\in\Omega_{\mathbf{j}}$.

Let the bilinear form $\alpha_\Omega(u,v)$, $\forall\ u,v \in \overset{\circ}{\mathcal{V}}(\Omega)$, induces the FE stiffness matrix $\mathbf{K}$ and $\mathbf{S} = \mathbf{K}_B - \mathbf{K}_{B,I}\mathbf{K}_I^{-1}\mathbf{K}_{I,B}$ be its Schur complement, which evidently may be viewed as assembled of the stiffness matrices $\mathbf{K}_\mathbf{j}$ and the corresponding Schur complements $\mathbf{S}_\mathbf{j}$ for subdomains $\Omega_\mathbf{j}$. The FE space is the sum $\mathcal{V}(\Omega) = \mathcal{V}_{\sum}(\Omega)\oplus\mathcal{V}_V(\Omega)$, where the latter is the decomposition (coarse) grid subspace, and it is assumed that the block $\mathbf{K}_V$ for vertices of the decomposition grid is induced by the nodal basis in the vertex subspace $\mathcal{V}_V(\Omega)$.

For the systems

$$\mathbf{K}\mathbf{u} = \mathbf{f}\,, \qquad\qquad \mathbf{S}\mathbf{u}_B = \mathbf{F}_B$$

algorithms of the close computational complexity can be implemented, and we start from the Schur complement algorithm. It is based on the use of two preconditioners $\boldsymbol{\mathcal{S}}_k$, $k=1,2$, for Schur complement $\mathbf{S}$ with different properties. It is implied that $\boldsymbol{\mathcal{S}}_1$ is as close as possible to $\mathbf{S}$ in the spectrum and at the same time is much cheaper than $\mathbf{S}$ for matrix-vector multiplications. The preconditioner $\boldsymbol{\mathcal{S}}_2$ may be less close to $\mathbf{S}$ in the spectral sense, but it is required that there exists a solver for $\boldsymbol{\mathcal{S}}_2$ much cheaper than for $\mathbf{S}$ and $\boldsymbol{\mathcal{S}}_1$. Namely, we solve the system $\mathbf{S}\mathbf{u}_B = \mathbf{F}_B$ by PCG with the preconditioner $\boldsymbol{\mathcal{S}}_1$, whereas arising at each PCG iteration systems $\boldsymbol{\mathcal{S}}_1\mathbf{v}_B = \boldsymbol{\mathcal{F}}_B$ are solved inexactly by means of the iterative processes

$$\mathbf{v}^{k+1} = \mathbf{v}^k - \sigma_k\boldsymbol{\mathcal{S}}_2^{-1}(\boldsymbol{\mathcal{S}}_1\mathbf{v}^k - \boldsymbol{\mathcal{F}}_B)\,, \qquad\qquad k = 1,2,..k_s\,, \tag{5.2}$$

with Chebyshev iteration parameters σ_k for some fixed number k_s of iterations. In other words, the system $\mathbf{S}\mathbf{u}_B = \mathbf{F}_B$ is solved by PCG with the preconditioner $\boldsymbol{\mathcal{S}}_{1,\,\mathrm{it}}$ which inverse is

$$\boldsymbol{\mathcal{S}}_{1,\,\mathrm{it}}^{-1} = [\mathbf{I} - \prod_{k=1}^{k_s}(\mathbf{I} - \sigma_k\boldsymbol{\mathcal{S}}_2^{-1}\boldsymbol{\mathcal{S}}_1)]\,\boldsymbol{\mathcal{S}}_1^{-1}\,.$$

Proposition 5.1. *Suppose that*

ι) the preconditioners $\boldsymbol{\mathcal{S}}_k$ satisfy

$$\underline{\gamma}_1\boldsymbol{\mathcal{S}}_1 \prec \mathbf{S} \prec \overline{\gamma}_1\boldsymbol{\mathcal{S}}_1\,, \qquad\qquad \underline{\gamma}_2\boldsymbol{\mathcal{S}}_2 \prec \boldsymbol{\mathcal{S}}_1 \prec \overline{\gamma}_2\boldsymbol{\mathcal{S}}_2\,,$$

ιι) matrix-vector multiplications by $\mathbf{S}$ and $\boldsymbol{\mathcal{S}}_1$ spend $\mathcal{N}_s$ and $\mathcal{N}_1$ a.o., respectively, and

ιιι) solving the system $\boldsymbol{\mathcal{S}}_2\mathbf{v}_B = \boldsymbol{\mathcal{F}}_B$, $\forall\,\boldsymbol{\mathcal{F}}_B$, requires $\mathcal{N}_2$ operations.

Then the computational cost of solving the system $\mathbf{S}\mathbf{u}_B = \mathbf{F}_B$ is

$$\mathcal{O}\left(\sqrt{\overline{\gamma}_1/\underline{\gamma}_1}\left[\mathcal{N}_s + \sqrt{\overline{\gamma}_2/\underline{\gamma}_2}(\mathcal{N}_1 + \mathcal{N}_2)\right]\right)$$

arithmetic operations.

Proof. For the proof, now standard, of similar statements we refer to, *e.g.,* Korneev/Langer [29]. □

It is left to define the preconditioners $\boldsymbol{\mathcal{S}}_k$.

`Preconditioner` $\boldsymbol{\mathcal{S}}_1$. First we turn to one of the subdomains $\Omega_{\mathbf{j}}$. In the preconditioner $\widetilde{\boldsymbol{\mathcal{Y}}} = \mathrm{diag}\,[\boldsymbol{\mathcal{D}}_B^E, \mathbb{Y}^U]$, see (4.18), we replace the block $\mathbb{Y}^U$ by the matrix $\mathbf{C}$, which is one of the finite-difference analogues of the shape dependent norm (see Section 2) defined on the rarefied mesh, and, according to Lemma 2.2, is spectrally equivalent to $\mathbb{Y}^U$. Therefore, using the notations $\boldsymbol{\mathcal{D}}_{B,\mathbf{j}}^E, \mathbf{C}_{\mathbf{j}}$ for the matrices $\boldsymbol{\mathcal{D}}_B^E, \mathbf{C}$ derived for the domain $\Omega = \Omega_{\mathbf{j}}$, we set

$$\boldsymbol{\mathcal{S}}_{1,\mathbf{j}} = \sqrt{\rho_{1,\mathbf{j}}\,\rho_{2,\mathbf{j}}}\,(1+\log\delta_{\mathbf{j}})\ \mathrm{diag}\,[\boldsymbol{\mathcal{D}}_{B,\mathbf{j}}^E, \mathbf{C}_{\mathbf{j}}]$$

and assemble the matrix $\boldsymbol{\mathcal{S}}_1$ from these matrices. As it follows from Corollary 4.1 and Lemma 2.2,

$$\boldsymbol{\mathcal{S}}_1 \prec \mathbf{S} \prec (\max_{\mathbf{j}}(1+\log\delta_{\mathbf{j}}))\ \boldsymbol{\mathcal{S}}_1\,, \tag{5.3}$$

where $\delta_{\mathbf{j}}$ is the value of δ in (4.16) for the subdomain $\Omega_{\mathbf{j}}$.

`Preconditioner` $\boldsymbol{\mathcal{S}}_2$. For each subdomain $\Omega_{\mathbf{j}}$, we consider the preconditioner $\boldsymbol{\mathcal{S}}_{2,\mathbf{j}} = \hat{\boldsymbol{\mathcal{Y}}}_{\mathbf{j}}$, where $\hat{\boldsymbol{\mathcal{Y}}}_{\mathbf{j}} = \mathrm{diag}\,[\hat{\boldsymbol{\mathcal{Y}}}_{E,\mathbf{j}}, \sqrt{\rho_{1,\mathbf{j}}\,\rho_{2,\mathbf{j}}}\ \mathbb{A}_{0,\mathbf{j}}^{\mathrm{v}}]$ is defined as $\hat{\boldsymbol{\mathcal{Y}}} = \mathrm{diag}\,[\hat{\boldsymbol{\mathcal{Y}}}_E, \sqrt{\rho_1\,\rho_2}\ \mathbb{A}_0^{\mathrm{v}}]$ figuring in Theorem 4.2, but for the domain $\Omega = \Omega_{\mathbf{j}}$. Then $\boldsymbol{\mathcal{S}}_2$ is assembled of the subdomain preconditioners $\boldsymbol{\mathcal{S}}_{2,\mathbf{j}}$ and has the block diagonal form $\boldsymbol{\mathcal{S}}_2 = \mathrm{diag}\,[\boldsymbol{\mathcal{S}}_E, \mathbf{K}_V]$, where $\mathbf{K}_V$ is the block of the FE matrix $\mathbf{K}$ for vertices. Obviously, $\boldsymbol{\mathcal{S}}_E$ is assembled from the matrices $\boldsymbol{\mathcal{S}}_E = \hat{\boldsymbol{\mathcal{Y}}}_{E,\mathbf{j}}$. According to Theorem 4.2

$$\min_{\mathbf{j}}\left\{\frac{1}{(1+\log\,\delta_{\mathbf{j}})(1+\log\underline{m}_{\mathbf{j}})}\ \min\big(\frac{1}{\theta_{\mathbf{j}}}, \frac{1}{1+\log\underline{m}_{\mathbf{j}}}\big)\right\}\ \boldsymbol{\mathcal{S}}_2 \prec \mathbf{S} \prec \boldsymbol{\mathcal{S}}_2\,. \tag{5.4}$$

Let us describe now the DD algorithm. If to represent the FE stiffness matrix in the block form

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_B \end{pmatrix}, \tag{5.5}$$

then the inverse to the DD preconditioner $\boldsymbol{\mathcal{K}}_{\mathrm{DD}}$ can be defined by the expression

$$\boldsymbol{\mathcal{K}}_{\mathrm{DD}}^{-1} = \begin{pmatrix} \mathbf{I} & -\mathbf{K}_I^{-1}\mathbf{K}_{IB} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{K}_I^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{S}}_{1,\,\mathrm{it}}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_{BI}\mathbf{K}_I^{-1} & \mathbf{I} \end{pmatrix}. \tag{5.6}$$

The block $\mathbf{K}_I$ is related to the internal for each subdomain unknowns and has the block diagonal structure

$$\mathbf{K}_I = \mathrm{diag}\,[\,\mathbf{K}_{I,\mathbf{j}}\,]_{j_1,j_2=1}^{J_1,J_2}\,,$$

and each system $\mathbf{K}_{I,\mathbf{j}}\mathbf{v}_{I,\mathbf{j}} = \mathbf{F}_{I,\mathbf{j}},\ \forall\,\mathbf{v}_{I,\mathbf{j}}$, can be solved by FDFT for $\mathcal{O}(n_{1,j_1}n_{2,j_2}\log\overline{n}_{\mathbf{j}})$ a.o.

Theorem 5.1. *Let $H_{k,j_k} \in (0,1)$, $n_{k,j_k} \geq 1$, $\rho_{k,\mathbf{j}} > 0$ be arbitrary in the pointed out ranges. Then the total arithmetical costs $Q_{\mathbf{K}}, Q_{\mathbf{S}}$ of the DD and Schur complement algorithms are bounded according to the estimates*

$$\begin{aligned} Q_{\mathbf{K}} \prec\ & N_1N_2(1+\log\overline{N}) + Q_{\mathbf{S}}\,, \\ Q_{\mathbf{S}} \prec\ & N_1N_2(1+\log\overline{N})^{1/2} + [(J_1N_2+J_2N_1)(1+\log\overline{N}) + \\ & +\Upsilon(J_1J_2)]\sqrt{\overline{N}}(1+\log\overline{N})^{3/2}(1+\log\underline{N})^{1/2}\,, \end{aligned} \tag{5.7}$$

where $\overline{N} = \max_k N_k$ *and* $\underline{N} = \min_k N_k$ *and* $\Upsilon(J_1, J_2)$ *stands for the cost of solution of the subsystem with the matrix* $\mathbf{K}_V$*.*

If to assume additionally that the decomposition mesh and, therefore, the number of subdomans fixed, the bound (5.7) simplifies to

$$Q_{\mathbf{S}} \prec N_1 N_2 (1 + \log \overline{N})^{1/2} + (N_2 + N_1)\sqrt{\overline{N}}(1 + \log \overline{N})^{5/2}(1 + \log \underline{N})^{1/2}\,, \tag{5.8}$$

First term in the right parts of (5.7) and (5.8) for $Q_{\mathbf{S}}$ accounts for the matrix-vector multiplications by $\mathbf{S}$ and for the calculation of the Schur complement, see Remark 4.1.

Proof. Let us list first main factors contributing the computational cost of the Schur complement algorithm:

– the number of external PCG iterations –

$$k_{\,\mathrm{PCG}} := \max_{\mathbf{j}} \sqrt{1 + \log \delta_{\mathbf{j}}}\,, \tag{5.9}$$

– the cost of the matrix-vector multiplications by $\mathbf{S}$ at each PCG iteration – $N_1 N_2$ a.o.,

– the number of secondary Richardson iterations (5.2) –

$$k_s := \max_{\mathbf{j}} (1 + \log \delta_{\mathbf{j}}) \max\Big((1 + \log \underline{m}_{\mathbf{j}})^2, \sqrt{\theta_{\mathbf{j}}(1 + \log \underline{m}_{\mathbf{j}})}\Big)\,, \tag{5.10}$$

· matrix-vector multiplication by $\boldsymbol{\mathcal{S}}_1$ –

$$k_1 := \sum_{\mathbf{j}} \big\{(m_{1,\mathbf{j}} + m_{2,\mathbf{j}})(1 + \log \underline{m}_{\mathbf{j}}) + \overline{n}_{\mathbf{j}}\big\}\,, \qquad \overline{m}_{\mathbf{j}} = \max(m_{1,\mathbf{j}}, m_{2,\mathbf{j}})\,,$$

· solving the systems with the preconditioner $\boldsymbol{\mathcal{S}}_2$ –

$$k_2 := [J_1 \mathcal{N}_2 (1 + \log \max_{j_2} n_{2,j_2}) + J_2 \mathcal{N}_1 (1 + \log \max_{j_1} n_{1,j_1})] + \Upsilon(J_1, J_2)\,.$$

From the bounds (5.3) and (5.4), it follows that

$$\min_{\mathbf{j}} \left\{ \frac{1}{(1 + \log \delta_{\mathbf{j}})^2 (1 + \log \underline{m}_{\mathbf{j}})} \min\left(\frac{1}{\theta_{\mathbf{j}}}, \frac{1}{(1 + \log \underline{m}_{\mathbf{j}})}\right) \right\} \boldsymbol{\mathcal{S}}_2 \prec \boldsymbol{\mathcal{S}}_1 \prec \boldsymbol{\mathcal{S}}_2\,, \tag{5.11}$$

and, therefore, the given above number k_s of secondary iterations is sufficient to provide the spectral equivalence

$$\boldsymbol{\mathcal{S}}_2 \asymp \boldsymbol{\mathcal{S}}_{1,\,\mathrm{it}}\,.$$

Now, the last relationship and (5.3) guarantee that $k_{\,\mathrm{PCG}} \log \varepsilon^{-1}$ PCG iterations provide the relative error in the norm $||\cdot||_{\mathbf{S}}$ bounded by the prescribed $\varepsilon > 0$. At the multiplication $\boldsymbol{\mathcal{S}}_{1,\mathbf{j}} \mathbf{v}_{B_{\mathbf{j}}}$ the multiplication by $\boldsymbol{\mathcal{D}}^E_{B,\mathbf{j}}$ and by $\boldsymbol{\nabla}$, entering expression (2.15) for $\mathbf{C}_{\mathbf{j}}$, require not more than $\mathcal{O}(\overline{n}_{\mathbf{j}})$ a.o. Multiplications by other matrices entering (2.15) with the use of FDFT spend $\mathcal{O}((m_{1,\mathbf{j}} + m_{2,\mathbf{j}})(1 + \log \underline{m}_j))$ a.o. Thus, ops $[\boldsymbol{\mathcal{S}}_1 \mathbf{v}_B] = \mathcal{O}(k_1)$.

In accordance with the description of the preconditioner $\boldsymbol{\mathcal{S}}_2$, the term in the square brackets of the expression for k_2 bounds the arithmetical cost of solving the system with the matrix $\boldsymbol{\mathcal{S}}_E$.

Thus, in the general situation, one can write

$$k_{\,\mathrm{PCG}} \prec (1+\log \max_k n_{k,j_k})^{1/2} \prec (1+\log \overline{N})^{1/2}\,, \qquad k_s \prec (1+\log \overline{N})(1+\log \underline{N})^{1/2}\sqrt{\overline{N}}\,,$$

$$k_1 \prec (J_1N_2+J_2N_1)(1+\log \overline{N})\,, \qquad k_2 \prec (J_1N_2+J_2N_1)(1+\log \overline{N}) + \Upsilon(J_1J_2)\,,$$

from where the bound (5.7) directly follows.

Taking into account the cost of solution of the subsystems with the matrices $\mathbf{K}_{I,\mathbf{j}}$ by FDFT and that the DD peconditioner $\boldsymbol{\mathcal{K}}_{\mathrm{DD}}$ is spectrally equivalent to the FE matrix $\mathbf{K}$, one comes to the estimate of the arithmetic cost for the DD algorithm. □

Suppose for simplicity that $N_1 = N_2 = N$, $J_1 = J_2 = J$ and the decomposition mesh and the number of subdomains, *i.e.,* J, are fixed. Then

$$Q_{\mathbf{K}} \prec N^2(1+\log N)\,, \qquad Q_{\mathbf{S}} \prec N^2(1+\log N)^{1/2}\,. \tag{5.12}$$

If the number of subdomains grows alongside with the growth of the numbers N_k of the source mesh lines, the contribution $\Upsilon(J_1J_2)$ of the solver for the vertex subproblem can compromise this bound. Assuming that a direct solver can be used for the systems with the matrix $\mathbf{K}_V$ from (5.7), we conclude that the bounds (5.12) are retained under the condition $J_k \le N_k^{1/2}$.

It is worth emphasizing that the above estimates are relatively crude, since we practically made no restrictions on H_{k,j_k}, n_{k,j_k}, and $\rho_{k,\mathbf{j}} > 0$ and how they change from subdomain to subdomain. As a consequence we used, *e.g.*, the bounds $n_{k,j_k} \le N_k$, which obviously as a rule will be too high. If the variation of these values can be characterized by some functions, the bounds can be immediately improved.

5.2 Possible improvements, examples

Until now we concentrated on the main for the Schur complement and DD algorithms problem of the Schur complement preconditioning, assuming FDFT for solving the systems with the matrices $\mathbf{K}_{I,\mathbf{j}}$. However there is known a variety of the optimal algorithms, which can be applied to these systems. We refer to Schieweck [39] and Pflaum [38] for multigrid solvers for orthotropic discretizations and to Griebel and Oswald [22] for different versions of BPX and MDS preconditioners, which they derive employing the tensor product nature the matrices $\mathbf{K}_{I,\mathbf{j}}$ and 1-d multilevel FE and prewavelet decompositions of the initial FE space. Therefore, one can replace $\mathbf{K}_I$ by

$$\boldsymbol{\mathcal{K}}_I = \mathrm{diag}\,[\boldsymbol{\mathcal{K}}_{I,\mathbf{j}}]_{j_1,j_2=1}^{J_1,J_2}$$

where $\boldsymbol{\mathcal{K}}_{I,\mathbf{j}}$, *e.g.,* are the BPX preconditioners of one of the types, considered by Griebel and Oswald [22]. Several wavelet bases, found in Dahmen [15], Cohen, Daubechies and Vial [14] and Schneuder [40], also can be used in optimal preconditioners-solvers for systems with the matrices $\boldsymbol{\mathcal{K}}_{I,\mathbf{j}}$. The preconditioner-solver $\boldsymbol{\mathcal{K}}_I$ can efficiently replace $\mathbf{K}_I$ inside of the second pair of brackets in (5.6) , thus removing one cause for appearing additional factor $(1+\log \overline{N})$ in the estimate of computational cost of DD solver. But such replacement is insufficient for removing this factor from the cost of the prolongation operator based, for instance, on the inexact iterative solver. Therefore, for improving performance of the DD algorithm one should also use a more efficient prolongation operator, which enters the representation of the DD preconditioner-solver in the form

$$\boldsymbol{\mathcal{K}}_{\mathrm{DD}}^{-1} = \boldsymbol{\mathcal{K}}_{I}^{+} + \boldsymbol{\mathcal{P}}_{V_B\to V}\boldsymbol{\mathcal{S}}_{1,\mathrm{it}}^{-1}\boldsymbol{\mathcal{P}}_{V_B\to V}^{\top}\,. \tag{5.13}$$

The operator $\boldsymbol{\mathcal{P}}_{V_B\to V}$ is defined subdomain wise, so that its restriction to a subdomain $\Omega_{\mathbf{j}}$ is the prolongation operator $\boldsymbol{\mathcal{P}}_{V_{B_{\mathbf{j}}}\to V_{\mathbf{j}}}$ on this subdomain from its boundary.

In the case of a more regular subdomains and discretizations, multilevel decompositions, used for obtaining BPX type preconditioners for a subdomain, as a rule can be used also for creation of optimal prolongation operators $\boldsymbol{\mathcal{P}}_{V_{B_{\mathbf{j}}}\to V_{\mathbf{j}}}$ and Schur complement preconditioners-solvers, see, *e.g.*, Griebel and Oswald [22] and Oswald [34]. In the latter work it is mentioned that in the case of orthotropic discretizations the discrete harmonic prolongation operators can still be derived, if the underlying BPX splitting is obtained by both full and semi-coarsening.

The last step, which allow to come to the optimal Schur complement and DD solvers, is replacing the preconditioner $\boldsymbol{\mathcal{S}}_1$, used above, by the spectrally equivalent preconditioner $\boldsymbol{\mathcal{S}}_1$, which is still cheap for the vector-matrix multiplications. Let us underline that the problem seem to be easier than commonly studied, since we need not a cheap Schur complement *preconditioner-solver*, but a cheap *preconditioner-multiplicator*.

Having completed the pointed out replacements, one comes to (5.12) without multipliers, containing $(1 + \log \overline{N})$, if the same conditions as for (5.12) are fulfilled.

As it is known, specific highly orthotrpic discretiations arise at the preconditioning of the stiffness matrices of hierarchical and spectral p-elements by the finite-difference or FE type (with triangular linear or rectangular bilinear elements) matrices. By the hierarchical reference p-element is understood the one, coordinate polynomials of which are defined by means of the tensor products of the integrated Legendre's polynomials. The finite-difference and FE systems for the pointed out discretizations can be efficiently solved by the DD solver, figuring in Theorem 5.1. In particular, for the stiffness matrix of the hierarchical reference p-element, Korneev [26, 27] introduced the spectrally equivalent uniformly in p block diagonal preconditioner $\boldsymbol{\mathcal{B}} = \mathrm{diag}\,[\mathbf{B}_{e,e}, \mathbf{B}_{e,e}, \mathbf{B}_{e,e}, \mathbf{B}_{e,e}]$, see in Korneev [27] relationships (3.2)–(3.4) for the definition of the typical block $\mathbf{B}_{e,e}$ and this paper and the paper of Korneev Langer Xanthis [30] for spectral equivalence inequalities. The preconditioner $\mathbf{B}_{e,e}$ is a particular case (indeed much simpler) of the stiffnees matrix for the discretization (1.2)-(1.4), and the systems $\mathbf{B}_{e,e}\mathbf{v} = \mathbf{f}$ can be solved by the presented in this section DD algorithm. Since the decomposition mesh (see for its definition (3.2) of [27]) used for generating $\mathbf{B}_{e,e}$ satisfies $J_k \le \log N_k$, the number of a.o., needed for solving this system, is estimated according Theorem 5.1 by $\mathcal{O}(N^2 \log N)$, where now $N = p/2$.

A quite similar situation takes place for the finite-difference or FE preconditioners for the stiffness matrices of the Lagrange reference p-elements with the nodes introduced by the GLC (Gauss-Lobatto-Chebyshev) and GLL (Gauss-Lobatto-Legendre) quadrature rules. Decomposition meshes for the finite-difference and FE discretizations, corresponding to these Lagrange elements, are created in the same, in essential, manner as in the case of the hierarchical reference element, see Proposition 6.1 in [31] or Proposition 1 in [32]. The systems with the generated on decomposition meshes FE preconditioners $\mathbf{B}$ for the matrix[1] may be solved by the DD solver of

[1] The notation $\mathbf{B}$ is used in the cited papers [31, 32].

Theorem 5.1 for $\mathcal{O}(N^2 \log N)$ a.o. Again, this bound follows from Theorem 5.1 and the bound $J_k \leq \log N_k$, see for the last bound the mentioned Propositions, in which for $J_k = J$ stands the notation ℓ_0.

In order to avoid confusion, let us emphasize that closely related, but different from this paper, DD preconditioners-solvers were studied in [27] and [31, 32, Section 6.1] for the systems with the matrices $\mathbf{B}_{e,e}$ and $\mathbf{B}$, respectively. In particular, the vertex unknowns in the DD preconditioner were not decoupled from the rest unknowns, that resulted in more complex DD algorithms and weaker bounds of computational cost.

6 Appendix

In the proof of Theorem2.2 we use a few basic facts, pertaining traces of functions from the space $H^{1/2}(\partial\Omega)$, which we present for completeness at the beginning.

Let Γ be a rectifiable curve in $\mathbf{R}^2$. Function $f \in L_2(\Gamma)$ belongs to the class $H^{1/2}(\Gamma)$ if the following norm is finite

$$\|f\|_{1/2,\Gamma} = \left(\|f\|_{0,\Gamma}^2 + [f]_\Gamma^2\right)^{1/2},$$

where $[f]_\Gamma$ is the seminorm defined by

$$[f]_\Gamma^2 = \iint\limits_{\Gamma\times\Gamma} |f(p) - f(q)|^2 \frac{ds_p ds_q}{|p-q|^2} \tag{6.1}$$

and ds_p, ds_q are the length elements of Γ.

The following "Poincaré inequality" holds for $f \in H^{1/2}(\Gamma)$:

$$\|f - \overline{f}\|_{0,\Gamma} \leq |\Gamma|^{-1/2}[f]_\Gamma. \tag{6.2}$$

Here $|\Gamma|$ is the length of Γ and

$$\overline{f} = \frac{1}{|\Gamma|}\int_\Gamma f(p) ds_p$$

is the mean value of f on Γ. Inequality (6.2) is a simple consequence of Hölder's inequality and above definition.

The theorem stated below is due to Aronszajn [4], Babich/Slobodetsky [5], and Gagliardo [18].

Theorem 1. *Let Ω be a planar simply connected Lipschitz domain. Then every function $u \in H^1(\Omega)$ has the trace $f = u|_{\partial\Omega}$, the space of all these traces coincides with $H^{1/2}(\partial\Omega)$. Moreover, the norm $\|f\|_{1/2,\partial\Omega}$ is equivalent to the norm*

$$\inf\{\|u\|_{1,\partial\Omega} : u \in H^1(\Omega),\ u|_{\partial\Omega} = f\},$$

and there exists a linear continuous extension operator $E : H^{1/2}(\partial\Omega) \to H^1(\Omega)$.

Under assumptions of Theorem 1 the following equivalence relation holds:

$$\inf\{\|\nabla u\|_{0,\Omega} : u \in H^1(\Omega),\ u|_{\partial\Omega} = f\} \asymp [f]_{\partial\Omega}. \tag{6.3}$$

Now we turn directly to the proof of Theorem 2.2. The similarity transformation with coefficient ε^{-1} transfers rectangle $\Omega = (0,1)\times\epsilon$ to a long rectangle $\Omega = (0,N)\times(0,1)$, $N=\varepsilon^{-1}$, and the desired equivalence relations take the form

$$\inf\{\|\nabla u\|_{0,\Omega} : u \in H^1(\Omega),\ u|_{\partial\Omega} = f\} \asymp \langle f\rangle, \tag{6.4}$$

$$\inf\{\delta^2\|u\|^2_{0,\Omega} + \|\nabla u\|^2_{0,\Omega} : u \in H^1(\Omega),\ u|_{\partial\Omega} = f\} \asymp \delta^2\|f\|^2_{0,\partial\Omega)} + \langle f\rangle^2, \tag{6.5}$$

where

$$\langle f\rangle^2 = \sum_{i=0}^{1} \iint\limits_{\{t,\tau\in(0,N):|t-\tau|<1\}} |f(t,i)-f(\tau,i)|^2 \frac{dtd\tau}{(t-\tau)^2} +$$

$$+\int_0^N |f(t,1)-f(t,0)|^2 dt + \sum_{i=0,N} \iint\limits_{\Gamma_i\times\Gamma_i} |f(p)-f(q)|^2 \frac{ds_p ds_q}{|p-q|^2},$$

$\Gamma_0 = \{(x,y)\in\partial\Omega : x<1\}$, $\Gamma_N = \{(x,y)\in\partial\Omega : x>N-1\}$ and ds_p, ds_q are the length elements of $\partial\Omega$.

We can assume without loss of generality that $N=\varepsilon^{-1}$ is an integer[2], $N>2$. **Below c is an absolute positive constant whose value can change within the same chain of inequalities.**

Let $u\in H^1(\Omega)$, $u|_{\partial\Omega}=f$. We now establish the estimate

$$\langle f\rangle \prec \|\nabla u\|_{0,\Omega}. \tag{6.6}$$

By using

$$f(t,1)-f(t,0) = \int_0^1 \frac{\partial u}{\partial\tau}(t,\tau)d\tau.$$

and Hölder's inequality, we find that

$$\int_0^N |f(t,1)-f(t,0)|^2 dt \le \int_0^N dt \int_0^1 \left|\frac{\partial u}{\partial\tau}(t,\tau)\right|^2 d\tau \le \|\nabla u\|^2_{0,\Omega}. \tag{6.7}$$

Next, (6.3) implies the estimate

$$\iint\limits_{\Gamma_0\times\Gamma_0} |f(p)-f(q)|^2\frac{ds_p ds_q}{|p-q|^2} \le c \iint\limits_{(0,1)\times(0,1)} |\nabla u(x,y)|^2 dxdy \tag{6.8}$$

The estimate which appears if the integration is made over $\Gamma_N\times\Gamma_N$ on the left of (8) and over $(N-1)\times(0,1)$ on the right of (8), is verified in the same way.

Now fix $i=0$ or $i=1$. Then

$$\iint\limits_{\{t,\tau\in(0,N):|t-\tau|<1\}} |f(t,i)-f(\tau,i)|^2\frac{dtd\tau}{(t-\tau)^2} =$$

[2]If N is not an integer and $[N]$ is its integer part, we can transform Ω on the rectangle $(0,[N]+1)\times(0,1)$ with the aid of transformation $\Omega(x,y)\mapsto(\xi,\eta)$, $\xi=([N]+1)N^{-1}x$, $\eta=y$ with coefficient $([N]+1)N^{-1}\in(1,1+\varepsilon)$.

$$= \sum_{k=0}^{N-1} \int_k^{k+1} dt \int_{\{\tau\in(0,N):|t-\tau|<1\}} |f(t,i)-f(\tau,i)|^2 \frac{dtd\tau}{(t-\tau)^2} \leq$$

$$\leq \sum_{k=1}^{N-1} \iint_{\Delta_k\times\Delta_k} |f(t,i)-f(\tau,i)|^2 \frac{dtd\tau}{(t-\tau)^2},$$

where $\Delta_1 = (0,2)$, $\Delta_k = (k-1,k+2)$ for $k = 2,\dots,N-2$ and $\Delta_{N-1} = (N-2,N)$. By Theorem 1 and Remark 2 the general term of the last sum is dominated by $c\,\|\nabla u\|^2_{L_2(\Omega_k)}$ with $\Omega_k = \Delta_k \times (0,1)$. The sum does not exceed $c\,\|\nabla u\|^2_{L_2(\Omega)}$ because every point of Ω belongs to at most three rectangles Ω_k. Unifying this with (6.7), (6.8) yields (6.6).

We now apply for $u \in H^1(\Omega)$, $u|_{\partial\Omega} = f$, the following obvious estimate

$$\|f\|^2_{0,\partial\Omega} \leq \sum_{k=0}^{N-1} \|f\|^2_{0,\partial Q_k},$$

where $Q_k = (k,k+1)\times(0,1)$. By using Theorem 1 for each square Q_k, we dominate the last sum by

$$c\sum_{k=0}^{N-1} \|u\|^2_{H^1(Q_k)} = c\|u\|^2_{1,\Omega}.$$

A combination of the latter with (6.6) gives

$$\delta^2\|f\|^2_{0,\partial\Omega} + \langle f\rangle^2 \leq c\left(\delta^2\|u\|^2_{0,\Omega} + \|\nabla u\|^2_{0,\Omega}\right). \tag{6.9}$$

To conclude the proof of (6.4) and (6.5), we should construct an extension $u \in H^1(\Omega)$ of a function $f \in H^{1/2}(\partial\Omega)$ for which the reverse inequalities (6.6) and (6.9) are valid. The required extension will be defined step by step. By $L^1_{\infty,\omega}(-1,1) = (v \in L^1_\infty(-1,1) : v = 0 \text{ for } x \notin (-1+\omega,1-\omega))$ we denote the subspace of functions from the space $L^1_\infty(-1,1)$, vanishing outside of the interval $(-1+\omega,1-\omega)$ for some fixed $\omega \in (0,1/2)$. First consider a function μ such that

$$\mu \in L^1_{\infty,\omega}(-1,1), \quad 0 \leq \mu \leq 1, \quad \sum_{k=-\infty}^{\infty} \mu(t-k) = 1, \quad t \in \mathbf{R}^1.$$

For instance, μ can be a continuous piece wise linear function. Let $\mu_k(t) = \mu(2t-k)$ for $t \in \mathbf{R}^1$, $k = 0,\pm1,\pm2,\dots$ Then the collection $\{\mu_k\}$ is a uniform partition of unity on $\mathbf{R}^1$ subordinate to the covering by intervals $\gamma_k = ((k-1)/2,(k+1)/2)$. In particular

$$\sum_{k=1}^{2N-1} \mu_k(t) = 1 \quad \text{for} \quad t \in [1/2, N-1/2].$$

We also introduce a function $\lambda \in L^1_{\infty,\omega/2}(-1,1))$ with the property[3] $\lambda\mu = \mu$. Let $\lambda_k(t) = \lambda(2t-k)$, $t \in \mathbf{R}^1$, $k = 0,\pm1,\dots$. It is clear that $\mu_k\lambda_k = \mu_k$ for all integer k.

[3]Absolute constants c below may depend on μ and λ.

Fix $i = 0$ or $i = 1$. Let $\gamma_{k,i} = \gamma_k \times i$ where $\gamma_k = ((k-1)/2, (k+1)/2)$, $k = 1, \ldots, 2N-1$. Let $\bar{f}_{k,i}$ denote the mean value of f on $\gamma_{k,i}$. Put $g_k = \gamma_k \times (0,1)$. The function $\mu_k(f - \bar{f}_{k,i})$ is defined on $\gamma_{k,i}$. We extend this function to be zero on $\partial g_k \setminus \gamma_{k,i}$. Since the support of μ_k is distant from the endpoints of γ_k, we have

$$\|\mu_k(f - \bar{f}_{k,i})\|_{1/2,\partial g_k} \leq c \,\|\mu_k(f - \bar{f}_{k,i})\|_{1/2,\gamma_{k,i}}.$$

By Theorem 1 there exists a linear continuous extension operator $E_k : H^{1/2}(\partial g_k) \to H^1(g_k)$. Combining this with the above inequality, we provide an extension $\mu_k(f - \bar{f}_{k,i}) \mapsto w_{k,i}$ from the boundary segment $\gamma_{k,i}$ into the square g_k such that

$$\|w_{k,i}\|_{1,g_k} \leq c \,\|\mu_k(f - \bar{f}_{k,i})\|_{1/2,\gamma_{k,i}}, \quad i = 0, 1, \quad k = 1, \ldots, 2N-1. \tag{6.10}$$

Let

$$u_i = v_i + w_i, \quad i = 0, 1, \tag{6.11}$$

where

$$v_i(x, y) = \sum_{k=1}^{2N-1} \bar{f}_{k,i} \mu_k(x), \quad (x, y) \in \Omega, \tag{6.12}$$

$$w_i(x, y) = \sum_{k=1}^{2N-1} \lambda_k(x) w_{k,i}(x, y), \quad (x, y) \in \Omega \tag{6.13}$$

(we accept the convention that $\lambda_k(x) w_{k,i}(x, y) = 0$ outside g_k). It easily follows from our construction that $u_i(x, i) = f(x, i)$ for $i = 0, 1$ and $x \in (1/2, N - 1/2)$.

Some corrections are required to satisfy boundary conditions near vertical segments $0 \times (0,1)$, $N \times (0,1)$. Let $\bar{f}_0$ be the mean value of f on the broken line Γ_0 and $\bar{f}_N$ the mean value on Γ_N. We extend the function $\mu_0(f - \bar{f}_0)$ by zero from the broken line Γ_0 to ∂Q_0, where $Q_0 = (0,1) \times (0,1)$, and apply Theorem 1 to extend this function from ∂Q_0 into Q_0. This way provides an extension

$$H^{1/2}(\Gamma_0) \supset \mu_0(f - \bar{f}_0) \mapsto U \in H^1(Q_0)$$

such that

$$\|U\|_{1,Q_0} \leq c \,\|\mu_0(f - \bar{f}_0)\|_{1/2,\Gamma_0}. \tag{6.14}$$

In the same way an extension $V \in H^1(Q_{N-1})$ of the function $\mu_{2N}(f - \bar{f}_N) \in H^{1/2}(\Gamma_N)$ is defined, where $Q_{N-1} = (N-1) \times (0,1)$ and the following estimate holds:

$$\|V\|_{1,Q_{N-1}} \leq c \,\|\mu_{2N}(f - \bar{f}_N)\|_{1/2,\Gamma_N}. \tag{6.15}$$

Furthermore, we define

$$W_0(x, y) = \bar{f}_0 \mu_0(x) + \lambda_0(x) U(x, y), \quad (x, y) \in \Omega \tag{6.16}$$

and

$$W_N(x, y) = \bar{f}_N \mu_{2N}(x) + \lambda_{2N}(x) V(x, y), \quad (x, y) \in \Omega, \tag{6.17}$$

where $\lambda_0 U = 0$ outside $(0, 1/2) \times (0,1)$ and $\lambda_{2N} V = 0$ outside $(N - 1/2, N) \times (0,1)$.

Now the function

$$\Omega(x,y) \mapsto u(x,y) = W_0(x,y) + W_N(x,y) + u_0(x,y) + y(u_1(x,y) - u_0(x,y))$$

can be considered as an appropriate candidate for the required extension u of f. We claim that it really is, i.e. $u|_{\partial\Omega} = f$ and the following estimates hold

$$\|\nabla u\|_{0,\Omega} \le c\,\langle f\rangle \tag{6.18}$$

and

$$\left(\delta^2\|u\|_{0,\Omega}^2 + \|\nabla u\|_{0,\Omega}^2\right) \le c\,(\delta^2\|f\|_{0,\partial\Omega}^2 + \langle f\rangle^2). \tag{6.19}$$

As we noted above, the functions u_0, u_1 introduced by (6.11)–(6.13), satisfy $u_i(x,i) = f(x,i)$ for $i = 0,1$, $x \in (1/2, N-1/2)$. Since $W_0(x,y) = W_N(x,y) = 0$ for the same x, one has $u(x,i) = f(x,i)$ for $x \in (1/2, N-1/2)$, $i = 0,1$.

Let $x \in [0,1/2)$. Then

$$u(x,0) = u_0(x,0) + W_0(x,0) = \bar f_{1,0}\mu_1(x) + \lambda_1(x)\mu_1(x)(f(x,0) - \bar f_{1,0}) +$$

$$+\bar f_0\mu_0(x) + \lambda_0(x)\mu_0(x)(f(x,0) - \bar f_0).$$

Because $\lambda_k\mu_k = \mu_k$, we obtain

$$u(x,0) = \mu_1(x)f(x,0) + \mu_0(x)f(x,0) = f(x,0).$$

Next, since $\lambda_0(0) = \mu_0(0) = 1$, it follows that

$$u(0,y) = W_0(0,y) = \mu_0(0)\bar f_0 + \lambda_0(0)\mu_0(0)(f(0,y) - \bar f_0) = f(0,y).$$

In an analogous way one can check that $u(x,1) = f(x,1)$ for $x \in (0,1/2)$ and $u(x,y) = f(x,y)$ for $(x,y) \in \partial\Omega$, $x > N-1/2$. Thus u satisfies the boundary condition $u|_{\partial\Omega} = f$.

Turning to the proof of (6.18), we observe that the definition of u implies the estimate

$$\|\nabla u\|_{0,\Omega} \le \|\nabla W_0 + \nabla W_N + \nabla v_0\|_{0,\Omega} + \|v_1 - v_0\|_{1,\partial\Omega} + \sum\nolimits_{i=0,1}\|w_i\|_{1,\partial\Omega}, \tag{6.20}$$

where v_i and w_i are given by (6.12), (6.13). Below we bound each term on the right of (20). Let

$$G_k = (k/2, (k+1)/2) \times (0,1), \quad k = 0,\dots,2N-1.$$

Since $W_N|_{G_0} = 0$, one obtains from definitions (6.12), (6.16) that

$$\|\nabla W_0 + \nabla W_N + \nabla v_0\|_{0,G_0} = \|\nabla W_0 + \nabla v_0\|_{0,G_0} \le$$

$$\le \|\bar f_{1,0}\mu_1' + \bar f_0\mu_0'\|_{0,G_0} + \|\nabla(\lambda_0 U)\|_{0,G_0} \le$$

$$\le \|\mu_0'(\bar f_0 - \bar f_{1,0})\|_{0,G_0} + c\,\|U\|_{1,G_0}.$$

At the last step we have used that $\mu_0(x) + \mu_1(x) = 1$ for $x \in [0,1/2]$. Next, the first term on the right is not greater than

$$c\,|\bar f_0 - \bar f_{1,0})| \le c\,\|\bar f - f(\cdot,0)\|_{0,(0,1/2)} + c\,\|f(\cdot,0) - \bar f_{1,0}\|_{0,(0,1/2)} \le$$

$$\leq c\,\|f-\bar f_0\|_{0,\Gamma_0}+c\,\|f(\cdot,0)-\bar f_{1,0}\|_{0,(0,1/2)}.$$

According to the "Poincaré inequality" (6.2), the sum of the last two summands does not exceed $c\,[f]_{\Gamma_0}$, where $[\cdot]_{\Gamma_0}$ is the seminorm (6.1). Thus, we established the estimate

$$\|\mu_0'(\bar f_0-\bar f_{1,0})\|_{0,G_0}\leq c\,[f]_{\Gamma_0}.$$

Furthermore, an application of estimate (6.14) gives

$$\|U\|_{H^1(G_0)}\leq c\,\|\mu_0(f-\bar f_0)\|_{1/2,\Gamma_0}.$$

Note that μ_0 is a smooth function with a compact support in Γ_0. Therefore, the last norm is not greater than $c\,\|f-\bar f_0\|_{1/2,\Gamma_0}$ which does not exceed $c\,[f]_{\Gamma_0}$ in view of (6.3). So we verified the estimate

$$\|\nabla W_0+\nabla W_N+\nabla v_0\|_{0,G_0}\leq c\,[f]_{\Gamma_0}. \tag{6.21}$$

The estimate

$$\|\nabla W_0+\nabla W_N+\nabla v_0\|_{0,G_{2N-1}}\leq c\,[f]_{\Gamma_N}. \tag{6.22}$$

is verified in an analogous way (one only should apply inequality (6.15) in place of (6.14) to bound the norm of the last term on the right part of (6.17)).

If $x\in(1/2,N-1/2)$, then $W_0(x,y)=W_N(x,y)=0$ so that

$$\|\nabla W_0+\nabla W_N+\nabla v_0\|^2_{0,((1/2,N-1/2)\times(0,1))}=\sum_{k=1}^{2N-2}\|\nabla v_0\|^2_{0,G_k}.$$

Clearly

$$v_0|_{G_k}=\bar f_{k,0}\mu_k+\bar f_{k+1,0}\mu_{k+1}=\bar f_{k,0}+(\bar f_{k+1,0}-\bar f_{k,0})\mu_{k+1}$$

and hence

$$|\nabla v_0|\leq c\,|\bar f_{k+1,0}-\bar f_{k,0}|\quad\text{on }\ G_k.$$

This implies that

$$\|\nabla v_0\|^2_{0,G_k}\leq c\sum_{j=k}^{k+1}\|f(\cdot,0)-\bar f_{j,0}\|^2_{0,\gamma_j},\quad \gamma_j=((j-1)/2,(j+1)/2).$$

By using again the "Poincaré inequality" (6.2), one arrives at

$$\|\nabla v_0\|^2_{0,G_k}\leq c\sum_{j=k}^{k+1}\iint\limits_{\gamma_j\times\gamma_j}|f(t,0)-f(\tau,0)|^2\frac{dtd\tau}{(t-\tau)^2}.$$

After summation over $k=1,\dots,2N-2$ we obtain

$$\int_{1/2}^{N-1/2}dx\int_0^1|\nabla v_0(x,y)|^2dxdy\leq c\iint\limits_{\{t,\tau\in(0,N):|t-\tau|<1\}}|f(t,0)-f(\tau,0)|^2\frac{dtd\tau}{(t-\tau)^2}. \tag{6.23}$$

To bound $\|v_1 - v_0\|_{1,\partial\Omega}$, we first write

$$v_1(x,y) - v_0(x,y) = \sum_{k=1}^{2N-1} (\bar{f}_{k,1} - \bar{f}_{k,0})\mu_k(x), \quad (x,y) \in \Omega,$$

which follows from (6.12). The last sum contains at most two nonzero summands for each fixed $x \in (0,1)$, therefore,

$$\|v_1 - v_0\|^2_{1,\partial\Omega} \leq c \sum_{k=1}^{2N-1} |\bar{f}_{k,1} - \bar{f}_{k,0}|^2 \leq$$

$$\leq c \sum_{k=1}^{2N-1} \int_{\gamma_k} \left(|f(x,0) - \bar{f}_{k,0}|^2 + |f(x,1) - f(x,0)|^2 + |f(x,1) - \bar{f}_{k,1}|^2\right) dx.$$

In view of (6.2) we have

$$\int_{\gamma_k} |f(x,i) - \bar{f}_{k,i}|^2 dx \leq c \iint_{\gamma_k \times \gamma_k} |f(t,i) - f(\tau,i)|^2 \frac{dt d\tau}{(t-\tau)^2}.$$

Consequently,

$$\begin{aligned} \|v_1 - v_0\|^2_{1,\partial\Omega} &\leq c \textstyle\int_0^1 |f(x,1) - f(x,0)|^2 dx + \\ &+ c \textstyle\sum_{i=0,1} \iint_{\{t,\tau \in (0,1): |t-\tau|<1\}} |f(t,i) - f(\tau,i)|^2 \frac{dt d\tau}{(t-\tau)^2}. \end{aligned} \tag{6.24}$$

We now bound the last term in (6.20). It follows from (6.10) and (6.13) that

$$\|w_i\|^2_{1,\partial\Omega} \leq c \sum_{k=1}^{2N-1} \|\lambda_k w_{k,i}\|^2_{1,\partial\Omega} \leq$$

$$\leq c \sum_{k=1}^{2N-1} \|w_{k,i}\|^2_{1,g_k} \leq c \sum_{k=1}^{2N-1} \|\mu_k (f - \bar{f}_{k,i})\|^2_{1/2,\gamma_{k,i}},$$

where $\gamma_{k,i} = \gamma_k \times i$, $i = 0,1$. Since μ_k has compact support in γ_k and μ'_k is uniformly bounded, the general term of the last sum does not exceed $c\,\|f - \bar{f}_{k,i}\|^2_{1/2,\gamma_{k,i}}$ which is dominated by $c\,[f]^2_{\gamma_{k,i}}$ because of inequality (6.2). Thus

$$\|w_i\|^2_{1,\partial\Omega} \leq c \sum_{k=1}^{2N-1} \iint_{\gamma_k \times \gamma_k} |f(t,i) - f(\tau,i)|^2 \frac{dt d\tau}{(t-\tau)^2} \leq$$

$$\leq \iint_{\{t,\tau \in (0,1): |t-\tau|<1\}} |f(t,i) - f(\tau,i)|^2 \frac{dt d\tau}{(t-\tau)^2}.$$

A combination of the last estimate with (6.20) –(6.24) gives (6.18).

It remains to check (6.19) to conclude the proof of the lemma. Clearly (6.19) follows from (6.18) and the estimate

$$\|u\|_{0,\Omega} \leq c\,(\|f\|_{0,\Omega} + \langle f \rangle), \tag{6.25}$$

where u is the extension of f constructed above. We have from the definition of u

$$\|u\|_{0,\Omega} \leq \|v_0\|_{0,\Omega} + \|W_0\|_{0,\Omega} + \|W_N\|_{0,\Omega} + \|v_1 - v_0\|_{0,\Omega} + \sum_{i=0,1} \|w_i\|_{0,\Omega}.$$

According to what has been said above, two last terms are dominated by the right part of (6.18), so (6.25) is a consequence of the estimate

$$\|v_0\|_{0,\Omega} + \|W_0\|_{0,\Omega} + \|W_N\|_{0,\Omega} \leq c\,(\|f\|_{0,\Omega} + \langle f \rangle) \tag{6.26}$$

to be checked below.

Indeed, it follows from (6.12) that

$$\|v_0\|^2_{0,\Omega} \leq c \sum_{k=1}^{2N-1} |\bar{f}_{k,0}|^2 \leq c \sum_{k=1}^{2N-1} \|f(\cdot,0)\|^2_{0,\gamma_k} \leq c \int_0^1 |f(x,0|^2 dx.$$

Next, by using (6.14) and (6.16), we obtain

$$\|W_0\|_{0,\Omega} \leq c\,|\bar{f}_0| + c\,\|\mu_0(f - \bar{f}_0)\|_{1/2,\Gamma_0}.$$

The first term on the right does not exceed $c\,\|f\|_{L_2(\Gamma_0)}$, while the last term is not greater than $c\,\|f - \bar{f}_0\|_{1/2,\Gamma_0}$ which is dominated by $c\,[f]_{\Gamma_0}$ in view of inequality (6.2). Therefore

$$\|W_0\|_{0,\Omega} \leq \|f\|_{1/2,\Gamma_0}.$$

In the same way from (6.15) and (6.17) we deduce the estimate

$$\|W_N\|_{0,\Omega} \leq \|f\|_{1/2,\Gamma_N}$$

thus concluding the proof of (6.26) and the Theorem 2.2.

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