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Stability of steady states in kinetic Fokker-Planck equations for Bosons and Fermions

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STABILITY OF STEADY STATES IN KINETIC FOKKER-PLANCK EQUATIONS FOR BOSONS AND FERMIONS

LUKAS NEUMANN AND CHRISTOF SPARBER

ABSTRACT. We study a class of nonlinear kinetic Fokker-Planck type equations modeling quantum particles which obey the Bose-Einstein and Fermi-Dirac statistics, respectively. We establish the existence of classical solutions in the perturbative regime and prove exponential convergence towards the equilibrium.

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1. INTRODUCTION AND MAIN RESULTS

In recent years the rigorous mathematical study of kinetic equations has been enlarged to a class of models which take into account also quantum effects, *cf.* [25] for a general overview. This so-called *quantum kinetic theory* can be seen as an attempt to incorporate certain properties of an underlying quantum systems into the framework of classical statistical mechanics. One might hope that these "hybrid models" on the one hand allow for a somewhat simpler description of the particle dynamics, while maintaining, on the other hand, some purely quantum mechanical features such as generalized statistics for *Bosons* and *Fermions*. Clearly such models can only be justified in a *semiclassical regime*, respectively, in situations where the transport properties of the particles are mainly governed by Newtonian mechanics. Indeed this point of view has already been adopted in the classical paper by Uehling and Uhlenbeck [24], in which they derived their celebrated nonlinear Boltzmann type equation for quantum particles.

Following their spirit most of the quantum kinetic models studied so far invoke nonlinear collision operators of Boltzmann type, see, *e.g.*, [5, 6, 16, 17]. For these kind of models, a particular focus of interest is the long time behavior of their solutions, in particular the convergence towards steady states, which generalize the classical *Maxwellian distribution*, *cf.* [8, 16, 18, 20]. Very often though, the simplified case of a spatially homogeneous gas is considered.

In what follows, we shall also be interested in such kind of relaxation-to-equilibrium phenomena, in the *spatially inhomogeneous* case. We shall not deal with a Boltzmann type equation, but rather study a *nonlinear Fokker-Planck* type model (FP). More precisely, we consider the following kinetic equation

$$(1.1) \qquad \partial_t f + p \cdot \nabla_x (f + \kappa f^2) = \operatorname{div}_p \left(\nabla_p f + p f (1 + \kappa f) \right),$$

where, for any $t \geq 0$, $f = f(t, x, p) \geq 0$ denotes the *particle distribution* on *phase space* $\Omega_x \times \mathbb{R}_p^d$. In what follows, the spatial domain is chosen to be $\Omega_x = \mathbb{T}^d$, the

d-dimensional torus. This setting can be seen as a convenient and mathematically simpler replacement for the incorporation of confining potentials $V(x)$, needed to guarantee the existence of nontrivial steady states in the whole space. In (1.1) we set $\kappa = -1$ for Fermions and $\kappa = 1$ for Bosons.

Remark 1.1. Obviously, for $\kappa = 0$ equation (1.1) simplifies to the classical *linear* Fokker-Planck equation (or *Kramer's equation*) on phase space, *i.e.*

$$\partial_t f^{\mathrm{lin}} + p \cdot \nabla_x f^{\mathrm{lin}} = \Delta_p f^{\mathrm{lin}} + \mathrm{div}_p(pf^{\mathrm{lin}}).$$

For this linear model, the convergence to equilibrium has been recently studied in [4, 11, 19], using several different approaches.

The FP type model (1.1) has been introduced in [14] and a formal derivation from the (spatially homogeneous) Uehling-Uhlenbeck equation is given in [21]. Different physical applications can be found in [9, 12, 13, 15] dealing with, both, the spatially homogeneous as well as the inhomogeneous case (see also [10] and the references given therein). More recently, a similar but somewhat simpler FP type model has been proposed in [22, 23] to describe self-gravitating particles and the formation of Bose-Einstein condensates in a kinetic framework. The authors consider

$$(1.2) \qquad \partial_t f + p \cdot \nabla_x f = \mathrm{div}_p\left(\nabla_p f + pf(1 + \kappa f)\right),$$

where, in contrast to (1.1), only the diffusive part of the equation includes a non-linearity. As we will see both equation however share the same steady states. In order to deal with both models, we study from now on the following initial value problem

$$(1.3) \qquad \begin{cases} \partial_t f + p \cdot \nabla_x(f + \sigma\kappa f^2) = \mathrm{div}_p\left(\nabla_p f + pf(1 + \kappa f)\right), \\ f\big|_{t=0} = f_0(x, p), \end{cases}$$

with $\sigma = 1$ or $\sigma = 0$, corresponding to the case (1.1) and (1.2), respectively. We note that the long time behavior of these models in the spatially homogeneous case (and for $d = 1$) has been rigorously investigated quite recently in [3] via an entropy-dissipation approach.

In what follows, the initial phase space distribution $f_0 \in L^1(\mathbb{T}^d_x \times \mathbb{R}^d_p)$ is assumed to be normalized according to

$$(1.4) \qquad \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, p)\, \mathrm{d}x\, \mathrm{d}p = M,$$

for some given *mass* $M > 0$. This normalization is conserved by the evolution. Moreover in the fermionic case, *i.e.* $\kappa = -1$, we require $f_0(x, p) < 1$, $\forall\, (x, p) \in \mathbb{T}^d \times \mathbb{R}^d$, as usual in the physics literature [9]. In particular the latter is needed to define the associated quantum mechanical *entropy functionals*, *i.e.*

$$H[f] := \iint_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{|p|^2}{2}f + f\ln f - \kappa(1 + \kappa f)\ln(1 + \kappa f)\right) \mathrm{d}x\, \mathrm{d}p,$$

which obviously requires $f(t, x, p) < 1$, if $\kappa = -1$. It is now straightforward to verify that (independent of the particular choice of σ) the unique *steady state* of (1.3) is given by

$$(1.5) \qquad f_\infty = \frac{1}{\exp\left(\frac{|p|^2}{2} + \theta\right) - \kappa},$$

where the constant θ is used to ensure that f_∞ satisfies the mass constraint (1.4). In the bosonic case $\theta \in \mathbb{R}_+$, whereas in the fermionic case we can allow for $\theta \in \mathbb{R}$,

cf. [3, 7] for more details. In the latter situation the distribution (1.5) is the well known *Fermi-Dirac equilibrium distribution*. On the other hand, for $\kappa = 1$, f_∞ is the so-called *regular Bose-Einstein distribution*. Finally, if $\kappa = 0$, formula (1.5) simplifies to the classical Maxwellian, *i.e.*

$$f_\infty^{\mathrm{lin}} = M \, \mathrm{e}^{-|p|^2/2},$$

where $\log M = -\theta$. Note that in any case the equilibrium state is *independent* of x since we have chosen $\mathbb{T}^d$ as our spatial domain.

In the bosonic case there is an additional difficulty, at least for $d \geq 3$, since one can *not* associate an arbitrarily large $M > 0$ to the steady state. More precisely, the maximum amount of mass comprised by f_∞ is determined via

$$\iint_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{\mathrm{e}^{|p|^2/2} - 1} \, \mathrm{d}x \, \mathrm{d}p =: M_{\mathrm{crit}} < \infty,$$

i.e. for $\theta = 0$. Due to mass conservation this induces a threshold on M. This problem, which does not appear in dimensions $d = 1, 2$, has led to the introduction of more general bosonic steady states, where an additional δ-distribution (appropriately normalized) is added to f_∞, *cf.* [6, 8]. This singular measure can then be interpreted as a so-called *Bose-Einstein condensate* (BEC). The formation of a δ-measure in finite or infinite time is a task of extensive research in quantum kinetic theory, see, *e.g.*, [2, 8]. For our nonlinear model though, including such generalized solutions on a rigorous mathematical level seems to be out of reach so far and we thus have to impose $\theta > 0$ in the bosonic case. (Indeed, as we shall see below, we also require $\theta > 0$ in the fermionic case, although for different and rather technical reasons.)

Our main task here is the description of the convergence for solutions to (1.3) towards the steady state (1.5). To this end we shall conceptually follow the approach given in [19] where the trend to equilibrium is studied for a wide class of kinetic models *close to equilibrium*. The difference in our case being mainly that we are dealing with *local* nonlinearities (in the Boltzmann type models studied in [19] the nonlinearities usually appear only within an integral kernel) which moreover are also allowed to enter in the transport part of the considered equation. We thus *linearize* the solution f of (1.3) around the steady state f_∞ in the form

$$(1.6) \qquad\qquad f = f_\infty + g\sqrt{\mu_\infty}.$$

where the new unknown $g(t, x, p) \in \mathbb{R}$ can be interpreted as a *perturbation* of the equilibrium state such that

$$\iint_{\mathbb{T}^d \times \mathbb{R}^d} g\sqrt{\mu_\infty} \, \mathrm{d}x \, \mathrm{d}p = 0.$$

In (1.6) we use the additional (time-independent) *scaling factor*

$$\mu_\infty := f_\infty + \kappa f_\infty^2,$$

which allows for an easier description of the functional framework given below.

Plugging (1.6) into (1.3) straightforward calculations formally yield the following equation for g

$$(1.7) \qquad\qquad \partial_t g + (1 + 2\sigma\kappa f_\infty)p \cdot \nabla_x g = L(g) + Q(g) \,,$$

Above the *linearized collision operator* L is defined by

$$L(g) = \frac{1}{\sqrt{\mu_\infty}} \operatorname{div}_p \left(\nabla_p \left(g \sqrt{\mu_\infty} \right) + p \, \eta_\infty g \, \sqrt{\mu_\infty} \right)$$

(1.8)

$$= \Delta_p g + g \left(\frac{d}{2} \, \eta_\infty - |p|^2 \left(\frac{1}{4} + 2\kappa\mu_\infty \right) \right),$$

where we use the short hand notation $\eta_\infty := 1 + 2\kappa f_\infty$. The *quadratic remainder* Q is given by

(1.9) $$Q(g) = \frac{\kappa}{\sqrt{\mu_\infty}} \left(\operatorname{div}_p \left(p \, \mu_\infty g^2 \right) - \sigma\mu_\infty p \cdot \nabla_x(g^2) \right).$$

The main result is as follows.

Theorem 1.2. *Let f_0 be of the form*

$$0 \le f_0 = f_\infty + g_0 \sqrt{\mu_\infty},$$

with $\theta > 0$. Moreover if $\kappa = \sigma = 1$, i.e. the bosonic case with nonlinear transport, assume that in addition be $\theta > \theta^*$, for a certain $\theta^* > 0$.

There exists an $\epsilon_0 > 0$, such that for all f_0 with $\| g \|_{H^k} \le \epsilon < \epsilon_0$, for $\mathbb{N} \ni k > 1 + d/2$, the equation (1.3) admits a unique solution $0 \le f \in C([0, \infty); H^k(\mathbb{T}_x^d \times \mathbb{R}_p^d))$. Moreover

$$\left\| \mu_\infty^{-1/2} \left(f(t) - f_\infty \right) \right\|_{H^k} \le C\epsilon_0 e^{-\tau t},$$

where C, τ are positive constants.

The proof of this result is given in Section 3. To this end we first collect several preliminary results in the Section 2.

Remark 1.3.

- Note that the assumption $\theta > 0$ is imposed for both cases, the bosonic as well as the fermionic one. Physically speaking this means that we have to consider initial data f_0 with sufficiently small mass M. For Bosons this is crucial in order to avoid formation of a BEC in $d \ge 3$. In the fermionic case the reason to impose $\theta > 0$ is to guarantee $\eta_\infty > 0$, which we will make use of several times. It might be possible to overcome this restriction, *cf.* [3], albeigth the authors follow a different approach.
- The additional restriction $\theta > \theta^*$ is *not* needed for the model (1.2) where only a nonlinear diffusion operator is present. The reason for this additional constraint when dealing with (1.1) is that we have to maintain a fundamental regularizing property in x of the transport part, *cf.* Section 3 for more details. Since θ^* is then determined by a transcendental nonlinear equation, we can not give an exact value for it but only perform numerical experiments which indicate that $\theta^* \approx 0.451$.
- Formal calculations given in [23] indicate that an analogous theorem as above does not hold if one includes the possibility of a BEC, *i.e.* if one allows $\theta = 0$ for $\kappa = 1$, in dimension $d \ge 3$. In particular it seems that in general one can not expect exponential convergence towards a BEC equilibirum state, *cf.* [23] or [8] where a different kinetic model is used.
- The theorem and its proof can be seen as a variant of a rather new type of mathematical approach in the study of kinetic models, based on the so-called *hypo-coercivity* property of the linearized equation. Here the phrase "hypo-" has to be understood as in the distinction between "hypo-elliptic"

and (classical) "elliptic" operators. For more details on that we refer to [26].

The theorem directly yields the following result for the quantum mechanical entropies.

Corollary 1.4. *Under the same assumptions as above*

$$H[f(t)] - H[f_\infty] \le C e^{-\tau t},$$

i.e. we have exponential decay in relative entropy.

Proof. Inserting $f = f_\infty + \sqrt{\mu_\infty} g$, where $\iint \sqrt{\mu_\infty} g = 0$, performing a Taylor expansion around the steady state f_∞, and finally using Theorem 1.2. yields the assertion of the corollary . $\square$

Remark 1.5. One might think of several extensions of our result. As already discussed in [19] one could take into account *weak* external potentials, *i.e.* potentials $V(x)$ such that $\|V\|_{C^2(\mathbb{T}^d)}$ small enough, or *self-consistent* potentials, which stem from a coupling to Poisson's equation. The latter case might be particularly interesting in semiconductor modeling, where the fermionic FP type equation could be used to describe the dynamical behavior of charge carries obeying the "physically correct" equilibrium statistics.

2. Study of the linearized collision operator

We shall now derive several properties on the linearized collision operator L to be used in the proof of the main result. First note that L is self adjoint on $L^2(\mathbb{R}_p^d)$ and, by partial integration, one obtains

$$(2.1) \quad \langle L(g), g \rangle_{L^2(\mathbb{R}_p^d)} = -\int_{\mathbb{R}^d} \left| \nabla_p g + \frac{p}{2}\, \eta_\infty g \right|^2 \, \mathrm{d}p = -\int_{\mathbb{R}^d} \left| \nabla_p \left(\frac{g}{\sqrt{\mu_\infty}} \right) \right|^2 \mu_\infty \, \mathrm{d}p \,.$$

Thus the kernel of the non-positive operator L is given by

$$\mathrm{Ker}(L) = \mathrm{span}\{ \sqrt{\mu_\infty} \}.$$

Let us define the orthogonal projection in $L^2(\mathbb{R}_p^d)$ onto this kernel via

$$\Pi(f) := \left(\frac{1}{\rho_\infty} \int_{\mathbb{R}^d} f \sqrt{\mu_\infty} \, \mathrm{d}p \right) \sqrt{\mu_\infty} \,,$$

where we set

$$\rho_\infty = \left(\int_{\mathbb{R}^d} \mu_\infty \, \mathrm{d}p \right) > 0$$

for reasons of normalization. Note that this is only a projection in the momentum variable $p \in \mathbb{R}^d$. Motivated by (2.1) we introduce the following weighted space

$$\Lambda_p := \left\{ f \in L^2(\mathbb{R}_p^d): \ \|f\|_{\Lambda_p} < \infty \right\},$$

where

$$\|f\|_{\Lambda_p}^2 := \|\nabla_p f\|_{L_p^2}^2 + \|p\,\eta_\infty f\|_{L_p^2}^2 \,,$$

Here, and in what follows, we write $L_p^2 \equiv L^2(\mathbb{R}_p^d)$ for simplicity. Moreover we denote for $f(x, \cdot) \in \Lambda_p$

$$\|f\|_\Lambda := \left\| \|f\|_{\Lambda_p} \right\|_{L^2(\mathbb{T}_x^d)} \,,$$

the induced norm on phase space. Obviously the Λ_p norm controls the L_p^2 norm for κ nonnegative. In the fermionic case ($\kappa = -1$) however this is not true in general,

since η_∞ may change sign. For our functional approach the control of L_p^2 via Λ_p is crucial and thus we have to guarantee that $\eta_\infty > 0$ by assumption. This implies that we need to impose for $\kappa = -1$ that $f_\infty < 1/2$, $\forall\, p \in \mathbb{R}^d$, or equivalently $\theta > 0$. This certainly is more restrictive than the usual bound *i.e.* $f_\infty < 1$, used in the physics literature. In summary we require $\theta > 0$ in the bosonic case to prevent BEC and in the fermionic case to ensure that η_∞ is globally bounded away from zero.

First we obtain a Poincaré inequality for the steady state of the linearized model.

Lemma 2.1. *If $\theta > 0$ then the (strictly positive and normalized) measure μ_∞/ρ_∞ satisfies a Poincaré inequality on $\mathbb{R}_p^d$, i.e.*

$$\int_{\mathbb{R}^d} g^2 \mu_\infty \,\mathrm{d}p - \frac{1}{\rho_\infty}\left(\int_{\mathbb{R}^d} g\mu_\infty \,\mathrm{d}p\right)^2 \leq C_{\mathrm{p}} \int_{\mathbb{R}^d} |\nabla_p g|^2 \mu_\infty \,\mathrm{d}p, \quad C_{\mathrm{p}} > 0.$$

Note that in the fermionic case $\kappa = -1$ this lemma indeed holds more generally for any $\theta \in \mathbb{R}$.

Proof. Let $\mathcal{A}$ be defined by $\mathcal{A} = -\ln\mu_\infty$. Having in mind the results of [1] it is enough to prove that $\mathcal{A}$ can be decomposed as as

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2,$$

where $\mathcal{A}_1(p)$ is uniformly convex and $\mathcal{A}_2(p)$ is a local perturbation, such that

$$\exists\, a, b > 0, \text{ s.t. } \forall\, p \in \mathbb{R}^d: \ a \leq \mathrm{e}^{-\mathcal{A}_2(p)} \leq b.$$

To this end note that $\mathcal{A}$ is given by

$$\mathcal{A} = -\log\left(\frac{e^{|p|^2/2+\theta}}{(e^{|p|^2/2+\theta} - \kappa)^2}\right) = \frac{|p|^2}{2} + \theta - 2\log\left(\frac{e^{|p|^2/2+\theta}}{e^{|p|^2/2+\theta} - \kappa}\right)$$

$$= \frac{|p|^2}{2} + \theta + 2\log\left(1 - \frac{1}{\kappa e^{|p|^2/2+\theta}}\right).$$

We now pick $\mathcal{A}_1 = |p|^2/2$ and $\mathcal{A}_2$ to be the rest of the terms appearing on the r.h.s.. Since $p \mapsto |p|^2/2$ is strictly convex it is larger than some positive constant outside a ball of finite radius. Outside this ball the function $\mathcal{A}_2$ is naturally bounded. On the other hand, inside the ball $\mathcal{A}_2$ is bounded because $\mathcal{A}$, as well as $\mathcal{A}_1$ are naturally bounded if $\kappa = -1$, or if $\kappa = 1$ and $\theta > 0$. $\square$

With the above lemma in hand we can now establish the *coercivity* of the linearized collision operator.

Lemma 2.2. *For $\theta > 0$ there exists a $\lambda > 0$ such that*

$$\langle L(g), g\rangle_{L_p^2} \leq -\lambda \|g - \Pi(g)\|_{\Lambda_p}^2, \quad \forall\, g \in \Lambda_p.$$

The coercivity property (in $p \in \mathbb{R}^d$) of the operator L is indeed an essential requirement to establish our main result. It would not hold for example in $L^2(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$ and induces the particular choice of Λ_p and its corresponding norm.

Proof. We start with

$$\langle L(g), g \rangle_{L_p^2} = - \rho_\infty \int \left| \nabla_p \left(\frac{g}{\sqrt{\mu_\infty}} \right) \right|^2 \frac{\mu_\infty}{\rho_\infty} \, \mathrm{d}p$$

$$\leq -C_{\mathrm{p}} \left(\int g^2 \, \mathrm{d}p - \frac{1}{\rho_\infty} \left(\int \sqrt{\mu_\infty} \, g \, \mathrm{d}p \right)^2 \right)$$

$$\leq -C \int \left(g - \frac{\sqrt{\mu_\infty}}{\rho_\infty} \int \sqrt{\mu_\infty} \, g \, \mathrm{d}p \right)^2 \, \mathrm{d}p = -C \, \| g - \Pi(g) \|_{L_p^2}^2 \,,$$

for some $C > 0$, where we used the fact that the measure μ_∞ / ρ_∞ satisfies a Poincaré inequality due to the previous lemma. Now to improve on the L_p^2 norm we use

$$\langle L(g), g \rangle_{L_p^2} = - \int \left(\nabla_p(g - \Pi(g)) + \frac{p}{2} \eta_\infty (g - \Pi(g)) \right)^2 \, \mathrm{d}p$$

$$\leq -K_1 \| g - \Pi(g) \|_{\Lambda_p}^2 + K_2 \| g - \Pi(g) \|_{L_p^2}^2 \,.$$

Adding the two inequalities above (multiplied by appropriate constants) finishes the proof. $\square$

We get a similar result for the derivatives w.r.t. $p \in \mathbb{R}^d$.

Lemma 2.3. *Let $\theta > 0$, then there exist positive constants C_1 and C_2, such that for any $g \in L_p^2$ with $\nabla_p g \in \Lambda_p$*

$$\langle \nabla_p L(g), \nabla_p g \rangle_{L_p^2} \leq -C_1 \| \nabla_p g \|_{\Lambda_p}^2 + C_2 \| g \|_{L_p^2}^2 \,.$$

Proof. A lengthy calculation yields

$$\langle \nabla_p L(g), \nabla_p g \rangle_{L_p^2} = \int -(\Delta_p g)^2 + |\nabla_p g|^2 \left(\frac{d}{2} \eta_\infty - \frac{|p|^2}{4} - 2\kappa \mu_\infty |p|^2 \right) \, \mathrm{d}p$$

$$+ \int \frac{g^2}{2} \left(d^2 \kappa \mu_\infty + \frac{d}{2} + 4\kappa d \mu_\infty \right) \, \mathrm{d}p$$

$$- \frac{\kappa}{2} \int g^2 |p|^2 \mu_\infty \left((d+10)\eta_\infty - 2|p|^2 \mu_\infty \eta_\infty \right) \, \mathrm{d}p.$$

The last integral on the r.h.s. is dominated by the L_p^2 norm, since μ_∞ decays exponentially fast as $|p| \to \infty$. We also have that $1/4 + 2\kappa\mu_\infty \geq C > 0$, $\forall p \in \mathbb{R}^d$. This obviously holds true for the bosonic case but is also guaranteed in the fermionic situation where $f_\infty < 1/2$. Thus we can estimate

$$\langle \nabla_p L(g), \nabla_p g \rangle_{L_p^2} \leq -C_1 \| \nabla_p g \|_{\Lambda_p}^2 + \frac{d}{2} \int \eta_\infty |\nabla_p g|^2 \, \mathrm{d}p + C_2 \| g \|_{L_p^2}^2$$

and a classical interpolation argument applied to the second term on the r.h.s. yields the assertion of the lemma (with different constants C_1, C_2). $\square$

Finally we need the following technical lemma.

Lemma 2.4. *For $g, h \in \Lambda_p$ it holds that*

$$\langle L(h), g \rangle_{L_p^2} \leq C \| g \|_{\Lambda_p} \| h \|_{\Lambda_p} \,, \quad C > 0.$$

Proof. We first note that

$$\|g\|_{\Lambda_p} \|h\|_{\Lambda_p} = \left(\|\nabla_p g\|_{L_p^2}^2 + \|p\,\eta_\infty\, g\|_{L_p^2}^2 \right)^{1/2} \left(\|\nabla_p h\|_{L_p^2}^2 + \|p\,\eta_\infty\, h\|_{L_p^2}^2 \right)^{1/2}$$

$$\geq \tilde{C} \left(\|\nabla_p g\|_{L_p^2}^2 + \int (1+|p|^2)\,\eta_\infty^2 g^2 \,\mathrm{d}p \right)^{1/2} \left(\|\nabla_p h\|_{L_p^2}^2 + \int (1+|p|^2)\,\eta_\infty^2 h^2 \,\mathrm{d}p \right)^{1/2}$$

for some $\tilde{C} > 0$ since the Λ_p norm dominates the L_p^2 norm. Using the following simple algebraic estimate

$$\left((a^2 + b^2)(c^2 + d^2) \right)^{1/2} \geq ac + bd \;,$$

(with $a, b, c, d \in \mathbb{R}$) we further obtain

$$C \|g\|_{\Lambda_p} \|h\|_{\Lambda_p} \geq \|\nabla_p g\|_{L_p^2} \|\nabla_p h\|_{L_p^2} + \left(\int (1+|p|^2)\,\eta_\infty^2 g^2 \,\mathrm{d}p \int (1+|p|^2)\,\eta_\infty^2 h^2 \,\mathrm{d}p \right)^{1/2} \;.$$

The proof then follows by applying the Cauchy Schwarz inequality to both terms on the right hand side and integrating by parts in the first one. $\qquad\square$

3. CONVERGENCE FOR THE LINEAR MODEL AND PROOF OF THEOREM 1.2

Now we are able to establish the long time asymptotics for the linearized equation, which eventually will be translated also to the nonlinear model (1.3) in the perturbative setting.

Proposition 3.1. *Consider the linearized Fokker Planck type equation*

$$(3.1) \qquad\qquad \partial_t g + (1 + 2\sigma\kappa f_\infty)p \cdot \nabla_x g = L(g) \;,$$

with L given by (1.8) and $\theta > 0$. Moreover if $\kappa = \sigma = 1$, assume in addition that $\theta > \theta^*$, for a certain $\theta^* > 0$. Then, if the intial data $g_0 \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$, for $k \in \mathbb{N}$, the solution $g(t)$ exists globally in time and

$$\| g(t) - g_\infty \|_{H^k} \leq C\mathrm{e}^{-\tau t} \;, \qquad \text{with } C,\,\tau > 0,$$

where the global equilibrium g_∞ is given by

$$g_\infty = \left(\frac{1}{\rho_\infty} \iint_{\mathbb{T}^d \times \mathbb{R}^d} g_0\,\sqrt{\mu_\infty}\,\mathrm{d}p\,\mathrm{d}x \right) \sqrt{\mu_\infty} \;.$$

Proof. The results of Section 2 show that our linearized collision operator L fits into the class of models studied in [19]. We will sketch the proof and stress the differences which occur due to the changes in the transport operator. Note that, since the equation is linear, we can w.r.o.g. consider the case where $g_\infty \equiv 0$. This can always be achieved by subtracting initially the projection onto the global equilibrium, *i.e.* by considering initial data $\widetilde{g}_0 = g_0 - g_\infty$.

We start with $k = 1$. The idea of the proof is to study the time evolution of a combination of derivatives w.r.t. x and p. More precisely we consider the following functional

$$\mathcal{F}[g(t)] := \alpha \|g\|^2 + \beta \|\nabla_x g\|^2 + \gamma \|\nabla_p g\|^2 + \delta \langle \nabla_x g, \nabla_p g \rangle \;,$$

where $\|\cdot\|$ denotes the standard norm on $L^2(\mathbb{T}_x^d \times \mathbb{R}_p^d)$ and $\alpha, \beta, \gamma, \delta$ are some positive constants. We will not go into details about the choice these coefficients but note that δ has to be small enough in comparison to β and γ such that $\mathcal{F}$ is positive and controlled from above and below by the square of the usual $H^1(\mathbb{T}_x^d \times \mathbb{R}_p^d)$ norm of

g. On the other hand δ has to be strictly positive, since we need it in order to close the argument (see blow). We aim to prove that

$$(3.2) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{F}[g(t)] \leq -C \left(\|g\|_\Lambda^2 + \|\nabla_{x,p}\, g\|_\Lambda^2 \right).$$

To this end we calculate the time derivatives of the various summands in $\mathcal{F}$. First, for the L^2 norm we have

$$(3.3) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\, \|g\|^2 = 2\, \langle L(g), g \rangle \leq -\lambda\, \|g - \Pi(g)\|_\Lambda^2,$$

where we have used that the transport part does not contribute (due to its divergence form) and the assertion of Lemma 2.2.

Next the spatial derivatives evolve according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \|\nabla_x g\|^2 = 2\, \langle \nabla_x L(g), \nabla_x g \rangle \leq -\lambda\, \|\nabla_x g - \Pi(\nabla_x g)\|_\Lambda^2,$$

where we again used that the contribution from the transport term vanishes and the fact that L commutes with ∇_x thus allowing us to apply Lemma 2.2 also on $\nabla_x g$.

For the derivatives w.r.t. p we get some additional terms by the coefficient of the transport operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \|\nabla_p g\|^2 = 2\, \langle \nabla_p L(g), \nabla_p g \rangle + \left\langle \left(4\kappa\sigma\mu_\infty |p|^2 - 2\eta_\infty(1 + 2\kappa\sigma f_\infty) \right) \nabla_x g, \nabla_p g \right\rangle$$
$$\leq 2\, \langle \nabla_p L(g), \nabla_p g \rangle + K_1 \|\nabla_x g\|^2 + K_2 \|\nabla_p g\|^2.$$

To estimate the term invoking the mixed derivative we used that f_∞ as well as $|p|^2 \mu_\infty$ are uniformly bounded in L^∞. Now the first term on the r.h.s. is estimated by Lemma 2.3, which yields some damping for $\nabla_p g$ in Λ. However this comes at the price of an additional term $\propto \|g\|$, with a positive sign.

We split $g = (g - \Pi(g)) + \Pi(g)$ and estimate

$$\|g\|_{L^2}^2 \leq \|g - \Pi(g)\|_{L^2}^2 + \|\Pi(g)\|_{L^2}^2 \leq \|g - \Pi(g)\|_{L^2}^2 + C_\mathbb{T} \|\nabla_x g\|_{L^2}^2, \quad C_\mathbb{T} > 0.$$

For the second inequality we used the classical Poincare inequality w.r.t $x \in \mathbb{T}^d$ and the fact that $\Pi(g)$ has zero mean on the torus, since $g_\infty = 0$. Note that this is an improvement in the sense that the term $\|g - \Pi(g)\|^2$ can be canceled by adjusting α, and having in mind (3.3).

Finally we look at the mixed derivatives w.r.t x and p, which evolve acording to

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \langle \nabla_x g, \nabla_p g \rangle = 2\, \langle L(\nabla_x g), \nabla_p g \rangle - \left\langle \nabla_x g, \left(1 + 2\sigma\kappa f_\infty - 2\sigma\kappa\mu_\infty |p|^2 \right) \nabla_x g \right\rangle.$$

For the first term on the r.h.s. we invoke Lemma 2.4, which together with the Cauchy-Schwarz inequality in x implies

$$\langle L(\nabla_x g), \nabla_p g \rangle \leq C \|\nabla_x g - \Pi(\nabla_x g)\|_\Lambda^2 + \tilde{C} \|\nabla_p g\|_\Lambda^2.$$

The second term on the r.h.s., which stems from the transport part, generates a damping for $\|\nabla_x g\|$ (which the operator L can not provide as it only acts in p), provided that

$$(3.4) \qquad 1 + 2\kappa\sigma f_\infty - 2\kappa\sigma\mu_\infty |p|^2 \geq K > 0.$$

From here we can extract a damping effect in the Λ norm, due to Poincaré's inequality and the fact that $\Pi(\nabla_x g)$ is well behaved in p

$$-\|\nabla_x g\|^2 \leq -K \|\nabla_x g\|_\Lambda^2 + C \|g - \Pi(g)\|_\Lambda^2 + C \|\nabla_x g - \Pi(\nabla_x g)\|_\Lambda^2.$$

Assuming for the moment that (3.4) is true it remains to add all the above terms and to find coefficients $\alpha, \beta, \gamma, \delta$ in $\mathcal{F}$, such such that all "bad" terms in the above given estimates (*i.e.* those which come with the wrong or without sign) can be controlled and the differential inequality (3.2) holds true. This can be done similarly to [19], although it is a somewhat lengthy procedure and we will not elaborate further on it. The functional $\mathcal{F}$ clearly induces a new norm on phase space, equivalent to $H^1(\mathbb{R}^d \times \mathbb{T}^d)$, via

$$\|\cdot\|_{\mathcal{H}^1}^2 := \mathcal{F}[\,\cdot\,]$$

and similarly we can define higher order norms $\|\cdot\|_{\mathcal{H}^k}$, $k \in \mathbb{N}$, *cf.* [19].

To retain the (fundamental) damping property in the spatial derivatives coming from the evolution of the mixed term it remains to show that the constraint (3.4) holds true. We denote

$$\Psi_\kappa(p\,;\theta) := 1 + 2\kappa f_\infty - 2\kappa\mu_\infty|p|^2 \ .$$

If $\kappa = -1$ and since $\theta > 0$, it is clear that $\Psi_{-1}(p\,;\theta) \geq K > 0$ because $\eta_\infty \geq 0$ in this case. For Bosons, *i.e.* $\kappa = 1$, however the situation is more difficult. Note that

$$\lim_{|p|\to\infty} \Psi_1(p\,;\theta) = \lim_{|p|\to\infty} (\eta_\infty - 2\mu_\infty|p|^2) = 1, \quad \forall\,\theta > 0,$$

and thus by continuity it is enough to make sure that $\Psi_1(p\,;\theta) \neq 0, \forall\,p \in \mathbb{R}^d, \theta > \theta^*$. Straightforward calculations show that $\Psi_1(p\,;\theta)$ can only be zero if

$$\left(e^{|p|^2/2+\theta}\right)^2 - 2|p|^2 e^{|p|^2/2+\theta} - 1 = 0 \ ,$$

which indeed implies

$$e^{|p|^2/2+\theta} = |p|^2 + \sqrt{|p|^4 + 1} \ .$$

Obviously this equality can not be true for θ larger than some critical value θ^*. Numerical experiments suggest that this critical value is approximately $\theta^* \approx 0.451$. In summary one obtains the final estimate (3.2), which finishes the proof for $k = 1$.

To proceed to higher order estimates in the Sobolev index $k \in \mathbb{N}$, we observe that the proof of Lemma (2.3) can be generalized in a straightforward way to obtain

$$\langle \partial_{x_\ell} \partial_{p_j} L(g), \partial_{x_\ell} \partial_{p_j} g \rangle_{L_p^2} \leq -\tilde{C}_1 \left\| \partial_{x_\ell} \partial_{p_j} g \right\|_{\Lambda_p}^2 + \tilde{C}_2 \left\| g \right\|_{H_p^{k-1}}^2 ,$$

for any multi-indices j, ℓ such that $k = |j| + |\ell|$, $|j| \geq 1$. An induction argument in $k \in \mathbb{N}$, analogously to the one given in [19] then yields the corresponding statement in H^k. We do not run into additional problems due to the coefficient of the transport operator, since the terms containing the highest order derivatives of g can be treated as before and the lower order terms (which contain derivatives of $\Psi_\kappa(p\,;\theta)$) can be handled by interpolation. This finishes the proof. $\square$

Now we apply the result for the linearized equation the nonlinear problem.

Proof of Theorem 1.2. We have to show that the quadratic nonlinearity does not change the estimates obtained for the linearized equation, as long as the deviation from the equilibrium is small. The function $g = (f - f_\infty)\mu_\infty^{-1/2}$ solves (1.7) from which we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \|g\|_{\mathcal{H}^k}^2 = 2 \langle Tg, g \rangle_{\mathcal{H}^k} + 2 \langle Q(g), g \rangle_{\mathcal{H}^k} ,$$

where $T := L - (1 + 2\sigma\kappa f_\infty) p \cdot \nabla_x$ and L, Q are given in (1.8), (1.9), respectively. From the proof of Proposition 3.1 we know that

$$\langle Tg, g \rangle_{\mathcal{H}^k} \leq -C \left(\sum_{|j|+|\ell|\leq k} \left\| \partial_{x_\ell} \partial_{p_j} g \right\|_\Lambda^2 \right).$$

Thus, if we can prove the following property for the nonlinear part

(3.5) $$\langle Q(g), g \rangle_{H^k} \leq C_Q \|g\|_{H^k}^2 \left(\sum_{|j|+|\ell|\leq k} \left\| \partial_{x_\ell} \partial_{p_j} g \right\|_\Lambda \right),$$

it follows, since $\| \cdot \|_{\mathcal{H}^k} \simeq \| \cdot \|_{H^k}$, that

$$\frac{\mathrm{d}}{\mathrm{d}t} \| g \|_{\mathcal{H}^k}^2 \leq -C \left(\sum_{|j|+|\ell|\leq k} \left\| \partial_{x_\ell} \partial_{p_j} g \right\|_\Lambda^2 \right) + C_Q \|g\|_{H^k}^2 \left(\sum_{|j|+|\ell|\leq k} \left\| \partial_{x_\ell} \partial_{p_j} g \right\|_\Lambda \right),$$

which concludes the proof of Theorem 1.2 by a Gronwall type argument.

In order to prove (3.5), we recall that $Q(g)$ is given by

$$Q(g) = \frac{\kappa}{\sqrt{\mu_\infty}} \left(\mathrm{div}_p(p\,\mu_\infty g^2) \right) - \frac{\kappa}{\sqrt{\mu_\infty}} \left(\sigma\mu_\infty p \cdot \nabla_x(g^2) \right) \equiv Q_1(g) - Q_2(g),$$

and note that $\mu_\infty^{-1/2} \partial_{x_\ell} \partial_{p_j}(p\,\mu_\infty) \in L^\infty(\mathbb{T}_x^d \times \mathbb{R}_p^d)$, for all multi-indices $\ell, j \in \mathbb{N}^d$. We shall now treat $Q_1(g)$ and $Q_2(g)$ separately, using that $H^k(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, for $k > d/2$, together with Leibniz' formula to differentiate $Q(g)$. It is then relatively easy to see that (3.5) holds for $Q_1(g)$ by Sobolev imbedding, as soon as $\mathbb{N} \ni k > d/2$, since the Λ norm incorporates an additional derivate w.r.t $p \in \mathbb{R}^d$. The estimate for $\langle Q_2(g), g \rangle_{H^k}$ is more complicated though, since Q_2 contains a derivate w.r.t. $x \in \mathbb{T}^d$ which is not taken into account for by the Λ norm.

Thus we have to estimate terms of the form

$$\left\langle \partial_{x_\ell} \partial_{p_j} \mu_\infty p \cdot \nabla_x(g^2), \, \partial_{x_\ell} \partial_{p_j} g \right\rangle_{L^2}, \quad |j| + |\ell| \leq k.$$

Since μ_∞ and all of its derivatives are well behaved the highest order terms are

$$\left\langle \mu_\infty p \cdot \nabla_x \partial_{x_\ell} \partial_{p_j}(g^2), \, \partial_{x_\ell} \partial_{p_j} g \right\rangle_{L^2}, \quad |j| + |\ell| = k,$$

and moreover, because of the additional derivative w.r.t. p in the Λ norm, the most problematic terms are those where $|\ell| = k$. Denoting $\partial_\ell \equiv \partial_{x_\ell}$ we compute

$$\left\langle \mu_\infty p \cdot \nabla_x \partial_\ell(g^2), \, \partial_\ell g \right\rangle_{L^2} = 2 \left\langle \mu_\infty g p \cdot \nabla_x \partial_\ell g, \, \partial_\ell g \right\rangle_{L^2}$$
$$+ \sum_{i=1}^d \left\langle \mu_\infty p_i \sum_{\substack{0 \leq r \leq \ell + \delta_i \\ 0 < |r| < |\ell|+1}} \binom{\ell + \delta_i}{r} \partial_r g\, \partial_{\ell + \delta_i - r}\, g, \, \partial_\ell g \right\rangle_{L^2},$$

where δ_i denotes the i-th standard basis vector in $\mathbb{R}^d$. Using divergence theorem, we obtain

$$\left\langle \mu_\infty p \cdot \nabla_x \partial_\ell(g^2), \, \partial_\ell g \right\rangle_{L^2} = - \left\langle \mu_\infty(p \cdot \nabla_x g)\, \partial_\ell g, \, \partial_\ell g \right\rangle_{L^2}$$
$$+ \sum_{i=1}^d \left\langle \mu_\infty p_i \sum_{\substack{0 \leq r \leq \ell + \delta_i \\ 1 \leq |r| < |\ell|}} \binom{\ell + \delta_i}{r} \partial_r g\, \partial_{\ell + \delta_i - r}\, g, \, \partial_\ell g \right\rangle_{L^2}.$$

We only estimate the first term on the r.h.s. side since those appearing in the summation can be treated similarly. This yields (remember $|\ell| = k$)

$$\left\langle \mu_\infty p \cdot \nabla_x \partial_\ell(g^2), \, \partial_\ell g \right\rangle_{L^2} \leq c_1 \left\| \partial_\ell g \right\|_{L^2}^2 \left\| \nabla_x g \right\|_{L^\infty} \leq c_2 \|g\|_{H^k}^2 \|g\|_{H^k},$$

as soon as $\mathbb{N} \ni k > 1 + d/2$ and since $\| g \|_{H^k} \simeq \left\| \sum_{|j|+|\ell| \leq k} \partial_{x_\ell} \partial_{p_j} g \right\|_{\Lambda}$ we obtain the desired estimate (3.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.2. The above given proof shows that we can not proceed completely analogous to [19] since the nonlinearity stemming from $Q_2(g)$ has to be carefully integrated by parts.

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