

# **Overlapping Additive Schwarz preconditioners for degenerated elliptic problems: Part II- locally anisotropic problems**

**S. Beuchler, S.V. Nepomnyaschikh**

**RICAM-Report 2006-33**

# Overlapping Additive Schwarz preconditioners for degenerated elliptic problems: Part II locally anisotropic problems

Sven Beuchler

Institute of Computational Mathematics

University of Linz

Altenberger Straße 69

A-4040 Linz, Austria

sven.beuchler@jku.at

Sergey V. Nepomnyaschikh

Institute for Computational Mathematics

and Computational Geophysics SD Russian Academy of Sciences

Novosibirsk, Russia

svnep@oapmg.ssc.ru

November 8, 2006

## Abstract

In this paper, we consider degenerated and locally anisotropic boundary value problems on the unit square. These problems are discretized by piecewise linear finite elements on a triangular mesh of isosceles right-angled triangles. The system of linear algebraic equations is solved by a preconditioned gradient method using a domain decomposition preconditioner with overlap. We prove that the condition number of the preconditioned system is bounded by a constant which independent of the discretization parameter. Moreover, the preconditioning operation requires  $O(N)$  operations, where  $N$  is the number of unknowns. Several numerical experiments show the performance of the proposed method.

## 1 Introduction

In this paper, we investigate the degenerated and locally anisotropic boundary value problem

$$\begin{aligned} -\omega_1^2(y)u_{xx} - \omega_2^2(x)u_{yy} &= f, & \text{in } \Omega = (0, 1)^2 \\ u &= 0, & \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

with some strongly monotonic increasing and bounded weight functions  $\omega_i : [0, 1] \mapsto \mathbb{R}$  satisfying  $\omega_1(0)\omega_2(0) = 0$ .

In the past, degenerated problems have been considered relatively rarely. One reason is the unphysical behavior of the partial differential equation (pde) which is quite unusual in technical applications. One work focusing on this type of partial differential equation is the book of Kufner and Sändig [13]. Nowadays, problems of this type become more and more popular because there are stochastic pde's of a similar structure. An example of an isotropic degenerated stochastic pde is the Black-Scholes pde, [18].

An example of a locally anisotropic degenerated elliptic problem is the solver related to the problem of the sub-domains for the  $p$ -version of the finite element method using quadrilateral elements. This solver can be interpreted as  $h$ -version fem-discretization matrix of problem (1.1) with  $\omega_1(\xi) = \omega_2(\xi) = \xi$ . We refer to [1], [12] for more details.

The discretization of (1.1) using the  $h$ -version of the finite element method (fem) leads to a linear system of algebraic equations

$$\mathcal{K}\underline{u} = \underline{f}. \quad (1.2)$$

It is well known from the literature that preconditioned conjugate gradient-methods (pcg-methods) with domain decomposition preconditioners are among the most efficient iterative solvers for systems of the type (1.2), see e.g. [7], [14], [10] [20], [15]. In this paper, we will propose and analyze overlapping Domain Decomposition (DD) preconditioners.

The type of overlapping DD-preconditioners presented in this paper is originally developed for problems with jumping coefficients in [17], see also the recent research for highly jumping coefficients in [19], [9]. In our recent paper [3], we analyzed these overlapping DD preconditioners for isotropic degenerated problems. In most cases, the optimality of this method has been shown. Here, we adapt these techniques to problem (1.1). For tensor product discretizations, we will prove the optimality of the method. Moreover, this method can easily be extended to more general  $h$ -version fem discretizations, too.

Only a limited number of papers have investigated fast solvers for degenerated elliptic problems. The paper [6] deals with the Laplacian in  $2D$  in polar coordinates. In the paper [8], multigrid methods for some other types of degenerated problems are proposed. Multigrid solvers for FE-discretizations of (1.1) have been investigated in [1], see also [5]. However, the convergence of the  $V$ -cycle was not yet proved. A similar approach can be found in [12]. The paper [4] proposes wavelet methods for several classes of degenerated elliptic problems on the unit square. One of them is problem (1.1) under the restriction  $\lim_{\xi \rightarrow 0+} \frac{\xi^3}{\omega_i^2(\xi)} = 0$ ,  $i = 1, 2$ , to the weight functions. Moreover, a fast direct solver based on eigenvalue computations combined with fast Fourier transform and solving tridiagonal systems can be designed if at least one of the weight functions  $\omega_i$ ,  $i = 1, 2$ , is assumed to be constant on  $(0, 1)$  and if a tensor product discretization is used.

The remaining part of this paper is organized as follows. In Section 2, we introduce the reader into our problem and into our notation. The preconditioners are defined in Section 3. Moreover, the main theorems with the condition number estimates are stated. In Section 4, we formulate some auxiliary results from the Additive Schwarz Method (ASM), which are required for the proofs of our main theorems given in Section 5. In Section 6, we present some numerical experiments which show the performance of the presented methods.

Throughout this paper, the integer  $k$  denotes the level number. For two real symmetric and positive definite  $n \times n$  matrices  $A, B$ , the relation  $A \preceq B$  means that  $A - cB$  is negative definite, where  $c > 0$  is a constant independent of  $n$ . The relation  $A \sim B$  means  $A \preceq B$  and  $B \preceq A$ , i.e. the matrices  $A$  and  $B$  are spectrally equivalent. The parameter  $c$  denotes a generic constant. The isomorphism between a function  $u = \sum_i u_i \psi_i \in L^2$  and the corresponding vector of coefficients  $\underline{u} = [u_i]_i$  in the basis  $[\Psi] = [\psi_1, \psi_2, \dots]$  is denoted by  $u = [\Psi]\underline{u}$ .

## 2 Setting of the problem

In this paper, we investigate the following boundary value problem: Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain. Find  $u \in \mathbb{H}_{\mathcal{D},0} := \{u \in L_2(\Omega) : \int_{\Omega} (\nabla u)^T \mathcal{D} \nabla u < \infty, u|_{\partial\Omega} = 0\}$  such that

$$a(u, v) := \int_{\Omega} (\nabla v)^T \mathcal{D} \nabla u = (f, v) \quad \forall v \in \mathbb{H}_{\mathcal{D},0} \quad (2.1)$$

with the coefficient matrix  $\mathcal{D}(x, y) = \begin{bmatrix} \omega_1^2(y) & 0 \\ 0 & \omega_2^2(x) \end{bmatrix}$  and the weight functions  $\omega_i$ ,  $i = 1, 2$ , which satisfy

**Assumption 2.1.** *The functions  $\omega_i : [0, 1] \mapsto \mathbb{R}$ ,  $i = 1, 2$*

- *are monotonic increasing,*
- *are continuous, and*
- *satisfy the estimate*

$$\omega_i(2\xi) \leq c_\omega \omega_i(\xi) \quad \forall \xi \in \left(0, \frac{1}{2}\right] \quad (2.2)$$

with some constants  $c_\omega > 0$ .

We discretize problem (2.1) by piecewise linear finite elements on the regular Cartesian grid consisting of congruent, isosceles, right-angled triangles. For this purpose, some notation is introduced. Let  $k$  be the level of approximation and  $n = 2^k$ . Let  $x_{ij}^k = (\frac{i}{n}, \frac{j}{n})$ , where  $i, j = 0, \dots, n$ . The domain  $\Omega$  is divided into congruent, isosceles, right-angled triangles  $\tau_{ij}^{s,k}$ , where  $0 \leq i, j < n$  and  $s = 1, 2$ , see Figure 1. The triangle  $\tau_{ij}^{1,k}$  has the three vertices  $x_{ij}^k, x_{i+1,j+1}^k$  and  $x_{i,j+1}^k$ ,  $\tau_{ij}^{2,k}$  has the three vertices  $x_{ij}^k, x_{i+1,j+1}^k$  and  $x_{i+1,j}^k$ , see Figure 1. Piecewise linear finite elements are used on the mesh  $T_k = \{\tau_{ij}^{s,k}\}_{i=0,j=0,s=1}^{n-1,n-1,2}$ . The

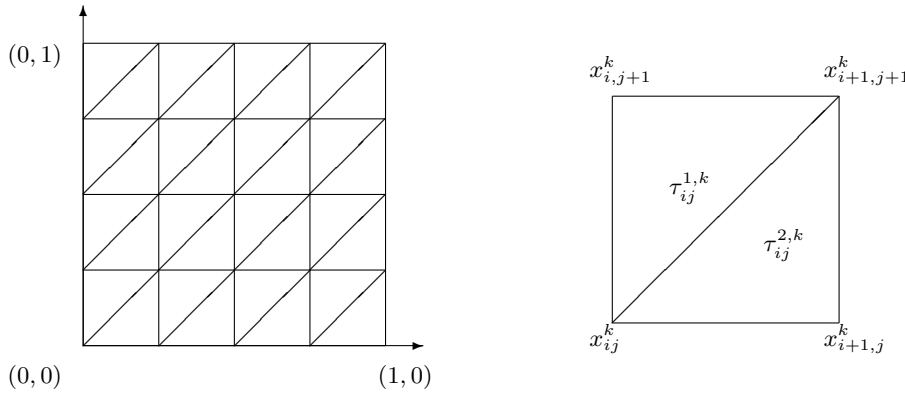


Figure 1: Mesh for the finite element method (left), Notation (right).

subspace of piecewise linear functions  $\phi_{ij}^k$  with

$$\phi_{ij}^k \in H_0^1(\Omega), \quad \phi_{ij}^k |_{\tau_{lm}^{s,k}} \in \mathbb{P}_1(\tau_{lm}^{s,k})$$

is denoted by  $\mathbb{V}_k$ , where  $\mathbb{P}_1$  is the space of polynomials of degree  $\leq 1$ . A basis of  $\mathbb{V}_k$  is the system of the usual hat-functions  $\Phi_k = \{\phi_{ij}^k\}_{i,j=1}^{n-1}$  uniquely defined by

$$\phi_{ij}^k(x_{lm}^k) = \delta_{il}\delta_{jm}$$

and  $\phi_{ij}^k \in \mathbb{V}_k$ , where  $\delta_{il}$  is the Kronecker delta. Now, we can formulate the discretized problem.

Find  $u^k \in \mathbb{V}_k$  such that

$$a(u^k, v^k) = (f, v^k) \quad \forall v^k \in \mathbb{V}_k \quad (2.3)$$

holds. Problem (2.3) is equivalent to solving the system of linear algebraic equations

$$K_k \underline{u}_k = \underline{f}_k, \quad (2.4)$$

where  $K_k = [a(\phi_{ij}^k, \phi_{lm}^k)]_{i,j,l,m=1}^{n-1}$ ,  $\underline{u}_k = [u_{ij}]_{i,j=1}^{n-1}$  and  $\underline{f}_k = [(f, \phi_{lm}^k)]_{l,m=1}^{n-1}$ . The size of the matrix  $K_k$  is  $N \times N$  with  $N = (n-1)^2$ .

### 3 Definition of the preconditioners

In this section, we define the overlapping preconditioners for the matrix  $K_k$  (2.3). We distinguish between two cases,

- the weight function  $\omega_1$  is assumed to be constant,
- both weight functions satisfy  $\omega_i(0) = 0$ ,  $i = 1, 2$ .

#### 3.1 The case $\omega_1(\xi) = 1$

We introduce the following notation. Let

- $\Omega_{i,x} = \{(x, y) \in \mathbb{R}^2, 2^{-i-1} < x < 2^{-i}, 0 < y < 1\}$ ,  $i = 0, \dots, k-2$ ,
- $\Omega_{k-1,x} = \{(x, y) \in \mathbb{R}^2, 0 < x < 2^{-k+1}, 0 < y < 1\}$ ,
- $\Gamma_{i,x} = \{(x, y) \in \mathbb{R}^2, x = 2^{-i}, 0 < y < 1\}$ ,  $i = 1, \dots, k-1$ ,
- $\tilde{\Omega}_{j,x} = \text{int} \left( \bigcup_{i=j}^{k-1} \overline{\Omega_{i,x}} \right)$ , and
- $n_j = 2^{k-j} - 1$  be the number of interior grid points in  $\tilde{\Omega}_{j,x}$  in  $x$ -direction and  $N_j = (n-1)n_j$  be the total number of interior grid points.
- Moreover, let

$$\varepsilon_{2,j} = (\omega_2(2^{-j}))^2.$$

Figure 2 displays a sketch with the notation for  $k = 4$ .

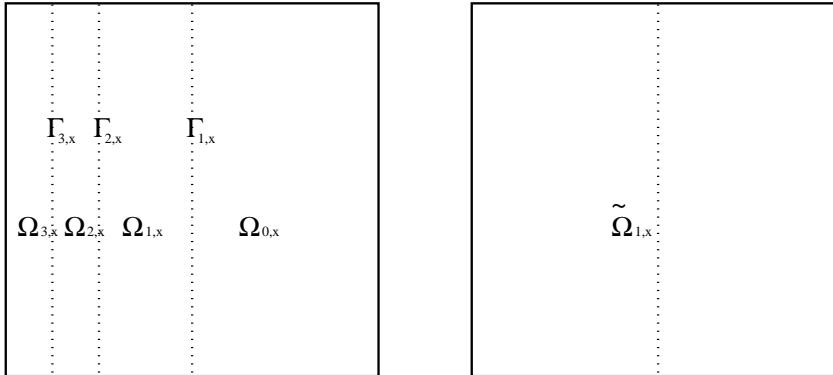


Figure 2: Notation for  $k = 4$ .

With this notation, we introduce the bilinear form

$$a_{j,B}(u, v) = \int_{\tilde{\Omega}_{j,x}} (\nabla u)^T \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_{2,j} \end{bmatrix} \nabla v, \quad j = 0, \dots, k-1.$$

This is a bilinear form with constant coefficients on  $\tilde{\Omega}_{j,x}$ . Let  $C_{j,B,D}$  be the stiffness matrix

$$C_{j,B,D} = [a_{j,B}(\phi_{ii'}, \phi_{ll'})]_{i,l=1;i',l'=1}^{n_j, n_0}, \quad j = 0, \dots, k-1$$

according to the bilinear form  $a_{j,B}(\cdot, \cdot)$ . Finally, let

$$\Theta_{j,D} = \begin{bmatrix} C_{j,B,D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-N_j} \end{bmatrix} \in \mathbb{R}^{N \times N} \quad (3.1)$$

be the corresponding global assembled stiffness matrix.

Then, we define

$$C_B^{-1} = \sum_{j=0}^{k-1} \Theta_{j,D}^+ \quad (3.2)$$

as a first preconditioner for  $K_k$ , where  $B^+$  denoted the pseudo inverse of a matrix  $B$ . Note that the locally anisotropic diffusion matrix  $\mathcal{D}(x, y)$  is hidden in the matrix  $\Theta_{j,D}$ .

This preconditioner turns out not to be optimal, see Theorem 3.2. To develop an optimal preconditioner, we have to modify  $C_B$ . Therefore, let

$$\hat{C}_{j,B,D} = \left[ \int_{\overline{\Omega_{j+1,x}} \cup \overline{\Omega_{j,x}}} \nabla \phi_{ii'}^T \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_{2,j} \end{bmatrix} \nabla \phi_{ll'} \right]_{i,l=n_{j+2}; i',l'=1}^{n_j, n_0}.$$

This is the discretized operator on  $\overline{\Omega_{j+1,x}} \cup \overline{\Omega_{j,x}}$  with Dirichlet boundary conditions at all edges. Moreover, let

$$\hat{\Theta}_{j,B,D} = \begin{bmatrix} \mathbf{0}_{N_{j+2}+n_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{C}_{j,B,D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{N-N_j} \end{bmatrix} \in \mathbb{R}^{N \times N}, \quad j = 0, \dots, k-2.$$

be the corresponding assembled matrix.

The second overlapping preconditioner for  $K_k$  is defined as

$$C_{mod,B}^{-1} = \sum_{j=0}^{k-2} \hat{\Theta}_{j,B,D}^+ + \Theta_{k-1,D}^+. \quad (3.3)$$

Then, we can formulate the following

**Theorem 3.1.** *Let  $C_{mod,B}$  be defined via (3.3) and let  $\mathcal{D}(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & \omega_2^2(x) \end{bmatrix}$ . Then, the matrix  $C_{mod,B}^{-1}$  is symmetric positive definite and satisfies  $K_k \sim C_{mod,B}$ .*

*Proof.* A detailed proof is given in subsection 5.2. □

Concerning the first preconditioner (3.2), we can prove now the following result.

**Theorem 3.2.** *Let  $C_B$  be defined via (3.2) and let  $\mathcal{D}(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & \omega_2^2(x) \end{bmatrix}$ . Then, the spectral equivalence relations  $\frac{1}{k} K_k \leq C_B \preceq K_k$  hold.*

*Proof.* The lower estimate is trivial. To prove the upper estimate, we use Theorem 3.1 and obtain

$$C_B \leq C_{mod,B} \preceq K_k,$$

which proves the result. □

### 3.2 The general case

In addition to the notation of subsection 3.1, we define

- $\Omega_{i,y} = \{(x, y) \in \mathbb{R}^2, 0 < x < 1, 2^{-1-i} < y < 2^{-i}\}, i = 0, \dots, k-1,$
- $\Omega_{k,y} = \{(x, y) \in \mathbb{R}^2, 0 < x < 1, 0 < y < 2^{-k}, 0 < y < 1\},$  and
- $\tilde{\Omega}_{j,y} = \text{int} \left( \bigcup_{i=j}^k \overline{\Omega_{i,y}} \right).$
- Moreover, let

$$\varepsilon_{1,j} = (\omega_1 (2^{-j}))^2.$$

Here, we introduce the bilinear form

$$a_{j,j',C}(u, v) = \int_{\tilde{\Omega}_{j,x} \cap \tilde{\Omega}_{j',y}} \nabla u \begin{bmatrix} \varepsilon_{1,j'} & 0 \\ 0 & \varepsilon_{2,j} \end{bmatrix} \nabla v, \quad j', j = 0, \dots, k-1.$$

For  $j = 0, \dots, k-1$ , let  $C_{j,j',C}$  be the stiffness matrices

$$C_{j,j',C} = [a_{j,j',C}(\phi_{ii'}, \phi_{ll'})]_{i,l=1; i',l'=1}^{n_j, n_{j'}},$$

according to the bilinear form  $a_{j,j',C}(\cdot, \cdot)$  with Dirichlet boundary conditions at all edges.

The corresponding global assembled stiffness matrices is denoted by the matrix  $\Upsilon_{j,j',C} \in \mathbb{R}^{N \times N}$ . Then, we define

$$C_C^{-1} = \sum_{j=0}^{k-1} \sum_{j'=0}^{k-1} \Upsilon_{j,j',C}^+ \quad (3.4)$$

as a first preconditioner for  $K_k$ . Note that the locally anisotropic diffusion matrix  $\mathcal{D}(x, y)$  is hidden in the matrix  $\Upsilon_{j,j',C}$ . This gives us a non-optimal preconditioner, see Theorem 3.4. Moreover, we introduce an optimal preconditioner. Therefore, let

$$\hat{C}_{j,j',C} = \left[ \int_{(\tilde{\Omega}_{j,x} \cup \tilde{\Omega}_{j+1,x}) \cap (\tilde{\Omega}_{j',y} \cup \tilde{\Omega}_{j'+1,y})} \nabla \phi_{ii'}^T \begin{bmatrix} \varepsilon_{1,j'} & 0 \\ 0 & \varepsilon_{2,j} \end{bmatrix} \nabla \phi_{ll'} \right]_{i,l=n_{j+2}+2; i',l'=n_{j'+2}+2}^{n_j, n_{j'}}, \quad (3.5)$$

$$0 \leq j, j' \leq k-2.$$

This matrix is the finite element discretization matrix of an operator with piecewise constant coefficients on  $(\tilde{\Omega}_{j+1,x} \cup \tilde{\Omega}_{j,x}) \cap (\tilde{\Omega}_{j'+1,y} \cup \tilde{\Omega}_{j',y})$  and Dirichlet boundary conditions at all edges. For  $j, j' \leq k-2$ , the corresponding global assembled stiffness matrices are denoted by the matrices  $\hat{\Upsilon}_{j,j',C} \in \mathbb{R}^{N \times N}$ . If  $j = k-1$  or  $j' = k-1$ , we set

$$\hat{\Upsilon}_{j,j',C} = \Upsilon_{j,j',C}.$$

The second overlapping preconditioner for  $K_k$  is defined as

$$C_{mod,C}^{-1} = \sum_{j=0}^{k-1} \sum_{j'=0}^{k-1} \hat{\Upsilon}_{j,j',C}. \quad (3.6)$$

**Theorem 3.3.** Let  $C_{mod,C}$  be defined via (3.6) and let  $\mathcal{D}(x, y) = \begin{bmatrix} \omega_1^2(y) & 0 \\ 0 & \omega_2^2(x) \end{bmatrix}$ . Then, the matrix

$C_{mod,C}^{-1}$  is symmetric positive definite and satisfies  $K_k \sim C_{mod,C}$ .

*Proof.* The proof is given in subsection 5.3. □

Concerning the first preconditioner (3.4), we can prove the following result.

**Theorem 3.4.** *Let  $C_C$  be defined via (3.4) and let  $\mathcal{D}(x, y) = \begin{bmatrix} \omega_1^2(y) & 0 \\ 0 & \omega_2^2(x) \end{bmatrix}$ . Then, the spectral equivalence relations  $\frac{1}{k^2}K_k \leq C_C \leq K_k$  hold.*

*Proof.* The lower estimate is trivial. To prove the upper estimate, we use Theorem 3.3 and obtain

$$C_C \leq C_{mod,C} \leq K_k,$$

which proves the result.  $\square$

**Remark 3.5.** *We can replace the  $\hat{C}_{j,j',C}$  in (3.5) by*

$$\tilde{C}_{j,j',C} = \left[ \int_{(\bar{\Omega}_{j,x} \cup \bar{\Omega}_{j+1,x}) \cap (\bar{\Omega}_{j',y} \cup \bar{\Omega}_{j'+1,y})} \nabla \phi_{ii'}^T \begin{bmatrix} \omega_1^2(y) & 0 \\ 0 & \omega_2^2(x) \end{bmatrix} \nabla \phi_{ll'} \right]_{i,l=n_{j+2}+2; i',l'=n_{j'+2}+2}^{n_j, n_{j'}}.$$

Let  $\tilde{\Upsilon}_{j,j',C}$  be the assembled matrices and

$$C_{var,C}^{-1} = \sum_{j=0}^{k-1} \sum_{j'=0}^{k-1} \tilde{\Upsilon}_{j,j',C}^+.$$

Due to (2.2), we have  $\tilde{C}_{j,j',C} \sim \hat{C}_{j,j',C}$  which gives  $C_{mod,C} \sim C_{var,C}$ . In the preconditioner  $C_{var,C}$ , we have now an operator with variable coefficients. However, the constants do not change too much since we

have the estimate  $\sup_{x_1, x_2 \in (\bar{\Omega}_{j,x} \cup \bar{\Omega}_{j+1,x})} \frac{\omega_2^2(x_1)}{\omega_2^2(x_2)} \leq c_\omega^4$  from our assumption (2.2).

### 3.3 Computational aspects

In this subsection, we investigate the preconditioning operation  $C^{-1}\underline{u}$  for the preconditioners (3.2)-(3.6).

We present algorithms to perform this preconditioning operation in optimal arithmetical complexity.

Let us start with the case  $\omega_1(\xi) = 1$ . Here, we have developed the preconditioners

$$C_B^{-1} = \sum_{j=0}^{k-1} \Theta_{j,D}^+,$$

see (3.2) and

$$C_{mod,B}^{-1} = \sum_{j=0}^{k-2} \hat{\Theta}_{j,B,D}^+ + \Theta_{k-1}^+,$$

see (3.3). In both cases, we have to solve systems of linear algebraic equations with the discretization of an operator with constant coefficients on a rectangle using triangular finite elements. The corresponding operators are now

$$-\varepsilon_j^2 u_{xx} - u_{yy} \tag{3.7}$$

with some numbers  $0 < \varepsilon_j \leq 1$ . The computational domains are displayed in Figures 3 and 4.

These domains are the same as for the preconditioners  $C$  and  $C_{mod}$  of [3].

Using multigrid preconditioners combined with a line smoother, optimal solvers for  $\Theta_{j,D}$  and  $\hat{\Theta}_{j,B,D}$  can easily be designed, see [11]. The line smoother is necessary to remove the anisotropy of the operator. It can be shown that the multigrid-preconditioner with line smoother and V-cycle is an optimal method



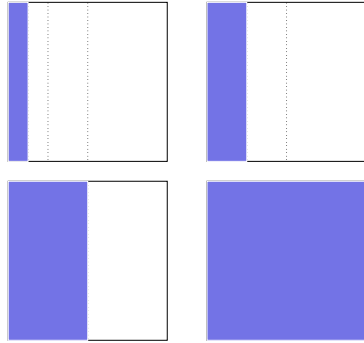


Figure 3: Computational domains for  $C_B$  (3.2):  $\Theta_{3,D}$  and  $\Theta_{2,D}$  above,  $\Theta_{1,D}$  and  $\Theta_{0,D}$  below.

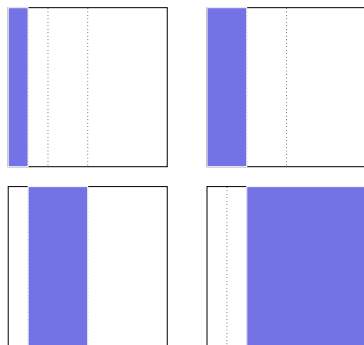


Figure 4: Computational domains for  $C_{mod,B}$  (3.3):  $\Theta_{3,D}$  and  $\hat{\Theta}_{2,D}$  above,  $\hat{\Theta}_{1,D}$  and  $\hat{\Theta}_{0,D}$  below.

independent of the parameter  $\varepsilon_j$ , [11]. With the same arguments as in the isotropic case, see [3], we can prove that the cost for the operations  $\underline{w} = C_B^{-1}\underline{r}$  and  $\underline{w} = C_{mod,B}^{-1}\underline{r}$  depends linearly on the number of unknowns.

In the general case, the application of the preconditioning operations  $C_C^{-1}\underline{r}$  (3.4) and  $C_{mod,C}^{-1}\underline{r}$  (3.6) implies again the solution of systems of linear algebraic equations with discretizations of operators of the type (3.7). However, these systems have to be solved on the smaller subdomains  $(\bar{\Omega}_{j,x} \cup \bar{\Omega}_{j+1,x}) \cap (\bar{\Omega}_{j',y} \cup \bar{\Omega}_{j'+1,y})$ , see Figure 5 for  $k = 4$ . The structure of the diffusion matrices are displayed below each of the 16 pictures for the weight functions  $\omega_i^2(\xi) = \xi^2$ ,  $i = 1, 2$ . The diffusion matrices are isotropic for  $j \approx j'$  and globally anisotropic elsewhere. Therefore, a multigrid algorithm or multigrid preconditioner with line smoother should be used as solution method for  $|j - j'| \gg 1$ . Similar as for the preconditioners  $C_B$  (3.2) and

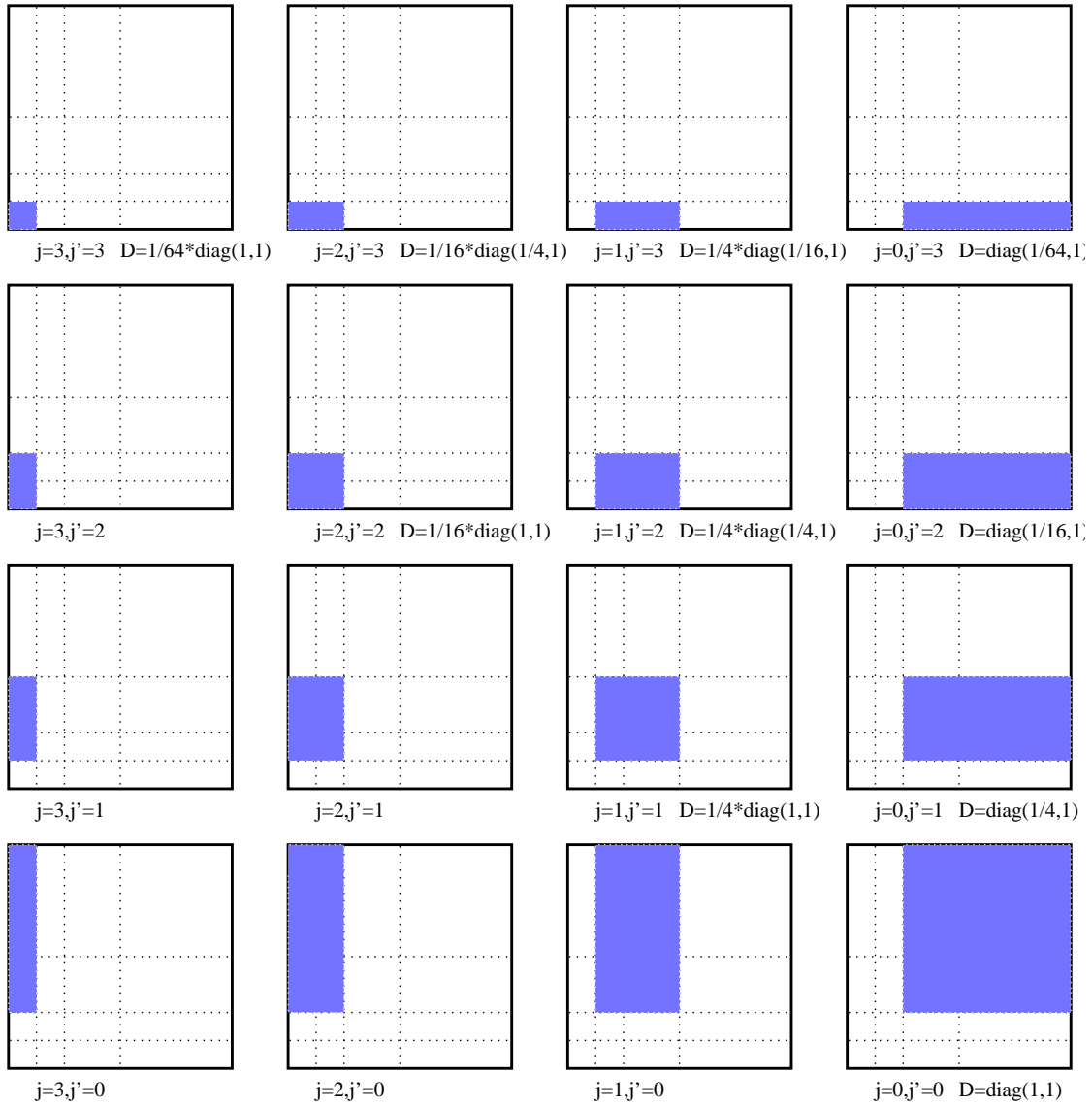


Figure 5: Corresponding domains and diffusion matrices with weight function  $\omega_i^2(\xi) = \xi^2$  for  $\hat{\Upsilon}_{j,j'}$ ,  $i = 1, 2$ .

$C_{mod,B}$  (3.3), the optimality of the preconditioning operations  $C_C^{-1}r$  and  $C_{mod,C}^{-1}r$  can be shown. Summarizing we have to solve now globally anisotropic problems with constant coefficients instead of locally anisotropic problems with changing directions of the anisotropy. This is much simpler than the original problem since solvers for the problem with constant coefficients are known in the literature. However, this method cannot remove the anisotropic behavior of the problem.

## 4 Some Preliminaries

In this section, we formulate some auxiliary results from ASM which are necessary to prove our main results. The proofs can be found in the literature.

### 4.1 Preliminaries from ASM

The first result is a general result for preconditioned ASM.

**Lemma 4.1.** *Let  $\mathbb{H}$  be a Hilbert space with the scalar product  $(\cdot, \cdot)$ . Moreover, let  $\mathbb{H}_i$ ,  $i = 1, \dots, m$  be subspaces of  $\mathbb{H}$  such that*

$$\mathbb{H} = \mathbb{H}_1 + \mathbb{H}_2 + \dots + \mathbb{H}_m.$$

*Let  $\mathcal{A} : \mathbb{H} \mapsto \mathbb{H}$  be a linear, self adjoint, bounded and positive definite operator and let*

$$(u, v)_{\mathcal{A}} = (\mathcal{A}u, v) \quad \forall u, v \in \mathbb{H}.$$

*We denote by  $P_i$ ,  $i = 1, \dots, m$ , the orthogonal projection operators from  $\mathbb{H}$  onto  $\mathbb{H}_i$  with respect to the scalar product  $(\cdot, \cdot)_{\mathcal{A}}$ . We assume that for any  $u \in \mathbb{H}$  there exists a decomposition  $u = u_1 + \dots + u_m$  such that*

$$c_1 \sum_{i=1}^m (u_i, u_i)_{\mathcal{A}} \leq (u, u)_{\mathcal{A}} \quad (4.1)$$

*with a positive constant  $c_1$ . Moreover, let  $c_2$  some positive constant such that*

$$\sum_{i=1}^m (P_i u, u)_{\mathcal{A}} \leq c_2 (u, u)_{\mathcal{A}} \quad \forall u \in \mathbb{H}. \quad (4.2)$$

*Also, let  $\mathcal{B}_i : \mathbb{H}_i \mapsto \mathbb{H}_i$ ,  $i = 1, \dots, m$  be some selfadjoint and surjective operators such that*

$$c_3 (\mathcal{B}_i u_i, u_i) \leq (\mathcal{A} P_i u_i, P_i u_i) \leq c_4 (\mathcal{B}_i u_i, u_i), \quad \forall u_i \in \mathbb{H}_i, \quad i = 1, \dots, m. \quad (4.3)$$

*Let  $\mathcal{B}^{-1} = \mathcal{B}_1^+ + \dots + \mathcal{B}_m^+$ , where  $\mathcal{B}_i^+$  denotes the pseudo-inverse operator of  $\mathcal{B}_i$ . Then,*

$$c_1 c_3 (\mathcal{A}^{-1} u, u) \leq (\mathcal{B}^{-1} u, u) \leq c_2 c_4 (\mathcal{A}^{-1} u, u) \quad \forall u \in \mathbb{H}.$$

*Proof.* The proof can be found in [16]. □

The second result is a technical result for some overlapping preconditioners, in which the domain is split into stripes as displayed in Figure 2. Before, we introduce some notation which is similar to the notation in Figure 2.

- Let

$$\bar{\Omega} = \bigcup_{j=0}^{k-1} \bar{\Omega}_j$$

be a domain  $\Omega$  which is decomposed into stripes  $\Omega_i$ , i.e.

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j, \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \begin{cases} \Gamma_i & i = j + 1 \\ \Gamma_j & i = j - 1 \\ \bar{\Omega}_i & i = j \\ \emptyset & |i - j| \geq 2 \end{cases}$$

and let  $\bar{\Omega}_{k-1} \cap \partial\Omega = \Gamma_k$ .

- Let  $\tau_k$  be a triangulation of  $\Omega$  which is admissible to the decomposition of  $\Omega$  into  $\Omega_i$ .
- Let  $\Phi_k = [\phi_i]_{i=1}^N$  be the basis of hat functions according to the triangulation  $\tau_k$  and let  $\mathbb{V}_k = \text{span}\Phi_k$  be the corresponding finite element space.
- Let  $a(\cdot, \cdot) : \mathbb{V}_k \times \mathbb{V}_k \mapsto \mathbb{R}$  be a symmetric and positive definite bilinear form and let

$$\|u\|_{a,\Omega} = a(u, u)$$

be the energetic norm. In the same way, let

$$\|u\|_{a,\bar{\Omega}} = a|_{\bar{\Omega}}(u, u)$$

be the restriction of the norm onto some subdomain  $\tilde{\Omega} \subset \Omega$ .

- For  $j = 0, \dots, k-2$ , let  $\mathbb{Y}_j = \{u \in \mathbb{V}_k : \text{supp } u \subset \bar{\Omega}_j \cup \bar{\Omega}_{j+1}\}$  be the restriction of the finite element space  $\mathbb{V}_k$  onto  $\bar{\Omega}_j \cup \bar{\Omega}_{j+1}$  with Dirichlet boundary conditions at the boundaries  $\Gamma_j$  and  $\Gamma_{j+2}$ . For  $j = k-1$ , we set  $\mathbb{Y}_{k-1} = \{u \in \mathbb{V}_k : \text{supp } u \subset \bar{\Omega}_{k-1}\}$ .
- Let

$$\|w\|_{\Gamma_j, \text{left}}^2 = \min_{\substack{u \in \mathbb{V}_k \\ u|_{\Gamma_j} = w \\ u|_{\Gamma_{j+1}} = 0}} \|u\|_{a,\Omega_j}^2 \quad \text{and} \quad \|w\|_{\Gamma_j, \text{right}}^2 = \min_{\substack{u \in \mathbb{V}_k \\ u|_{\Gamma_j} = w \\ u|_{\Gamma_{j-1}} = 0}} \|u\|_{a,\Omega_{j-1}}^2 \quad (4.4)$$

be the left and right trace norm on  $\Gamma_j$ .

- Let  $\mathcal{T}_{j, \text{left}} : \mathbb{V}_k|_{\Gamma_j} \mapsto \mathbb{V}_k|_{\Omega_j}$  and  $\mathcal{T}_{j, \text{right}} : \mathbb{V}_k|_{\Gamma_j} \mapsto \mathbb{V}_k|_{\Omega_{j-1}}$  be the minimal energetic extension operators from  $\Gamma_j$  to  $\Omega_j$  and  $\Omega_{j-1}$ , i.e.

$$\|w\|_{\Gamma_j, \text{left}} = \|\mathcal{T}_{j, \text{left}} w\|_{a,\Omega_j} \quad \text{and} \quad \|w\|_{\Gamma_j, \text{right}} = \|\mathcal{T}_{j, \text{right}} w\|_{a,\Omega_{j-1}}.$$

**Theorem 4.2.** *In addition to the above assumptions, let us assume the following: There exists an integer  $j_0$  such that:*

- *There exists a constant  $\gamma < 1$  which is independent of the discretization parameter and  $j$  such that*

$$a(\mathcal{T}_{j, \text{left}} u, \mathcal{T}_{j+1, \text{right}} v) \leq \gamma \|u\|_{\Gamma_j, \text{left}} \|v\|_{\Gamma_{j+1}, \text{right}} \quad (4.5)$$

$$\forall j = 0, \dots, j_0, \quad \forall u \in \mathbb{Y}_j|_{\Gamma_j}, \quad \forall v \in \mathbb{Y}_{j+1}|_{\Gamma_{j+1}}.$$

- *There exists a constant  $q_0 < 1$  and a constant  $c_2$  which are independent of  $j$  and the discretization parameter such that*

$$q_0^{-1} \|w\|_{\Gamma_j, \text{left}}^2 \leq \|w\|_{\Gamma_j, \text{right}}^2 \leq c_2 \|w\|_{\Gamma_j, \text{left}}^2 \quad \forall w, \quad j = j_0. \quad (4.6)$$

- There exists a constant  $c_1$  which is independent of discretization parameter such that

$$c_1^{-1} \|w\|_{\Gamma_j, \text{left}}^2 \leq \|w\|_{\Gamma_j, \text{right}}^2 \leq c_2 \|w\|_{\Gamma_j, \text{left}}^2 \quad \forall w, \quad \forall j = j_0 + 1, \dots, k-1. \quad (4.7)$$

Then, there exists a decomposition  $u = \sum_{j=0}^{k-1} u_j$  with  $u_j \in \mathbb{Y}_j$  such that

$$c_L^2 \sum_{j=0}^{k-1} a(u_j, u_j) \leq a(u, u) \quad \forall u \in \mathbb{V}_k.$$

The constant  $c_L > 0$  depends only on  $\gamma, c_1, c_2$  and  $q_0$ . Moreover, for all decompositions of  $u = \sum_{j=0}^{k-1} u_j$  with  $u_j \in \mathbb{Y}_j$ , the estimate

$$a(u, u) \leq 2 \sum_{j=0}^{k-1} a(u_j, u_j) \quad \forall u \in \mathbb{V}_k$$

holds.

*Proof.* The proof can be found in [3]. □

Next, we construct a bilinear form  $a_p(\cdot, \cdot)$  with piecewise constant coefficients which is spectrally equivalent to the original bilinear form  $a(\cdot, \cdot)$ , cf. (2.1). This idea has originally been developed in [12]. For  $i = 1, 2$ , let

$$\chi_i^2(\xi) = \varepsilon_{i,j}, \quad \xi \in (2^{-j-1}, 2^{-j}) \quad \text{with} \quad \varepsilon_{i,j} := \omega_i^2(2^{-j}). \quad (4.8)$$

be an piecewise constant coefficient function and

$$a_p(u, v) := \int_{\Omega} (\nabla v)^T \begin{bmatrix} \chi_1^2(y) & 0 \\ 0 & \chi_2^2(x) \end{bmatrix} \nabla u \quad (4.9)$$

be the corresponding bilinear form. Moreover, we define the energetic norm

$$\|u\|_p^2 := a_p(u, u) \quad \forall u \in \mathbb{V}_k \quad (4.10)$$

with respect to the bilinear form  $a_p(\cdot, \cdot)$ . The stiffness matrix with respect to the basis  $\Phi_k$  is denoted by  $K_{k,p}$ , i.e.

$$K_{k,p} = [a_p(\phi_{il}, \phi_{i'l'})]_{(i,i'), (l,l')=(1,1)}^{n_0, n_0}. \quad (4.11)$$

**Lemma 4.3.** *Let us assume that the weight functions  $\omega_i, i = 1, 2$ , satisfy Assumption 2.1. Then, we have*

$$a(u, u) \leq a_p(u, u) \leq 2c_\omega^2 a(u, u) \quad \forall u \in \mathbb{V}_k.$$

The constant  $c_\omega$  is from (2.2).

*Proof.* The proof is similar to the proof of Lemma 4.3 in [3]. □

**Remark 4.4.** • In the case of the weight function  $\omega_i^2(\xi) = \xi^\alpha$  with  $\alpha > 0$ , we have  $c_\omega^2 = 2^\alpha$ .

- A direct consequence of (2.4), (4.11) and Lemma 4.3 is the spectral equivalence estimate

$$\frac{1}{2} c_\omega^{-2} K_{k,p} \leq K_k \leq K_{k,p}.$$

## 4.2 Some estimates for tridiagonal matrices

Finally, some estimates for tridiagonal matrices with constant main- and subdiagonals are required. For a fixed integer  $m$  and some positive parameter  $\kappa$ , we introduce

$$F_m = \begin{bmatrix} 2 + \kappa & -1 & & & \mathbf{0} \\ -1 & 2 + \kappa & -1 & & \\ & & \ddots & \ddots & \ddots \\ \mathbf{0} & & & -1 & 2 + \kappa & -1 \\ & & & & -1 & 2 + \kappa \end{bmatrix} \in \mathbb{R}^{m-1 \times m-1}$$

$$\tilde{F}_m = \begin{bmatrix} 1 + \frac{\kappa}{2} & e_1^T \\ e_1 & F_m \end{bmatrix} \in \mathbb{R}^{m \times m}, e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^{m-1 \times 1} \quad (4.12)$$

and the real number

$$q = 1 + \frac{\kappa}{2} + \frac{1}{2}\sqrt{\kappa(\kappa + 4)}. \quad (4.13)$$

We prove the following lemma.

**Lemma 4.5.** *Let  $F_m$  and  $\tilde{F}_m$  be defined via (4.12). Then, the following assertions are valid:*

- *The determinant of  $F_m$  satisfies the equation*

$$\det F_{m+1} = \frac{q^{m+2} - q^{-m}}{q^2 - 1} = q^{-m} \sum_{i=0}^m q^{2i}. \quad (4.14)$$

- *Let  $s_m = 1 + \frac{\kappa}{2} - e_1^T F_m^{-1} e_1$  be the Schur complement of  $\tilde{F}_m$  with respect to the first row and column. Then,*

$$s_m = \frac{\kappa}{2} + \frac{1 + q^{2m-1}}{1 + q + \dots + q^{2m-1}}. \quad (4.15)$$

Moreover, the estimate

$$\frac{\kappa}{2} + \frac{1}{m} \leq s_m \leq \frac{\kappa}{2} + \frac{1}{m} \frac{1 + q^{2m-1}}{2\sqrt{q^{2m-1}}} \quad (4.16)$$

holds.

- *Let  $\hat{s}_m = e_1^T F_m^{-1} e_m$ ,  $e_m = (0, \dots, 0, 1)^T$ . Then,*

$$|\hat{s}_m| = \frac{1}{\det F_{m-1}}. \quad (4.17)$$

- *Let  $\gamma_m = \frac{|\hat{s}_m|}{s_m}$ . Then,*

$$\gamma_m \leq \frac{2}{q^{m-1}}. \quad (4.18)$$

*Proof.* Relation (4.14) is a consequence of the following recursion:

$$\begin{aligned} \det F_m &= (2 + \kappa)\det F_{m-1} - \det F_{m-2}, \\ \det F_0 &= 1, \quad \det F_1 = 2 + \kappa. \end{aligned}$$

The solution of this linear recursion of second order gives the first assertion. The second relation follows from the first one by the geometric series

$$1 + q^2 + \dots + q^{2m} = \frac{q^{2m+2} - 1}{q^2 - 1}.$$

To prove relation (4.15), we compute the Schur complement by using Cramers rule and (4.14) explicitly. Since

$$e_1^T F_M^{-1} e_1 = (F_m^{-1})_{(1,1)} = \frac{\det F_{m-1}}{\det F_m},$$

we conclude

$$s_m = \frac{\kappa}{2} + \frac{\det F_m - \det F_{m-1}}{\det F_m}.$$

We simplify the second summand with (4.14) and obtain

$$\det F_m - \det F_{m-1} = q^{-m+1} \sum_{i=0}^{2m-2} (-q)^i = q^{-m+1} \frac{q^{2m-1} + 1}{1 + q}. \quad (4.19)$$

Hence,

$$\frac{\det F_m - \det F_{m-1}}{\det F_m} = \frac{1 + q^{2m-1}}{1 + q + \dots + q^{2m-1}}, \quad (4.20)$$

which proves (4.15). To prove (4.16), we start from (4.15) and use the convexity of the function  $f : (1, \infty) \mapsto \mathbb{R}$  given by  $f(x) = q^x$  for  $q > 1$ . Then, we have

$$\begin{aligned} 1 + q^{2m-1} &\geq q + q^{2m-2}, \\ 1 + q^{2m-1} &\geq q^2 + q^{2m-3}, \\ &\vdots \\ 1 + q^{2m-1} &\geq q^{m-1} + q^m. \end{aligned}$$

Summing up over all inequalities yields to

$$\frac{1 + q^{2m-1}}{1 + q + \dots + q^{2m-1}} \geq \frac{1}{m},$$

which proves the lower estimate. For the upper estimate, the inequality of the mean values between arithmetical and geometrical mean is used. Then we have

$$1 + q + \dots + q^{2m-1} \geq 2m \sqrt[2m]{q \cdot q^2 \cdot \dots \cdot q^{2m-1}} = 2m \sqrt{q^{2m-1}}.$$

This proves the lower estimate of (4.16).

The proof of (4.17) is similar to the proof of (4.15).

For the proof of (4.18), we use relations (4.15),(4.17) and equation (4.19). We obtain

$$\gamma_m = \frac{|\hat{s}_m|}{s_m} \leq \frac{1}{\det F_m - \det F_{m-1}} = \frac{q^{m-1}(1+q)}{q^{2m-1} + 1} \leq \frac{1+q}{qq^{m-1}}.$$

Since  $\frac{1+q}{q} \leq 2$  for  $q \geq 1$ , the assertion (4.18) follows, which proves the Lemma.  $\square$

The next lemma gives some asymptotical estimates for the Schur-complement  $s_m$  and the constant  $\gamma_m$ .

**Lemma 4.6.** • *Let  $m \geq \max\left\{\frac{1}{\sqrt{\kappa}}, 2\right\}$ . Then, we have*

$$\gamma_m < \frac{20}{21}. \quad (4.21)$$

- Let  $2 \leq m \leq \frac{1}{\sqrt{\kappa}}$  and  $m \in \mathbb{N}$ . Then, the estimate

$$\frac{1}{m} \leq s_m \leq \frac{9}{5} \frac{1}{m} \quad (4.22)$$

is valid.

*Proof.* To prove the first assertion, we use (4.13) and obtain

$$q^{m-1} = \left(1 + \frac{\kappa}{2} + \frac{1}{2} \sqrt{\kappa(\kappa+4)}\right)^{m-1} \geq \left(1 + \frac{\kappa}{2} + \sqrt{\kappa}\right)^{m-1}.$$

With  $m \geq \frac{1}{\sqrt{\kappa}}$ , we can conclude that

$$q^{m-1} \geq \left(1 + \frac{1}{2m^2} + \frac{1}{m}\right)^{m-1}.$$

The series  $\{a_m\}_m$  given by  $a_m = \left(1 + \frac{1}{2m^2} + \frac{1}{m}\right)^{m-1}$  is monotonic increasing and satisfies  $\lim_{m \rightarrow \infty} a_m = e$ . Moreover,  $a_m \geq \frac{21}{10}$  for  $m \geq 4$ . This gives

$$q^{m-1} \geq \frac{21}{10}.$$

Using (4.18), the assertion follows for  $m \geq 4$ . The case  $m = 2$  implies that  $\kappa \geq \frac{1}{4}$  and  $q \geq \frac{13}{8}$ . A direct computation shows

$$\gamma_2 \leq \frac{5}{6} < \frac{20}{21}.$$

A similar proof can be given for  $m = 3$ .

To prove the second assertion, we start with  $\kappa < m^{-2}$ . With the arguments we used above, we have

$$q^{2m-1} \leq \left(1 + \frac{1}{2m^2} + \frac{1}{m}\right)^{2m-1} \leq e^2.$$

Moreover, the function  $f : [1, \infty) \mapsto \mathbb{R}$  given by

$$f(x) = \frac{1+x}{\sqrt{x}} = \left(\sqrt[4]{x} - \frac{1}{\sqrt[4]{x}}\right)^2 - 2$$

is monotonic increasing for  $x \geq 1$ . Hence, we can estimate

$$\frac{1+q^{2m-1}}{\sqrt{q^{2m-1}}} \leq \frac{1+e^2}{e}.$$

Now, we insert this estimate into (4.16) and can conclude that

$$\frac{1}{m} \leq s_m \leq \frac{\kappa}{2} + \frac{1}{m} \frac{1+q^{2m-1}}{2\sqrt{q^{2m-1}}} \leq \frac{1}{m} \left(\frac{1}{2m} + \frac{1+e^2}{2e}\right) \leq \frac{1}{m} \left(\frac{1}{4} + \frac{31}{20}\right) = \frac{9}{5m}.$$

This proves the lemma. □

## 5 Condition number estimates

In this section, we prove the central theorems of this paper. The proof exploits the tensor product structure of the problems and uses some auxiliary results from the one-dimensional case. The results for the 1D case are presented in subsection 5.1, the proofs of Theorems 3.1 and 3.3 are presented in subsections 5.2 and 5.3 respectively.



## 5.1 Some one-dimensional auxiliary results

In this subsection, we prove some auxiliary results for the corresponding one-dimensional case. We start with the definition of a corresponding bilinear form and discretization matrices.

For  $n = 2^k$  and  $i = 1, 2$ , let  $I_{n-1} \in \mathbb{R}^{n-1 \times n-1}$ ,

$$T_{n-1} = \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots \\ \mathbf{0} & -1 & 2 & -1 & \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n-1 \times n-1} \quad \text{and} \quad (5.1)$$

$$M_{\omega_i} = \text{diag}(d_s)_{s=1}^{n-1} \quad \text{with} \quad d_s = \begin{cases} \varepsilon_{i,j} & \text{if } 2^{k-j-1} < s < 2^{k-j} \\ \frac{1}{2}(\varepsilon_{i,j} + \varepsilon_{i,j-1}) & \text{if } 2^{k-j} = s \end{cases} \quad (5.2)$$

be the identity, the unweighted Laplacian in 1D and a scaled weighted mass matrix with piecewise constant coefficients, respectively. The coefficients  $\varepsilon_{i,j}$  are defined via (4.8).

Since  $T_{n_0}$  is the 1D-Laplacian, we introduce linear finite elements on the equidistant mesh  $\mathcal{M}_n = \bigcup_{s=0}^{n-1} \tau_s^n$ , where  $\tau_s^n = (\frac{s}{n}, \frac{s+1}{n})$ . The one-dimensional hat functions on this mesh,

$$\phi_s^n(x) = \begin{cases} nx - (s-1) & \text{on } \tau_{s-1}^n, \\ (s+1) - nx & \text{on } \tau_s^n, \\ 0 & \text{otherwise,} \end{cases} \quad s = 1, \dots, n-1, \quad (5.3)$$

are a basis of the finite element space  $\mathbb{X}_n = \text{span}[\phi_s^n]_{s=1}^{n-1} = \text{span}[\Phi_1]$ . Then, the matrix  $T_{n-1} + \lambda M_{\omega_i}$  defines a bilinearform  $a_1(\cdot, \cdot)$  on  $\mathbb{X}_n$ , i.e.

$$\underline{u}^T (T_{n-1} + \lambda M_{\omega_i}) \underline{v} = a_1([\Phi_1] \underline{u}, [\Phi_1] \underline{v}) := \frac{1}{n} \int_0^1 u'(x)v'(x) dx + \sum_{s=1}^{n-1} \rho_{i,s} u\left(\frac{s}{n}\right) v\left(\frac{s}{n}\right) \quad (5.4)$$

with  $\rho_{i,s} = \frac{1}{2} \lambda (\chi_i^2 |_{\tau_{s-1}^n} + \chi_i^2 |_{\tau_s^n})$ ,  $i = 1, 2$ . Due to the symmetry and positive definiteness of the matrix  $T_{n-1} + \lambda M_{\omega_i}$ , the bilinearform  $a_1(\cdot, \cdot)$  is symmetric and coercive.

For  $j = 0, \dots, k-2$ , let  $\Omega_j = (2^{-j+1}, 2^{-j})$  and  $\Omega_{k-1} = (0, 2^{k-1})$ . Moreover, we introduce

$$\begin{aligned} \tilde{\mathbb{W}}_j &= \text{span}\{\phi_i^n\}_{i=n_{j+1}+2}^{n_j}, \quad j = 0, \dots, k-1, \quad \text{and} \\ \mathbb{W}_j &= \text{span}\{\phi_i^n\}_{i=n_{j+2}+2}^{n_j}, \quad j = 0, \dots, k-2, \quad \mathbb{W}_{k-1} = \tilde{\mathbb{W}}_{k-1}, \end{aligned}$$

where  $n_j$  is defined in subsection 3.1. Due to this definition, the spaces  $\mathbb{W}_j$  and  $\tilde{\mathbb{W}}_j$  are formed by those finite element functions of  $\mathbb{X}_n$  which have a support inside  $\overline{\Omega_{j+1,x}} \cup \overline{\Omega_{j,x}}$  and  $\overline{\Omega_{j,x}}$ , respectively.

**Lemma 5.1.** *There exists a decomposition  $u = \sum_{j=0}^{k-1} u_j$  with  $u_j \in \mathbb{W}_j$  such that*

$$a_1(u, u) \geq c^2 \sum_{j=0}^{k-1} a_1(u_j, u_j) \quad \forall u \in \mathbb{X}_n.$$

The constant  $c^2 > 0$  is constant which does not depend on  $n$  and  $\rho_i$ .

*Proof.* We adapt the notation of Theorem 4.2, i.e. let  $\Gamma_{j+1} = \overline{\Omega_{j+1}} \cup \overline{\Omega_j}$  and

$$\|w\|_{\Gamma_j, \text{left}}^2 = \min_{\substack{u \in \mathbb{X}_n \\ u|_{\Gamma_j} = w \\ u|_{\Gamma_{j+1}} = 0}} \|u\|_{a_1, \Omega_j}^2 \quad \text{and} \quad \|w\|_{\Gamma_j, \text{right}}^2 = \min_{\substack{u \in \mathbb{X}_n \\ u|_{\Gamma_j} = w \\ u|_{\Gamma_{j-1}} = 0}} \|u\|_{a_1, \Omega_{j-1}}^2 \quad (5.5)$$

be the left and right trace norm on  $\Gamma_j$ . Moreover, the minimal energy extension operators with respect to  $a_1(\cdot, \cdot)$  from  $\Gamma_j$  to  $\Omega_j$  and  $\Omega_{j-1}$  are denoted by  $\mathcal{T}_{j,\text{left}}$  and  $\mathcal{T}_{j,\text{right}}$ , respectively. Since the coefficients of the bilinear form  $a_1(\cdot, \cdot)$  are constant on  $\Omega_j$  and the discretization is symmetric with respect to the left and right boundary, we have

$$\|w\|_{\Gamma_j, \text{left}}^2 = \|w\|_{\Gamma_{j+1}, \text{right}}^2 \quad \forall w \in \mathbb{R}. \quad (5.6)$$

We fix now a stripe  $\Omega_j$ . A simple computation shows

$$[a_1|_{\Omega_j}(\phi_l^n, \phi_{l'}^n)]_{l, l' = n_{j+1}+2}^{n_j} = T_{m_j-1} + \kappa_{i,j} I_{m_j-1}, \quad i = 1, 2. \quad (5.7)$$

with  $m_j = 2^{k-j-1}$  and  $\kappa_{i,j} = \varepsilon_{i,j} \lambda$ , i.e. this matrix has the structure of the matrix  $F_m$  (4.12) with  $m = m_j$  and  $\kappa = \kappa_{i,j}$ ,  $i = 1, 2$ . So, it is possible to apply the results about the matrix  $F_m$ . Due to the properties of the Schur-complement, we have

$$\|w\|_{\Gamma_j, \text{left}}^2 = \|w\|_{\Gamma_{j+1}, \text{right}}^2 = w^2 s_{m_j} \quad \forall w \in \mathbb{R} \quad (5.8)$$

with  $s_{m_j}$  be defined via (4.15). A simple computation shows

$$a_1(\mathcal{T}_{j,\text{left}} u, \mathcal{T}_{j+1,\text{right}} v) = u \hat{s}_{m_j} v \quad \forall u, v \in \mathbb{R} \quad (5.9)$$

with  $\hat{s}_{m_j}$  of (4.17). Hence, we can conclude that

$$\gamma_{m_j}^2 = \max_{\substack{u, v \in \mathbb{R} \\ u, v \neq 0}} \frac{a_1(\mathcal{T}_{j,\text{left}} u, \mathcal{T}_{j+1,\text{right}} v)}{\|u\|_{\Gamma_j, \text{left}} \|v\|_{\Gamma_{j+1}, \text{right}}} = \frac{\hat{s}_{m_j}}{s_{m_j}}. \quad (5.10)$$

The series  $\{m_j\}_{j=0}^{k-1}$  is monotonic decreasing by definition. The series  $\{\kappa_{i,j}\}_{j=0}^{k-1}$  is monotonic decreasing by Assumption 2.1. Hence, the series  $\{m_j \kappa_{i,j}^2\}_{j=0}^{k-1}$  is monotonic decreasing, too. So, there exists a number  $j_0 \in \{-1, 0, \dots, k\}$  such that

$$m_j \geq \kappa_{i,j}^{-2} \quad \forall j \leq j_0, \quad \text{and} \quad m_j \leq \kappa_{i,j}^{-2} \quad \forall j > j_0, \quad i = 1, 2. \quad (5.11)$$

Now, we verify the assumptions of Theorem 4.2. Using (5.10) and (4.21), we have

$$\gamma_{m_j}^2 \leq \frac{20}{21} \quad \text{for} \quad j < j_0.$$

This gives (4.5). Using (5.8) and (4.22), the lower estimate in (4.7) follows with

$$q = \frac{9}{10} < 1 \quad \text{for} \quad j > j_0.$$

The estimates (4.6) and the upper estimate in (4.7) are a consequence of Assumption 2.1, i.e. the weight function before the mass matrix is not changing too much on two neighboring stripes. Using Theorem 4.2, the assertion follows.  $\square$

We define now an overlapping preconditioner of the type (3.3), (3.6) for the matrix

$$A_\lambda = \lambda M_{\omega_i} + T_{n_0}, \quad \lambda > 0. \quad (5.12)$$

Before, we have to introduce some auxiliary matrices. For  $j = 0, \dots, k-2$ , let

$$M_{i,j} = \begin{bmatrix} \mathbf{0}_{n_{j+2}+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon_{i,j} I_{n_j - n_{j+2} - 1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n_0 - n_j} \end{bmatrix} \in \mathbb{R}^{n_0 \times n_0}, \quad i = 1, 2,$$

$$T_{j,n_0} = \begin{bmatrix} \mathbf{0}_{n_{j+2}+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T_{n_j - n_{j+2} - 1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n_0 - n_j} \end{bmatrix} \in \mathbb{R}^{n_0 \times n_0},$$

where  $\varepsilon_{i,j}$  is defined via (4.8). For  $j = k - 1$ , we set

$$M_{i,k-1} = \begin{bmatrix} \varepsilon_{i,k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_0-1} \end{bmatrix} \in \mathbb{R}^{n_0 \times n_0}, \quad i = 1, 2, \quad T_{j,n_0} = \begin{bmatrix} 2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_0-1} \end{bmatrix} \in \mathbb{R}^{n_0 \times n_0}.$$

Now, we can define

$$C_1^{-1} = \sum_{j=0}^{k-1} (\lambda M_{i,j} + T_{j,n_0})^+ \quad (5.13)$$

as preconditioner for  $A_\lambda$ . Now, we are able to formulate the final lemma.

**Lemma 5.2.** *For  $i = 1, 2$  and  $\lambda > 0$ , let  $A_\lambda$  and  $C_1$  be defined via (5.12) and (5.13), respectively. Then,  $c_1 C_1 \leq A_\lambda \leq c_2 C_1$ . The constants do not depend on the structure of the weight functions  $\omega_i$  and the parameter  $\lambda$ .*

*Proof.* We apply Lemma 4.1 with the bilinear form  $(\cdot, \cdot)_{\mathcal{A}} = a_1(\cdot, \cdot)$  and verify the assumptions (4.1), (4.2) and (4.3). The space splitting implies  $\beta = 2$ , cf. Theorem 4.2, which proves (4.2). Relation (4.1) follows from Lemma 5.1.

The bilinear form  $a_1(\cdot, \cdot)$  (5.4) is sum of two terms, a stiffness term and a mass term. The coefficient before the stiffness term is constant. The coefficient before the mass term is piecewise constant, i.e.  $\varepsilon_{i,j}$  on  $\Omega_j$ ,  $i = 1, 2$ . So, the maximum of the coefficients on  $\bar{\Omega}_j \cup \bar{\Omega}_{j+1}$  is  $\varepsilon_{i,j}$  and the minimum is  $\varepsilon_{i,j+1}$ . In the preconditioner  $C_1$  (5.13), the coefficient on  $\bar{\Omega}_j \cup \bar{\Omega}_{j+1}$  is replaced by  $\varepsilon_{i,j}$ . Assumption 2.1 implies that the ratio of coefficients  $\varepsilon_{i,j+1}^{-1} \varepsilon_{i,j}$  is bounded. This gives (4.3) and proves the lemma for the matrix  $C_1$ .  $\square$

## 5.2 The proof of Theorem 3.1

Now, we prove Theorem 3.1.

*Proof.* Due to Lemma 4.3, it suffices to show the result for the matrix  $K_{k,p}$  (4.11). A simple computation shows that

$$K_{k,p} = T_{n_0} \otimes M_{\omega_2} + I_{n_0} \otimes T_{n_0},$$

where the matrices  $T_n$  and  $M_{\omega_2}$  are defined via (5.1) and (5.2). Since the matrix  $T_{n_0}$  is symmetric and positive definite, we have

$$T_{n_0} = Q^T \Lambda Q \quad \text{with} \quad Q^T Q = I_{n_0}, \quad \Lambda = \text{diag}[\lambda_i]_i, \quad \lambda_i > 0$$

Hence,

$$\begin{aligned} K_{k,p} &= (Q^T \otimes I_{n_0})(\Lambda \otimes M_{\omega_2} + I_{n_0} \otimes T_{n_0})(Q \otimes I_{n_0}) \\ &= (Q^T \otimes I_{n_0}) \text{blockdiag}[\lambda_i M_{\omega_2} + T_{n_0}]_i (Q \otimes I_{n_0}). \end{aligned}$$

We apply now Lemma 5.2 and obtain

$$\begin{aligned}
K_{k,p}^{-1} &= (Q^T \otimes I_{n_0}) \text{blockdiag} [(\lambda_i M_{\omega_2} + T_{n_0})^{-1}]_i (Q \otimes I_{n_0}) \\
&\sim (Q^T \otimes I_{n_0}) \text{blockdiag} \left[ \sum_{j=0}^{k-1} (\lambda_i M_j + T_{j,n_0})^+ \right]_i (Q \otimes I_{n_0}) \\
&= (Q^T \otimes I_{n_0}) \sum_{j=0}^{k-1} (\Lambda \otimes M_j + I_{n_0} \otimes T_{j,n_0})^+ (Q \otimes I_{n_0}) \\
&= \sum_{j=0}^{k-1} ((Q^T \otimes I_{n_0})(\Lambda \otimes M_j + I_{n_0} \otimes T_{j,n_0})(Q \otimes I_{n_0}))^+ \\
&= \sum_{j=0}^{k-1} (T_{n_0} \otimes M_j + I_{n_0} \otimes T_{j,n_0})^+ = C_{mod,B}^{-1},
\end{aligned}$$

which proves the result.  $\square$

### 5.3 The proof of Theorem 3.3

*Proof.* As in the previous case, we use the tensor product structure of the stiffness matrices  $K_k$  and  $K_{k,p}$  (4.11). Due to Lemma 4.3, it suffices to prove  $C_{mod,C}^{-1} \sim K_{k,p}$ . A simple computation shows that

$$K_{k,p} = T_{n_0} \otimes M_{\omega_2} + M_{\omega_1} \otimes T_{n_0},$$

see (5.1) and (5.2) for the definition of the involved matrices. Since  $M_{\omega_1}$  and  $T_{n_0}$  are symmetric and positive definite matrices, we can conclude

$$M_{\omega_1}^{-1/2} T_{n_0} M_{\omega_1}^{-1/2} = \tilde{Q}^T \tilde{\Lambda} \tilde{Q} \quad \text{with} \quad \tilde{Q}^T \tilde{Q} = I_{n_0}, \quad \tilde{\Lambda} = \text{diag}[\tilde{\lambda}_i]_i, \quad \tilde{\lambda}_i > 0.$$

This gives

$$\begin{aligned}
K_{k,p} &= (M_{\omega_1}^{1/2} \tilde{Q} \otimes I_{n_0})(\tilde{\Lambda} \otimes M_{\omega_2} + I_{n_0} \otimes T_{n_0})(\tilde{Q}^T M_{\omega_1}^{1/2} \otimes I_{n_0}) \\
&= (M_{\omega_1}^{1/2} \tilde{Q} \otimes I_{n_0}) \text{blockdiag} \left[ \tilde{\lambda}_i M_{\omega_2} + T_{n_0} \right]_i (\tilde{Q}^T M_{\omega_1}^{1/2} \otimes I_{n_0})
\end{aligned}$$

Using Lemma 5.2, we can conclude

$$(\tilde{\lambda}_i M_{\omega_2} + T_{n_0})^{-1} \sim \sum_{j=0}^{k-1} (\tilde{\lambda}_i M_{2,j} + T_{j,n_0})^+.$$

Hence, we can proceed with the estimates

$$\begin{aligned}
K_{k,p}^{-1} &\sim (M_{\omega_1}^{-1/2} \tilde{Q} \otimes I_{n_0}) \text{blockdiag} \left[ \sum_{j=0}^{k-1} (\tilde{\lambda}_i M_{2,j} + T_{j,n_0})^+ \right]_i (\tilde{Q}^T M_{\omega_1}^{-1/2} \otimes I_{n_0}) \\
&= (M_{\omega_1}^{-1/2} \tilde{Q} \otimes I_{n_0}) \left[ \sum_{j=0}^{k-1} \tilde{\Lambda} \otimes M_{2,j} + I_{n_0} \otimes T_{j,n_0} \right] (\tilde{Q}^T M_{\omega_1}^{-1/2} \otimes I_{n_0}) \\
&= \sum_{j=0}^{k-1} (T_{n_0} \otimes M_{2,j} + M_{\omega_1} \otimes T_{j,n_0})^+ := \sum_{j=0}^{k-1} C_{3,j}^+ \tag{5.14}
\end{aligned}$$

In a second step, we derive a preconditioner for  $C_{3,j}$ . With the same tensor product arguments as above, we obtain

$$C_{3,j}^+ \sim \sum_{j'=0}^{k-1} (T_{j',n_0} \otimes M_{2,j} + M_{1,j'} \otimes T_{j,n_0})^+ \quad (5.15)$$

uniformly for all  $j = 0, \dots, k-1$ . Combining (5.14) and (5.15), we have

$$K_{k,p}^{-1} \sim \sum_{j=0}^{k-1} \sum_{j'=0}^{k-1} (T_{j',n_0} \otimes M_{2,j} + M_{1,j'} \otimes T_{j,n_0})^+ = C_{mod,C}^{-1}$$

which proves the result.  $\square$

## 6 Numerical Experiments

In this section, we present some numerical experiments.

### 6.1 The case $\omega_1(\xi) = 1$ .

In a first experiment, we investigate the preconditioner  $C_B$  (3.2). Figure 6 displays the maximal and minimal eigenvalue of the matrix  $C_B^{-1}K_{k,p}$  for different weight functions. The minimal eigenvalue of the matrix

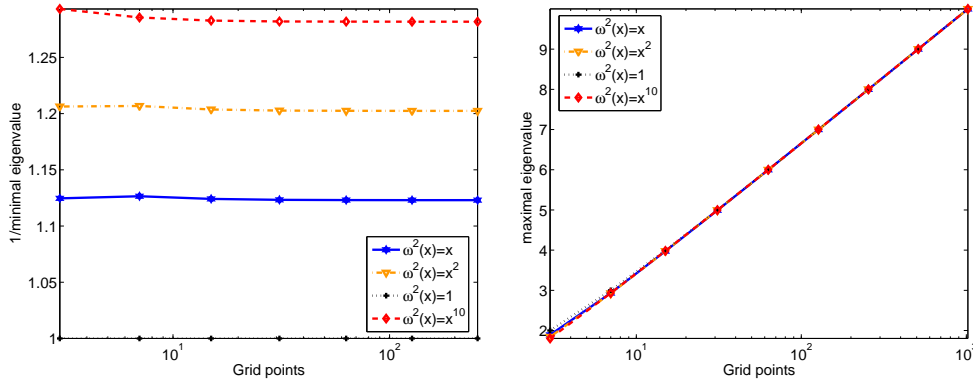


Figure 6: Eigenvalue bounds with the preconditioner (3.2), minimal eigenvalue left, maximal eigenvalue right.

$C_B^{-1}K_{k,p}$  is bounded from below by a positive constant for all types of investigated weight functions. The constant are very close to 1. The maximal eigenvalue is about  $k$  on level  $k$ .

In a second experiment, we investigate the preconditioner  $C_{mod,B}$  (3.3). In comparison to the first preconditioner, this preconditioner is optimal. Figure 7 displays the maximal and minimal eigenvalue for the matrix  $C_{mod,B}^{-1}K_{k,p}$  with the modified preconditioner (3.3) for different weight functions. The minimal eigenvalue of the matrix  $C_{mod,B}^{-1}K_{k,p}$  is bounded from below by a positive constant for all types of investigated weight functions. The maximal eigenvalue is bounded from above by a constant of 2. The asymptotic optimally behavior can only be seen for relatively high level numbers. So, the condition number of  $C_B^{-1}K_{k,p}$  is lower for  $k \leq 10$  than the condition number of  $C_{mod,B}^{-1}K_{k,p}$ , although the condition number of  $C_{mod,B}^{-1}K_{k,p}$  grows logarithmically whereas the condition number of  $C_B^{-1}K_{k,p}$  is bounded.

Finally, we investigated the preconditioners for the matrix  $K_k$ . The results for the minimal eigenvalue of the matrices  $C_B^{-1}K_k$  and  $C_{mod,B}^{-1}K_k$  are displayed in Figures 8 and 9, respectively. For the maximal

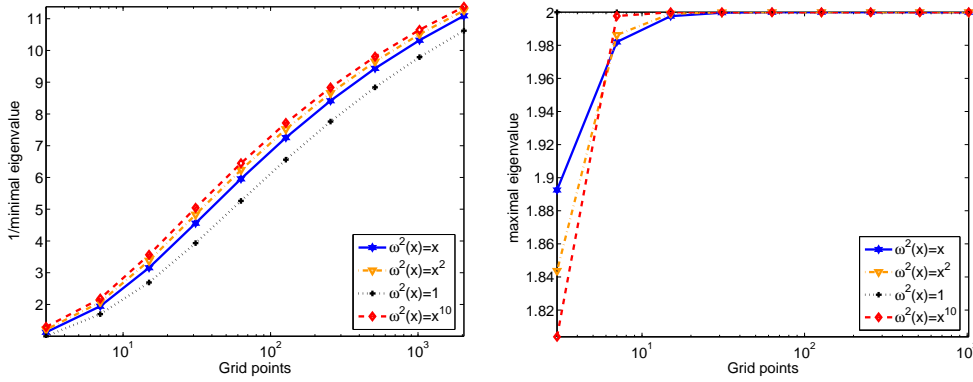


Figure 7: Eigenvalue bounds with the modified preconditioner (3.3), minimal eigenvalue left, maximal eigenvalue right.

eigenvalues, the results are the same as for the matrix  $K_k$ , i.e. the maximal eigenvalue of the matrix  $C_B^{-1}K_k$  is about  $k$ , whereas the maximal eigenvalue of the matrix  $C_{mod,B}^{-1}K_k$  is about 2.

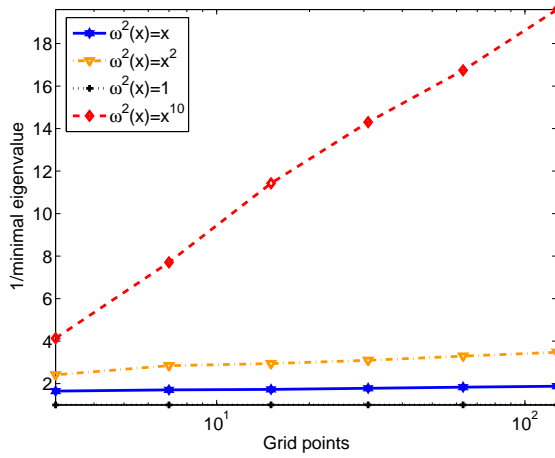


Figure 8: Minimal eigenvalue of the matrix  $C_B^{-1}K_k$ .

Considering the minimal eigenvalue, the results are different. The results for the matrix  $C_{mod,B}^{-1}K_k$  are comparable with the results for the matrix  $C_{mod,B}^{-1}K_{k,p}$  if the weight function is not  $\omega_2^2(\xi) = \xi^{10}$ . Then, an additional factor of about 2.5 can be seen. The minimal eigenvalue  $C_B^{-1}K_k$  has the expected (pessimistic) additional factor of  $2 \cdot 2^\alpha$  of Lemma 4.3 compared with the minimal eigenvalue  $C_B^{-1}K_{k,p}$ . Summarizing, the preconditioner  $C_{mod,B}$  (3.3) should be preferred for the matrix  $K_k$  with an weight function  $\omega_2^2(\xi) = \xi^\alpha$ ,  $\alpha > 1$ .

## 6.2 The general case

In this subsection, we consider the systems

$$K_k \underline{u} = \underline{f} \quad \text{and} \quad K_{k,p} \underline{u} = \underline{f}.$$

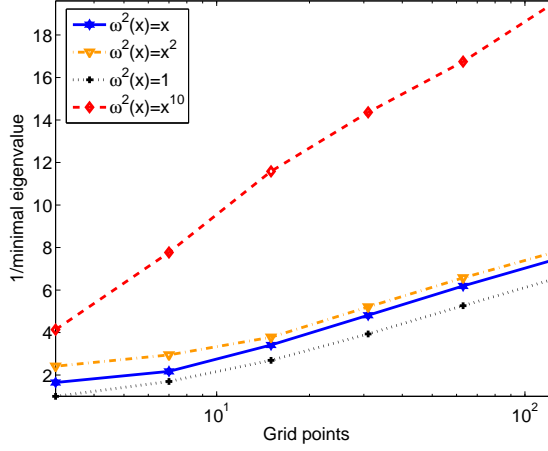


Figure 9: Minimal eigenvalue of the matrix  $C_{mod,B}^{-1}K_k$ .

In all experiments, we choose the weight functions  $\omega_i^2(\xi) = \xi^2$ ,  $i = 1, 2$ . Figure 10 displays the condition number of  $C^{-1}K$  for five combinations of  $C = \{C_C, C_{C,mod}, C_{C,var}\}$  and  $K = \{K_k, K_{k,p}\}$ . The best

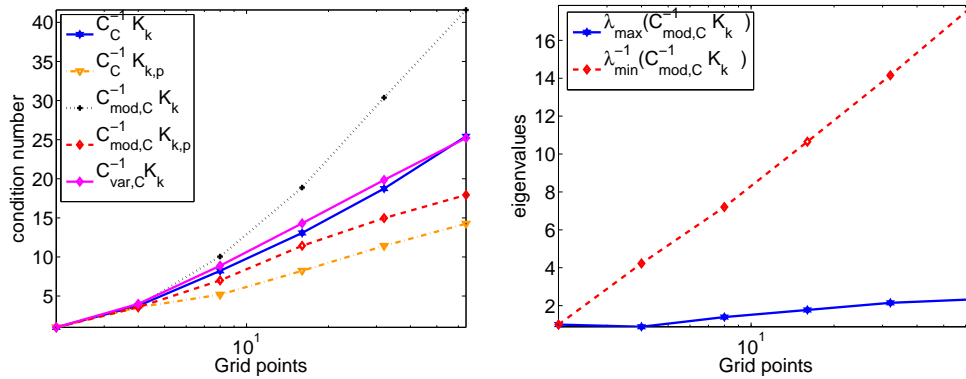


Figure 10: Condition numbers for the general case (left), eigenvalue bounds for  $C_{mod,C}^{-1}K_k$  (right).

results are obtained for the matrix  $K_{k,p}$  with a piecewise constant coefficient function. Then, the condition number is moderately increasing for both preconditioners. For the matrix  $K_k$  with the smooth coefficient function, the results are not as good as in the previous case if we take a preconditioner with constant coefficients. The preconditioner with variable but smooth coefficients  $C_{var,C}$  of Remark 3.5 behaves better. In particular for  $C_{mod,C}^{-1}K_k$  (3.6), an additional factor of about 4 can be seen which arises from the estimates of Lemma 4.3. The eigenvalue bounds of  $C_{mod,C}^{-1}K_k$  and  $C_{var,C}^{-1}K_k$  are similar as for the preconditioner  $C_{mod,B}$  (3.3), cf. Figures 9 and 7, where the asymptotics can be seen only for relatively high level numbers. Hence, the nonoptimal preconditioner  $C_C$  (3.4) should be preferred for moderate level numbers of  $k = 5, 6, 7$ .

Finally, we investigate the pcg-iterations of preconditioned conjugated gradient method with the preconditioner  $C_C$  (3.4). In all experiments, we have taken a randomly chosen right hand side, a relative accuracy of  $10^{-5}$ . The results for both systems are displayed in Table 1. From the results, a slight increase of the iteration numbers can be seen. Since this preconditioner is not optimal, cf. Theorem 3.4, the growth of

Level	3	4	5	6	7	8	9	10
$\widetilde{K}_k$	16	21	25	29	33	37	40	43
$\widetilde{K}_{k,p}$	16	20	25	30	34	38	41	43

Table 1: PCG-Iterations for the systems  $K_k \underline{u} = \underline{f}$  and  $K_{k,p} \underline{u} = \underline{f}$  with the preconditioner  $C_C$  (3.4).

the pcg-number is not surprising. The pcg-iterations are about the same for the matrix  $K_k$  with continuous weight function and the matrix  $K_{k,p}$  with piecewise constant weight function.

## 7 Concluding remarks

We will conclude the paper with a remark about an application for the  $p$ -version of the Finite Element Method in 3D. Using the basis of the integrated Legendre polynomials  $\{\hat{L}_i\}_{i=2}^p$ , it has been proved in [2] that the element stiffness matrix for odd polynomial degree  $p$  with respect to the interior bubbles is spectrally equivalent to the matrix

$$\begin{aligned} K_{pv} &= P^T \text{blockdiag} [T_n \otimes T_n \otimes M_\omega + T_n \otimes M_\omega \otimes T_n + M_\omega \otimes T_n \otimes T_n]_{i=1}^8 P \\ &=: P^T \text{blockdiag} [K_3]_{i=1}^8 P, \end{aligned}$$

where  $T_n$  is the matrix (5.1) of dimension  $\frac{p-1}{2}$ ,  $M_\omega$  is the matrix (5.2) with the weight function  $\omega^2(\xi) = \xi^2$  and  $P$  is a permutation matrix. In [4], an optimal solver for  $K_{pv}$  based on wavelets has been derived.

Another preconditioner  $C_3$  for  $K_3$  can be developed in the same way as for  $K_k$  in (3.6) and (3.4). With the same tensor product arguments as in the proof of Theorem 3.3 presented in subsection 5.3, the optimality of the estimate  $C_3 \sim K_{3,k}$  is proved. Using fast Fourier transform for the remaining problem, we obtain a second fast solver for the block of the interior bubbles in the  $p$ -version of the FEM using hexahedral elements.

**Acknowledgement:** The paper was written during the Special Semester on Computational Mechanics in Linz 2005. The second author thanks the RICAM for the hospitality during his stay in Linz.

## References

- [1] S. Beuchler. Multi-grid solver for the inner problem in domain decomposition methods for  $p$ -FEM. *SIAM J. Numer. Anal.*, 40(3):928–944, 2002.
- [2] S. Beuchler and D. Braess. Improvements for some condition number estimates in p-fem. *Numerical Linear Algebra with Applications*, 2006. to appear.
- [3] S. Beuchler and S. Nepomnyaschikh. Overlapping additiv Schwarz preconditioners for degenerated elliptic problems: Part i isotropic problems. Technical report, RICAM, 2006.
- [4] S. Beuchler, R. Schneider, and C. Schwab. Multiresolution weighted norm equivalences and applications. *Numer. Math.*, 98(1):67–97, 2004.
- [5] Sven Beuchler. Multilevel solvers for a finite element discretization of a degenerate problem. *SIAM J. Numer. Anal.*, 42(3):1342–1356 (electronic), 2004.
- [6] S. Börm and R. Hiptmair. Analysis of tensor product multigrid. *Numer. Algorithms*, 26(3):219–234, 2001.
- [7] J. Bramble, J. Pasciak, and A. Schatz. The construction of preconditioners for elliptic problems by substructuring I. *Math. Comp.*, 47(175):103–134, 1986.
- [8] J. Bramble and X. Zhang. Uniform convergence of the multigrid V-cycle for an anisotropic problem. *Math. Comp.*, 70(234):453–470, 2001.



- [9] I. G. Graham, P. Lechner, and R. Scheichl. Domain decomposition for multiscale pdes. Technical report, University of Bath, 2006.
- [10] G. Haase, U. Langer, and A. Meyer. The approximate Dirichlet domain decomposition method. Part I: An algebraic approach. *Computing*, 47:137–151, 1991.
- [11] W. Hackbusch. *Multigrid Methods and Applications*. Springer-Verlag, Heidelberg, 1985.
- [12] V. G. Korneev. Почти оптимальный метод решения задач Дирихле на подобластях декомпозиции иерархической  $hp$ -версии. *Дифференциальные Уравнения*, 37(7):1008–1018, 2001. An almost optimal method for Dirichlet problems on decomposition subdomains of the hierarchical  $hp$ -version.
- [13] A. Kufner and A.M. Sändig. *Some applications of weighted Sobolev spaces*. B.G.Teubner Verlagsgesellschaft, Leipzig, 1987.
- [14] P. L. Lions. On the Schwarz alternating method I. In R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, editors, *Proc. 1st Int. Symp. on Domain Decomposition Methods*. SIAM, Philadelphia, pages 1–42, Philadelphia, 1988. SIAM.
- [15] A. M. Matsokin and S. V. Nepomnyaschikh. The Schwarz alternation method in a subspace. *Iz. VUZ Mat.*, 29(10):61–66, 1985.
- [16] S. V. Nepomnyaschikh. Method of splitting into subspaces for solving elliptic boundary value problems in complex-form domains. *Soviet J. Numer. Anal. Math. Modelling*, 6(2):151–168, 1991.
- [17] S. V. Nepomnyaschikh. Preconditioning operators for elliptic problems with bad parameters. In *Eleventh International Conference on Domain Decomposition Methods (London, 1998)*, pages 82–88 (electronic). DDM.org, Augsburg, 1999.
- [18] O. Pironneau and F. Hecht. Mesh adaption for the Black and Scholes equations. *East-West Journal of Numerical Mathematics*, 8(1):25–36, 2000.
- [19] R. Scheichl and E. Vainikko. Additive schwarz and aggregation-based coarsening for elliptic problems with highly variable coefficients. Technical report, University of Bath, 2006.
- [20] A. Toselli and O. Widlund. *Domain Decomposition Methods- Algorithms and Theory*. Springer, 2005.