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A. Gabih, J. Sass, R. Wunderlich

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Utility Maximization Under Bounded Expected Loss*

Abdelali Gabih [†], Jörn Sass [‡], Ralf Wunderlich[§]

We consider the optimal selection of portfolios for utility maximizing investors under joint budget and shortfall risk constraints. The shortfall risk is measured in terms of the expected loss. Depending on the parameters of the risk constraint we show existence of an optimal solution and uniqueness of the corresponding Lagrange multipliers. Using Malliavin calculus we also provide the optimal trading strategy.

Key Words: Portfolio optimization, utility maximization, expected shortfall, risk constraint, Malliavin calculus.

1 Introduction

In this paper we investigate for a finite time horizon the optimal selection of portfolios for utility maximizing investors under a shortfall risk constraint. Without risk constraint it is known that the distribution of the optimal terminal wealth is quite skew and causes losses with a high probability. That is not very attractive e.g. from the viewpoint of a pension fund manager. A strict corrective would be to put simply a lower bound on the terminal wealth (portfolio insurer). Better performance can be expected when allowing for some freedom, putting only a bound on the expected losses. However, if using a value at risk (VaR) based criterion, i.e. requiring that the loss which is exceeded with a given probability remains below a certain level, it can happen that the losses may be larger than for the unconstrained optimization problem, since the magnitude of the losses below the threshold plays no role for the risk measure. So one also wants to take the magnitude of the losses into account, e.g. by averaging the possible losses leading to an expected shortfall constraint. And in

*Supported by the Austrian Science Fund FWF, project P17947-N12

†Department of Mathematics, University of Leipzig, Augustusplatz 10/11, D-04109 Leipzig, Germany, e-mail:abdelali.gabih@math.uni-leipzig.de

‡RICAM, Austrian Academy of Sciences, Altenberger Str. 69, A-4040 Linz, Austria, e-mail: joern.sass@oeaw.ac.at

§Mathematics Group, Zwickau University of Applied Sciences, PSF 201037, D-08056 Zwickau, Germany, e-mail: ralf.wunderlich@fh-zwickau.de

fact, see e.g. [2], the distribution of the resulting optimal terminal wealth has more desirable properties.

We consider a complete financial market consisting of a money market and n risky assets. P and $\widetilde{P}$ denote the physical and the unique equivalent martingale or risk neutral measures, respectively. For the exact definitions see Section 2.

The shortfall risk is measured in terms of the expected loss. Comparing the portfolio value X_T at terminal time $T > 0$ with some given benchmark or shortfall level $q > 0$ a shortfall occurs in the random event $\{X_T < q\}$. We measure this risk by averaging the loss $(X - q)^-$ w.r.t. some probability measure Q equivalent to P, e.g. $Q = P$ or $Q = \widetilde{P}$. The non-negative number $E_Q[(X - q)^-]$ is called expected loss.

Let us fix a utility function $U : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$, initial capital $x_0 > 0$, shortfall level $q > 0$, a bound $\varepsilon \geq 0$ for the expected loss, and a probability measure Q equivalent to P for averaging the loss. Then the static optimization problem with risk constraint reads as

$$\max_{X_T \geq 0} E[U(X_T)] \qquad \text{subject to} \tag{1}$$

$$\widetilde{E}[\beta_T X_T] \leq x_0 \quad \text{(budget constraint)}, \tag{2}$$

$$E_Q[\gamma(X_T - q)^-] \leq \varepsilon \quad \text{(risk constraint)}, \tag{3}$$

where E, $\widetilde{E}$ and E_Q denote expectation with respect to P, $\widetilde{P}$ and Q, respectively, β_T is the discount factor corresponding to the money market, and γ a discount factor for the risk constraint, e.g. $\gamma = \beta_T$. For given initial capital $x_0 > 0$ any terminal wealth X_T has to satisfy the budget constraint (2) in order to be attainable by a trading strategy.

If we drop the risk constraint (3), the problem is known as the so-called Merton problem while the limiting case $\varepsilon = 0$ corresponds to the risk constraint $X \geq q$. This case is known as the so-called portfolio insurer problem, cf. [11, 12, 16]. Both problems are considered in more detail in Section 3. If $\varepsilon > 0$ and the risk constraint is binding, there are two typical choices for Q. First $Q = \widetilde{P}$: If we choose $\gamma = \beta_T$ the expected loss corresponds to the price of a derivative to hedge against the shortfall. So we may call this choice present expected loss (PEL), since by paying ε now, one can hedge against the risk to fall short of q. This criterion was called limited expected loss in [2] and analyzed thoroughly in comparison to a value at risk (VaR) based risk measure showing that the law of the optimal terminal wealth for PEL is more attractive than it is for VaR, the latter allowing for large losses. Another choice would be $Q = P$, termed as future expected loss (FEL), since it corresponds to the (discounted) amount we have to pay at T to cover the loss. In addition any other measure Q equivalent to P might be used in (3).

The special feature of this paper is the detailed investigation of conditions for the existence and uniqueness of optimal solutions. Here we extend the findings of our papers [7, 8, 9] and of Basak et al. [2, 3] which concentrate on the construction of optimal solutions to the static problem by combining methods from martingale and convex duality theory. For more motivation concerning expected loss we refer to [2]. Assuming that an optimal solution exists they state in Proposition 4 the optimal wealth for $Q = \widetilde{P}$ and compare with the VaR based optimization. Also see [16] for a recent account of the portfolio insurer problem and [15] for a stochastic control approach to optimal consumption with HARA utility and modifications of the VaR and expected shortfall risk measures.

The present paper has been inspired by the paper of Gundel and Weber [13] who investigate the problem for a risk constraint containing a risk measure out of a family of convex risk measures which they call utility-based shortfall risk, in detail: For a loss function $\ell : \mathbb{R} \to [0, \infty]$ (increasing, non-constant), the functional $\rho(X) = \inf\{m \in \mathbb{R} : E_Q[\gamma\, \ell(-X - m)] \leq \varepsilon\}$ defines a convex risk measure on a suitable class of payoffs, i.e. it is monotone, translation invariant and convex. But it is not positively homogeneous hence not coherent, cf. [1], and [6] for a motivation and a concise overview of risk measures. In (3) we use the loss function $\ell(x) = (-x - q)^- = (x + q)^+$ and ρ might be defined on the class of positive random variables. In [13] a loss function is used which is strictly convex on $(-q, \infty)$. So the piecewise linear loss function considered in our paper can be embedded as a limiting case only. But then the conditions for the existence of optimal solutions imposed in [13] are in some cases violated. So our paper extends the findings of [13] to the case of expected loss which covers the important examples PEL and FEL.

The problem of existence and uniqueness of optimal solutions is also considered in Gandy [10] who considers two particular cases of the family of expected loss risk measures used in our paper, namely the cases PEL and FEL. Contrary to [10] we allow for stochastic coefficients and give a more explicit answer to the question of uniqueness of the Lagrange multipliers associated to the maximization problem. We proceed by an analysis of the budget and risk equation which we use to derive properties of the Lagrange multipliers. These properties allow us to show uniqueness and they are extremely helpful when setting up a numerical scheme to find the optimal Lagrange multipliers. While we provide the optimal trading strategies in Section 7 we shall defer their thorough (numerical) analysis to a future publication.

We proceed as follows: In Section 2 we introduce the market model and state the problem. In preparation for our main results we investigate in Section 3 the existence of the so called Lagrange multipliers solving the budget equations for both Merton's problem and for the portfolio insurer problem; and in Section 4 we determine the minimal shortfall risk which we use to find the set of all admissible terminal positions for our static optimization problem. In Section 5 we then deal with the portfolio optimization problem under the additional risk constraint. Here we derive in Theorem 9 the form of the optimal terminal wealth depending on the bound of the shortfall risk. In case that the risk constraint is binding the solution can be described by two Lagrange multipliers associate with the constraints (2) and (3). The proof of their existence and uniqueness, which may be viewed as one of the main results of this paper, is given in Section 6: We look at the budget and risk equations as functions of the Lagrange multipliers and derive in Section 6.1 the properties we need. In Section

6.2 we solve the budget equation if the first Lagrange multiplier is given, and the main result in Section 6.3 provides the solution of the joint budget and risk constraints. While we give the proofs of the key theorems in the corresponding sections, the proofs of many of the auxiliary results are deferred to the appendix. All proofs are self contained. For deriving the properties of the budget and risk equations we partly need very elementary arguments. But these properties are also valuable when setting up a numerical scheme to find the multipliers and hence the optimal terminal wealth, cf. Remark 17. In Section 7 we derive the corresponding optimal strategies comparing martingale representation for the wealth and Clark's formula under $\widetilde{P}$.

2 A class of shortfall constraints

In this section we formulate the market model and specify the optimization problem. We shall consider a market with terminal trading time $T > 0$, consisting of one money market with interest rates $r_t \geq 0$, $t \in [0,T]$, and n stocks. The return process $(R_t)_{t\in[0,T]}$ of the stock prices is given by

$$R_t = \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \quad R_0 = \mathbf{0},$$

where $W = (W_t)_{t\in[0,T]}$ is an n-dimensional standard Brownian motion on a suitable filtered probability space $(\Omega, \mathcal{F}_T, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $\mathcal{F}$ satisfies the usual conditions. The n-dimensional drift process $(\mu_t)_{t\in[0,T]}$, the interest rates, and the $n \times n$-dimensional volatility process $(\sigma_t)_{t\in[0,T]}$ are progressively measurable. For the latter σ_t is non-singular for all $t \in [0,T]$. The processes satisfy

$$\int_0^T \left(\|\mu_t\| + \|r_t\| + \|\sigma_t\|^2\right) dt < \infty, \tag{4}$$

where $\|\sigma_t\| = (\sum_{i,j=1}^n (\sigma_t^{i,j})^2)^{1/2}$, as well as

$$\int_0^T \|\sigma_t^{-1}(\mu_t - r_t 1_n)\|^2 \, dt < \infty, \tag{5}$$

all inequalities understood to hold almost surely. The stock prices $S = (S_t)_{t\in[0,T]}$ are defined by $dS_t^i = S_t^i dR_t^i$ with constant $S_0^i > 0$, and thus evolve according to

$$dS_t^i = S_t^i dR_t^i = S_t^i \mu_t^i \, dt + S_t^i \sum_{j=1} \sigma_t^{i,j} \, dW_t^j, \quad i = 1, \ldots, n,$$

and we can identify investment in the money market with investment in a riskless asset S^0,

$$S_t^0 = \exp\left\{\int_0^t r_s \, ds\right\}, \quad t \in [0,T].$$

By $\beta_t = 1/S_t^0$, $t \in [0,T]$, we shall denote the corresponding discount factor.

The market price of risk is defined as

$$\vartheta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_n), \quad t \in [0, T].$$

Due to (5) we can introduce the density process $(Z_t)_{t\in[0,T]}$,

$$Z_t = \exp\left\{ -\int_0^t \vartheta_s^\top dW_s - \frac{1}{2}\int_0^t \|\vartheta_s\|^2 ds \right\},$$

which is a local martingale. We shall always make the

Assumption 1 $(Z_t)_{t\in[0,T]}$ is a martingale.

We need this to introduce a new probability measure $\widetilde{P}$ by

$$\widetilde{P}(A) = E[Z_T 1_A], \quad A \in \mathcal{F}_T \,.$$

So $\widetilde{P}$ is equivalent to P. Girsanov's theorem guarantees that $\widetilde{W} = (\widetilde{W}_t)_{t\in[0,T]}$,

$$\widetilde{W}_t = W_t + \int_0^t \vartheta_s \, ds,$$

is a Brownian motion under this risk neutral measure $\widetilde{P}$.

We describe the self-financing trading of an investor by his initial capital $x_0 > 0$ and by his n-dimensional trading strategy $\pi = (\pi_t)_{t\in[0,T]}$ where π_t^i is the amount of money invested in stock i at time t. The corresponding wealth process $(X_t^\pi)_{t\in[0,T]}$ satisfies

$$dX_t^\pi = \left(X_t^\pi - \pi_t^\top \mathbf{1}_n\right) r_t \, dt + \pi_t^\top dR_t = \left(X_t^\pi r_t + \pi^\top (\mu_t - r_t \mathbf{1}_n)\right) dt + \pi_t^\top \sigma_t dW_t. \qquad (6)$$

So for the discounted wealth process we find

$$d(\beta_t X_t^\pi) = \beta_t \pi^\top (\mu_t - r_t \mathbf{1}_n) dt + \beta_t \pi_t^\top \sigma_t dW_t = \beta_t \pi_t^\top \sigma_t d\widetilde{W}_t. \qquad (7)$$

Since the interest rates are positive, β_t is uniformly bounded. Hence it is enough for (6) and (7) to be well defined that we require

$$\int_0^T \left(\|\pi_t^\top (\mu_t - r_t \mathbf{1}_n)\| + \|\pi_t^\top \sigma_t\|^2 \right) dt < \infty. \qquad (8)$$

A trading strategy satisfying this condition and $X_T^\pi \geq 0$ starting with $x_0 > 0$ will be called admissible. By $\mathcal{A}(x_0)$ we denote the corresponding class of trading strategies.

A utility function $U : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, twice continuously differentiable and satisfies the Inada conditions

$$\lim_{x\to\infty} U'(x) = 0 \quad \text{and} \quad \lim_{x\to 0} U'(x) = \infty. \qquad (9)$$

The inverse function of U' is denoted by I. The function I is defined on $(0,\infty)$, continuously differentiable and strictly decreasing with limits

$$\lim_{x\to\infty} I(x) = \infty \quad \text{and} \quad \lim_{x\to 0} I(x) = 0. \tag{10}$$

Given some initial capital $x_0 > 0$ any terminal wealth X_T^π has to satisfy the so-called budget constraint $\widetilde{E}[\beta_T X_T^\pi] = E[\beta_T Z_T X_T^\pi] \le x_0$, because by (7) $(\beta_t X_t^\pi)_{t\ge 0}$ is a $\widetilde{P}$-supermartingale. Here, $\widetilde{E}$ denotes the expectation w.r.t. the equivalent martingale measure $\widetilde{P}$.

As motivated in the introduction we we measure the risk by averaging for shortfall level $q > 0$ the loss $(X_T^\pi - q)^-$ w.r.t. some probability measure Q which is equivalent to P. By Z^Q we denote the Radon-Nikodym derivative of Q w.r.t. P. Further we use a strictly positive, $\mathcal{F}_T$-measurable factor γ for discounting the loss and call the non-negative number $E_Q[\gamma(X_T^\pi - q)^-] = E[\gamma\, Z^Q (X_T^\pi - q)^-]$ expected loss.

Let us fix some initial capital $x_0 > 0$, shortfall level $q > 0$, a bound for the expected loss $\varepsilon \ge 0$, and a strictly positive, $\mathcal{F}_T$-measurable discount factor γ. The dynamic optimization problem under risk constraints is

$$\begin{array}{rcll} \max\limits_{\pi\in\mathcal{A}(x_0)} E[U(X_T^\pi)] & & & \text{subject to} \\ \widetilde{E}[\beta_T X_T^\pi] & \le & x_0 & \text{(budget constraint)}, \\ E_Q[\gamma(X_T^\pi - q)^-] & \le & \varepsilon & \text{(risk constraint)}. \end{array} \tag{11}$$

The dynamic portfolio optimization problem can be splitted into two problems - the static and the representation problem. While the static problem is concerned with the form of the optimal terminal wealth the representation problem consists in the computation of the optimal trading strategy.

We shall first consider the static problem and use in Sections 3 through 6 for convenience a shorter notation: Simply X for the $\mathcal{F}_T$-measurable terminal wealth X_T^π, $Z_1 = \beta_T Z_T$ and $Z_2 = \gamma Z^Q$. Then the static optimization problem under risk constraints reads as

$$\begin{array}{rcll} \max\limits_{X\ge 0} E[U(X)] & & & \text{subject to} \\ E[Z_1 X] & \le & x_0 & \text{(budget constraint)}, \\ E[Z_2(X-q)^-] & \le & \varepsilon & \text{(risk constraint)}. \end{array} \tag{12}$$

As a technical assumption we impose the following conditions.

Assumption 2 For all $y > 0$ it holds

(A1) $E[Z_1 I(yZ_1)] < \infty$.

(A2) $E[Z_1^2 |I'(yZ_1)|] < \infty$.

(A3) $\quad Z_1 = Z_2$ or $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$.
Further $P(Z_1 > 0) > 0$ for all $c > 0$.

Here and in the following all (in)equalities are understood to hold almost surely. Throughout this paper we assume that the conditions of Assumption 2 are satisfied. They are not very restrictive, in particular (A3) comprises the important cases PEL and FEL.

3 The Merton and the portfolio insurer problem

If we consider the optimization problem (12) without the risk constraint, i.e.

$$\max_{X \geq 0} E[U(X)] \qquad \text{subject to} \quad E[Z_1 X] \leq x_0$$

we are faced with the so-called Merton problem. Its optimal solution is known to be (see [4, 5]) $X^M = f_M(y^M Z_1)$, where $f_M(x) = I(x)$ for $x > 0$ and $y^M > 0$ solves $E[Z_1 f_M(y Z_1)] = x_0$. And for the solution of the latter equation it is well known that the following theorem holds.

Theorem 3 *There exists a unique solution $y^M > 0$ of the equation*

$$E[Z_1 f_M(y Z_1))] = x_0.$$

If we consider the optimization problem (12) for the limiting case $\varepsilon = 0$, which corresponds to the risk constraint $X \geq q$, then we are faced with the so-called portfolio insurer (PI) problem. This problem reads as

$$\max_{X \geq 0} E[U(X)] \qquad \text{subject to} \quad E[Z_1 X] \leq x_0 \quad \text{and } X \geq q. \tag{13}$$

For $qE[Z_1] > x_0$ there are no admissible solutions. If $qE[Z_1] = x_0$, the only admissible choice is $X = q$. For $qE[Z_1] < x_0$ its optimal solution is known to be (see [2]) $X^{PI} = f_{PI}(y^{PI} Z_1)$, where

$$f_{PI}(x) = \begin{cases} I(x), & \text{for} \quad x \in (0, U'(q)], \\ q, & \text{for} \quad x \in (U'(q), \infty), \end{cases} \tag{14}$$

and $y^{PI} > 0$ solves $E[Z_1 f_{PI}(y Z_1)] = x_0$. For the solution of the latter equation there holds the following theorem. The proof is given in the appendix.

Theorem 4 *For $q \in (0, x_0/E[Z_1])$ there exists a unique solution $y^{PI} > y^M$ of the equation*

$$E[Z_1 f_{PI}(y Z_1)] = x_0,$$

and it holds $\quad y^{PI} \uparrow \infty \quad$ *for* $\quad q \uparrow x_0/E[Z_1]$.

For $qE[Z_1] = x_0$ the solution of the portfolio insurer problem is $X^{PI} = q$. Since $f_{PI}(\infty) = \lim_{x \to \infty} f_{PI}(x) = q$ and we have from the above theorem $y^{PI} \uparrow \infty$ for $q \uparrow x_0/E[Z_1]$, we can set $y^{PI} = \infty$ for $qE[Z_1] = x_0$ in order to incorporate this case into the above representation $X^{PI} = f_{PI}(y^{PI} Z_1)$.

4 Minimal shortfall risk

Dealing with the optimization problem (12) which arises from the Merton problem by imposing an additional risk constraint, one has to take care about selecting the bound ε for the shortfall risk. Choosing a value which is too small, then there is no admissible solution of the problem, because the risk constraint cannot be satisfied.

In this section we will find the smallest value $\underline{\varepsilon}$ of the shortfall risk measure to given initial capital x_0, shortfall level q and measures Q and $\widetilde{P}$. If $qE[Z_1] \leq x_0$ the bound $\varepsilon \geq 0$ can be chosen arbitrarily small since $X \equiv q$ satisfies the budget constraint $E[Z_1 X] \leq x_0$ and yields a risk measure of zero. Hence, for $qE[Z_1] \leq x_0$ we can set the minimal shortfall risk to $\underline{\varepsilon} = 0$.

Remark 5 Note that the condition $qE[Z_1] \leq x_0$ means that the payout q at time T can be hedged with initial capital x_0. For deterministic interest rates this would read as $\beta_T q \leq x_0$, so a simple investment in the riskless asset yields the payoff. In general under certain integrability conditions like those in Section 7 a hedging strategy can be computed using Malliavin calculus.

In order to find the minimal shortfall risk $\underline{\varepsilon}$ for $qE[Z_1] > x_0$ we consider the following risk minimization problem

$$\underline{\varepsilon} := \inf_{X \geq 0} E[Z_2 (X-q)^-] \quad \text{subject to} \quad E[Z_1 X] \leq x_0. \tag{15}$$

First we investigate the case $Z_1 = Z_2$ which holds in particular if the measures $\widetilde{P}$ and Q coincide and we use β_T also as discount factor γ for the risk constraint. The following theorem is proven in the appendix.

Theorem 6 *Let $qE[Z_1] > x_0$ and $Z_1 = Z_2$. Then the minimum value of the shortfall risk measure of problem (15) is $\underline{\varepsilon} = qE[Z_1] - x_0$.*

This minimum value is attained for all random variables $\underline{X}$ with

$$\underline{X} \in [0, q] \quad \textit{and} \quad E[Z_1 \underline{X}] = x_0.$$

Next, we consider a case which covers e.g. $Q = P$ and $\gamma = \beta_T$.

Theorem 7 *Let $qE[Z_1] > x_0$ and $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$.*

The unique optimal solution of problem (15) is

$$\underline{X} = q\mathbf{1}_{\{\lambda^* Z_1 \leq Z_2\}},$$

where λ^* solves $E[Z_1 \underline{X}] = x_0$ which has a unique solution. The minimum value of the shortfall risk measure is $\underline{\varepsilon} = qE[Z_2 \mathbf{1}_{\{\lambda^* Z_1 > Z_2\}}]$.

The proof of the above theorem is given in the appendix. It is based on the following lemma which follows from the consideration of the piecewise linear function

$$g(s) = \begin{cases} (z_1 - z_2)s + z_2 q, & \text{for} \quad s < q, \\ \quad z_1 s, & \text{for} \quad s \geq q. \end{cases}$$

Lemma 8 *Let $z_1, z_2 > 0$ and $g : [0, \infty) \to \mathbb{R}$ be defined by*

$$g(s) = g(s; z_1, z_2) = z_1 s + z_2 (s - q)^-.$$

Then it holds

for	$\inf g(s)$	infimum is attained for
$z_1 < z_2$	$z_1 q$	$s = q$
$z_1 = z_2$	$z_1 q = z_2 q$	all $s \in [0, q]$
$z_1 > z_2$	$z_2 q$	$s = 0$

5 The constrained optimization problem

In this section we deal with the portfolio optimization problem with an additional risk constraint given in (12). The corresponding terminal wealth is denoted by X^*. We denote by ε^M the expected loss that can occur in the case of Merton's problem, i.e. $\varepsilon^M := E[Z_2(X^M - q)^-]$. Obviously we have $\varepsilon^M > \underline{\varepsilon}$, where $\underline{\varepsilon}$ is the minimal shortfall risk obtained in the last section.

The following theorem gives the form of the optimal terminal wealth depending on the bound of shortfall risk ε. Part (iii) corresponds to [2, Proposition 4] where it was shown under the assumption that an optimal solution exists. The different cases are similar to those in [13, Theorem 3.3] for the strict convex loss functions, cf. the introduction. We call the risk constraint binding, if it holds with equality for the optimal terminal wealth.

Theorem 9

The optimal terminal wealth X^* of problem (12) satisfies:

(i) If $\varepsilon \geq \varepsilon^M$, then $X^* = X^M$ and if $E[|U(X^*)|] < \infty$, then X^* is the unique optimal solution. For $\varepsilon > \varepsilon^M$ the risk constraint is not binding.

(ii) If $0 < \varepsilon < \underline{\varepsilon}$, then there is no solution.

(iii) If $\underline{\varepsilon} < \varepsilon < \varepsilon^M$, then the optimal terminal wealth is given by $X^* = f(y_1^* Z_1, y_2^* Z_2)$ where

$$f(x_1, x_2) = \begin{cases} I(x_1), & \text{for} \quad x_1 \leq U'(q), \\ q, & \text{for} \quad U'(q) < x_1 \leq U'(q) + x_2, \\ I(x_1 - x_2), & \text{for} \quad x_1 > U'(q) + x_2. \end{cases}$$

Here, $y_1^*, y_2^* > 0$ solve the following system of equations

$$\begin{aligned} E[Z_1 f(y_1 Z_1, y_2 Z_2)] &= x_0 \\ E[Z_2 (f(y_1 Z_1, y_2 Z_2) - q)^-] &= \varepsilon. \end{aligned}$$

There exist unique solutions of this system of equations and if $E[|U(X^*)|] < \infty$, then X^* is the unique optimal solution. The risk constraint is binding.

(iv) Let $\varepsilon = \underline{\varepsilon}$. For $qE[Z_1] \le x_0$ it holds $\underline{\varepsilon} = 0$ and

$$X^* = \begin{cases} X^{PI}, & \text{for} \quad qE[Z_1] < x_0, \\ q, & \text{for} \quad qE[Z_1] = x_0. \end{cases}$$

For $qE[Z_1] > x_0$ it holds

(a) if $Z_1 = Z_2$ (in particular for $Q = \widetilde{P}$, $\gamma = \beta_T$), then $X^* = f_0(y_0^* Z_1)$, where

$$f_0(x) = \begin{cases} q, & \text{for} \quad x \le U'(q), \\ I(x), & \text{for} \quad x > U'(q), \end{cases}$$

and $y_0^* > 0$ is the unique solution of $E[Z_1 f_0(y_0 Z_1)] = x_0$. If $E[|U(X^*)|] < \infty$, then X^* is unique;

(b) if $P(Z_1 = Z_2) = 0$, then the optimal terminal wealth is equal to the risk minimizing terminal wealth given in Theorem 7, i.e.

$$X^* = \underline{X} = q \mathbf{1}_{\{\lambda^* Z_1 \le Z_2\}}$$

where λ^* is the unique solution of $qE[Z_1 \mathbf{1}_{\{\lambda Z_1 \le Z_2\}}] = x_0$.

Proof: (i) Let $\varepsilon \ge \varepsilon^M$ and let us denote by X^* the optimal solution of

$$\max E[U(X)] \ \text{ subject to } \quad E[Z_1 X] \le x_0 \text{ and } E[Z_2 (X - q)^-] \le \varepsilon.$$

Obviously, we have $E[Z_2 (X^M - q)^-] = \varepsilon^M \le \varepsilon$ and this implies

$$E[U(X^M)] \le \sup_{E[Z_2(X-q)^-] \le \varepsilon, \ E[Z_1 X] \le x_0} E[U(X)] = E[U(X^*)].$$

On the other hand we have $E[U(X^*)] \le E[U(X^M)]$, since one can consider the terminal wealth X^* as an admissible solution for the Merton problem. This implies that $E[U(X^M)] = E[U(X^*)]$. Hence $X^* = X^M$ and X^M is known to be the unique optimal solution of the Merton problem. Since X^M yields a risk measure ε^M, for $\varepsilon > \varepsilon^M$ the risk constraint is not binding.

(ii) In this case there is no admissible solution since the risk constraint cannot be fulfilled for any X satisfying the budget constraint $E[Z_1 X] \le x_0$, see Section 4.

(iii) Assume that there are unique solutions $y_1, y_2 > 0$ of the following system of equations

$$\begin{aligned} E[Z_1 f(y_1 Z_1, y_2 Z_2)] &= x_0 \\ E[Z_2(f(y_1 Z_1, y_2 Z_2) - q)^-] &= \varepsilon. \end{aligned}$$

The proof of the existence and uniqueness of the solutions y_1, y_2 is given in Section 6. First, we prove that the optimal terminal wealth is

$$X^* = f(y_1 Z_1, y_2 Z_2),$$

and after that we prove that X^* is the unique optimal solution. We use the following lemma which is proven in the appendix.

Lemma 10 *Let $q, z_1, z_2 > 0$ be fixed and*

$$g(x) = g(x; z_1, z_2) = U(x) - z_1 x - z_2(x-q)^- \quad \textit{for} \quad x \geq 0.$$

Then $x^* = f(z_1, z_2)$ is the unique maximizer for $g(x)$.

Applying the above lemma pointwise for all $z_1 = y_1 Z_1(\omega)$ and $z_2 = y_2 Z_2(\omega)$ it follows that $\xi^* = f(y_1 Z_1, y_2 Z_2)$ is the unique maximizer of $g(\xi; y_1 Z_1, y_2 Z_2)$.

If y_1, y_2 are chosen as solutions of the system of equations given in the theorem, i.e., $y_1 = y_1^*$ and $y_2 = y_2^*$, then it follows $\xi^* = X^*$ and the budget as well as the risk constraint are binding.
Let Y be any admissible solution satisfying the budget constraint and the risk constraint with $\underline{\varepsilon} < \varepsilon < \varepsilon^M$. We have

$$\begin{aligned} E[U(X^*)] - E[U(Y)] &= E[U(X^*) - E[U(Y)] - y_1^* x_0 + y_1^* x_0 - y_2^* \varepsilon + y_2^* \varepsilon \\ &\geq E[U(X^*)] - y_1^* E[Z_1 X^*] - y_2^* E[Z_2(X^* - q)^-] \\ &\quad - \Big(E[U(Y)] - y_1^* E[Z_1 Y] - y_2^* E[Z_2(Y-q)^-] \Big) \\ &= E[g(X^*; y_1 Z_1, y_2 Z_2) - g(Y; y_1 Z_1, y_2 Z_2)] \geq 0, \end{aligned} \tag{16}$$

where the first inequality follows from the budget constraint and the constraint for the risk holding with equality for X^*, while holding with inequality for Y. The last inequality is a consequence of the above lemma. Hence we obtain that X^* is optimal. In order to prove the uniqueness of X^* we assume that there exists another admissible solution Y with $Y \neq X^*$ on the set D with $P(D) > 0$ and $E[U(X^*)] = E[U(Y)]$. Lemma 10 implies on D the strict inequality

$$g(X^*) = g(X^*; y_1^* Z_1, y_2^* Z_2) > g(Y; y_1^* Z_1, y_2^* Z_2) = g(Y).$$

Under the assumption $E[|U(X^*)|] < \infty$ we conclude that

$$\infty > E[g(X^*)] > E[g(Y)] > -\infty.$$

Hence the last inequality (16) is strict. Together with the uniqueness of y_1^*, y_2^* which we prove in Section 6 this proves the uniqueness of the optimal terminal wealth X^*.

(iv) For $qE[Z_1] \le x_0$ there is a minimal risk of $\underline{\varepsilon} = 0$ and the constrained optimization problem (12) coincides with the portfolio insurer problem (13) whose solution is known to be $X^* = X^{PI}$.

For $qE[Z_1] > x_0$ there is a minimal risk of $\underline{\varepsilon} > 0$. To prove the case (a) we consider the function $g(x) = g(x; z) = U(x) - zx$ for $x \in [0, q]$ and fixed $z > 0$. We can prove that $x^* = f_0(z)$ is the unique maximizer of g(x) on $[0, q]$. To this end we extend the domain of definition of g from $[0, q]$ to $[0, \infty)$. On the extended domain $x_1 = I(z)$ is the unique maximizer of g. If $q > x_1$ then $x^* = x_1$ is the unique maximizer of g on $[0, q]$. Since the function I is strictly decreasing from $q > x_1 = I(z)$ it follows $z > U'(q)$.

Otherwise, if $q \le x_1$ the strict concavity of U implies that g is strictly increasing on $[0, q]$, hence $x^* = q$ is the unique maximizer of g on $[0, q]$. In this case $q \le x_1 = I(z)$ implies $z \le U'(q)$.

Assume Y is any terminal wealth satisfying $E[Z_1 Y] \le x_0$ and $0 < Y \le q$. Then it holds

$$
\begin{aligned}
E[U(X^*)] - E[U(Y)] &\ge E[U(X^*)] + y_0^*\Big(x_0 - E[Z_1 X^*]\Big) \\
&- \Big(E[U(Y)] + y_0^*(x_0 - E[Z_1 Y])\Big) \\
&= E\Big[U(X^*) - y_0^* Z_1 X^* - (U(Y) - y_0^* Z_1 Y)\Big] \\
&= E[g(X^*; Z_1) - g(Y; Z_1)] \ge 0,
\end{aligned}
$$

where the last inequality follows from the fact that $X^* = f_0(Z_1)$ is the pointwise maximizer of $g(X; Z_1)$.

The proof of the uniqueness is analogous to the corresponding proof in (iii) and relies on the uniqueness of the solution y_0 of the equation $E[Z_1 f_0(y Z_1)] = x_0$. This is the assertion of the following lemma which is proven in the appendix.

Lemma 11 *There exists a unique solution $y_0^* > 0$ of the equation*

$$E[Z_1 f_0(y_0 Z_1)] = x_0.$$

To prove case (b) we observe that the set of admissible solutions consists of random variables which coincide with the risk minimizing terminal wealth $\underline{X}$, hence $\underline{X}$ is optimal. □

6 Properties, existence and uniqueness of the solution of the budget and risk equation

In the proof of Theorem 9 (iii) we assumed that a unique solution y_1^*, y_2^* of the equations

$$\begin{aligned} E[Z_1 f(y_1 Z_1, y_2 Z_2)] &= x_0 \\ E[Z_2 (f(y_1 Z_1, y_2 Z_2) - q)^-] &= \varepsilon. \end{aligned}$$

exists, i.e. for the optimal solution we have equality in the budget as well as risk constraint. This section is devoted to the proof of this assertion.

We first study some properties of the left-hand sides of these equations and define the *budget function*

$$F(y_1, y_2) \quad = \quad E[Z_1 f(y_1 Z_1, y_2 Z_2)] \tag{17}$$

and the *risk function*

$$G(y_1, y_2) \quad = \quad E[Z_2 (f(y_1 Z_1, y_2 Z_2) - q)^-] \tag{18}$$

for $(y_1, y_2) \in \mathcal{D} \subset \mathbb{R}_+^2$. For the subsequent considerations it will be sufficient to choose the domain

$$\mathcal{D} = \{(y_1, y_2) \in \mathbb{R}_+^2 \, : \, y_1 \in [y^M, \overline{y}_1), \; y_2 \in [0, \overline{y}_2(y_1))\},$$

where

$$\begin{aligned} \overline{y}_1 \quad &= \quad \begin{cases} y^{PI}, & \text{if} \quad qE[Z_1] < x_0, \\ \infty, & \text{if} \quad qE[Z_1] \geq x_0, \end{cases} \\ \text{and} \quad \overline{y}_2 = \overline{y}_2(y_1) \quad &= \quad \begin{cases} y_1, & \text{if} \quad Z_1 = Z_2, \\ \infty, & \text{if} \quad P(\lambda Z_1 = Z_2) = 0 \text{ for all } \quad \lambda > 0. \end{cases} \end{aligned} \tag{19}$$

Here y^M and y^{PI} denote the unique solutions of the budget equation for the Merton and portfolio insurer problem given in Theorem 3 and 4, respectively. The restriction $y_2 < y_1$ in case of $Z_1 = Z_2$ ensures that $f(y_1 Z_1, y_2 Z_2) = I(y_1 Z_1 - y_2 Z_2) = I((y_1 - y_2) Z_1$ for $y_1 Z_1 > U'(q) + y_2 Z_2$ and $Z_1 = Z_2$ is well defined.

It will be convenient to use the following notation for random events occurring in the definition of the optimal terminal wealth. We define

$$\begin{aligned} A \quad &= \quad A(y_1) := \{y_1 Z_1 \leq U'(q)\}, \\ B \quad &= \quad B(y_1, y_2) := \{U'(q) < y_1 Z_1 \leq U'(q) + y_2 Z_2\}, \\ C \quad &= \quad C(y_1, y_2) := \{y_1 Z_1 > U'(q) + y_2 Z_2\}. \end{aligned} \tag{20}$$

Then $f(y_1 Z_1, y_2 Z_2)$ can be written as

$$f(y_1 Z_1, y_2 Z_2) = \begin{cases} I(y_1 Z_1) & \text{on} \quad A(y_1), \\ q & \text{on} \quad B(y_1, y_2), \\ I(y_1 Z_1 - y_2 Z_2) & \text{on} \quad C(y_1, y_2). \end{cases} \tag{21}$$

6.1 Properties of the budget and risk functions

The next lemma contains some properties of the budget function $F(y_1, y_2)$. The proof can be found in the appendix. Note that the partial derivatives of F at the boundary of $\mathcal{D}$, i.e. for $y_1 = y^M$ or $y_2 = 0$, are treated as the corresponding one-sided derivatives.

Lemma 12 *For the budget function $F(y_1, y_2)$ defined in (17) we have*

(F1) $F(y_1, y_2)$ *is continuous and continuously differentiable on* $\mathcal{D}$.

(F2) $F_1(y_1, y_2) := \frac{\partial}{\partial y_1} F(y_1, y_2) < 0 \quad$ *on* $\mathcal{D}$.

(F3) $F_2(y_1, y_2) := \frac{\partial}{\partial y_2} F(y_1, y_2) > 0 \quad$ *on* $\mathcal{D}$.

(F4) $F(y^M, 0) = x_0$ *and* $F(y_1, 0) < x_0$ *for* $y_1 \in (y^M, \overline{y}_1)$.

(F5) There exists $F(y_1, \overline{y}_2) := \lim\limits_{y_2 \uparrow \overline{y}_2} F(y_1, y_2)$ *and* $F(y_1, \overline{y}_2) > x_0$ *for* $y_1 \in (y^M, \overline{y}_1)$.

The lemma below, which is proven in the appendix, gives some properties of the risk function $G(y_1, y_2)$ defined in (18) which represents the left-hand side of the risk equation. The monotonicity of the function $f(x_1, x_2)$ w.r.t. the first variable implies that for all $x_1 \le U'(q) + x_2$ it holds $f(x_1, x_2) \ge q$ while for $x_1 > U'(q) + x_2$ it holds $f(x_1, x_2) < q$. Using these properties, the function G can be written as

$$G(y_1, y_2) = E[Z_2(q - f(y_1 Z_1, y_2 Z_2))\mathbf{1}_{C(y_1, y_2)}]. \tag{22}$$

As before, derivatives of G at the boundary of $\mathcal{D}$ are treated as the corresponding one-sided derivatives.

Lemma 13 *For the risk function $G(y_1, y_2)$ defined in (18) and (22) there hold the following properties.*

(G1) $G(y_1, y_2)$ *is continuous and continuously differentiable on* $\mathcal{D}$.

(G2) $G_1(y_1, y_2) := \frac{\partial}{\partial y_1} G(y_1, y_2) > 0 \quad$ *on* $\mathcal{D}$.

(G3) $G_2(y_1, y_2) := \frac{\partial}{\partial y_2} G(y_1, y_2) < 0 \quad$ *on* $\mathcal{D}$.

(G4) $G(y^M, 0) = \varepsilon^M$ *and* $G(y_1, 0) > \varepsilon^M$ *for* $y_1 \in (y^M, \overline{y}_1)$.

6.2 Solution of the budget equation

Next we study the solution of the budget equation $F(y_1, y_2) = x_0$. The next theorem describes properties of the set of solutions $\{(y_1, y_2) \in \mathcal{D} : F(y_1, y_2) = x_0\}$ in particular we study the dependence of y_2 on y_1 in this set and the asymptotic behavior of $y_2(y_1)$ if y_1 reaches the boundary values y^M and $\overline{y}_1 = y^{PI}$ or $\overline{y}_1 = \infty$. The proof is provided in appendix.

Theorem 14

(1) For all $y_1 \in (y^M, \overline{y}_1)$ there exists a unique $y_2 = y_2(y_1) \in (0, \overline{y}_2)$ satisfying $F(y_1, y_2) = x_0$, where F is given in (17).

(2) The function $y_1 \mapsto y_2(y_1)$ is strictly increasing, continuous and continuously differentiable on $(y^M, \overline{y}_1)$.

(3) For the limit of $y_2(y_1)$ for $y_1 \downarrow y^M$ it holds $y_2(y^M) := \lim_{y_1 \downarrow y^M} y_2(y_1) = 0.$

(4) For $Z_1 = Z_2$ we have the following limits:

		limit for $y_1 \to \overline{y}_1$ of		
	$\overline{y}_1$	$y_2(y_1)$	$y_1 - y_2(y_1)$	$f(y_1 Z_1, y_2(y_1) Z_2)$
$qE[Z_1] < x_0$	y^{PI}	y^{PI}	0	$f_{PI}(y^{PI} Z_1) = X^{PI}$
$qE[Z_1] = x_0$	∞	∞	0	q
$qE[Z_1] > x_0$	∞	∞	y_0^*	$f_0(y_0^* Z_1)$

where f_{PI} is given in (14), f_0 is given in Theorem 9, (iv,a) and $y_0^* > 0$ is the unique solution of the equation $E[Z_1 f_0(y_0 Z_1)] = x_0$.

(5) If $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$ we have:

		limit for $y_1 \to \overline{y}_1$ of		
	$\overline{y}_1$	$y_2(y_1)$	$\frac{y_2(y_1)}{y_1}$	$f(y_1 Z_1, y_2(y_1) Z_2)$
$qE[Z_1] < x_0$	y^{PI}	∞	∞	$f_{PI}(y^{PI} Z_1) = X^{PI}$
$qE[Z_1] = x_0$	∞	∞	∞	q
$qE[Z_1] > x_0$	∞	∞	λ^*	$q\mathbf{1}_{\{\lambda^* Z_1 \le Z_2\}} = \underline{X}$

where $\lambda^* > 0$ is the unique solution of the equation (see Theorem 7)
$E[Z_1 \underline{X}] = q\, E[Z_1 \mathbf{1}_{\{\lambda Z_1 \le Z_2\}}] = x_0$.

Remark 15

(i) The assertions in (4) and (5) for the case $qE[Z_1] = x_0$ can be embedded as limiting cases into the case $qE[Z_1] > x_0$ with $y_0^* = 0$ and $\lambda^* = 0$, respectively.

(ii) Note that the the limits $\lim_{y_1 \to \overline{y}_1} f(y_1 Z_1, y_2(y_1) Z_2)$ given in (4) and (5) yield the minimal shortfall risk $\underline{\varepsilon}$, where $\underline{\varepsilon} = 0$ for $qE[Z_1] \le x_0$ while for $qE[Z_1] > x_0$ the minimal shortfall risk $\underline{\varepsilon}$ is given in Theorems 6 and 7.

6.3 Solution of the budget and risk equations

After investigating the set of solutions $(y_1, y_2(y_1))$ of the budget equation now we study the solutions of the system of the budget as well as the risk equation in the form $(y_1^*, y_2^*) = (y_1^*, y_2(y_1^*))$. We require $\varepsilon \in (\underline{\varepsilon}, \varepsilon^M)$, i.e. the bound ε in the risk constraint is bounded from above by the risk measure $\varepsilon^M = G(y^M, 0)$ associated to the optimal terminal wealth of the Merton problem, and bounded from below by the minimal risk $\underline{\varepsilon}$ obtained in Section 4.

Theorem 16 *Let $q, x_0 > 0$ and $\varepsilon \in (\underline{\varepsilon}, \varepsilon^M)$. Then there exist unique solutions $y_1^* \in (y^M, \overline{y}_1)$, $y_2^* \in (0, \overline{y}_2(y_1^*))$ of the system of equations*

$$\begin{aligned} F(y_1, y_2) &= x \\ G(y_1, y_2) &= \varepsilon. \end{aligned}$$

Proof: We consider the risk function G on the set of solutions for the budget equation studied in Theorem 14 and define the function

$$h(y_1) = G(y_1, y_2(y_1)) \quad \text{for} \quad y_1 \in [y^M, \overline{y}_1),$$

with the function $y_2(y_1)$ defined in Theorem 14, (2). For $y_1 = y^M$ the corresponding limit $y_2(y^M) = 0$ given in Theorem 14, (3) is used and from property (G4) it follows

$$h(y^M) = G(y^M, 0) = \varepsilon^M > \varepsilon,$$

due the assumptions of this theorem. Next, we study the limit of $h(y_1)$ for $y_1 \to \overline{y}_1$. From Theorem 14, (4) and (5) and Remark 15, it is known, that the limit $\lim_{y_1 \to \overline{y}_1} f(y_1 Z_1, y_2(y_1) Z_2)$ yields the minimal shortfall risk $\underline{\varepsilon}$, where $\underline{\varepsilon} = 0$ for $qE[Z_1] \le x_0$ while for $qE[Z_1] > x_0$ the minimal shortfall risk $\underline{\varepsilon}$ is given in Theorem 6 and 7. Hence

$$h(\overline{y}_1) := \lim_{y_1 \to \overline{y}_1} h(y_1) = \lim_{y_1 \to \overline{y}_1} G(y_1, y_2(y_1)) = \underline{\varepsilon} < \varepsilon.$$

Property (G1) and Theorem 14, (2) imply that $h(y_1)$ is continuous and continuously differentiable on $(y^M, \overline{y}_1)$. Below we prove, that the derivative h' is negative and it follows that $h(y_1)$ is strictly decreasing. This property yields that there is a unique solution $y_1^* \in (y^M, \overline{y}_1)$ of the equation $h(y_1) = \varepsilon$. Finally, Theorem 14 implies that there is a unique solution $(y_1^*, y_2^*) = (y_1^*, y_2(y_1^*))$ of the system of equations given in the theorem.

It remains to prove $h'(y_1) < 0$. We apply the chain rule to derive

$$\begin{aligned} h'(y_1) &= \frac{\partial}{\partial y_1} G(y_1, y_2(y_1)) + \frac{\partial}{\partial y_2} G(y_1, y_2(y_1)) y_2'(y_1) \\ &= \frac{\partial}{\partial y_1} G(y_1, y_2(y_1)) - \frac{\partial}{\partial y_2} G(y_1, y_2(y_1)) \frac{\frac{\partial}{\partial y_1} F(y_1, y_2(y_1))}{\frac{\partial}{\partial y_2} F(y_1, y_2(y_1))}. \end{aligned} \tag{23}$$

Abbreviating the partial derivatives $\frac{\partial}{\partial y_k} F(y_1, y_2(y_1))$ by F_k and $\frac{\partial}{\partial y_k} G(y_1, y_2(y_1))$ by G_k, $k = 1, 2$, we rewrite the derivative function as

$$h'(y_1) = G_1 - G_2 \frac{F_1}{F_2} = \frac{1}{F_2}(G_1 F_2 - G_2 F_1)$$

and it suffices to prove that $G_1 F_2 - G_2 F_1 < 0$.
We denote by $V = I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C$. Then we have

$$\begin{array}{rcll} G_1 & = & -E[Z_2 I'(y_1 Z_1 - y_2 Z_2) Z_1 \mathbf{1}_C] & = -E[Z_1 Z_2 V] \\ G_2 & = & E[Z_2 I'(y_1 Z_1 - y_2 Z_2) Z_2 \mathbf{1}_C] & = E[Z_2^2\, V] \\ F_1 & = & E[Z_1 I'(y_1 Z_1) Z_1 \mathbf{1}_A] & \\ & & +E[Z_1 I'(y_1 Z_1 - y_2 Z_2) Z_1 \mathbf{1}_C] & > E[Z_1^2\, V] \\ F_2 & = & -E[Z_1 I'(y_1 Z_1 - y_2 Z_2) Z_2 \mathbf{1}_C] & = -E[Z_1 Z_2 V]. \end{array}$$

Substituting the above expressions yields

$$G_1 F_2 - G_2 F_1 < E[Z_1 Z_2 V]\; E[Z_1 Z_2 V] - E[Z_2^2\, V]\; E[Z_1^2\, V]. \tag{24}$$

Defining $U_1 = Z_1 \sqrt{-V}$ and $U_2 = Z_2 \sqrt{-V}$, substituting the above expressions into (24) and applying Cauchy-Schwarz inequality yields

$$\begin{array}{rcl} G_1 F_2 - G_2 F_1 & < & \Big(E[Z_1 Z_2 V]\Big)^2 - E[Z_2^2 V]\; E[Z_1^2 V] \\ & = & \Big(E[U_1 U_2]\Big)^2 - E[U_2^2]\; E[U_1^2] \quad \leq 0, \end{array}$$

hence $h'(y_1) = \frac{1}{F_2}(G_1 F_2 - G_2 F_1) < 0$ because $F_2 > 0$.

□

Remark 17 Usually the solutions y_1^* and y_2^* of the budget and risk equation cannot be computed explicitly because the dependence of F and G on y_1 and y_2 can be quite involved. Therefore iterative numerical solution procedures have to be applied. The findings of this section ensure the existence and uniqueness of the solutions y_1^* and y_2^*. Further the structure of the set $\mathcal{D}$ gives hints for choosing the initial guess and the monotonicity properties derived in Lemmas 12, 13 can simplify an iterative solution procedure considerably.

For the construction of a numerical scheme one can proceed as follows. Consider the function $h(y_1) = G(y_1, y_2(y_1))$ which has been used in the proof of Theorem 16. There it has been shown that $h(y^M) = \varepsilon^M > \varepsilon,\ h(y_1) \to \underline{\varepsilon} < \varepsilon$ for $y_1 \to \overline{y}_1$ and the derivative $h'(y_1)$ which is given in (23) is strictly negative. This can be used for a Newton iteration method for the solution of the risk equation $h(y_1) = G(y_1, y_2(y_1)) = \varepsilon$ with the initial guess $y_1^0 = y^M$ for which we have $y_2(y^M) = 0$. For the Newton method the iteration sequence $y_1^0, y_1^1, y_1^2, \ldots$ is defined by $y_1^{k+1} = y_1^k - (h(y_1^k) - \varepsilon)/h'(y_1^k)$.

In each iteration step we need the value of $y_2(y_1^k)$ for the computation of the values of h and its derivative h'. The value of $y_2(y_1^k)$ can be computed from the solution of the budget equation $l(y_2) = F(y_1^k, y_2) = x_0$, where we can again use the Newton iteration method. For the derivative of l it is known from Lemma 12, (F3) that $l'(y_2) = \frac{\partial}{\partial y_2} F(y_1, y_2) > 0$.

For simple financial market models, e.g. the classical Black Scholes model with constant drift, volatility and risk-free interest rate and for logarithmic or power utility functions one can find analytical expressions for the expectations contained in the functions F and G and their derivatives. Hence to given y_1 and y_2 the values of F and G and their derivatives can be computed exactly. This is the case in [2, 7, 8, 10]. For more sophisticated market models, e.g. in case of a hidden Markov model for the drift as in [9, 18] the expectations involved in F and G usually cannot be evaluated analytically but have to be approximated e.g. by using Monte Carlo simulation.

7 Optimal trading strategies

We shall derive optimal trading strategies for the most interesting case when both constraints are binding, i.e. we assume $\varepsilon \in (\underline{\varepsilon}, \varepsilon^M)$. Then by Theorem 9 (iii)

$$X_T = f(y_1 Z_1, y_2 Z_2),$$

where $Z_1 = \beta_T Z_T$, $Z_2 = \gamma\, Z^Q$ and y_1, y_2 are the optimal unique parameters given by

$$E[Z_1 X_T] = x_0, \quad E[Z_2 (X_T - q)^-] = \varepsilon.$$

Let us denote by $\mathcal{F}^{\widetilde{W}}$ the augmented filtration generated by $\widetilde{W}$. For random variables $Y \in \mathbb{D}_{1,p} \subset L^p(\widetilde{P}, \mathcal{F}_T^{\widetilde{W}})$ we introduce the Malliavin derivatives $\mathrm{D}Y = (\mathrm{D}_t Y)_{t\in[0,T]}$ w.r.t. $\widetilde{W}$ as introduced in [17]. For details and suitable chain rules we also refer to [18]. Our aim is to compute the strategies by comparing martingale representation and Clark's formula in $\mathbb{D}_{1,1}$ (cf. [14]): For $Y \in \mathbb{D}_{1,1}$

$$Y = \widetilde{E}[Y] + \int_0^T \widetilde{E}[\mathrm{D}_t Y \,|\, \mathcal{F}_t^{\widetilde{W}}]^\top d\widetilde{W}_t.$$

Note that we use the convention that for m-dimensional Y, the matrix $\mathrm{D}_t Y$ is $n \times m$-dimensional with $(\mathrm{D}_t Y)_{i,j} = \mathrm{D}_t^i Y^j$, where D^i denotes the operator w.r.t. $\widetilde{W}^i$.

We cannot apply standard chain rules directly since f is not differentiable on the set $\{(z_1, z_2) \,:\, z_1 = U'(q) \text{ or } z_1 - z_2 = U'(q)\}$. But similar as in Section 5.1 of [16] we can prove a chain rule if we use the following piecewise derivatives. Having certain applications in mind, cf. Remark 21 below, we need different conditions than [16, Proposition 5.2] and thus include the proof. As substitute for the derivative w.r.t the first component we use

$$f_1(z_1, z_2) = \begin{cases} I'(z_1), & \text{for} \quad z_1 \le U'(q), \\ 0, & \text{for} \quad U'(q) < z_1 \le U'(q) + z_2, \\ I'(z_1 - z_2), & \text{for} \quad z_1 > U'(q) + z_2, \end{cases} \qquad F_1 = f_1(y_1 Z_1, y_2 Z_2)$$

and for the second component:

$$f_2(z_1,z_2)=\begin{cases}0, & \text{for} \quad z_1\le U'(q),\\ 0, & \text{for} \quad U'(q)<z_1\le U'(q)+z_2,\\ -I'(z_1-z_2), & \text{for} \quad z_1>U'(q)+z_2,\end{cases} \qquad F_2=f_2(y_1Z_1,y_2Z_2).$$

Theorem 18 *Suppose that the conditions on the coefficients r, μ, σ in Section 2 are satisfied and that*

(i) $\mathcal{F}=\mathcal{F}^{\widetilde{W}}$,

(ii) β_T^{-1} *is bounded,*

(iii) $\beta_T, Z_1, Z_2\in\mathbb{D}_{1,\frac{p}{p-1}}$ *and* $I'(y_1Z_1), I'(y_1Z_1-y_2Z_2)\mathbf{1}_C\in L^p(\widetilde{P})$ *for some* $p\in(1,2]$,

(iv) $\beta_TX_T\in L^2(\widetilde{P})$.

Then an optimal trading strategy is given by

$$\pi_t=\beta_t^{-1}(\sigma_t^\top)^{-1}\widetilde{E}[(D_t\beta_T)X_T+y_1\beta_T(D_tZ_1)F_1+y_2\beta_T(D_tZ_2)F_2\,|\,\mathcal{F}_t].$$

Proof:

(a) We want to apply Clark's formula to

$$\beta_TX_T=\beta_Tf(y_1Z_1,y_2Z_2)=g(\beta_T,y_1Z_1,y_2Z_2),$$

where $g(z_0,z_1,z_2)=z_0f(z_1,z_2)$.

(b) We shall denote by $\|\cdot\|$, $\|\cdot\|_p$ and $\|\cdot\|_{L^2}$ the Euclidean norm, and the norms in $L^p=L^p(\Omega,\widetilde{P})$ and $L^2((0,T))$, respectively. We can choose $f^{(n)}\in C^1((0,\infty)^2)$, $n\in\mathbb{N}$, such that

$$0<f^{(n)}\le f,\quad f^{(n)}\to f\quad(n\to\infty) \tag{25}$$

and (denoting the partial derivatives of $f^{(n)}$ by $f_1^{(n)}$, $f_2^{(n)}$)

$$|f_i^{(n)}(z_1,z_2))|<K(1+|I'(z_1)|+|I'(z_1-z_2)\,\mathbf{1}_{(0,\infty)}(z_1-z_2)|),\quad i=1,2, \tag{26}$$

for some constant K, and

$$f_i^{(n)}(z_1,z_2)\to f_i(z_1,z_2)\quad(n\to\infty)\qquad\text{for all } z_1,z_2>0,\quad i=1,2. \tag{27}$$

From conditions (ii) and (iv) follows $f(y_1Z_1,y_2Z_2)\in L^2\subseteq L^p\subseteq L^1$, hence by (25) $f^{(n)}(y_1Z_1,y_2Z_2)\in L^p$, $n\in\mathbb{N}$. So Lebesgue's theorem on dominating convergence implies

$$\|f^{(n)}(y_1Z_1,y_2Z_2)-X_T\|_p\to 0\quad(n\to\infty). \tag{28}$$

By the boundedness of β_T we also get

$$\|\beta_T f^{(n)}(y_1 Z_1, y_2 Z_2) - \beta_T X_T\|_1 \to 0 \quad (n \to \infty). \tag{29}$$

From the conditions in the theorem we have $I'(y_1 Z_1), I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C \in L^p$. Thus $\beta_T F_1, \beta_T F_2 \in L^p$ since β_T is bounded. And by (26) also $\beta_T f_i^{(n)}(y_1 Z_1, y_2 Z_2) \in L^p$, so

$$\|\beta_T f_i^{(n)}(y_1 Z_1, y_2 Z_2) - \beta_T F_i\|_p \to 0 \quad (n \to \infty), \quad i = 1, 2. \tag{30}$$

(c) Now let $g^{(n)}(z_0, z_1, z_2) = z_0 f^{(n)}(z_1, z_2)$. Then $g^{(n)}$ has derivatives

$$\nabla g^{(n)}(z_0, z_1, z_2) = \begin{pmatrix} f^{(n)}(z_1, z_2) \\ z_0 f_1^{(n)}(z_1, z_2) \\ z_0 f_2^{(n)}(z_1, z_2) \end{pmatrix}.$$

We shall denote

$$Z = (\beta_T, y_1 Z_1, y_2 Z_2)^\top, \quad G^{(n)} = g^{(n)}(Z), \quad \nabla G^{(n)} = \nabla g^{(n)}(Z).$$

So by (25), (28), (30)

$$G^{(n)} \to g(Z) = \beta_T X_T \quad (\text{in } L^1), \quad \nabla G^{(n)} \to (X_T, \beta_T F_1, \beta_T F_2)^\top \quad (\text{in } L^p). \tag{31}$$

By condition (iii) we know that $Z \in (\mathbb{D}_{1,\frac{p}{p-1}})^3$. So we can use for $G^{(n)}$ a chain rule like Proposition 8.4 in [18] yielding $G^{(n)} \in \mathrm{D}_{1,1}$ with $\mathrm{D}G^{(n)} = (\mathrm{D}Z)\nabla G^{(n)}$. Further $Z \in (\mathbb{D}_{1,\frac{p}{p-1}})^3$ implies $\| \|\mathrm{D}Z\|_{L^2}\|_{\frac{p}{p-1}} < \infty$. Using Hölder's Inequality and (30) we obtain

$$\begin{aligned}
&\Big\|\|\mathrm{D}G^{(n)} - (\mathrm{D}Z)(X_T, \beta_T F_1, \beta_T F_2)^\top\|_{L^2}\Big\|_1 \\
&\quad \le \Big\| \|\nabla G^{(n)} - (X_T, \beta_T F_1, \beta_T F_2)^\top\| \, \|\mathrm{D}Z\|_{L^2}\Big\|_1 \\
&\quad \le \Big\|\|\nabla G^{(n)} - (X_T, \beta_T F_1, \beta_T F_2)^\top\|\Big\|_p \; \| \|\mathrm{D}Z\|_{L^2}\|_{\frac{p}{p-1}}
\end{aligned}$$

which converges for $n \to \infty$ to 0 because of (31). Therefore $\mathrm{D}G^{(n)}$ converges to $(\mathrm{D}Z)(X_T, \beta_T F_1, \beta_T F_2)^\top$ in $\|\|\cdot\|_{L^2}\|_1$ and $G^{(n)}$ to $\beta_T X_T$ in L^1. Since D is a closed operator on $\mathbb{D}_{1,1}$ we find $\beta_T X_T \in \mathbb{D}_{1,1}$ with $\mathrm{D}(\beta_T X_T) = (\mathrm{D}Z)(X_T, \beta_T F_1, \beta_T F_2)^\top$.

(d) Thus we can use Clark's formula and by condition (iv) also the Martingale Representation Theorem can be applied, yielding

$$x_0 + \int_0^T \beta_t \pi_t^\top \sigma \, d\widetilde{W}_t = \beta_T X_T = x_0 + \int_0^T \widetilde{E}[D_t(\beta_T X_T) \,|\, \mathcal{F}_t]^\top \, d\widetilde{W}_t.$$

Due to uniqueness we get π by comparison of the integrands. Using (c) we obtain

$$\pi_t = \beta_t^{-1}(\sigma_t^\top)^{-1} \widetilde{E}[\mathrm{D}_t(\beta_T X_T) \,|\, \mathcal{F}_t].$$

With $\mathrm{D}(y_1 Z_1) = y_1 \mathrm{D} Z_1$, $\mathrm{D}(y_2 Z_2) = y_2 \mathrm{D} Z_2$ the representation in the theorem follows. Finally the martingale representation theorem guarantees $\widetilde{E}[\int_0^T \|\beta_t \sigma_t^\top \pi_t\|^2 dt] < \infty$. Thus by equivalence of $\widetilde{P}$ and P we also have

$$\int_0^T \|\beta_t \sigma_t^\top \pi_t\|^2 dt < \infty \quad \text{and} \quad \int_0^T \|\beta_t \pi_t^\top (\mu_t - r_t^\top \mathbf{1}_n)\| dt < \infty,$$

where the second inequality follows from the first, using (5) and the Cauchy-Schwarz Inequality. So by the boundedness of β_T^{-1} also (8) holds. □

We shall now provide the strategies in more detail for the two important cases which we discussed in the introduction. First we look at PEL, corresponding to a bound on the hedging price for the shortfall.

Corollary 19 *If $Q = \widetilde{P}$ and $\gamma = \beta_T$ (i.e. $Z_2 = Z_1$) and the conditions of Theorem 18 are satisfied we have*

$$X_T = f_P(Z_1), \qquad \text{where } f_P(z) = \begin{cases} I(y_1^P z), & z \le \frac{U'(q)}{y_1^P}, \\ q, & \frac{U'(q)}{y_1^P} < z \le \frac{U'(q)}{y_1^P - y_2^P}, \\ I((y_1^P - y_2^P) z), & z > \frac{U'(q)}{y_1^P - y_2^P}, \end{cases}$$

and $y_1^P > 0$, $y_2^P > 0$ are given as the unique solutions of

$$\widetilde{E}[\beta_T f_P(Z_1)] = x_0, \quad \widetilde{E}[\gamma (f_P(Z_1) - q)^-] = \varepsilon.$$

The optimal trading strategy is given by

$$\pi_t = \beta_t^{-1} (\sigma_t^\top)^{-1} \widetilde{E}[X_T \, D_t \beta_T + \beta_T G_P \, D_t Z_1 \,|\, \mathcal{F}_t].$$

where

$$G_P = y_1^P I'(y_1^P Z_1) \mathbf{1}_{\{y_1^P Z_1 \le U'(q)\}} + (y_1^P - y_2^P) I'((y_1^P - y_2^P) Z_1) \mathbf{1}_{\{(y_1^P - y_2^P) Z_1 \ge U'(q)\}}.$$

Next we look at FEL, corresponding to putting a bound on the expected shortfall.

Corollary 20 *If $Q = P$ (i.e. $Z_2 = \gamma$) and the conditions of Theorem 18 are satisfied, then it holds*

$$X_T = f_F(Z_1), \qquad \text{where } f_F(z) = \begin{cases} I(y_1^F z), & y_1^F z \le U'(q), \\ q, & U'(q) < y_1^F z \le U'(q) + y_2^F \gamma, \\ I(y_1^F z - y_2^F), & z > U'(q) + y_2^F \gamma, \end{cases}$$

and $y_1^F > 0$, $y_2^F > 0$ are given as the unique solutions of

$$\widetilde{E}[\beta_T f_F(Z_1)] = x_0, \quad E[\gamma (f_F(Z_1) - q)^-] = \varepsilon.$$

The optimal trading strategy is given by

$$\pi_t = \beta_t^{-1}(\sigma_t^\top)^{-1}\widetilde{E}[X_T\, D_t\beta_T + \beta_T G_F\, D_t Z_1 \,|\, \mathcal{F}_t]$$

where

$$G_F = y_1^F I'(y_1^F Z_1)\mathbf{1}_{\{y_1^F Z_1 \le U'(q)\}} + y_1^F I'(y_1^F Z_1 - y_2^F\gamma)\mathbf{1}_{\{y_1^F Z_1 \ge U'(q)+y_2^F\gamma\}}.$$

Remark 21 The model in Section 2 and the conditions of Theorem 18 are general enough to incorporate a model with partial information like e.g. presented in [18] after a suitable transformation to a model with full information. The details and the numerical analysis of the corresponding strategies are deferred to a future publication.

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A Appendix

Proof of Theorem 4
For $y \in [y^M, \infty)$ we define the function $g(y) = E[Z_1 f_{PI}(yZ_1)]$. The function $g(y)$ is continuous and differentiable. This follows from Assumptions (A1) and (A2) and the fact that f_{PI} is a continuous and differentiable function of y. Moreover, g is strictly decreasing since

$$\frac{\partial}{\partial y} f_{PI}(yz) = \begin{cases} I'(yz)z < 0, & yz \in (0, U'(q)], \\ 0, & yz \in (U'(q), \infty), \end{cases}$$

hence

$$g'(y) = E\Big[\frac{\partial}{\partial y}(Z_1\, f_{PI}(yZ_1))\Big] = E\Big[Z_1^2 I'(yZ_1)\mathbf{1}_{\{yZ_1 \le U'(q)\}}\Big] < 0. \tag{A.1}$$

For the value of $g(y)$ at $y = y^M$ it holds

$$g(y^M) = E[Z_1 f_{PI}(y^M Z_1)] > E[Z_1 f_M(y^M Z_1)] = x_0$$

since $f_{PI}(x) \ge f_M(x) = I(x)$ where the strict inequality holds on $(U'(q), \infty)$. On the other hand we find for $y \to \infty$

$$g(y) = E[Z_1 I(yZ_1)\mathbf{1}_{\{yZ_1 \le U'(q)\}}] + qE[Z_1 \mathbf{1}_{\{yZ_1 > U'(q)\}}] \to qE[Z_1] < x_0$$

since $\frac{1}{y}U'(q) \downarrow 0$ and the expectation in the first expression converges to 0 while $E[Z_1 \mathbf{1}_{\{yZ_1 > U'(q)\}}] \to E[Z_1 \mathbf{1}_\Omega] = E[Z_1]$ for $y \to \infty$. Thus, $g(y)$ is a continuous function on $[y^M, \infty)$ with $g(y^M) > x_0$ and $g(\infty) = \lim_{y\to\infty} g(y) = qE[Z_1] < x_0$. Hence the equation $g(y) = E[Z_1 f_{PI}(yZ_1)] = x_0$ possesses a unique solution $y = y^{PI} > y^M$.

The last assertion of the theorem is a consequence of the fact g is a continuous and bijective function from (y^M, ∞) to $(qE[Z_1], x_0)$. $\square$

Proof of Theorem 6
For the function $g(x) = (x-q)^- + x$ defined on $[0,\infty)$ it holds $g(x) \geq q$ with equality on $[0,q]$ and strict inequality on (q,∞). Let X be any non-negative random variable satisfying the budget constraint, i.e. $E[Z_1 X] \leq x_0$, then it holds

$$\begin{aligned} E[Z_2(X-q)^-] = E[Z_1(X-q)^-] &\geq E[Z_1(X-q)^-] + E[Z_1X] - x_0 \\ &= E[Z_1 g(X)] - x_0 \\ &\geq E[Z_1 q] - x_0 = qE[Z_1] - x_0. \end{aligned}$$

Here, the last inequality holds with equality for all random variables X with values in $[0,q]$. The first inequality holds with equality if X satisfies the budget constraint with equality. $\square$

Proof of Theorem 7
Let X be any non-negative random variable satisfying the budget constraint. Then we have for $\lambda > 0$

$$\begin{aligned} E[Z_2(X-q)^-] &\geq E[Z_2(X-q)^-] + \lambda(E[Z_1X] - x_0) && \text{(A.2)} \\ &= E[Z_2(X-q)^- + \lambda Z_1 X] - \lambda x_0 \\ &= E[g(X;\lambda Z_1, Z_2)] - \lambda x_0 \\ &= E[g(X;\lambda Z_1, Z_2)\mathbf{1}_{\{\lambda Z_1 \leq Z_2\}} + g(X;\lambda Z_1, Z_2)\mathbf{1}_{\{\lambda Z_1 > Z_2\}}] - \lambda x_0 \\ &\geq \lambda\, q\, E[Z_1 \mathbf{1}_{\{\lambda Z_1 \leq Z_2\}}] \;+\; q\, E[Z_2 \mathbf{1}_{\{\lambda Z_1 > Z_2\}}] - \lambda x_0. && \text{(A.3)} \end{aligned}$$

For Inequality (A.3) we have applied Lemma 8 pointwise with $s = X(\omega), z_1 = Z_1(\omega)$ and $z_2 = Z_2(\omega)$. From that lemma it also follows that the inequality holds with equality for

$$X(\omega) = \begin{cases} q, & \text{for } \lambda Z_1(\omega) < Z_2(\omega), \\ \xi(\omega) \in [0,q]\,, & \text{for } \lambda Z_1(\omega) = Z_2(\omega), \\ 0, & \text{for } \lambda Z_1(\omega) > Z_2(\omega). \end{cases}$$

Using the condition that $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$, the above random variable coincides with $\underline{X}$ given in the theorem. Inequality (A.2) holds with equality if X satisfies the budget constraint with equality. For $X = \underline{X}$ this leads to

$$x_0 = E[Z_1 \underline{X}] = q\, E[Z_1 \mathbf{1}_{\{\lambda Z_1 \leq Z_2\}}] =: h(\lambda).$$

Since $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$ the function $h(\lambda)$ is a continuous and strictly decreasing function on $(0,\infty)$ with limits

$$h(0) := \lim_{\lambda \to 0} h(y) = qE[Z_1] > x_0, \quad \text{and} \quad h(\infty) := \lim_{\lambda \to \infty} h(y) = 0 < x_0.$$

Hence, there is a unique solution $\lambda^* \in (0,\infty)$ of the equation $h(\lambda) = E[Z_1\underline{X}] = x_0$. Since for $\underline{X}$ we have equality in (A.2) and (A.3), $\underline{X}$ solves problem (15), i.e. $\underline{X}$ is risk minimizing.

In order to find the minimum value $\underline{\varepsilon}$ of the shortfall risk measure we use that $\underline{X}$ satisfies the budget constraint with equality, i.e. $E[Z_1\underline{X}] = q\,E[Z_1\mathbf{1}_{\{\lambda^* Z_1 \le Z_2\}}] = x_0$. Substituting into (A.3) we find

$$\underline{\varepsilon} = E[Z_2(\underline{X}-q)^-] = \lambda^* q\,E[Z_1\mathbf{1}_{\{\lambda Z_1 \le Z_2\}}] + q\,E[Z_2\mathbf{1}_{\{\lambda^* Z_1 > Z_2\}}] - \lambda^* x_0 = E[Z_2\mathbf{1}_{\{\lambda^* Z_1 > Z_2\}}].$$

□

Proof of Lemma 10

Let us define for $x \ge 0$

$$\begin{aligned} g_1(x) &= U(x) - z_1 x \\ g_2(x) &= U(x) - (z_1 - z_2)x - z_2 q. \end{aligned}$$

Then $g(x)$ can be written as

$$g(x) = \begin{cases} g_1(x), & \text{for} \quad x \ge q, \\ g_2(x), & \text{for} \quad x < q, \end{cases}$$

The functions g_1 and g_2 are strictly concave and the unique maximizer are $x_1^* = I(z_1)$ and $x_2^* = I(z_1 - z_2)$ respectively. Since I is strictly decreasing and $z_2 > 0$ we have $x_1^* < x_2^*$. Moreover it holds

$$\begin{aligned} g(x) = g_1(x) &< g_2(x) \quad \text{on} \quad (q,\infty) \\ g(x) = g_2(x) &< g_2(x) \quad \text{on} \quad [0,q) \\ g(q) = g_1(q) &= g_2(q), \end{aligned}$$

and $g(x) = \min(g_1(x), g_2(x))$. For deriving the maximizer of g we consider the following three cases.

$\boldsymbol{q \le x_1^* \le x_2^*}$: In this case from $q \le x_1^* = I(z_1)$ it follows $z_1 \le U'(q)$. For $x \ge q$ and $x \ne x_1^*$ it holds

$$g(x) = g_1(x) < g_1(x_1^*) = g(x_1^*)$$

while for $x < q$ we have

$$g(x) = g_2(x) < g_2(q) = g(q) < g(x_1^*)$$

since g_2 is strictly increasing for $x < q \le x_2^*$. Hence, $x^* = x_1^* = I(z_1)$ is the unique maximizer for $z_1 \le U'(q)$.

$\boldsymbol{x_1^* < q \le x_2^*}$: Here form $x_1^* = I(z_1) < q \le x_2^* = I(z_1 - z_2)$ it follows $z_1 > U'(q) \ge z_1 - z_2$ or equivalently $U'(q) < z_1 \le U'(q) + z_2$. For $x < q$ we find

$$g(x) = g_2(x) < g_2(q) = g(q)$$

since g_2 is strictly increasing for $x < q \le x_2^*$. On the other hand for $x > q$ it holds

$$g(x) = g_1(x) < g_1(q) = g(q)$$

since g_1 is strictly decreasing for $x > q > x_1^*$. Hence, $x^* = q$ is the unique maximizer for $U'(q) < z_1 \le U'(q) + z_2$.

$\boldsymbol{x_1^* \le x_2^* < q}$: Now from $x_2^* = I(z_1 - z_2) < q$ we find $z_1 > U'(q) + z_2$. For $x < q$ and $x \ne x_2^*$ it holds

$$g(x) = g_2(x) < g_2(x_2^*) = g(x_2^*)$$

and for $x > q$ we find

$$g(x) = g_1(x) < g_1(q) = g_2(q) \le g_2(x_2^*) = g(x_2^*)$$

since g_1 is strictly decreasing for $x \ge q > x_1^*$. Hence, $x^* = x_2^* = I(z_1 - z_2)$ is the unique maximizer for $z_1 > U'(q) + z_2$. □

Proof of Lemma 11

For the function $h(y) := E[Z_1 f_0(yZ_1)]$ defined on $(0, \infty)$ it holds

$$h(y) \;=\; E[Z_1(q\mathbf{1}_{A(y)} + I(yZ_1)\mathbf{1}_{\overline{A}(y)})],$$

with the events $A(y) := \{yZ_1(\omega) \le U'(q)\} = \{Z_1(\omega) \le \frac{U'(q)}{y}\}$ and $\overline{A}(y) = \Omega \setminus A(y)$. The function $h(y)$ is continuous and differentiable on $(0, \infty)$. For its derivative we find

$$h'(y) = E[Z_1(0 \cdot \mathbf{1}_{A(y)} + I'(yZ_1)Z_1\mathbf{1}_{\overline{A}(y)})] < 0,$$

since $I'(s) < 0$ for $s > 0$. Hence, $h(y)$ is strictly decreasing on $(0, \infty)$ with the limits

$$\begin{aligned} h(0) &:= \lim_{y\to 0} h(y) = E[Z_1(q\mathbf{1}_\Omega + 0 \cdot \mathbf{1}_\varnothing)] = qE[Z_1] \\ h(\infty) &:= \lim_{y\to\infty} h(y) = E[Z_1(q\mathbf{1}_\varnothing + 0 \cdot \mathbf{1}_\Omega)] = 0 \end{aligned}$$

since $I(s) \to 0$ for $s \to \infty$, $A(y) \to \Omega$ for $y \to 0$ while $A(y) \to \varnothing$ for $y \to \infty$. Thus, $h(y)$ is a continuous and strictly decreasing function on $(0, \infty)$ with limits $h(0) = qE[Z_1] > x_0 > 0 = h(\infty)$ and there exists a unique solution of $h(y) = x_0$. □

Proof of Lemma 12

(F1) These properties follow from Assumptions (A1) and (A2) and the fact that f is a continuously differentiable function of y_1, y_2.

(F2) Evaluating the derivative we find

$$\begin{aligned} F_1(y_1, y_2) &= \frac{\partial}{\partial y_1} F(y_1, y_2) = E\Big[\frac{\partial}{\partial y_1}(Z_1\, f(y_1Z_1, y_2Z_2))\Big] \\ &= E[Z_1^2 I'(y_1Z_1)\mathbf{1}_A + 0 \cdot \mathbf{1}_B + Z_1^2\, I'(y_1Z_1 - y_2Z_2)\mathbf{1}_C] \\ &= -D_1(y_1) - D_2(y_1, y_2) < 0, \end{aligned}$$

since

$$D_1(y_1) := -E[Z_1^2 I'(y_1 Z_1)\mathbf{1}_A] \quad \text{and} \quad D_2(y_1,y_2) := -E[Z_1^2 I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C]$$

are both strictly positive for $(y_1, y_2) \in \mathcal{D}_1$ because of $I'(s) < 0$.

(F3) Evaluating the derivative yields

$$\begin{aligned} F_2(y_1,y_2) &= \frac{\partial}{\partial y_2} F(y_1,y_2) = E\Big[\frac{\partial}{\partial y_2}(Z_1\, f(y_1 Z_1, y_2 Z_2))\Big] \\ &= E[Z_1 \cdot 0 \cdot \mathbf{1}_A + Z_1 \cdot 0 \cdot \mathbf{1}_B - Z_1 Z_2\, I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C] \\ &= -E[Z_1 Z_2\, I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C] > 0, \end{aligned}$$

because of $I'(y_1 Z_1 - y_2 Z_2) < 0$ for all $(y_1, y_2) \in \mathcal{D}_1$.

(F4) For $y_2 = 0$ it holds $f(y_1 z, 0) = I(y_1 z)$ and $F(y_1, 0) = E[Z_1 I(y_1 Z_1)]$. From the definition of y^M it follows $F(y^M, 0) = E[Z_1 I(y^M Z_1)] = x_0$. (F1) and (F2) imply $F(y_1, 0) < F(y^M, 0) = x_0$ for $y_1 \in (y^M, \overline{y}_1]$ since for fixed $y_2 = 0$ the function $F(y_1, 0)$ is strictly decreasing.

(F5) For the case $Z_1 = Z_2$ the definition of $\overline{y}_2$ in (19) yields $\overline{y}_2 = y_1$. Since

$$\lim_{x_2 \uparrow x_1} f(x_1,x_2) = f_{PI}(x_1) = \begin{cases} I(x_1), & \text{for} \quad x_1 \le U'(q), \\ q, & \text{for} \quad x_1 > U'(q), \end{cases}$$

and $f(x_1, x_2) \le f_{PI}(x_1)$ it follows from Lebesgue's Theorem on dominating convergence that

$$F(y_1, \overline{y}_2) \;:=\; \lim_{y_2 \uparrow \overline{y}_2} F(y_1,y_2) = \lim_{y_2 \uparrow y_1} E[Z_1 f(y_1 Z_1, y_2 Z_1)] = E[Z_1 f_{PI}(y_1 Z_1)].$$

For the case $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$ we have $\overline{y}_2 = \infty$. From $\lim_{x_2 \to \infty} f(x_1, x_2) = f_{PI}(x_1)$ and again by using Lebesgue's Theorem on dominating convergence we find the limit

$$F(y_1, \overline{y}_2) \;:=\; \lim_{y_2 \uparrow \overline{y}_2} F(y_1,y_2) = \lim_{y_2 \uparrow \infty} E[Z_1 f(y_1 Z_1, y_2 Z_2)] = E[Z_1 f_{PI}(y_1 Z_1)].$$

For $qE[Z_1] < x_0$ we have $\overline{y}_1 = y^{PI}$ and for all $y_1 < y^{PI}$ we find from $I(y_1 z) > I(y^{PI} z)$ that $f_{PI}(y_1 z) \ge f_{PI}(y^{PI} z)$ with strict inequality for $y_1 z < U'(q)$ and

$$F(y_1, \overline{y}_2) = E[Z_1 f_{PI}(y_1 Z_1)] > E[Z_1 f_{PI}(\overline{y}_1 Z_1)] = x_0.$$

For $qE[Z_1] \ge x_0$ it holds $\overline{y}_1 = \infty$. Since $f_{PI}(y_1 z) \ge q$ with strict inequality for $y_1 z < U'(q)$ it follows that for all $y_1 < \overline{y}_1$ we have

$$F(y_1, \overline{y}_2) = E[Z_1 f_{PI}(y_1 Z_1)] > qE[Z_1] \ge x_0.$$

□

Proof of Lemma 13

(G1) These properties follow from Assumptions (A1) and (A2) and the fact that f is a continuously differentiable function of y_1, y_2.

(G2) Using (22) and evaluating the derivative we find

$$\begin{aligned} G_1(y_1,y_2) &= \frac{\partial}{\partial y_1} G(y_1,y_2) = E\Big[\frac{\partial}{\partial y_1}\big(Z_2(q - f(y_1 Z_1, y_2 Z_2))\big)\mathbf{1}_C\Big] \\ &= -E[Z_2 Z_1\, I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C] > 0, \end{aligned}$$

since $I'(s) < 0$ for all s.

(G3) Evaluating the derivative yields

$$\begin{aligned} G_2(y_1,y_2) &= \frac{\partial}{\partial y_2} G(y_1,y_2) = E\Big[\frac{\partial}{\partial y_2}\big(Z_2(q - f(y_1 Z_1, y_2 Z_2))\big)\mathbf{1}_C\Big] \\ &= -E[Z_2(-Z_2)\, I'(y_1 Z_1 - y_2 Z_2)\mathbf{1}_C] < 0. \end{aligned}$$

(G4) For $y_2 = 0$ it holds $f(y_1 z_1, 0) = I(y_1 z)$ and from the definition of G and ε^M it follows $G(y^M, 0) = \varepsilon^M$. (G1) and (G2) imply $G(y_1, 0) > G(y^M, 0) = \varepsilon^M$ for $y_1 \in (y^M, \overline{y}_1)$ since for fixed $y_2 = 0$ the function $G(y_1, 0)$ is strictly increasing. □

Proof of Theorem 14

(1) From Lemma 12 it is known that for fixed $y_1 \in (y^M, \overline{y}_1)$ the function $F(y_1, .)$ is continuous and strictly increasing with $F(y_1, 0) < x_0$ and $F(y_1, \overline{y}_2) > x_0$. Hence, there exists a unique $y_2 = y_2(y_1)$ in $(0, \overline{y}_2)$ with $F(y_1, y_2) = x_0$.

(2) The assertion follows from implicit function theorem which implies that the derivative $y_2'(y_1)$ is given by

$$y_2'(y_1) = -\frac{\frac{\partial}{\partial y_1} F(y_1,y_2)}{\frac{\partial}{\partial y_2} F(y_1,y_2)} = -\frac{F_1(y_1,y_2)}{F_2(y_1,y_2)} > 0$$

where properties (F1), (F2) and (F3) have been used.

(3) From assertions (1) and (2) it is known that $y_2(y_1)$ is strictly increasing on $(y^M, \overline{y}_1)$ and bounded from below by 0. Hence there exist the limit

$$y_2(y^M) = \lim_{y_1 \downarrow y^M} y_2(y_1)$$

and it remains to prove that $y_2(y^M) = 0$. The continuity of F implies

$$F(y^M, y_2(y^M)) = \lim_{y_1 \downarrow y^M} F(y_1, y_2(y_1)) = x_0.$$

Moreover, it holds $F(y^M, 0) = x_0$. Property (F3) states that $\frac{\partial}{\partial y_2} F(y_1, y_2) > 0$ for $y_2 \in (0, \overline{y}_2)$, hence we find

$$x_0 = F(y^M, 0) = F(y^M, y_2(y^M)) < F(y^M, y_2) \quad \text{for} \quad y_2 \in (0, \overline{y}_2)$$

which yields $y_2(y^M) = 0$.

(4) For $Z_1 = Z_2$ it holds $y_2(y_1) < y_1$ and consequently $y_1 - y_2(y_1) > 0$ for all $y_1 \in (y^M, \overline{y}_1)$. For the proof of the limits of $y_2(y_1)$ for $y_1 \to \overline{y}_1$ given in the assertion of the theorem it is sufficient to prove the corresponding limits of the difference $y_1 - y_2(y_1)$. Since $y_2(y_1)$ is on $(y^M, \overline{y}_1)$ continuous and strictly increasing the relation $y_2(y_1) < y_1$ implies that the difference $y_1 - y_2(y_1)$ for $y_1 \to \overline{y} = y^{PI}$ either tends to some strictly positive real number $\alpha \in (0, y^{PI})$ or to 0. For $y_1 \to \overline{y} = \infty$ there are three candidates for the limit which are 0, $\alpha \in (0, \infty)$ and $+\infty$.

In order to find out the correct candidate for the limit we study the asymptotic behavior of the events A, B, C introduced in (20) which occur in the representation (21) for f. For $Z_1 = Z_2$ it holds

$$\begin{aligned}
A(y_1) &= \{y_1 Z_1 \le U'(q)\} \\
B(y_1, y_2(y_1)) &= \{U'(q) < y_1 Z_1 \le U'(q) + y_2(y_1) Z_1\} \\
&= \left\{ \frac{U'(q)}{y_1} < Z_1 \le \frac{U'(q)}{y_1 - y_2(y_1)} \right\} \\
C(y_1, y_2(y_1)) &= \{y_1 Z_1 > U'(q) + y_2(y_1) Z_1\} = \left\{ Z_1 > \frac{U'(q)}{y_1 - y_2(y_1)} \right\}.
\end{aligned}$$

(4.i) The case $\mathbf{qE[Z_1] < x_0}$: In this case we have to study the limits for $y_1 \to \overline{y}_1 = y^{PI}$ and we consider two candidates for the limit of $y_1 - y_2(y_1)$ which are 0 and $\alpha \in (0, y^{PI})$. For these two candidates the following table gives the limits of the events A, B, C for $y_1 \to y^{PI}$ which follow from the representation of the events given above.

	limits for $y_1 \to y^{PI}$	
$y_1 - y_2(y_1)$	0	$\alpha \in (0, y^{PI})$
$A(y_1)$	$\{y^{PI} Z_1 \le U'(q)\} = A(y^{PI})$	
$B(y_1, y_2(y_1))$	$\{y^{PI} Z_1 > U'(q)\}$	$\left\{ \frac{U'(q)}{y^{PI}} < Z_1 \le \frac{U'(q)}{\alpha} =: B_\alpha \right\}$
$C(y_1, y_2(y_1))$	$\varnothing$	$\left\{ Z_1 > \frac{U'(q)}{\alpha} \right\} =: C_\alpha$

Assuming the second candidate $\alpha \in (0, y^{PI})$ is true then from representation (21) of f and the limits of A, B, C for $y_1 \to y^{PI}$ given in the above table it follows

$$f(y_1 Z_1, y_2(y_1) Z_2) \to I(y^{PI} Z_1)\, \mathbf{1}_{A(y^{PI})} + q \mathbf{1}_{B_\alpha} + I(\alpha Z_1)\, \mathbf{1}_{C_\alpha}$$

From the definition of f_{PI} given in (14), i.e. $f_{PI}(z) = I(z)\, \mathbf{1}_{\{z \le U'(q)\}} + q\, \mathbf{1}_{\{z > U'(q)\}}$ and from $B_\alpha \cup C_\alpha = \{y_1 Z_1 > U'(q)\}$ we find

$$f(y_1 Z_1, y_2(y_1) Z_2) \quad \to \quad f_{PI}(y^{PI} Z_1) + (I(\alpha Z_1) - q) \mathbf{1}_{C_\alpha} < f_{PI}(y_1 Z_1) = X^{PI}$$

since on C_α it holds $I(\alpha Z_1) < I(U'(q)) = q$. For the budget function then it follows

$$\lim_{y_1 \to y^{PI}} F(y_1, y_2(y_1)) = \lim_{y_1 \to y^{PI}} E[Z_1 f(y_1 Z_1, y_2(y_1) Z_2)] < E[Z_1 X^{PI}] = x_0.$$

This is a contradiction to (A.5). Hence the limit is 0, i.e. $y_1 - y_2(y_1) \to 0$ for $y_1 \to y^{PI}$ and

$$\begin{aligned} f(y_1 Z_1, y_2(y_1) Z_2) &\to I(y^{PI} Z_1)\, \mathbf{1}_{\{y^{PI} Z_1 \le U'(q)\}} + q\, \mathbf{1}_{\{y^{PI} Z_1 > U'(q)\}} \\ &= f_{PI}(y^{PI} Z_1) = X^{PI}. \end{aligned}$$

(4.ii) The case $\mathbf{qE[Z_1] = x_0}$: In this case we have to consider the limits for $y_1 \to \overline{y}_1 = \infty$. Since $y_2(y_1)$ is on (y^M, ∞) continuous, strictly increasing with values in $(0, y_1)$ the difference $y_1 - y_2(y_1)$ for $y_1 \to \infty$ either tends to some finite positive real number $\alpha \in (0, \infty)$, to 0 or to $+\infty$. The following table gives the limits of the events A, B, C for $y_1 \to \infty$ depending on the three possible limits of $y_1 - y_2(y_1)$. These limits follow immediately from the above representations of the events A, B, C in the proof for the case $qE[Z_1] < x_0$. Moreover, the table contains the corresponding limits of the random variables $f(y_1 Z_1, y_2(y_1) Z_2)$ and of the budget function $F(y_1, y_2(y_1)) = E[Z_1 f(y_1 Z_1, y_2(y_1) Z_2)]$.

	limits for $y_1 \to \infty$		
$y_1 - y_2(y_1)$	0	$\alpha \in (0,\infty)$	∞
$A(y_1)$	$\varnothing$	$\varnothing$	$\varnothing$
$B(y_1, y_2(y_1))$	Ω	$\{\alpha Z_1 \le U'(q)\}$	$\varnothing$
$C(y_1, y_2(y_1))$	$\varnothing$	$\{\alpha Z_1 > U'(q)\}$	Ω
$f(y_1 Z_1, y_2(y_1) Z_2)$	q	$f_0(\alpha Z_1)$	0
$F(y_1, y_2(y_1))$	$qE[Z_1] = x_0$	$E[Z_1 f_0(\alpha Z_1)]$	0

The limits of f follow from the representation

$$\begin{aligned} f(y_1 Z_1, y_2(y_1) Z_2) &= I(y_1 Z_1)\, \mathbf{1}_{A(y_1)} + q\, \mathbf{1}_{B(y_1, y_2(y_1))} \\ &\quad + I((y_1 - y_2(y_1)) Z_1)\, \mathbf{1}_{C(y_1, y_2(y_1))} \end{aligned} \tag{A.4}$$

the relations $I(s) \to 0$ for $s \to \infty$ and $I((y_1 - y_2(y_1))Z_1) < I(U'(q)) = q$ on $C(y_1, y_2(y_1))$ by using the limits of the events A, B, C.

Since $y_2(y_1)$ for all $y_1 > y^M$ satisfies the budget equation $F(y_1, y_2(y_1)) = x_0$ it follows

$$\lim_{y_1 \to \infty} F(y_1, y_2(y_1)) = x_0. \tag{A.5}$$

With the above relation we can exclude the second and third candidate for the limit of $y_1 - y_2(y_1)$. Assuming this limit is ∞ then it yields $F(y_1, y_2(y_1)) \to 0 < x_0$ while assuming a finite limit $\alpha \in (0, \infty)$ we find $F(y_1, y_2(y_1)) \to E[Z_1 f_0(\alpha Z_1)] <$

$qE[Z_1] = x_0$, since $f_0(z) = q\,\mathbf{1}_{\{z \le U'(q)\}} + I(z)\,\mathbf{1}_{\{z > U'(q)\}} < q$ and $I(z) < I(U'(q)) = q$ for $z > U'(q)$. Thus both assumptions lead to a contradiction to (A.5). Hence the limit is 0, i.e. $y_1 - y_2(y_1) \to 0$ for $y_1 \to \infty$ and it holds $f(y_1 Z_1, y_2(y_1) Z_2) \to q$.

(4.iii) The case $\mathbf{qE[Z_1] > x_0}$: As in the above case it holds $\overline{y}_1 = \infty$ and we first prove $y_1 - y_2(y_1) \to y_0^*$. This implies $y_2(y_1) \to \infty$. Again, there are three candidates for the limit of $y_1 - y_2(y_1)$, namely 0, a finite positive limit α and ∞. We can exclude 0 and ∞ since then according to the table given in (4.ii) the limit of the budget function $F(y_1, y_2(y_1))$ would be $qE[Z_1] > x_0$ and $0 < x_0$, respectively. This is a contradiction to (A.5).

Hence there is a finite limit $\alpha \in (0, \infty)$ and $f(y_1 Z_1, y_2(y_1) Z_2) \to f_0(\alpha Z_1)$. In order to determine the value of α we use $(A.5)$ which gives $F(y_1, y_2(y_1)) \to E[Z_1 f_0(\alpha Z_1)] = x_0$. Hence, α coincides with the solution y_0^* of the budget equation given in the assertion of the theorem.

(5) For the proof of $y_2(y_1) \to \infty$ for $y_1 \to \overline{y}_1$ it is sufficient to prove the limits for $y_2(y_1)/y_1$ given in the assertion. Since $y_2(y_1)$ is positive and strictly increasing the limit of $y_2(y_1)/y_1$ for $y_1 \to y^{PI}$ is either a finite and positive real number $\alpha \in (0, \infty)$ or ∞. For $y_1 \to \infty$ there is a third candidate which is 0. As in the proof of assertion (4) we study the asymptotic behavior of the events A, B, C. While the results for $A = A(y_1)$ remain unchanged the dependence of B and C on y_2 and the assumption $P(\lambda Z_1 = Z_2) = 0$ for all $\lambda > 0$ which implies $P(Z_1 = Z_2) = 0$ require a different approach. It holds

$$\begin{aligned} B(y_1, y_2(y_1)) &= \{U'(q) < y_1 Z_1 \le U'(q) + y_2(y_1) Z_2\} \\ &= \left\{ \frac{U'(q)}{y_1} < Z_1 \le \frac{U'(q)}{y_1} + \frac{y_2(y_1)}{y_1} Z_2 \right\} \qquad \text{(A.6)} \\ C(y_1, y_2(y_1)) &= \{y_1 Z_1 > U'(q) + y_2(y_1) Z_2\} = \left\{ Z_1 > \frac{U'(q)}{y_1} + \frac{y_2(y_1)}{y_1} Z_2 \right\}. \end{aligned}$$

(5.i) The case $\mathbf{qE[Z_1] < x_0}$: Here we have to consider the limits for $y_1 \to y^{PI}$. The following table gives the limits of the events A, B, C depending on the two possible limits for $y_2(y_1)/y_1$.

$\frac{y_2(y_1)}{y_1}$	limits for $y_1 \to y^{PI}$: $\alpha \in (0, \infty)$	∞
$A(y_1)$	$\{y^{PI} Z_1 \le U'(q)\} = A(y^{PI})$	
$B(y_1, y_2(y_1))$	$\{U'(q) < y^{PI} Z_1 \le U'(q) + \alpha y^{PI} Z_2\}$ $=: B_\alpha$	$\{y^{PI} Z_1 > U'(q)\}$
$C(y_1, y_2(y_1))$	$\{y^{PI} Z_1 > U'(q) + \alpha y^{PI} Z_2\} =: C_\alpha$	$\varnothing$

Assume that there is a finite limit $\alpha \in (0, \infty)$. Then from representation (21) of f

and the limits of A, B, C for $y_1 \to y^{PI}$ given in the table it follows

$$f(y_1 Z_1, y_2(y_1) Z_2) \to I(y^{PI} Z_1)\, \mathbf{1}_{A(y^{PI})} + q\, \mathbf{1}_{B_\alpha} + I(y^{PI} Z_1 - \alpha y^{PI} Z_2)\, \mathbf{1}_{C_\alpha}$$

Using the definition of f_{PI} given in (14), i.e. $f_{PI}(z) = I(z)\, \mathbf{1}_{\{z \le U'(q)\}} + q\, \mathbf{1}_{\{z > U'(q)\}}$ and $B_\alpha \cup C_\alpha = \{y_1 Z_1 > U'(q)\}$ we deduce

$$\begin{aligned} f(y_1 Z_1, y_2(y_1) Z_2) &\to f_{PI}(y^{PI} Z_1) + (I(y^{PI} Z_1 - \alpha y^{PI} Z_2) - q)\mathbf{1}_{C_\alpha} \\ &< f_{PI}({}^{PI} Z_1) = X^{PI} \end{aligned}$$

since on C_α it holds $I(y^{PI} Z_1 - \alpha y^{PI} Z_2) < I(U'(q)) = q$. For the budget function then it follows

$$\lim_{y_1 \to y^{PI}} F(y_1, y_2(y_1)) = \lim_{y_1 \to y^{PI}} E[Z_1 f(y_1 Z_1, y_2(y_1) Z_2)] < E[Z_1 X^{PI}] = x_0.$$

This is a contradiction to (A.5). Hence the limit ist ∞, i.e. $\frac{y_2(y_1)}{y_1} \to \infty$ for $y_1 \to y^{PI}$ and

$$\begin{aligned} f(y_1 Z_1, y_2(y_1) Z_2) &\to I(y^{PI} Z_1)\, \mathbf{1}_{\{y^{PI} Z_1 \le U'(q)\}} + q\, \mathbf{1}_{\{y^{PI} Z_1 > U'(q)\}} \\ &= f_{PI}(y^{PI} Z_1) = X^{PI}. \end{aligned}$$

(5.ii) The case $\mathbf{qE[Z_1] = x_0}$: In this case it holds $\overline{y}_1 = \infty$ and we have to check whether the limit of $y_2(y_1)/y_1$ is $0, \alpha \in (0, \infty)$ or ∞. The next table gives the limits of A, B, C as well as of the random variables $f(y_1 Z_1, y_2(y_1) Z_2)$ and of the budget function $F(y_1, y_2(y_1))$ for $y_1 \to \infty$.

	limits for $y_1 \to \infty$		
$\frac{y_2(y_1)}{y_1}$	0	$\alpha \in (0, \infty)$	∞
$A(y_1)$	$\varnothing$	$\varnothing$	$\varnothing$
$B(y_1, y_2(y_1))$	$\varnothing$	$\{Z_1 \le \alpha Z_2\}$	Ω
$C(y_1, y_2(y_1))$	Ω	$\{Z_1 > \alpha Z_2\}$	$\varnothing$
$f(y_1 Z_1, y_2(y_1) Z_2)$	0	$q\, \mathbf{1}_{\{Z_1 \le \alpha Z_2\}}$	q
$F(y_1, y_2(y_1))$	0	$qE[Z_1 \mathbf{1}_{\{Z_1 \le \alpha Z_2\}}]$	$qE[Z_1]$

The limits for the events A, B, C follow immediately from representation (A.6) of these events. In order to prove $f(y_1 Z_1, y_2(y_1) Z_2) \to 0$ for the case $y_2(y_1)/y_1 \to 0$ we use the relation (21) and deduce

$$\begin{aligned} f(y_1 Z_1, y_2(y_1) Z_2) = I(y_1 Z_1)\, \mathbf{1}_{A(y_1)} &+ q\, \mathbf{1}_{B(y_1, y_2(y_1))} + I(y_1 Z_1 - y_2(y_1) Z_2)\, \mathbf{1}_{C(y_1, y_2(y_1))} \\ \to \quad 0 \cdot \mathbf{1}_{\varnothing} \quad &+ \quad q\, \mathbf{1}_{\varnothing} \quad + \quad 0 \cdot \mathbf{1}_{\Omega} = 0, \end{aligned}$$

since $I(s) \to 0$ for $s \to \infty$ and

$$y_1 Z_1 - y_2(y_1) Z_2 = y_1 \Big(Z_1 - \frac{y_2(y_1)}{y_1} Z_2 \Big) \to \infty$$

for $y_1 \to \infty$. Similarly, in case of $y_2(y_1)/y_1 \to \alpha \in (0,\infty)$ we obtain

$$\begin{aligned} f(y_1 Z_1, y_2(y_1) Z_2) &\to 0 \cdot \mathbf{1}_{\varnothing} \quad + \quad q\,\mathbf{1}_{\{Z_1 \le \alpha Z_2\}} \quad + \quad 0 \cdot \mathbf{1}_{\{Z_1 > \alpha Z_2\}} \\ &= q\,\mathbf{1}_{\{Z_1 \le \alpha Z_2\}}, \end{aligned}$$

where we use that on $C(y_1, y_2(y_1))$ it holds

$$U'(q) < y_1 Z_1 - y_2(y_1) Z_2 \; = y_1\Big(Z_1 - \frac{y_2(y_1)}{y_1} Z_2\Big) \to \infty$$

for $y_1 \to \infty$. Looking at the limits of the budget function $F(y_1, y_2(y_1))$ given in the above table we can exclude the cases $y_2(y_1)/y_1 \to 0$ and $\alpha \in (0,\infty)$ since in both cases it holds $\lim\limits_{y_1 \to \infty} F(y_1, y_2(y_1)) < qE[Z_1] = x_0$ which contradicts relation (A.5). Hence it holds $y_2(y_1)/y_1 \to \infty$, consequently $y_2(y_1) \to \infty$ and $f(y_1 Z_1, y_2(y_1) Z_2) \to q$ for $y_1 \to \infty$.

(5.iii) The case $\mathbf{qE[Z_1] > x_0}$: Considering the three candidates $0, \alpha \in (0,\infty)$ and ∞ for the limit of $y_2(y_1)/y_1$ we can exclude 0 for the same reason as in the above case $qE[Z_1] = x_0$. Assuming the limit is ∞ then according to the table given in (5.ii) for the budget function it follows $F(y_1, y_2(y_1)) \to qE[Z_1] > x_0$ which contradicts relation (A.5). Hence there is a finite limit $\alpha \in (0,\infty)$ and again from the table given in (5.ii) we find $f(y_1 Z_1, y_2(y_1) Z_2) \to q\,\mathbf{1}_{\{Z_1 \le \alpha Z_2\}}$. From (A.5) then it follows $F(y_1, y_2(y_1)) \to qE[Z_1 \mathbf{1}_{\{Z_1 \le \alpha Z_2\}}] = x_0$.

Comparing with Theorem 7 it follows that α is equal to $1/\lambda^*$ where λ^* is the solution of $qE[Z_1 \mathbf{1}_{\{\lambda Z_1 \le Z_2\}}] = x_0$. Moreover, the limit of $f(y_1 Z_1, y_2(y_1) Z_2)$ for $y_1 \to \infty$ is equal to the risk minimizing random variable $\underline{X} = q\,\mathbf{1}_{\{\lambda^* Z_1 \le Z_2\}}$. □