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S. Margenov, J. Synka

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Generalized aggregation-based multilevel preconditioning of Crouzeix-Raviart FEM elliptic problems

Svetozar MARGENOV[1)], **Josef SYNKA**[2)]

[1)] Institute for Parallel Processing, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 25-A, 1113 Sofia, Bulgaria

[2)] Industrial Mathematics Institute, Joh. Kepler University, Altenberger Str. 69, A-4040 Linz, Austria

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Abstract

Preconditioners based on various multilevel extensions of two-level finite element methods (FEM) are well-known to yield iterative methods of optimal order complexity with respect to the size of the system, as was first shown by Axelsson and Vassilevski [4]. The derivation of optimal convergence rate estimates in this context is mainly governed by the constant $\gamma \in (0,1)$ in the so-called Cauchy-Bunyakowski-Schwarz (CBS) inequality, which is associated with the angle between the two subspaces obtained from a (recursive) two-level splitting of the finite element space. Accurate quantitative bounds, especially for the upper bound of γ are required for the construction of proper multilevel extensions of the related two-level methods.

In this paper, an improved algebraic preconditioning algorithm for second-order elliptic boundary value problems is presented, where the discretization and formulation is based on Crouzeix-Raviart linear finite elements and the two-level splitting is based on differentiation and aggregation (DA). A uniform estimate of γ in the strengthened CBS inequality for anisotropic problems using the aforementioned finite element type was given by Blaheta, Margenov and Neytcheva [5]. In this study we show that the estimates of the constant γ can significantly be improved for the DA-algorithm by utilizing a minimum angle condition, where the latter is naturally used in mesh generators for practical problems. The results obtained herein can be used to set up a self-adaptive aggregation based mulitlevel preconditioner for elliptic problems based on Crouzeix-Raviart finite elements.

Keywords: Multilevel preconditioners, hierarchical basis, CBS constant, differentiation and aggregation, finite elements.

1 Introduction

In our study we consider self-adjoint elliptic boundary problems of the form

$$\begin{aligned} \mathcal{L}u \equiv -\nabla \cdot (a(\mathbf{x})\,\nabla u(\mathbf{x})) &= f(x) \quad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \Gamma_D \\ (a(\mathbf{x})\,\nabla u(\mathbf{x})) \cdot \mathbf{n} &= 0 \qquad \text{on} \quad \Gamma_N \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ denotes a convex polygonal domain, $f(\mathbf{x})$ is a given function in $L^2(\Omega)$, $a(\mathbf{x}) := (a_{ij}(\mathbf{x}))_{i,j\in\{1,2\}}$ designates a bounded, symmetric and uniformly positive definite (SPD) matrix on Ω with piecewise smooth functions $a_{ij}(\mathbf{x})$ in $\overline{\Omega} := \Omega \cup \partial\Omega$, and $\mathbf{n}$ is the outward unit normal vector onto the boundary $\Gamma := \partial\Omega$ with $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$.

The associated weak formulation of such problems then reads as:

Given $f \in L^2(\Omega)$, find $u \in \mathcal{V} \equiv H^1_D(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, which satisfies

$$\mathcal{A}(u, v) = (f, v) \quad \forall\, v \in H^1_D(\Omega)\,, \tag{2}$$

where the bilinear form and linear functional are respectively given as

$$\mathcal{A}(u, v) := \int_\Omega a_{ij}(\mathbf{x})\,\nabla_j\, u(\mathbf{x})\,\nabla_i\, v(\mathbf{x})\,d\mathbf{x} \qquad \text{and} \qquad (f, v) := \int_\Omega f\, v\, d\mathbf{x}. \tag{3}$$

Thereby, $\nabla_i := \frac{\partial}{\partial x_i}$ and we make use of Einstein's summation convention, where repeated indices in a term are summed over their index set. As is well-known, the bilinear form $\mathcal{A}(.,.)$ for problems, as defined in (1), is symmetric, bounded and $H^1_D(\Omega)$ elliptic.

Let us assume that the domain Ω is discretized using triangular elements and that the fine-grid partitioning, denoted by $\mathcal{T}_h$, is obtained by a uniform refinement of a given coarser triangulation $\mathcal{T}_H$ (in two dimensions, e.g., by dividing each triangle into four congruent ones). Further, if the coefficient functions $a_{ij}(\mathbf{x})$ is discontinuous along some polygonal interfaces, we assume that the partitioning $\mathcal{T}_H$ is aligned with these lines to ensure that over each element $E \in \mathcal{T}_H$ the function $a(\mathbf{x})$ is sufficiently smooth. The variational problem (2) is now discretized by utilizing non-conforming Crouzeix-Raviart finite elements, where the nodal basis functions are set up at the midpoints along the edges rather than at the vertices of an element. The corresponding discrete problem can thus be written as:

Given $f \in L^2(\Omega)$, find $u_h \in \mathcal{V}_h$, satisfying

$$\mathcal{A}_h(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in \mathcal{V}_h, \quad \text{where } \mathcal{A}_h(u_h, v_h) := \sum_{e\in\mathcal{T}_h} \int_e a(e)\,\nabla u_h(\mathbf{x})\,\nabla v_h(\mathbf{x})\,d\mathbf{x} \tag{4}$$

and $\mathcal{V}_h := \{v \in L^2(\Omega) : v|_e$ is linear on each $e \in \mathcal{T}_h,\ v$ is continuous at the midpoints of the edges of triangles from $\mathcal{T}_h$ and $v = 0$ at the midpoints on $\Gamma_D\}$.

The discrete linear functional is of the form (3) with Ω and v now being replaced by e and v_h, respectively. In (4), $a(e)$ is a piecewise constant coefficient matrix, defined by the integral averaged values of $a(\mathbf{x})$ over each triangle from the coarse-grid triangulation $\mathcal{T}_H$. Note that this allows for arbitrary large jumps in the coefficients across the boundaries between adjacent finite elements from $\mathcal{T}_H$.

In our study we will restrict ourselves to piecewise linear finite elements. In contrast to standard conforming FEM, where the interpolation nodes are simply the vertices of the triangles, the nodal basis for the Crouzeix-Raviart finite elements is now associated with the midpoints of the edges. After having assembled the contributions resulting from the discrete problem for each finite element, the following system of linear equations is obtained:

$$A_h \, \mathbf{u}_h \; = \; \mathbf{b}_h \, . \tag{5}$$

Thereby, $A_h, \mathbf{u}_h$, and $\mathbf{b}_h$ denote the corresponding global stiffness matrix, the vector of degrees of freedom, and the load vector, respectively, and the subscript h indicates the discretization (mesh-size) parameter of the corresponding triangulation $\mathcal{T}_h \subset \Omega$.

The main goal of this paper is to show that the uniform bounds for the constant in the strengthened CBS inequality, derived for the non-conforming Crouzeix-Raviart finite elements in Blaheta, Margenov and Neytcheva [5], can be improved under the assumption of a minimum angle condition. This condition forms an integral element in any numerical realisations of practical problems. The improved values of γ then enable the set up of more problem-adapted multilevel preconditioners with faster convergence rates.

The next two subsections briefly summarize the two-level setting and the role of the constant γ in the CBS inequality. In Section 2 we discuss the two-level splitting by differences and aggregates (DA) and the main results on the estimates of the CBS constant for the non-conforming Crouzeix-Raviart finite element without restriction to mesh and coefficient anisotropy. Section 3 deals with a generalization of two-level DA-decomposition by enhancing the aggregates involved in the formulation of the vector spaces. In Section 4 the effect of this generalization on γ w.r.t. a minimum angle condition, as commonly used in numerical implementations, is studied. In Section 5 we show that the application of the generalized DA-splitting in a multilevel approach results in a sequence of element stiffness matrices, all of which correspond to a Crouzeix-Raviart linear finite element, and which converges to the element stiffness matrix corresponding to the equilateral triangle for which the best CBS constant is obtained. Hence, the generalized DA-splitting yields a CBS constant, which is continuously improved in a multilevel preconditioning approach at each level. The paper is concluded in Section 6 with a summary of the main results.

1.1 The construction of a two-level hierarchy

The construction of multilevel hierarchical preconditioner is based on a two-level framework. Thereby, one seeks a preconditioner M for A_h such that the spectral condition number $\kappa(M^{-1}A_h)$ of the precondtioned matrix $M^{-1}A_h$ is uniformly bounded with respect to the discretization parameter h, the shape of triangular finite elements and arbitrary coefficient anisotropy. Further, it requires two nested finite element spaces $\mathcal{V}_H$ and $\mathcal{V}_h$ with $\mathcal{V}_H \subset \mathcal{V}_h$ that correspond to two successive (regular) mesh refinements $\mathcal{T}_H$ and $\mathcal{T}_h$ of the domain Ω. Let $\Phi_H := \{\phi_H^{(i)}, i = 1, 2, \ldots, N_H\}$ and $\Phi_h := \{\phi_h^{(i)}, i = 1, 2, \ldots, N_h\}$ be the related sets of finite element basis functions of cardinality N_H and N_h, respectively. For standard finite elements, where the basis functions are defined at the vertices of the finite element, the condition $\mathcal{V}_H \subset \mathcal{V}_h$ holds and a so-called set of hierarchical basis functions can be obtained as follows:

1. Split the nodal points on the fine mesh into two groups, viz., into a set of coarse grid nodes $S_2 := \{1, \ldots, N_H\} \in \mathcal{T}_H$ with vector space $\mathcal{V}_2 := \mathcal{V}_H$ and into a set of the remaining nodes on the fine grid, $S_1 := \{1, \ldots, N_h\} \backslash S_2 \in \mathcal{T}_h \backslash \mathcal{T}_H$ with the corresponding vector space $\mathcal{V}_1 := \mathcal{V}_{h \backslash H}$. For standard finite elements this splitting naturally satisfies the condition that the original vector space can now be written as the direct sum of the two partial vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., as $\mathcal{V} = \mathcal{V}_1 \bigoplus \mathcal{V}_2$.

2. The set of hierarchical basis functions is then obtained as

$$\tilde{\Phi}_h = \{\tilde{\phi}_h^{(i)}\}_{i=1,2,\ldots,N_h} := \{\phi_H^{(k)}\}_{k \in S_2} \cup \{\phi_h^{(k)}\}_{k \in S_1} . \tag{6}$$

Let $\tilde{A}_h$ denote the corresponding hierarchical stiffness matrix. The sets S_1 and S_2, as defined in the procedure above, then provide us with a natural two-by-two splitting of the stiffness matrix A_h on the fine grid and of $\tilde{A}_h$, viz.,

$$A_h = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad \tilde{A}_h = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \tag{7}$$

where the blocks A_{ii} and $\tilde{A}_{ii}$ are of size n_i^2 with n_i being the cardinality of the set S_i and $i = 1, 2$. It is well-known (cf., e.g., Axelsson and Vassilevski [4]), that there exists a transformation matrix J, which relates the nodal point vectors for the standard and the hierarchical basis in the following way:

$$\tilde{v} := \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ J_{21} v_1 + v_2 \end{pmatrix}, \text{ viz.,} \quad J := \begin{pmatrix} I_1 & 0 \\ J_{21} & I_2 \end{pmatrix}. \tag{8}$$

Remark 1.1 *Since the hierarchical matrix is related to the fine grid matrix by $\tilde{A}_h = J\, A_h\, J^T$ and A_h is less dense than $\tilde{A}_h$ it is recommended to work with A_h in practical applications in order to reduce the computational effort.*

1.2 The role of the constant in the CBS inequality

For a general symmetric and positive definite (SPD) matrix with a 2×2-block structure, as given above, the quality of such a partitioning can be characterized by the corresponding CBS inequality constant γ, which is defined as follows:

$$\gamma := \sup_{\mathbf{v}_1 \in \mathbb{R}^{n_1}, \mathbf{v}_2 \in \mathbb{R}^{n_2}} \frac{\mathbf{v}_1^T A_{12} \mathbf{v}_2}{\left[\left(\mathbf{v}_1^T A_{11} \mathbf{v}_1 \right) \left(\mathbf{v}_2^T A_{22} \mathbf{v}_2 \right) \right]^{1/2}}, \tag{9}$$

where again $n_i := |S_i|$ for $i = 1, 2$.

If a preconditioner M_a of a block-diagonal (additive) form is used, where $M_a = \mathrm{diag}\,(A_{11}, A_{22})$ is utilized in the simplest case, then the estimate of the spectral condition number of $M_a^{-1} A$ reads

$$\kappa \left(M_a^{-1} A \right) \le \frac{1 + \gamma}{1 - \gamma}. \tag{10}$$

Alternatively, one could use a full block-matrix factorization preconditioner (multiplicative type or of block Gauss-Seidel form), which is based on the exact block-matrix factorization of $A = C(A_{22})\, D$ with $M_m = C(B_{22})\, D$, where

$$C(T) := \begin{pmatrix} A_{11} & 0 \\ A_{21} & S(T) \end{pmatrix}, \qquad D := \begin{pmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{pmatrix}, \tag{11}$$

and $S(T) := T - A_{21} A_{11}^{-1} A_{12}$ is the exact Schur complement of A iff $T = A_{22}$, while it is an approximation thereof, otherwise. In this case and if B_{22} satisfies the relation

$$\delta_0 A_{22} \le B_{22} \le \delta_1 A_{22} \quad \text{with} \quad \gamma^2 < \delta_0 \le 1 \le \delta_1,$$

the estimate of the spectral condition number of $M_m^{-1} A$ is obtained as

$$\kappa \left(M_m^{-1} A \right) \le \frac{\delta_1 - \gamma^2}{\delta_0 - \gamma^2} \le \frac{\delta_1}{\delta_0 - \gamma^2}. \tag{12}$$

Note that in the context of hierarchical bases, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the subspaces of the finite element space $\mathcal{V}$, which are respectively spanned by the corresponding set of basis functions, S_1 and S_2, the strengthened CBS inequality constant again represents the cosine of the angle between these two subspaces. That means,

$$\gamma = \cos(\mathcal{V}_1, \mathcal{V}_2) = \sup_{u \in \mathcal{V}_1,\, v \in \mathcal{V}_2} \frac{\mathcal{A}(u, v)}{\left[\mathcal{A}(u, u)\, \mathcal{A}(v, v) \right]^{1/2}} \tag{13}$$

where $\mathcal{A}(.,.)$ denotes the bilinear form which appears in the variational formulation (2) of the original problem. Note that $\gamma = 0$ holds for orthogonal subspaces and is strictly less than one if $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$. For further details on types of preconditioners and on proofs of various estimates, the reader is referred to, e.g., Axelsson and Gustafsson [2] and Axelsson [1].

It was shown in [2] that under certain assumptions the constant γ for hierarchical basis splitting of the conforming finite elements can be estimated locally over each finite macro-element $E \in \mathcal{T}_H$, which means that $\gamma := \max_{E \in \mathcal{T}_H} \gamma_E$, where

$$\gamma_E = \sup_{u \in \mathcal{V}_{1,E},\, v \in \mathcal{V}_{2,E}} \frac{\mathcal{A}_E(u, v)}{\left[\mathcal{A}_E(u, u)\, \mathcal{A}_E(v, v) \right]^{1/2}} \tag{14}$$

and $\mathcal{V}_{i,E} := \mathcal{V}_i|_E$ is simply the restriction of the functions of the vector space $\mathcal{V}_i$ $(i = 1, 2)$ to an element E and, analogously, $\mathcal{A}_E(.,.) := \mathcal{A}(.,.)|_E$. With this local definition it can then be shown that γ depends on the construction of the subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ (i.e. on the type of basis functions chosen), but is independent of the mesh-size parameter h if, e.g., the refinement is performed by congruent triangles, is independent of the geometry of the domain Ω and also of any discontinuities of the coefficients involved in the bilinear form $\mathcal{A}_E(.,.)$, as long as they do not occur within the element E itself (cf. Blaheta et al. [5]).

2 The standard two-level DA-splitting and theoretical results for the Crouzeix-Raviart finite element

In order to derive estimates for the CBS-constant, it is known, that it suffices to consider an isotropic (Laplacian) problem in an arbitrarily shaped triangle T. Let us denote the angles in this triangle, as illustrated in Fig. 1, by θ_1, θ_2 and $\theta_3 := \pi - \theta_1 - \theta_2$ and let $a := \cot\theta_1$, $b := \cot\theta_2$ and $c := \cot\theta_3$ be the corresponding cotangens of the angles. Without loss of generality, for each such triangle T, we assume $\theta_1 \geq \theta_2 \geq \theta_3$. Then the following condition holds in the triangle (cf. Axelsson and Margenov [3]):

$$|a| \leq b \leq c\,. \tag{15}$$

A simple computation shows that the standard nodal basis element stiffness matrix for a non-conforming Crouzeix-Raviart (CR) linear finite element A_e^{CR} coincides with that for the conforming linear element A_e^c up to a factor 4 and can be written as

$$A_e^{CR} = 2\begin{pmatrix} b+c & -c & -b \\ -c & a+c & -a \\ -b & -a & a+b \end{pmatrix}. \tag{16}$$

The hierarchical stiffness matrix at macro-element level is then obtained by assembling four such matrices according to the numbering of the nodal points, as shown in Fig. 1(b). But, first we have to know on how to obtain a proper decomposition of the vector space $\mathcal{V}_h$, which is associated with the basis functions at the fine grid. This will be dealt with next.

For the non-conforming Crouzeix-Raviart finite element, where the nodal basis functions are defined at the midpoints along the edges of the triangle rather that at its vertices (cf. Fig. 1), the natural vector spaces $\mathcal{V}_H := \text{span}\,\{\phi_I, \phi_{II}, \phi_{III}\}$ and $\mathcal{V}_h := \text{span}\,\{\phi_i\}_{i=1}^9$ (cf. the macro-element in Fig.1(b)) are no longer nested, i.e. $\mathcal{V}_H \not\subseteq \mathcal{V}_h$. This renders a direct construction with $\mathcal{V}_2 := \mathcal{V}_H$, as used for conforming elements, impossible. Consequently, the hierarchical basis functions in Step 1 of the construction algorithm (see Section 1.1) have to be chosen such that the resulting vector subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ again satisfy the direct sum condition.

In the present paper, we will restrict ourselves to the splitting performed by differences and aggregates, the so-called DA-algorithm. The convergence behaviour and the constant in the

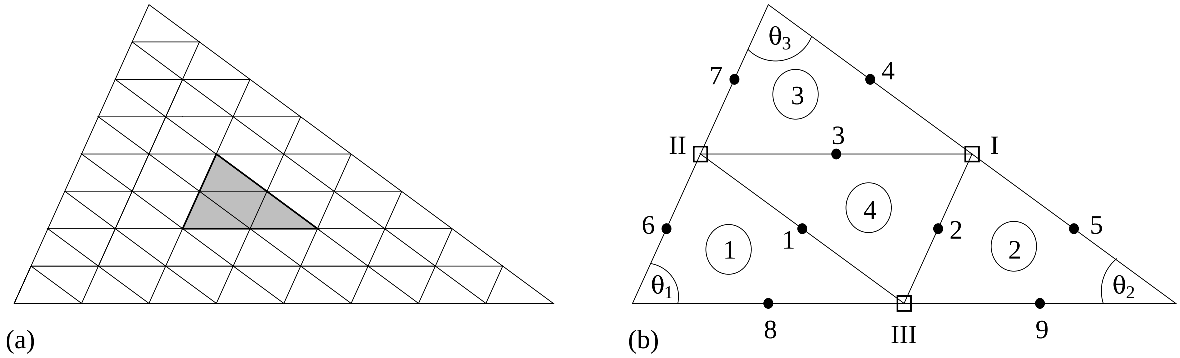


Figure 1: Crouzeix-Raviart finite element. (a) Discretization. (b) Macro-element in detail.

strengthened CBS inequality for this algorithm and of a second approach, the "first reduce" (FR) decomposition, were studied in detail in Blaheta et al., [5] and [6], for the case of no mesh restrictions. We will thus only present the main results for the standard DA-algorithm - as we will refer to it henceforth -, before we study possible improvements for practical applications where the mesh is usually restricted.

The decomposition by differences and aggregates (DA) on the macro-element level is based on the splitting $\mathcal{V}(E) := \text{span}\,\{\Phi_E\} = \mathcal{V}_1(E) \bigoplus \mathcal{V}_2(E)$, where $\Phi_E := \{\phi_E^{(i)}\}_{i=1}^9$ denotes the set of the "midpoint" basis functions of the four congruent elements in the macro-element E, as depicted in Fig. 1(b). For the standard DA-splitting the subspaces are defined as

$$\begin{aligned} \mathcal{V}_1(E) &:= \text{span}\,\{\phi_1,\, \phi_2,\, \phi_3,\, \phi_4 - \phi_5,\, \phi_6 - \phi_7,\, \phi_8 - \phi_9\} \quad \text{and} \\ \mathcal{V}_2(E) &:= \text{span}\,\{\phi_1 + \phi_4 + \phi_5,\, \phi_2 + \phi_6 + \phi_7,\, \phi_3 + \phi_8 + \phi_9\}\,. \end{aligned} \tag{17}$$

The transformation matrix corresponding to this DA-splitting is of the form

$$J_{DA} = \begin{pmatrix} I_3 & 0 \\ 0 & J^- \\ \frac{1}{2}\, I_3 & J^+ \end{pmatrix} , \tag{18}$$

where I_3 denotes the 3×3 unity matrix,

$$J^- := \frac{1}{2} \begin{pmatrix} 1 & -1 & & & & \\ & & 1 & -1 & & \\ & & & & 1 & -1 \end{pmatrix} , \quad \text{and} \quad J^+ := \frac{1}{2} \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \end{pmatrix} . \tag{19}$$

The matrix J_{DA} transforms the vector of the macro-element basis functions $\varphi_E := (\phi_E^{(i)})_{i=1}^9$ to the hierarchical basis vector $\tilde{\varphi}_E := (\tilde{\varphi}_E^{(i)})_{i=1}^9 = J_{DA}\, \varphi_E$ and the hierarchical stiffness matrix at macro-element level is obtained in a cost-saving way from its standard counterpart (cf. Remark 1.1) as

$$\tilde{A}_E \;=\; J_{DA}\, A_E\, J_{DA}^T, \tag{20}$$

which can be decomposed into a 2×2 block-matrix according to the basis functions of $\mathcal{V}_1$ and $\mathcal{V}_2$. Similarly, a two-by-two block structure for the hierarchical global stiffness matrix

$$\sum_{E \in \mathcal{T}_H} \tilde{A}_E =: \tilde{A}_h = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \tag{21}$$

can be obtained, whereby the upper-left block $\tilde{A}_{11}$ corresponds to the interior degrees of freedom (such as to element 4 in the macro-element, as depicted in Fig. 1(b)) plus the differences of the nodal unknowns along the edges of the macroelement $E \in \mathcal{T}_H$. The lower-right block on the other hand corresponds to the aggregates of nodal unknowns, as defined by $\mathcal{V}_2(E)$ for the macro-elements.

The analysis of the related two-level method can again be performed locally by considering the corresponding problems at macro-element level. The main result on the CBS constant was obtained in Blaheta et al. [5], and reads:

Theorem 2.1 *Let us consider the two-level DA-splitting, as defined in* (17)*, without restrictions on the mesh and/or coefficient anisotropy. Then, the CBS constant is uniformly bounded w.r.t. both coefficient and mesh anisotropy, by*

$$\gamma^2 \leq \frac{3}{4}. \tag{22}$$

The latter estimate is independent of the discretisation parameter h and possible jumps aligned with the finite element partitioning $\mathcal{T}_H$.

The extension of the two-level algorithm to the multilevel case is based on successive uniform refinement of a given triangulation $\mathcal{T}^{(k)}$ at level k and requires the recursive application of the two-level splitting of the finite element spaces $\mathcal{V}^{(k)}$ or $\mathcal{V}_2^{(k+1)}$, where $k = 0, 1, \ldots, n$, and $\mathcal{V}^{(0)}$ denotes the vector space corresponding to the coarsest mesh. This extension is ensured for the DA-algorithm by the following useful result, as shown by Blaheta, Margenov and Neytcheva [5]:

Theorem 2.2 *Let $\tilde{A}_{22}$ be the stiffness matrix, as defined by* (21)*, and let A_H be the stiffness matrix corresponding to the finite element space $\mathcal{V}_H$ of the coarse grid discretization $\mathcal{T}_H$, equipped with the standard nodal basis $\{\phi_H^{(k)}\}_{k=1,\ldots,N_H}$. Then,*

$$\tilde{A}_{22} = 4\, A_H\,. \tag{23}$$

3 Generalization of the 2-level splitting by differences and aggregates (GDA-splitting)

The DA-decomposition, as presented in Section 2, can be generalized by enhancing the aggregates in the vector space $\mathcal{V}_2(E)$: Instead of utilizing only the nodal basis function along the corresponding edge of the inner element, numbered as element 4 in the macro-element, as depicted in Fig. 1, one can use a linear combination of all three inner basis functions. In the generalized DA-splitting (GDA) we thus use:

$$\begin{aligned}\mathcal{V}_1(E) &:= \text{span}\,\{\phi_1,\,\phi_2,\,\phi_3,\,\phi_4-\phi_5,\,\phi_6-\phi_7,\,\phi_8-\phi_9\} \quad \text{and} \\ \mathcal{V}_2(E) &:= \text{span}\,\{\phi_{123}+\phi_4+\phi_5,\;\; \phi_{312}+\phi_6+\phi_7,\;\; \phi_{231}+\phi_8+\phi_9\}\,, \qquad (24)\end{aligned}$$

where $\phi_{ijk} := c_i\phi_i + c_j\phi_j + c_k\phi_k$ and $i,j,k \in \{1,2,3\}$. Note that the vector space $\mathcal{V}_1(E)$ remains unaltered compared to its definition in (17). The transformation matrix, which corresponds to this generalized DA-splitting, reads

$$J_{DA} = \begin{pmatrix} I_3 & 0 \\ 0 & J^- \\ \frac{1}{2}I_3 + C & J^+ \end{pmatrix}, \qquad (25)$$

where solely the lower-left block $\frac{1}{2}I_3$ of the representation in (18) is now replaced by $\frac{1}{2}I_3 + C$ with

$$C := \begin{pmatrix} c_1 - \frac{1}{2} & c_2 & c_3 \\ c_3 & c_1 - \frac{1}{2} & c_2 \\ c_2 & c_3 & c_1 - \frac{1}{2} \end{pmatrix}, \qquad (26)$$

and the coefficients c_1, c_2 and c_3 are yet to be determined. Note that for $c_1 = \frac{1}{2}$ and $c_2 = c_3 = 0$ the settings in the standard DA-algorithm are obtained.

In order to ensure the extension of the generalized two-level splitting to the multilevel case, we have to show that $\tilde{A}_{22}$ is properly related to A_H, the stiffness matrix at the coarser level. For the standard DA-approach we could show in Theorem 2.2 that this relation is given by the constant factor 4, which results in the same γ - estimate at each level. For the generalized DA-splitting, however, the relation is given by a matrix R in the form

$$\tilde{A}_{22} = R \cdot A_H\,. \qquad (27)$$

We will show in Section 5 that R has the property that a successive application in a multilevel context yields convergence to the equilateral case for which the minimal CBS-constant is obtained. Hence, this generalized splitting yields a self-improvement of the constant γ with each level of refinement.

In order to analyze the constant $\gamma = \cos(\mathcal{V}_1, \mathcal{V}_2)$ for the generalized two-level DA-splitting, as defined in (24), we again consider the corresponding problems at the macro-element level. Such a local analysis requires to have satisfied the condition

$$\ker\left(\tilde{A}^{(k+1)}_{22,E}\right) = \ker\left(A^{(k)}_e\right) \qquad (28)$$

at each level k, which can be shown to be satisfied iff

$$c_1 = \frac{1}{2} - c_2 - c_3\,. \tag{29}$$

Since the macro-element was obtained by uniform refinement of a triangle at the coarser level the condition

$$c_3 = c_2 \tag{30}$$

should hold by symmetry reasons. This latter setting is verified in Subsection 4.1 (see Figs. 2 and 3) by using symbolic computations with Mathematica in the case of fixed angles in the triangle, but varying values for c_2 and c_3. Note that for $c_2 = 0$ the standard DA-splitting is obtained.

Combining the settings in (29) and (30) yields $c_1 = \frac{1}{2} - 2\,c_2$. Hence, the generalized DA-splitting only depends on $c_2 \in (0, \frac{1}{4}]$, which can now be used to optimize the CBS-constant. In the following subsections we will study the behaviour of the CBS-constant for varying values of c_2 and varying angles θ_i, $i = 1, 2$, with $\theta_3 = \pi - (\theta_1 + \theta_2)$ in the triangle and show how these findings can be exploited for an improvement of an aggregation-based multilevel preconditioning.

4 Study of the CBS constant in the GDA-splitting

4.1 The effect of c_2 on the CBS-constant γ

In a first study we look at the behaviour of γ and of the c_i-values ($i = 1, 2, 3$) for various fixed angles θ_1 in the triangle. The estimate of the CBS-constant γ can be performed locally by considering the corresponding problems at macro-element level for the two-level method. The local analysis is based on the fact that $(1 - \gamma_E^2)$ is the maximal constant satisfying the inequality

$$1 - \gamma_E^2 \;\leq\; \frac{\mathbf{v}^T\,\tilde{S}_E\,\mathbf{v}}{\mathbf{v}^T\,\tilde{A}_{22}\,\mathbf{v}} \tag{31}$$

for all $\mathbf{v} \neq (c, c, c)^T$ with $c \in \mathbb{R}$. That means, $(1 - \gamma_E^2)$ is the minimal eigenvalue of the generalized eigenvalue problem

$$\tilde{S}_E\,\mathbf{v} \;=\; \lambda\,\tilde{A}_{22}\,\mathbf{v} \tag{32}$$

with $\mathbf{v} \notin \mathcal{N}(\tilde{A}_{22}) = \mathcal{N}(\tilde{S}_E)$, where

$$\tilde{S}_E := \tilde{A}_{22} - \tilde{A}_{21}\,\tilde{A}_{11}^{-1}\,\tilde{A}_{12} \tag{33}$$

denotes the Schur complement related to the hierarchical basis. Using relation (20) it can be shown that $\tilde{S}_E$ is equal to the Schur complement S_E based on the standard basis functions.

Figures 2 and 3 depict the results for γ_E^2, c_1, c_2 and c_3 for fixed values of $\theta_1 \in \{75°, 90°, 105°\}$ and $\theta_3 := 180° - (\theta_1 + \theta_2)$, but varying values of θ_2 in the ranges specified in the figures. The value of γ_E^2, thereby, is computed as

$$\gamma_E^2 := \max_{c_2, c_3 \in [0, \frac{1}{4}]} \gamma_E^2(c_2, c_3)\,, \tag{34}$$

and thus provides an upper bound for the true CBS-constant, where $1 - \gamma_E^2(c_2, c_3)$ denotes the minimal eigenvalue of the corresponding eigenproblem of the form (32). In each figure, the standard value of $\gamma_E^2 = \frac{3}{4}$ is indicated by a solid straight line.

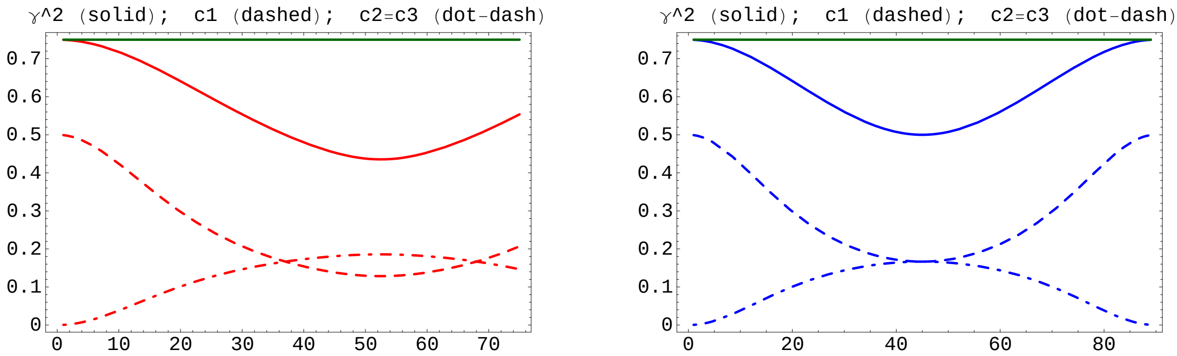


Figure 2: Distribution of γ_E^2, c_1, c_2 and c_3 for $\theta_1 = 75°$ (left) and $90°$ (right).

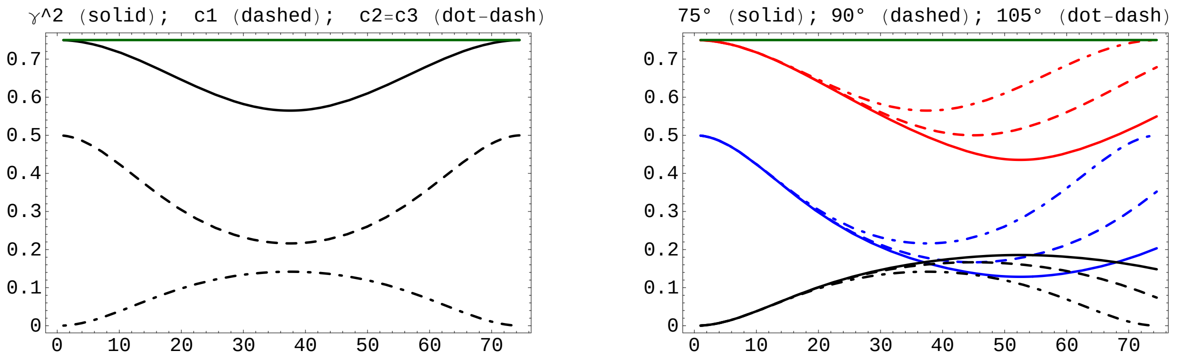
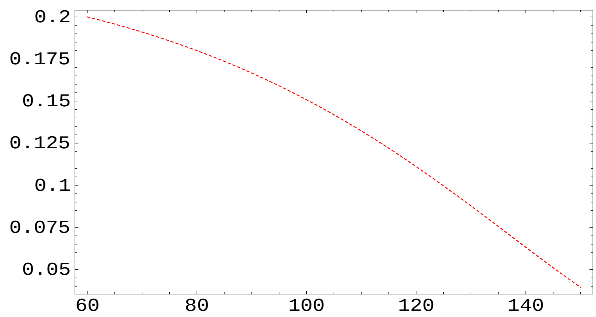


Figure 3: Distribution of γ_E^2, c_1, c_2 and c_3 for $\theta_1 = 105°$ (left) and summarized for all three θ_1-settings (right; from top to bottom: distributions of γ_E^2, of c_1, and of $c_2 = c_3$).

The numerical results obtained for c_2 and c_3 verify the setting in (30), which was obtained from symmetry considerations. It can be seen that throughout γ_E^2 and c_1 attain their minimum at $\theta_2 = \theta_3$ and that all distributions exhibit a symmetric behaviour about this setting. Note that for a better illustration of the behaviour of γ_E^2, c_1 and c_2, the values of θ_2 were always started at $0°$ rather than chosen in the range satisfying the condition $\theta_1 \geq \theta_2 \geq \theta_3$.



θ_1	$\gamma^2_{E,min}$	$\theta_2 = \theta_3$	$c_2 = c_3$
60°	0.375	60°	0.2
75°	0.435	52.5°	0.186
90°	0.5	45°	0.167
105°	0.565	37.5°	0.142
120°	0.625	30°	0.111
135°	0.677	22.5°	0.075

Figure 4: Sketch of the course of the minimal values of $\gamma_E^2(c_2, c_3)$ for $\theta_1 \in [60°, 150°]$, as attained at $\theta_2 = \theta_3$, and summary of γ_E^2 and c_2 for some values thereof in the table.

In Figure 4 the course of the minimal values of $\gamma_E^2(c_2, c_3)$ is sketched for different values of the angle θ_1 (x-axis) versus c_2 (y-axis) and in the enclosed table the minimal values of γ_E^2 are summarized for some discrete θ_1-values together with the corresponding values for θ_2 and c_2. It can be seen that for a fixed value of θ_1 the minimal value of γ_E^2 is always attained for the isosceles triangle and that the overall minimum of γ_E^2 at a value of 3/8 is obtained for the equilateral triangle. Note that for this part of our study no restrictions on the angles in the triangular mesh were posed.

4.2 The behaviour of the CBS constant γ_E for fixed c_2-values subject to a minimum angle condition in the isotropic case

As mentioned in the previous subsection, γ_E^2 attains its minimum for a fixed angle θ_1 at $\theta_2 = \theta_3$ and its overall minimum of 3/8 for the equilateral triangle. This means, that γ_E^2 can be improved compared to its value of 3/4, as obtained for the standard DA-algorithm, where $c_2 = 0$, if the angles in the triangles are chosen sufficiently large.

Therefore, we want to investigate the effect of c_2 on the CBS constant subject to a minimum angle condition. Together with the assumptions of Section 2, we now use

$$\theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_{min}\,, \tag{35}$$

where $\theta_3 = \pi - \theta_1 - \theta_2$, which renders Eq. (15) to become

$$|a| \leq b \leq c \leq d := \cot(\theta_{min})\,. \tag{36}$$

Figure 5 depicts the distribution of the maximum values of $\gamma_E^2(c_2)$ w.r.t. θ_1 and θ_2 for different settings of $\theta_{min} \in \{20°, 25°, 30°\}$, as commonly used in commercial mesh generators. The maximum value of $\gamma_E^2(c_2)$ for a given setting is thereby always obtained for the isosceles triangle with two angles equal to θ_{min}. It can be seen that the CBS constant can significantly be improved for larger minimum angles and that the larger θ_{min} the larger the c_2-value, where $\gamma_E^2(c_2)$ attains its minimum. While for values of c_2 less than about 0.05, the γ_E^2-curves, which correspond to three minimum angles chosen, stay close together, they spread out significantly for larger values of c_2. This effect is summarized in Table 1, where the values of γ_E^2 are presented at each minimum location (for a given curve) for all three settings of θ_{min} together with the corresponding c_2-value.

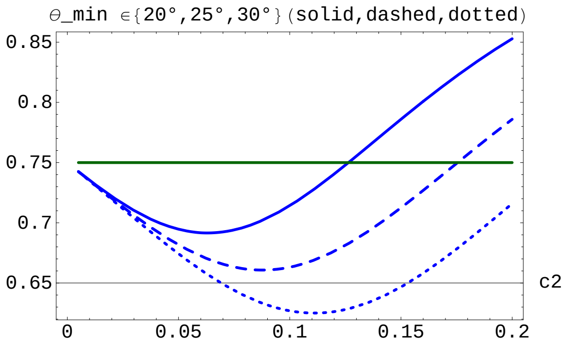


Figure 5: Distribution of $\max_{\theta_k \geq \theta_{min},\, k=1,2,3} \gamma_E^2(c_2)$ for different values of θ_{min}, compared to its distribution for the standard DA-algorithm (solid straight line).

c_2	0.0628	0.0876	0.1111
$\gamma_E^2(.,20°)$	0.692	0.702	0.728
$\gamma_E^2(.,25°)$	0.670	0.661	0.669
$\gamma_E^2(.,30°)$	0.657	0.633	0.625

Table 1: Values of $\max_{\theta_k \geq \theta_{min},\, k=1,2,3} \gamma_E^2(c_2)$, evaluated at the minimum locations of the three curves (the respective value of θ_{min} is given in parantheses).

Note that for larger minimum angles the CBS constant is also less sensitive to c_2-settings, i.e., γ_E^2 remains below the standard DA-threshold of 3/4 for a wider range of c_2-values. This underlines the importance of using a proper choice of c_2 besides a well-structured mesh in numerical computations of elliptic problem based on finite element methods.

As is known, for the standard DA-approach, where $c_2 = 0$ is used, no restrictions on the mesh are posed. Consequently, if one does not know anything about the underlying mesh it is recommended (based on our results in Figs. 4 and 5) to start with a relatively small value of c_2 of about 0.06 and increase it during the multilevel approach since a successive application of the generalized DA-algorithm in a multilevel framework continuously improves the minimum angle condition, as will be shown in the next section. If, on the other hand, one knows the minimum angle condition used in the mesh generator, one can determine a proper c_2-value at the first level in a multilevel algorithm (e.g., from Fig. 4), and use a slightly increased value of c_2 at the subsequent levels. This yields a self-adaptive algorithm, which continuously improves the CBS-constant.

5 On the properties of the stiffness matrix $A_e^{(k)}$ in a multilevel preconditioning approach

Applied in a multilevel context, Eq. (27), stated at elemental level, takes the form

$$A_e^{(k+1)} = R_e^{(k)} \cdot A_e^{(k)} . \tag{37}$$

Thereby, $A_e^{(k)}$ denotes the element stiffness matrix at level k, with the stiffness matrix at the coarsest level given by $A_e^{(0)}$, which is assembled by element stiffness matrices of the form (16) up to a certain constant factor, and $A_e^{(k+1)} := \tilde{A}_{E,22}^{(k+1)}$. Since we will normalize the matrix with regard to one of its entries in the following, this constant factor will cancel out.

The matrix $R_e^{(k)} := R_e^{(k)}(a, b, c)$ designates the relation matrix between the stiffness matrices at levels k and $k + 1$. In the general setting, by utilizing the condition $ab + ac + bc = 1$, which holds in any triangle by elementary trigonometric relations if a, b, and c denote the cotangens of the angles θ_1, θ_2, and θ_3, one obtains:

$$R_e^{(k)} = \begin{pmatrix} 4[(2a+c)\,p(c_2) + g(c_2)] + d_{13} & -4(b-c)\,p(c_2) + d_{13} & d_{13} \\ -4(a-c)\,p(c_2) + d_{23} & 4[(2b+c)\,p(c_2) + g(c_2)] + d_{23} & d_{23} \\ -4[(2c+a)\,p(c_2) + g(c_2)] + d_{33} & -4[(2c+b)\,p(c_2) + g(c_2)] + d_{33} & d_{33} \end{pmatrix}, \tag{38}$$

where $g(c_2) := 1 - 6c_2 + 12c_2^2$ and $p(c_2) := c_2^2(a + b + c)$. Since both local stiffness matrices, $A_e^{(k)}$ and $A_e^{(k+1)}$, are only of rank 2, the parameters d_{13}, d_{23} and d_{33} can be chosen arbitrarily. In the standard DA-approach, the setting $d_{13} = d_{23} = 0$ and $d_{33} = 4$ together with $c_2 = 0$ yields the relation stated in Eq. (23) of Theorem 2.2. For the generalized DA-algorithm, however, a setting of $d_{13} = d_{23} = d_{33} = 0$ at each level k is more favourable since under the assumption of a minimum angle condition this yields convergence to the equilateral case, which exhibits the best value for the CBS constant, viz. $\gamma_E^2 = 3/8$.

Note that the element stiffness matrix $A_e^{(k+1)}$ is used as the coarse grid matrix at the next finer level and can again be cast in the form (16). This can easily and without loss of generality be achieved by redefining the entries in the stiffness matrix as results of modified angles in a triangle before applying Eqs. (37) and (38). Hence, the only difference compared to $A_e^{(k)}$ is that it corresponds to a triangle with modified angles, as will be shown next. Consequently, the stiffness matrix $A_e^{(k)}$ remains positive semi-definite at all levels.

Lemma 5.1 *Let $a^{(0)} := a$, $b^{(0)} := b$, and $c^{(0)} := c$ denote the cotangens of the angles θ_1, θ_2, and θ_3 in an arbitrary triangle at level* 0 *with the corresponding element stiffness matrix being of the form* (16)*. Then, at any level $k \in \mathbb{N}_0$, the element stiffness matrices generated in a multilevel algorithm according to Eq.* (37)*, with the $R_e^{(k)}$ entries given by Eq.* (38)*, correspond to a Crouzeix-Raviart stiffness matrix of the form* (16)*, i.e.,*

$$(\alpha^{(k)}, \beta^{(k)}) := \left(\frac{a^{(k)}}{c^{(k)}}, \frac{b^{(k)}}{c^{(k)}} \right) \in D \tag{39}$$

holds at any level $k \in \mathbb{N}_0$*, where*

$$D := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \in (-1/2,\, 1],\;\; \beta \in (0,\, 1],\;\; \alpha+\beta > 0, \text{ and } \alpha \le \beta\}\,. \tag{40}$$

Proof. For $k = 0$ the validity of the statement was shown by Axelsson and Margenov [3]. Let Condition (39) be satisfied at an arbitrary level k and let, without loss of generality, $(a^{(k)}, b^{(k)}, c^{(k)}) =: (a, b, c)$. Then, from Eq. (37) the entries in the element stiffness matrix at level $k+1$ read as follows:

$$\left(a^{(k+1)},\; b^{(k+1)},\; c^{(k+1)}\right) := \left(\frac{a\,g(c_2)+p(c_2)}{c\,g(c_2)+p(c_2)}\,,\;\; \frac{b\,g(c_2)+p(c_2)}{c\,g(c_2)+p(c_2)}\,,\;\; c\right)\,, \tag{41}$$

where $p(c_2) := c_2^2(a+b+c)$ and

$$g(x_2) := 1 - 6c_2 + 12c_2^2 \ge \frac{1}{4} > 0 \quad \text{for all} \quad c_2 \in \mathbb{R}\,. \tag{42}$$

If $c_2 = 0$, then $(\alpha^{(k+1)},\, \beta^{(k+1)}) = (\alpha^{(k)},\, \beta^{(k)})$ and we are done. Let us therefore assume $c_2 \ge c_{2,min} > 0$ in the following: The bounds $\alpha^{(k+1)} \le 1$, $\beta^{(k+1)} \le 1$, and $\alpha^{(k+1)} \le \beta^{(k+1)}$ then follow immediately from Eq. (42) and Condition (15).

The inequalities $\beta^{(k+1)} > 0$ and $\alpha^{(k+1)} + \beta^{(k+1)} > 0$, which can equivalently be written as $b\,g(c_2) + c_2^2(a+b+c) > 0$ and $(a+b)\,g(c_2) + 2c_2^2(a+b+c) > 0$, respectively, hold by Eq. (42), Condition (15), and the fact that $a + b > 0$ is true in any triangle (cf. Axelsson and Margenov [3]).

Finally, the condition $\alpha^{(k+1)} > -\frac{1}{2}$ is satisfied iff $(2a+c)\,g(c_2) + 3c_2^2(a+b+c) > 0$ or, equivalently, iff $2a + c > 0$. If $\theta_1 \le \pi/2$, the result follows directly from the inequality $0 < a \le b \le c$. If, on the other hand, $\pi/2 < \theta_1 < \pi$, then $\theta_2 + \theta_3 < \pi/2$. From $\theta_1 \ge \theta_2 \ge \theta_3$ and $a := \cot\theta_1$, $b := \cot\theta_2$ and $c := \cot\theta_3$, we obtain $\theta_3 \le \theta_{3,max} = \theta_2$, which implies $c \ge c_{min} := \cot\theta_{3,max} = \cot\dfrac{\pi-\theta_1}{2}$. Let $\varphi := \theta_1/2$, then by elementary trigonometric relations we obtain

$$2a + c \ge 2a + c_{min} = 2\cot\theta_1 + \cot\left(\frac{\pi}{2} - \varphi\right) = 2\,\frac{\cos 2\varphi}{\sin 2\varphi} + \frac{\cos(\frac{\pi}{2}-\varphi)}{\sin(\frac{\pi}{2}-\varphi)} = \cot\varphi = \cot\frac{\pi}{2} > 0$$

since $\theta_1/2 \in (\pi/4,\, \pi/2)$. Hence, $(\alpha^{(k+1)},\, \beta^{(k+1)}) \in D$, which completes the proof. ∎

Lemma 5.2 *Let* $A_e^{(k)}$ *be the element stiffness matrix at level* k *and let* $A_e^{(k+1)}$ *be the corresponding stiffness matrix at level* $k+1$*, which is defined by* (21) *as* $A_e^{(k+1)} := \tilde{A}_{E,22}^{(k+1)}$ *and which is related to* $A_e^{(k)}$ *according to Eq.* (37)*. Let*

$$B^{(k)} := c\,A_e^{(k)}/A_{e,12}^{(k)} \tag{43}$$

be the normalized local stiffness matrix at level k*, where* $A_{e,12}^{(k)}$ *denotes the (1,2)-entry of the matrix* $A_e^{(k)}$*. Then, under the assumption of convergence and* $c_2 \ne 0$*, the limiting value of* $B^{(k)}$ *is given by the normalized stiffness matrix corresponding to the equilateral triangle, denoted as* B_{eq}*, with* $a = b = c = 1/\sqrt{3}$*.*

Proof. Due to the normalization given in the Lemma and under the assumption of convergence the following condition (obtained from Eq. (37)) must hold in the limit:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c\,[a\,g(c_2)+p(c_2)]\;/\;[c\,g(c_2)+p(c_2)] \\ c\,[b\,g(c_2)+p(c_2)]\;/\;[c\,g(c_2)+p(c_2)] \\ c \end{pmatrix} . \tag{44}$$

It can be shown easily that this holds true iff $(a = c \,\wedge\, b = c) \,\vee\, p(c_2) := c_2^2(a+b+c) = 0$. Consequently, if $c_2 \neq 0$, this can only be satisfied for the equilateral triangle since $a+b+c$ is positive in any triangle. ∎

Remark 5.1 *If $c_2 = 0$, Condition (44) holds for any triangle, where the latter remains unaltered at all levels. In this case, Condition (44) simply produces a constant sequence of the cotangens of the angles in the initial triangle.*

For $c_2 > 0$, where the limiting case is given by the equilateral triangle, we can show the following convergence result:

Theorem 5.1 *Let $B^{(k)}$ be the normalized stiffness matrix at a given level $k \in \mathbb{N}_0$, as defined in (43). Further, let c_2 be bounded away from zero, i.e., $c_2 \in [c_{2,min},\, 1/4]$ with $c_{2,min} > 0$. Then,*

$$\|B^{(k+1)} - B_{eq}\|_F \;\leq\; q\,\|B^{(k)} - B_{eq}\|_F \tag{45}$$

is satisfied for some positive $q < 1$, where $\|.\|_F$ denotes the Frobenius norm of a given matrix.

Proof. Using the normalization of the stiffness matrices, as given by Eqs. (43) and (44), and $(a_{eq},\, b_{eq},\, c_{eq}) := (c, c, c)$, we can define

$$G := \frac{\|B^{(k+1)} - B_{eq}\|_F}{\|B^{(k)} - B_{eq}\|_F} \;=\; \frac{c\,g(c_2)}{c\,g(c_2)+p(c_2)} \tag{46}$$

with the already introduced functions $g(c_2) = 1 - 6c_2 + 12c_2^2$ and $p(c_2) = c_2^2\,(a+b+c)$. Relation (45) then means:

$$G \leq q \iff c\,g(c_2) \;\leq\; q\;[c\,g(c_2)+p(c_2)]\;. \tag{47}$$

Since $a + b > 0$ (cf. Axelsson and Margenov [3]) the relation $a+b+c > c$ (> 0 by Eq. (15)) holds true in any triangle. Together with our assumption that $c_2 > 0$ this yields $p(c_2) > c\,c_2^2 > 0$. Since $g(c_2) \geq 1/4$ is satisfied for any $c_2 \in \mathbb{R}$ the term on the right-hand side in Eq. (47) is always positive and one obtains

$$q \;=\; \frac{c\,g(c_2)}{c\,g(c_2)+p(c_2)} \;<\; \frac{g(c_2)}{g(c_2)+c_2^2} \;=:\; \bar{q}(c_2)\,, \tag{48}$$

which immediately yields $0 < q < 1$. This completes our proof. ∎

Corollary 5.1 *By induction, one immediately obtains from Relation* (45) *that*

$$\|B^{(k)} - B_{eq}\|_F \;\leq\; q^k \,\|B^{(0)} - B_{eq}\|_F \tag{49}$$

and, consequently, since $0 < q < 1$ *and* $\|B^{(0)} - B_{eq}\|_F$ *is bounded*

$$\|B^{(k)} - B_{eq}\|_F \longrightarrow 0 \quad \text{if} \quad k \to \infty\,. \tag{50}$$

Therefore, the limiting matrix is given by the normalized element stiffness matrix B_{eq}*, which corresponds to the equilateral triangle.*

Remark 5.2 *Note that due to Eqs.* (37) *and* (43) *the element matrices* $A_e^{(k)}$ *and* $B^{(k)}$ *only differ by the constant scaling factor* $\alpha^{(k)} := c\,/A_{e,12}^{(k)}$ *at each level. Hence, the convergence result, as stated in Theorem 5.1 for* $B^{(k)}$*, also holds for the corresponding stiffness matrices* $A_e^{(k)}$*. Consequently, the same result is obtained if normalization is used at each level or only at the end of the computations. The normalization at each level only ensures that the entries in the corresponding stiffness matrices remain small during the calculation process.*

Remark 5.3 *A closer look at the estimate, as given by Eq.* (48)*, shows that* $\bar{q}(c_2)$ *attains its maximum at* $c_2 = 0$ *with* $\bar{q}(0) = 1$ *and its minimum at* $c_2 = 1/3$ *with* $q(1/3) = 3/4$*. For* $c_2 \in [c_{2,min},\, 1/4]$ *with* $c_{2,min} > 0$*, as used in Theorem 5.1, however, we obtain* $\bar{q} \in [4/5,\, 1)$*. That means, the larger the value of* c_2 *the better the convergence rate in Eq.* (45) *since* $q\,(< \bar{q})$*.*

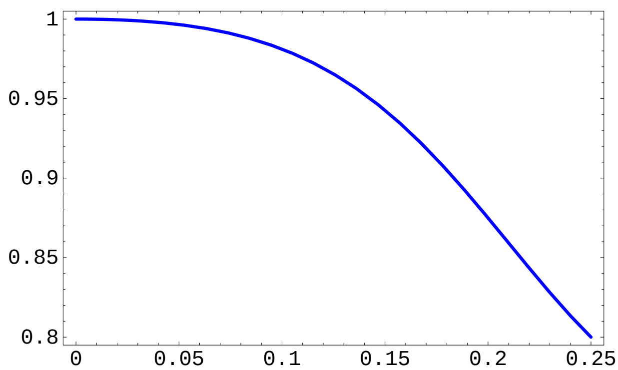


Figure 6: Distribution of the upper bound of the convergence factor q (viz. $\bar{q}$) w.r.t. c_2.

In Figure 6 the behaviour of the upper bound of the proportionality factor q is depicted for c_2 varying in the interval of interest $(0,\, 1/4]$. It can be seen that for small values of c_2, i.e. for $c_2 \in (0,\, 0.08]$, the factor $\bar{q}$ is only between 0.99 and unity, and decreases quite rapidly for values of $c_2 > 0.08$. The upper bound of q exhibits a nearly linear behaviour for $c_2 \geq 0.16$, approximately, with $\bar{q}$ decreasing from about 0.94 at $c_2 = 0.16$ down to a value of 0.8 at $c_2 = 0.25$. For such larger settings of c_2 we can thus expect a fast convergence of the stiffness matrices $A^{(k)}$ towards the stiffness matrix for the equilateral triangle within a reasonably small number of levels and all the more since the true factor q is even smaller than $\bar{q}$.

This expectation is confirmed in the following example, where we studied the behaviour of the true convergence factor q (as obtained from Eq. (46)) and of the Frobenius norm of the distance between $B^{(k)}$ and B_{eq} in the triangle with an angle setting of $(\theta_1, \theta_2, \theta_3) = (100^o, 60^o, 20^o)$ and an initial value of $\|B^{(0)} - B_{eq}\|_F = 1.623$ at level 0. In Table 2 the results for q and $B^{(k)}$ and B_{eq} are summarized for different levels k and different settings of c_2.

	$c_2 = 0.10$		$c_2 = 0.15$		$c_2 = 0.20$		$c_2 = 0.25$	
k	q	D	q	D	q	D	q	D
1	0.978	1.588	0.935	1.517	0.859	1.395	0.777	1.262
2	0.978	1.553	0.932	1.409	0.846	1.161	0.748	0.908
3	0.978	1.517	0.928	1.299	0.833	0.937	0.721	0.609
5	0.977	1.444	0.922	1.081	0.809	0.561	0.680	0.236
10	0.975	1.261	0.907	0.610	0.767	0.115	0.629	0.016
20	0.971	0.906	0.885	0.141	0.734	0.003	0.600	5.9e-5

Table 2: Study of the convergence factor q and of D:=$\|B^{(k)} - B_{eq}\|_F$ for different c_2-values.

It can be seen that the true value of the convergence factor q is indeed smaller than $\bar{q}$ and that q is truly smaller for larger values of c_2, which yields the expected faster convergence towards the (normalized) equilateral case. Consequently, the CBS-constant decreases faster for larger values of c_2 since the stiffness matrix resembles more and more its equilateral counterpart. This behaviour of γ can now be exploited in a multilevel preconditioning approach based on the generalized DA-splitting.

6 Concluding Remarks

In the present study we derived a generalized version of the DA-splitting by enhancing the aggregates in the decomposition, which can be shown to result in one additional parameter, termed c_2, in the end. Setting $c_2 = 0$ yields the standard DA-approach with the constant γ^2 in the Cauchy-Bunyakowski-Schwarz (CBS) inequality being uniformly bounded by 3/4 w.r.t. to both coefficient and mesh anisotropy. It was shown that the CBS constant can be significantly improved if the angles in the triangular mesh satisfy a minimum angle condition with θ_{min} being reasonably large (i.e. at least larger than 20 degrees) as commonly assumed in practical applications. For a given value of the largest angle in the triangle, θ_1, the CBS constant attains its minimum for the isosceles triangle and the overall best value of 3/8 is obtained for the equilateral one.

Further, it could be shown that all element stiffness matrices, generated by the generalized DA-algorithm in a multilevel setting, correspond to a Crouzeix-Raviart stiffness matrix of the form (16) and that their entries converge (with the number of levels) towards the entries of the stiffness matrix corresponding to the equilateral triangle. This means that the angles in the corresponding triangle are continuously improved with each level in a multilevel GDA-setting since the entries of the stiffness matrix stem from the cotangens of these angles. Moreover, and what is even more important, the CBS-constant in such a multilevel GDA-preconditioning can continuously be improved with each level and converges towards the optimal γ^2 value of 3/8, whereby the convergence can be accelerated for larger values of c_2, particularly for c_2-values greater than 0.15, i.e. for well-structured meshes.

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Authors' addresses: J. Synka, Industrial Mathematics Institute, Johannes Kepler University, Altenberger Str. 69, A-4040 Linz, Austria (E-mail: josef.synka@jku.at); S. Margenov, Institute for Parallel Processing, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 25-A, 1113 Sofia, Bulgaria (E-mail: margenov@parallel.bas.bg).