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Fast domain decomposition algorithm for discretizations of 3-d elliptic equations by spectral elements

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Abstract

The main obstacle for obtaining fast domain decomposition solvers for the spectral element discretizations of the 2-nd order elliptic equations was the lack of fast solvers for local internal problems on subdomains of decomposition and their faces. As was recently shown by Korneev/Rytov, such solvers can be derived on the basis of the specific interrelation between the stiffness matrices of the spectral and hierarchical p reference elements. The coordinate polynomials of the latter are tensor products of the integrated Legendre's polynomials. This interrelation allows to apply to the spectral element discretizations fast solvers which in some basic features are quite similar to those developed for the discretizations by the hierarchical elements. Using these facts and the preceding results on the wire basket preconditioners, we present an almost optimal in total arithmetical cost domain decomposition preconditioner-solver for the spectral element discretizations of the 2-nd order elliptic equations in 3-d domains.

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1 Introduction

In DD (domain decomposition) methods, the main contribution to the computational work is due to the two major components. They are related to solution of local Dirichlet problems

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on subdomains of decomposition and problems on their faces. In the case of hp methods, a good choice for subdomains of decomposition is the domains occupied by finite elements of the discretization. For this reason, under the conditions of the shape regularity of finite elements of a FE (finite element) assemblage, optimization of these components in the respect of the computational work is reduced to obtaining fast preconditioners-solvers for the stiffness matrix of the p reference element and the Schur complement related to its boundary.

In spite of the developed general theory of DD preconditioners for hp discretizations, providing almost optimal condition and a deep parallelization, see Babuska/Craig/Mandel/Pitkaranta [2], Ainsworth [1], Pavarino/Widlund [37], Ivanov/Korneev [20], Oden/Patra/Feng [35], Korneev/Flaherty/Oden/Fish [24], Toselli/Widlund [40] and other publications, optimization (especially in 3-d case) of the pointed out two main components in the respect of the total computational work started recently. As a starting point for obtaining such optimized components for the two major types of hp discrtetizations, there were primarily used the finite-difference preconditioners, suggested by Ivanov/Korneev [20], see also Korneev/Jensen [25], and Orzag [36] for the respective reference element stiffness matrices. In this paper, the reference element with the form functions, produced by the tensor products of the integrated Legendre's polynomials, are termed *hierarchical*. For such reference elements, especially in 2-d, a number of fast preconditioners-solvers for the internal stiffness matrices have been justified theoretically and tested numerically. For instance, the DD type solver of the second stage of Korneev [22, 23], the multilevel solver of Beuchler [8] and the multilevel solver of Beuchler/Schneider/Schwab [9], which is based on the multiresolution wavelet decompositions and is also optimal in 3-d. One suggestion on efficient solvers for faces of 3-d hierarchical reference elements was made by Korneev/Langer/Xanthis [27, 28]. For the spectral elements, there is known, *e.g.,* the multilevel solver of Shen/Wang/Xu [38], which efficiency was well approved numerically.

In outward appearance hierarchical and spectral elements seem rather different. However, very recently Korneev/Rytov [29, 30] established an interrelation between three the most popular types of the cubic p reference elements, including the mentioned two, showing that in computations they can be treated with a great measure of similarity. On one hand, it is established by a diagonal transformation of the known basic finite-difference preconditioners, neglecting some minor terms and easy estimates, and as a consequence its use is computationally cheap. On the other hand, it allows to adapt all solvers, known for reference elements of one type, into the solvers for stiffness matrices of other types reference elements. It is important, that the adapted solvers have the same in the order of p computational cost as the source solvers. For one of the instances, Korneev/Rytov [29] justified the fast multilevel solver for spectral elements, which is of the same type with one suggested by Beuchler [8] for the 2-d hierarchical p element. In this paper, first of all we obtain fast solvers for the internal finite element and face subproblems, arising in DD algorithms for 3-d discretizations by spectral elements. For the internal Dirichlet problem on the spectral element, the fast preconditioner-solver is based on multiresolution wavelet preconditioners-solvers for 1-d stiffness and mass matrices, which are similar to those used by Beuchler/Schneider/Schwab [9] in the case of hierarchical reference elements. The set of admissible wavelets satisfy even easier conditions in comparison with the conditions arising for hierarchical elements. The fast preconditioner-solver for a typical face is designed basically by means of the K-interpolation.

Inefficient prolongations from the interface boundary can also compromise optimality of DD algorithm. In the paper, we approve an almost optimal in the computational cost prolongations by means of the inexact iterative solver for the inner problems with the pointed out above multiresolution wavelet preconditioner. With these three main fast DD components in hands, it is left to look for a good preconditioner for the wire basket subproblem, which has though a relatively small dimension $\mathcal{O}(\mathcal{R}p)$, where $\mathcal{R}$ is the number of finite elements. We use exactly the one studied for h discretizations by Smith [39] and Dryja/Smith/Widlund [18] and expanded to spectral discretzations by Pavarino/Widlund [37] and Casarin [16]. The wire basket solver is described in these papers up to the solver for some $\mathcal{O}(\mathcal{R}) \times \mathcal{O}(\mathcal{R})$ subsystem. Without study of special algorithms, we make a mild assumption on the existence of an algorithm for solving this small subsystem which do not compromise the optimality of DD algorithm in a whole. The concluding result of the paper is that the DD preconditioner-solver, based on the pointed out components, provides the relative condition number $\mathcal{O}((1+\log p)^2)$, while solving the system of algebraic equations with the DD preconditioner for the matrix requires $\mathcal{O}(N(1+\log p))$ arithmetic operations, where N is the order of the system of the FE algebraic equations.

Apart from analysis of efficiency of the suggested components, for the proofs of these bounds we naturally attract the results on the wire basket preconditioning of the papers cited above. Among them the most important is the reduction of the face subproblem preconditioning to good matrix representations of the norms ${}_{00}|\cdot|_{1/2,F_i}$ for separate faces. In this and other relations, we should also refer to Bramble/Pasciak/Schatz [10]-[13], where some basic ideas were developed.

The functional space on the cubic reference elements of all types, considered in the paper, is the space $\mathcal{Q}_{p,\mathbf{x}}$ of the polynomials of order $p \geq 1$ in each variable of $\mathbf{x} = (x_1, x_2, .., x_d)$. For the reason of their similarity in our context, the Lagrange elements with the GLL and GLC nodes are included in the paper in one type and for convenience called *spectral*. By GLL and GLC nodes are assumed the nodes of the Gauss-Lobatto-Legendre and Gauss-Lobatto-Chebyshev quadrature formulas, respectively. In the literature the term *spectral* is commonly applied only to the elements of the latter subtype. Elements of the other type, called hierarchical, have for coordinate polynomials tensor products of the integrated Legendre polynomials. Similar to the presented in this paper DD solvers, but for the 3-d discretizations by means of the latter elements, may be found in Korneev/Langer/Xanthis [27, 28]. In this paper, they play a subsidiary role providing guidelines and better understanding of common features of the two types of *hp* discretizations.

The paper is organized as follows. In Section 2, we briefly describe the reference elements under consideration and the preconditioners of finite-difference and first order finite element types for their stiffness matrices. Although this material is well presented in the literature, we repeat some definitions crucial for the main content of the paper. The factored preconditioners of a new type for spectral elements are derived in Section 3. In the same section, this result is used for designing a fast multiresolution wavelet 3-d solver and the face solver for the spectral elements. Section 4 presents a fast DD solver for discretizations of the 3-d elliptic equations by spectral elements and results of the analysis of its relative condition and numerical complexity.

Let us list some notations. Signs $\prec$, $\succ$, $\asymp$ are used for the inequalities and equalities hold up to positive absolute constants; $\mathbf{A}^+$ – pseudo-inverse to a matrix $\mathbf{A}$; $\mathbf{A} \prec \mathbf{B}$ with nonnegative matrices $\mathbf{A}, \mathbf{B}$ implies $\mathbf{v}^\top \mathbf{A}\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}\mathbf{v}$ for any vector $\mathbf{v}$ and similarly for signs $\succ$, $\asymp$, $<$, $>$; for a symmetric nonnegative matrix $\mathbf{A}$ and $\forall\, \mathbf{v}$ it is assumed $||\mathbf{v}||^2_{\mathbf{A}} := \mathbf{v}^\top \mathbf{A}\mathbf{v}$; $\tau_0 = (-1,1)^d$ is the reference cube of dimension d. Notations $|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ stand for the semi-norm and the norm in the Sobolev space $H^k(\Omega)$, *i.e.*,

$$| v |^2_{k,\Omega} = \sum_{|q|=k} \int_{\Omega} (D^q_x v)^2 d\mathbf{x} \,, \qquad \| v \|^2_{k,\Omega} = \|v\|^2_{0,\Omega} + \sum_{l=1}^{k} |v|^2_{l,\Omega} \,,$$

where

$$D^q_x v := \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2}, ..\partial x_d^{q_d}, \quad q = (q_1, q_2, .., q_d), \quad q_k \geq 0, \quad |q| = q_1 + q_2 + .. + q_d \,;$$

$\overset{\circ}{H}{}^1(\Omega) := (v \in H^1(\Omega) : v|_{\partial\Omega} = 0)$. The relationship $\mathbf{v} \leftrightarrow v$ implies that $\mathbf{v}$ is the vector representation of a finite function v in a chosen basis.

2 Spectral and hierarchical p elements and preconditioners for stiffness matrices

2.1 Preconditioners for spectral elements

For the reference elements introduced in this section, we use the notation $\mathcal{E}_{\rm sp}$. The coordinates η_i of the GLL nodes on the segment [-1,1] are defined as the roots of the polynomial $(1 - s^2)P'_p(s)$, *i.e.*,

$$(1 - \eta_i^2) P'_p(\eta_i) = 0 \,, \qquad i = 0, 1, .., p \,. \tag{2.1}$$

The GLC nodes are the extremal points of the Chebyshev polynomials

$$\eta_i = \cos\left(\frac{\pi(p - i)}{p}\right), \qquad i = 0, 1, .., p \,. \tag{2.2}$$

The orthogonal meshes with the nodes

$$\mathbf{x} = \boldsymbol{\eta}_{\boldsymbol{\alpha}} = (\eta_{\alpha_1}, \eta_{\alpha_2}, .., \eta_{\alpha_d}) \,, \qquad \boldsymbol{\alpha} \in \omega = (\boldsymbol{\alpha} = (\alpha_1, \alpha_2, .., \alpha_d) : 0 \leq \alpha_1, \alpha_2, .., \alpha_d \leq p) \,,$$

having the coordinates (2.1) or (2.2), we term *Gaussian* for brevity. For the coordinate polynomials of the spectral reference elements, we use the notation $L_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathcal{L}_{\alpha_1}(x_1)\mathcal{L}_{\alpha_2}(x_2)...\mathcal{L}_{\alpha_d}(x_d)$, and $\mathcal{L}_i(s)$ is the 1-d polynomial of the order p satisfying the equalities $\mathcal{L}_i(\eta_j) = \delta_{i,j}$, $0 \leq j \leq p$, where $\delta_{i,j}$ – Kronecker's delta.

Without loss of generality, it is convenient to assume here $p = 2N$. For $i \leq N$, the steps $\hbar_i := \eta_i - \eta_{i-1}$ of the both Gaussian meshes have the asymptotic behavior $\hbar_i \asymp i/p^2$. One can define a more general class of meshes, which on the segment [-1,0] satisfy the relationships

$$\eta_0 = -1 \,, \quad \eta_i = \eta_{i-1} + \hbar_i \,, \qquad \eta_N = 0 \,, \qquad c_1 \frac{i^\gamma}{\aleph} \leq \hbar_i \leq c_2 \frac{i^\gamma}{\aleph} \,, \qquad \aleph = \sum_{i=1}^{N} i^\gamma \,, \tag{2.3}$$

with some fixed $c_k > 0$ and $\gamma \geq 0$ and are continued on [0,1] by the symmetry. For $\gamma = 0$, we have quasiuniform mesh with $\aleph = N$ and for $\gamma = 1$ – the mesh, which will be termed *pseudospectral*, with $\aleph = N(N + 1)/2$. In the particular case of $c_1 = c_2 = 1$, one has for the steps of the pseudospectral mesh $\hbar_i = i/\aleph = 2i/(N^2+N) = \beta\, i/p^2$, where $\beta \in [4,8]$. For the stiffness matrices of the reference elements with the Gaussian and pseudospectral nodes, we introduce the notations $\mathbf{A}_{\rm sp}$ $\mathbf{A}_{\rm p/s}$, respectively, assuming that they are induced by the Dirichlet integral

$$a_{\tau_0}(u,v) = \int_{\tau_0} \nabla u \cdot \nabla v \, d\mathbf{x}\,,$$

Let the orthogonal mesh $x_k = \eta_i$ be given on τ_0 and $\mathcal{H}(\tau_0)$ be the space functions continuous on $\overline{\tau}_0$ and belonging to $\mathcal{Q}_{1,\mathbf{x}}$ on each nest. Via $\boldsymbol{\mathcal{A}}_{\rm sp}, \boldsymbol{\mathcal{A}}_{\rm p/s}$, we denote the preconditioners, which are the FE matrices, induced by the Dirichlet integral a_{τ_0} on the spaces $\mathcal{H}(\tau_0)$ corresponding to the Gaussian and the pseudospectral meshes, respectively. As a preconditioner for the stiffness matrices $\mathbf{A}_{\rm sp}, \mathbf{A}_{\rm p/s}$ in 3-d, it can be also considered the simpler matrix

$$\mathbb{A}_\hbar = \boldsymbol{\Delta}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \boldsymbol{\Delta}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \boldsymbol{\Delta}_\hbar\,, \tag{2.4}$$

defined by means of the matrices generated for 1-d case. Namely, $\mathbb{D}_\hbar$ is the diagonal matrix

$$\mathbb{D}_\hbar = \mathrm{diag}\,[\widetilde{h}_i = \frac{1}{2}(\hbar_i + \hbar_{i+1})]_{i=0}^p\,, \qquad \widetilde{h}_i = 0 \;\; \text{for} \;\; i = 0, p+1\,, \tag{2.5}$$

and $\boldsymbol{\Delta}_\hbar$ is the FE matrix, induced by the bilinear form $(v', w')_{(-1,1)}$ on the space $\mathcal{H}(-1,1)$ of continuous and piece wise linear on the mesh η_i:

$$\begin{array}{lll} (\boldsymbol{\Delta}_\hbar\, \mathbf{u})|_{i=0} & = -\frac{1}{\hbar_1}(u_1 - u_0)\,, & (\boldsymbol{\Delta}_\hbar\, \mathbf{u})|_{i=p} = \frac{1}{\hbar_p}(u_p - u_{p-1})\,, \\ (\boldsymbol{\Delta}_\hbar\, \mathbf{u})|_{i} & = -\frac{1}{\hbar_i}u_{i-1} + (\frac{1}{\hbar_i} + \frac{1}{\hbar_{i+1}})u_i - \frac{1}{\hbar_{i+1}}u_{i+1}\,, & i = 1, 2, .., p-1\,. \end{array} \tag{2.6}$$

Lemma 2.1. *Let at the same p matrices $\boldsymbol{\mathcal{A}}_{\rm sp}$ and $\boldsymbol{\mathcal{A}}_{\rm p/s}$ be obtained on the Gaussian mesh and on the pseudospectral mesh at $\gamma = 1$, respectively, whereas $\mathbb{A}_\hbar$ be obtained on either of these meshes. Then they are spectrally equivalent to the stiffness matrix $\mathbf{A}_{\rm sp}$ of the reference element $\mathcal{E}_{\rm sp}$,* i.e.,

$$\mathbb{A}_\hbar, \boldsymbol{\mathcal{A}}_{\rm p/s}, \boldsymbol{\mathcal{A}}_{\rm sp} \prec \mathbf{A}_{\rm sp} \prec \boldsymbol{\mathcal{A}}_{\rm sp}, \boldsymbol{\mathcal{A}}_{\rm p/s}, \mathbb{A}_\hbar \tag{2.7}$$

uniformly in p. Under the same conditions similar inequalities

$$\mathbb{M}_\hbar, \boldsymbol{\mathcal{M}}_{\rm p/s}, \boldsymbol{\mathcal{M}}_{\rm sp} \prec \mathbf{M}_{\rm sp} \prec \boldsymbol{\mathcal{M}}_{\rm sp}, \boldsymbol{\mathcal{M}}_{\rm p/s}, \mathbb{M}_\hbar \tag{2.8}$$

hold for the mass matrix $\mathbf{M}_{\rm sp}$ of the spectral element, its FE preconditioners $\boldsymbol{\mathcal{M}}_{\rm p/s}, \boldsymbol{\mathcal{M}}_{\rm sp}$ obtained with the use of the space $\mathcal{H}(\tau_0)$, and $\mathbb{M}_\hbar = \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar$.

Proof. As it is well known, the inequalities (2.7) for $\boldsymbol{\mathcal{A}}_{\rm sp}$ in 1-d are due to Bernardi/Maday [6, 7], who proved $L_2(-1,1)$ and $H^1(-1,1)$ stability of the Lagrange interpolation over the set of Gaussian nodes. The step to a greater dimension may be found in Canuto [15] and Casarin [16]. The spectral equivalence inequalities for $\boldsymbol{\mathcal{A}}_{\rm p/s}$ are the consequence of the identical asymptotic behavior of the Gaussian and the pseudospectral meshes, see Lemma 6.1 in Korneev/Xanthis/Anoufriev [31] or [32]. For getting the inequalities with $\mathbb{A}_\hbar$, it is sufficient to take into account the spectral equivalence of the mass matrix of the 1-d linear element to its diagonal. Similar facts for the mass matrices of the spectral and pseudospectral elements result in (2.8). □

2.2 Preconditioners for hierarchical p elements

We introduce the set $\mathcal{M}_{1,p} = (\mathcal{L}_i(s)\,,\ i = 0, 1, \ldots, p)$ of polynomials on the interval (-1,1)

$$\begin{array}{c}\mathcal{L}_0(s) = \frac{1}{2}(1+s)\,, \qquad \mathcal{L}_1(s) = \frac{1}{2}(1-s)\,, \\ \mathcal{L}_i(s) := \beta_i \int_{-1}^{s} P_{i-1}(t)\,dt = \gamma_i[P_i(s) - P_{i-2}(s)]\,, \quad i \geq 2\,,\end{array} \tag{2.9}$$

where P_i are Legendre's polynomials and

$$\beta_i = \frac{1}{2}\sqrt{(2j-3)(2j-1)(2j+1)}\,, \qquad \gamma_i = 0.5\sqrt{(2i-3)(2i+1)/(2i-1)}\,.$$

Hence, $\mathcal{L}_i$ for $i = 0, 1$ are the "nodal" linear coordinate functions and for $i \geq 2$ are integrated Legendre's polynomials with β_i providing equalities $\|\mathcal{L}_i(s)\|_{0,(-1,1)} = 1$ for the norms in the space $L_2(-1,1)$. The choice of such multipliers was made in Korneev/Jensen [25] with the purpose to obtain the finite-difference like preconditioner (2.12) described below.

The hierarchical reference element $\mathcal{E}_{\text{hi}}$ is the cube τ_0 with the coordinate polynomials forming the set

$$\mathcal{M}_{d,p} = \big(L_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathcal{L}_{\alpha_1}(x_1)\mathcal{L}_{\alpha_2}(x_2)...\mathcal{L}_{\alpha_d}(x_d)\,,\ \boldsymbol{\alpha} \in \omega\big)\,,$$

which is the basis in $\mathcal{Q}_{p,\mathbf{x}}$. The reference element stiffness matrix $\mathbf{A}$ is induced by the set $\mathcal{M}_{d,p}$ and the Dirichlet integral $a_{\tau_0}(\,,\,)$, whereas by $\mathbf{A}_I$ is understood the *internal* stiffness matrix, corresponding to the subset $\overset{\circ}{\mathcal{M}}_{d,p} = (L_{\boldsymbol{\alpha}},\ 2 \leq \alpha_k \leq p)$ of the coordinate polynomials vanishing on $\partial\tau_0$. If to reorder the set $\overset{\circ}{\mathcal{M}}_{d,p}$ in a definite way, see [25], the matrix $\mathbf{A}_I$ and the reference element internal mass matrix $\mathbf{M}_I$ take the block diagonal forms, which for $d = 3$ are

$$\mathbf{A}_I = \text{diag}\,[\mathbf{A}_{eee}, \mathbf{A}_{eeo}, ..., \mathbf{A}_{ooe}, \mathbf{A}_{ooo}]\,, \qquad \mathbf{M}_I = \text{diag}\,[\mathbf{M}_{eee}, \mathbf{M}_{eeo}, ..., \mathbf{M}_{ooe}, \mathbf{M}_{ooo}]\,. \tag{2.10}$$

In the case $p = 2N+1$, each of 8 independent blocks in $\mathbf{A}_I$ and $\mathbf{M}_I$ is $N^3 \times N^3$ matrix. The entries of the block $\mathbf{A}_{eee}$ are the integrals $a_{\tau_0}(L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}})$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ with the even $\alpha_i, \beta_j,\ i, j = 1, 2, 3$; the entries of the block $\mathbf{A}_{eoe}$ correspond to even α_1, α_3, β_1, β_3 and odd α_2, β_2 etc. In other words, e or o at the k-th place correspond to the even and odd, respectively, powers α_k, β_k of polynomials $\mathcal{L}_{\alpha_k}(x_k), \mathcal{L}_{\beta_k}(x_k)$ entering the coordinate functions $L_{\boldsymbol{\alpha}}$, $L_{\boldsymbol{\beta}}$ in $a_{\tau_0}(L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}})$. For what follows, it is important that the blocks of matrices (2.10) are represented by the sums of the Kronecker's products

$$\begin{array}{c}\mathbf{A}_{abc} = \mathbb{K}_{1,a} \otimes \mathbb{K}_{0,b} \otimes \mathbb{K}_{0,c} + \mathbb{K}_{0,a} \otimes \mathbb{K}_{1,b} \otimes \mathbb{K}_{0,c} + \mathbb{K}_{0,a} \otimes \mathbb{K}_{0,b} \otimes \mathbb{K}_{1,c}\,, \\ \mathbf{M}_{abc} = \mathbb{K}_{0,a} \otimes \mathbb{K}_{0,b} \otimes \mathbb{K}_{0,c}\,, \qquad a, b, c = e, o\,,\end{array} \tag{2.11}$$

of the $N \times N$ matrices $\mathbb{K}_{1,a}\mathbb{K}_{0,b}$, which, respectively, may be preconditioned by the simple matrices

$$\boldsymbol{\mathcal{D}} = \text{diag}\,[4i^2]_{i=1}^{N}\,, \qquad \boldsymbol{\Delta} = \frac{1}{2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & & \cdots & \cdots & & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

Lemma 2.2. *For the preconditioners* $\boldsymbol{\mathcal{D}}, \boldsymbol{\Delta}$ *and*

$$\boldsymbol{\Lambda}_{\rm e} = \boldsymbol{\mathcal{D}} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} + \boldsymbol{\Delta} \times \boldsymbol{\mathcal{D}} \otimes \boldsymbol{\Delta} + \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{D}}\,, \qquad \boldsymbol{\mathcal{M}} = \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\Delta}\,, \tag{2.12}$$

there are hold the inequalities

$$\boldsymbol{\Delta} \prec \mathbb{K}_{0,a} \prec \boldsymbol{\Delta}\,,\;\; \boldsymbol{\mathcal{D}} \prec \mathbb{K}_{1,a} \prec \boldsymbol{\mathcal{D}}\,,\;\; \boldsymbol{\Lambda}_{\rm e} \prec \mathbf{A}_{abc} \prec \boldsymbol{\Lambda}_{\rm e}\,,\;\; \boldsymbol{\mathcal{M}} \prec \mathbf{M}_{abc} \prec \boldsymbol{\mathcal{M}}\,, \tag{2.13}$$

with $\mathbf{A}_{abc}$ *and* $\mathbf{M}_{abc}$ *defined in (2.11).*

Proof. In the case of d dimensions, inequalities for $\mathbf{A}_{abc}$ $\mathbf{M}_{abc}$ follow from the inequalities for $\mathbb{K}_{0,a}$ and $\mathbb{K}_{1,a}$, representing the case of 1-d, and (2.11),(2.12). The inequalities for $\mathbb{K}_{0,a}$ with $\boldsymbol{\Delta}$ replaced by the close to it preconditioner $\boldsymbol{\Delta} + \boldsymbol{\mathcal{D}}^{-1}$ and the inequalities for $\mathbb{K}_{1,a}$ were proved by Ivanov/Korneev [20], see also Korneev/Jensen [25]; in the form (2.13) the first pair of these inequalities was obtained by Korneev/Langer/Xanthis [28], see there (4.10). □

The way of derivation of fast preconditioners-solvers for the hierarchical and spectral p elements is motivated, firstly, by the finite-difference interpretation of the preconditioners, introduced above, and, secondly, by the properties of the corresponding differential operators. For this reason, it is useful to present these differential operators. Let $\pi_1 = (0,1)^d$. In the case of $d = 2$ the preconditioner

$$\boldsymbol{\Lambda}_{\rm e} = \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{D}} \otimes \boldsymbol{\Delta}$$

is the finite-difference approximation of the differential operator

$$Lu \equiv -2\left(x_1^2 \frac{\partial^2 u}{\partial x_2^2} + x_2^2 \frac{\partial^2 u}{\partial x_1^2}\right)\,, \qquad \mathbf{x} \in \pi_1\,, \qquad u|_{\partial \pi_1} = 0\,, \tag{2.14}$$

on the uniform square mesh of the size $\hbar = 1/(N+1)$. The form and even the order of the differential operator depends on d, and, *e.g.,* for $d = 3$ the matrix $\hbar^{-2}\boldsymbol{\Lambda}_{\rm e}$ is the finite-difference approximation of the 4-th order differential operator

$$Lu \equiv x_3^2 u_{,1,1,2,2} + x_2^2 u_{,1,1,3,3} + x_1^2 u_{,2,2,3,3}\,, \qquad \mathbf{x} \in \pi_1\,, \qquad u|_{\partial \pi_1} = 0\,, \tag{2.15}$$

on the uniform square mesh of the size $\hbar = 1/(N+1)$. Here, *e.g.,* $u_{,1,1,2,2} = \partial^4 u / \partial x_1^2 \partial x_2^2$.

The finite element preconditioners, introduced by the first order elements, are more adapted to analysis of fast solving procedures. They are easily derived, if to take into account the form of the finite-difference preconditioners and the corresponding differential operators. Let $d = 3$ and $\overset{\circ}{\mathcal{V}}(\pi_1)$ be the space of continuous on $\overline{\pi}_1$ and trilinear on each cubic nest of the mesh functions, vanishing on $\partial \pi_1$, and let in the bilinear form

$$b_{\pi_1}(u,v) = \sum_{k,l,m=1}^{3} \int_{\pi_1} \varphi_k u_{,l,m} v_{,l,m} d\mathbf{x}\,, \qquad \varphi_k = x_k^2\,, \tag{2.16}$$

all k, l, m are different. We define the matrix $\boldsymbol{\Lambda}_{\rm e,fem}$ as the matrix of this bilinear form on the space $\overset{\circ}{\mathcal{V}}(\pi_1)$ with the nodal basis.

Lemma 2.3. *Matrix* $\frac{1}{\hbar}\mathbf{\Lambda}_{\rm e,fem}$ *is spectrally equivalent to matrices* $\mathbf{A}_{abc}, \mathbf{\Lambda}_{\rm e}$ *uniformly in* p.

Proof. Lemma is proved in a standard way. It is sufficient to establish the spectral equivalence $\frac{1}{\hbar}\mathbf{\Lambda}_{\rm e,fem} \asymp \mathbf{\Lambda}_{\rm e}$. Each of the both matrices is the result of the assemblage of the 8×8 matrices defined for each nest of the mesh. For this reason, the proof is reduced to the easy proof of the spectral equivalence of 8×8 matrices. In doing this, we can replace 8×8 matrices corresponding to $\mathbf{\Lambda}_{\rm e,fem}$ by the spectrally equivalent matrices, obtained after replacement the varying coefficients in the bilinear form b_{π_1} by the ones constant on each mesh nest. Then for each pair of 8×8 matrices, it is left to compare three coefficients. □

In 2-d, one can use the FE space $\overset{\circ}{\mathcal{V}}_{\vartriangle}(\pi_1)$ of continuous and vanishing on $\partial\pi_1$ piece wise linear functions on the triangulation which is obtained by subdivision of each square nest of the mesh in two triangles. The preconditioner $\mathbf{\Lambda}_{\rm e,fem}$ is defined as the matrix of the bilinear form

$$b_{\pi_1}(u,v) = \sum_{k=1}^{2} \int_{\pi_1} \varphi_k u_{,3-k} v_{,3-k} d\mathbf{x} \tag{2.17}$$

on the space $\overset{\circ}{\mathcal{V}}_{\vartriangle}(\pi_1)$. We have $\mathbf{\Lambda}_{\rm e,fem} \asymp \hbar^2\mathbf{A}_{abc}, \hbar^2\mathbf{\Lambda}_{\rm e}$.

The finite element preconditioner may be simplified in particular by the replacement of the coefficients in (2.14) – (2.16) by piece wise constant coefficients. Such coefficients may be defined by the constants different for each nest of the mesh or for each nest of the specially designed coarse (decomposition) mesh, see Korneev [23].

It is worth emphasizing one essential difference between the preconditioners for the spectral and hierarchical p elements. In the preconditioner $\mathbf{\Lambda}_{\rm e}$ (2.12), matrices $\mathbf{\Delta}$ and $\boldsymbol{\mathcal{D}}$ are preconditioners for the "even" blocks of the mass and the stiffness matrices, respectively, of the 1-d reference element $\mathcal{E}_{\rm hi}$. On the contrary, in $\mathbb{A}_\hbar$ similar matrices $\mathbf{\Delta}_\hbar$ and $\mathbb{D}_\hbar$ are preconditioners for the stiffness and mass matrices of 1-d reference element $\mathcal{E}_{\rm sp}$.

3 Fast preconditioners-solvers for spectral elements

The introduced preconditioners may be termed *source preconditioners.* This is for the reason that they allow, as in Section 2.2, a simple finite-difference/FE interpretation or, as in Section 2.1, are finite-difference/FE by the definition and, due to this property, served a starting point for deriving modified preconditioners, more adapted to the specific fast solvers. By this moment, there is a number of fast multilevel algorithms applicable to the systems with the matrix $\mathbf{\Lambda}_{\rm e}$, which are not directly applicable to the systems with the preconditioners for the spectral elements, figuring in Lemma 2.1. In this Section, we establish a simple but rather important fact. The matrix $\mathbb{A}_{I,\hbar}$ may be transformed into the matrix, from which another matrix $\mathbf{\Lambda}_{I,\rm sp}$ can be obtained by neglecting some secondary terms. The latter matrix, though being quite different from $\mathbf{\Lambda}_{\rm e}$, has some basic properties with it in common, which allow to adapt to $\mathbf{\Lambda}_{I,\rm sp}$ fast solvers known for $\mathbf{\Lambda}_{\rm e}$. It is important that the transformation is cheap and indeed is defined by a diagonal matrix. Below, for simplifying notations, we sometimes omit index I in the notations of matrices related to the internal unknowns. For instance, by $\mathbb{D}_\hbar$, $\mathbf{\Delta}_\hbar$ are implied the blocks, which are obtained from the matrices (2.5),(2.6) by deleting the first and the last rows and columns.

3.1 Factored preconditioners for Dirichlet problems on spectral elements

The change of variables $\widetilde{\mathbf{v}} = \mathbf{C}\mathbf{v}$ by means of the diagonal matrix $\mathbf{C} = p^{-4}\,\mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2}$ (for 2-d $\mathbf{C} = p^{-2}\,\mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2}$) transforms $\mathbb{A}_{I,\hbar}$ (2.4) as the matrix of a quadratic form into the matrix

$$\begin{gathered}\widetilde{\mathbb{A}}_{I,\hbar} := \mathbf{C}^{-1}\mathbb{A}_\hbar\mathbf{C}^{-1} = p^8\,\mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2}\mathbb{A}_\hbar\,\mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} = \\ p^8\,\Big(\widetilde{\mathbf{\Delta}}_\hbar \otimes \mathbb{D}_\hbar^2 \otimes \mathbb{D}_\hbar^2 + \mathbb{D}_\hbar^2 \otimes \widetilde{\mathbf{\Delta}}_\hbar \otimes \mathbb{D}_\hbar^2 + \mathbb{D}_\hbar^2 \otimes \mathbb{D}_\hbar^2 \otimes \widetilde{\mathbf{\Delta}}_\hbar\Big)\,, \quad \widetilde{\mathbf{\Delta}}_\hbar = \mathbb{D}_\hbar^{1/2}\mathbf{\Delta}_\hbar\,\mathbb{D}_\hbar^{1/2}\,.\end{gathered} \tag{3.1}$$

Let us introduce also the matrices

$$\begin{aligned}\mathbf{\Delta}_{\rm sp} = &\quad \mathrm{tridiag}\,[-1,2,-1]\,, \quad \boldsymbol{\mathcal{D}}_{\rm sp} = \mathrm{diag}\,[1,4,..,N^2,(N-1)^2,(N-2)^2,..,4,1]\,,\\ \widetilde{\mathbf{\Lambda}}_{I,\rm sp} = &\quad (\mathbf{\Delta}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp}^{-1}) \otimes \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp} \otimes (\mathbf{\Delta}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp}^{-1}) \otimes \boldsymbol{\mathcal{D}}_{\rm sp} + \\ &\quad \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} \otimes (\mathbf{\Delta}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp}^{-1})\,,\\ \mathbf{\Lambda}_{I,\rm sp} = &\quad \mathbf{\Delta}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \mathbf{\Delta}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \mathbf{\Delta}_{\rm sp}\,,\end{aligned} \tag{3.2}$$

from which the two first have the dimension $(p-1)\times(p-1)$ and the two others $(p-1)^3\times(p-1)^3$.

Theorem 3.1. *Let the matrices $\widetilde{\mathbb{A}}_{I,\hbar}$, $\mathbf{\Lambda}_{I,\rm sp}$, $\widetilde{\mathbf{\Lambda}}_{I,\rm sp}$ be obtained on the Gaussian mesh or the pseudospectral mesh at $\hbar_i \asymp i/p^2$ for $1 \le i \le N$. Then these matrices are spectrally equivalent uniformly in p.*

Proof. It is sufficient to consider the case of the pseudospectral mesh with the steps $\hbar_i = \beta i/p^2$ and to prove the spectral equivalence of the matrix $\widetilde{\mathbf{\Delta}}_\hbar$ to $(\mathbf{\Delta}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp}^{-1})$, the matrix $p^4\mathbb{D}_\hbar^2$ to $\boldsymbol{\mathcal{D}}_{\rm sp}$ and the matrix $(\mathbf{\Delta}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp}^{-1})$ to $\mathbf{\Delta}_{\rm sp}$. In the proof we can use $\hbar_i = i/p^2$, since $4 \le \beta \le 8$ and we do not pay attention to absolute constants in asymptotic bounds. Taking this and the relation $\widetilde{h}_i = \frac{1}{2}(\hbar_i + \hbar_{i+1}) = \frac{1}{2p^2}(2i+1)$ into account, we immediately conclude that

$$\frac{i^2}{2p^4} \le \widetilde{h}_i^2 \le \frac{5i^2}{2p^4}\,, \qquad i = 1,2,..,N\,,$$

and we get the equivalence $\boldsymbol{\mathcal{D}}_{\rm sp} \asymp p^4\mathbb{D}_\hbar^2$. The matrix $\widetilde{\mathbf{\Delta}}_\hbar$ is tridiagonal, *i.e.*, $\widetilde{\mathbf{\Delta}}_\hbar = \mathrm{tridiag}\,[k_{i,i-1}, k_{i,i}, k_{i,i+1}]$. Taking into account explicit expressions for $k_{i,j}$ for $i = 1,2..,N$, we obtain

$$k_{i,i} = \frac{1}{2}(\hbar_i + \hbar_{i+1})(\frac{1}{\hbar_i} + \frac{1}{\hbar_{i+1}}) = 2 + \frac{1}{2i(i+1)}\,,$$

$$1 \ge |k_{i,i-1}| = \frac{1}{2\hbar_i}\sqrt{(\hbar_{i-1}+\hbar_i)(\hbar_i+\hbar_{i+1})} = \frac{1}{2i}\sqrt{4i^2-1} = \sqrt{1-\frac{1}{4i^2}} \ge 1 - \frac{1}{4i^2}\,,$$

$$1 \ge |k_{i,i+1}| \ge 1 - \frac{1}{4(i+1)^2}\,.$$

It is easy to see that the matrix $\widetilde{\mathbf{\Delta}}_\hbar$ is represented by the sum

$$\widetilde{\mathbf{\Delta}}_\hbar = \mathbf{K}_{1,I} + \mathbf{D}\,,$$

where

$$\mathbf{K}_{1,I} = \mathrm{tridiag}\,[k_{i,i-1}, (k_{i,i-1} + k_{i,i+1}), k_{i,i+1}]\,, \qquad \mathbf{D} = \mathrm{diag}\;[k_{i,i} - k_{i,i-1} - k_{i,i+1}]\;.$$

From the bounds for $k_{i,j}$ obtained above, we have

$$\frac{3}{4}\mathbf{\Delta}_{\mathrm{sp}} \leq \widetilde{\mathbf{K}}_{1,I} \leq \mathbf{\Delta}_{\mathrm{sp}}\,, \qquad \frac{1}{4}\boldsymbol{\mathcal{D}}_{\mathrm{sp}}^{-1} \leq \mathbf{D} \leq \boldsymbol{\mathcal{D}}_{\mathrm{sp}}^{-1}\,,$$

by means of which one comes to the inequalities

$$\frac{1}{4}\widetilde{\mathbf{\Lambda}}_{I,\mathrm{sp}} \leq \widetilde{\mathbb{A}}_{I,\hbar} \leq \widetilde{\mathbf{\Lambda}}_{I,\mathrm{sp}}\,. \tag{3.3}$$

For completing the proof of Theorem, it is left to establish the spectral equivalence $(\mathbf{\Delta}_{\mathrm{sp}} + \boldsymbol{\mathcal{D}}_{\mathrm{sp}}^{-1}) \asymp \mathbf{\Delta}_{\mathrm{sp}}$, which, in turn, requires the proof of the inequality

$$\boldsymbol{\mathcal{D}}_{\mathrm{sp}}^{-1} \leq \mathbf{\Delta}_{\mathrm{sp}}\,. \tag{3.4}$$

It may be completed with the use of Hardy's inequality. Indeed, for $k = 1, 2, ...\infty$, $s \neq k$, and $\mathcal{F}$, defined by

$$\mathcal{F}(x) = \begin{cases} \int_0^x f(t)\,, & s > 1/k\,, \\ \int_x^\infty f(t)\,, & s < 1/k\,, \end{cases}$$

according to Hardy's inequality [34] we have

$$\|x^{-s}\mathcal{F}\|_{L_k(0,\infty)} \leq \frac{1}{|s - \frac{1}{k}|}\,\|x^{1-s}f\|_{L_k(0,\infty)}\,.$$

For any $v \in H_0^1(-1,1)$, the inequalities

$$\|(1+x)^{-1}v\|^{\,2}_{\,0,(-1,1)} \leq 8\,|\,v\,|^2_{1,(-1,1)}\,, \qquad \|(1-x)^{-1}v\|^{\,2}_{\,0,(-1,1)} \leq 8\,|\,v\,|^2_{1,(-1,1)}$$

are a direct consequence of Hardy's inequality. From these, we conclude that for the weight function

$$\phi(x) = \begin{cases} 1 + x\,, & x \in [-1, 0]\,, \\ 1 - x\,, & x \in [0, 1]\,, \end{cases}$$

it holds the inequality

$$\|\phi^{-1}v\|^{\,2}_{\,0,(-1,1)} \leq 8\,|\,v\,|^2_{1,(-1,1)}\,, \qquad \forall\, v \in H_0^1(-1,1)\,. \tag{3.5}$$

Let $\mathcal{V}(-1,1)$ be the space of the continuous on $[-1,1]$ functions, which are piece wise linear on the uniform grid of the size $\hbar = 1/N$. We can substitute in (3.5) any $v \in \mathcal{V}(-1,1)$ vanishing at the ends of the interval (-1,1). Then (3.5) becomes the "matrix" inequality $\boldsymbol{\mathcal{D}}_\phi^{-1} \prec \mathbf{\Delta}_\phi$, in which $\boldsymbol{\mathcal{D}}_\phi$ and $\mathbf{\Delta}_\phi$ are the matrices of the quadratic forms in the left and the right parts of (3.5). By comparison of the matrix $\boldsymbol{\mathcal{D}}_\phi$ with $\boldsymbol{\mathcal{D}}_{\mathrm{sp}}$ and the matrix $\mathbf{\Delta}_\phi$ with $\mathbf{\Delta}_{\mathrm{sp}}$, we come to (3.4). □

The following consequence of Theorem 3.1 is obvious.

Corollary 3.1. *The matrices* $\mathbf{\Lambda}_{I,C} := \mathbf{C}\mathbf{\Lambda}_{I,\mathrm{sp}}\mathbf{C}$ *and* $\widetilde{\mathbf{\Lambda}}_{I,C} := \mathbf{C}\widetilde{\mathbf{\Lambda}}_{I,\mathrm{sp}}\mathbf{C}$*, defined on the pseudospectral at* $\gamma = 1$ *and the Gaussian meshes, are equivalent in the spectrum to the internal stiffness matrix* $\mathbf{A}_{I,\mathrm{sp}}$ *of the spectral reference element* $\mathcal{E}_{\mathrm{sp}}$ *uniformly in* p*,* i.e.,

$$\mathbf{\Lambda}_{I,C}, \widetilde{\mathbf{\Lambda}}_{I,C} \prec \mathbf{A}_{I,\mathrm{sp}} \prec \mathbf{\Lambda}_{I,C}, \widetilde{\mathbf{\Lambda}}_{I,C}\,. \tag{3.6}$$

The matrix $\mathbf{C}$ is diagonal, and thus the arithmetical cost of solving the systems with the preconditioners $\mathbf{\Lambda}_{I,C}, \widetilde{\mathbf{\Lambda}}_{I,C}$ coincides in the order with that for the preconditioners $\mathbf{\Lambda}_{I,\mathrm{sp}}$, $\widetilde{\mathbf{\Lambda}}_{I,\mathrm{sp}}$.

The matrix $\mathbf{\Lambda}_{I,\mathrm{sp}}$ can be viewed as the matrix of 7-point finite difference approximation of the differential operator on τ_0

$$L_{\mathrm{sp}}u = -\left[\phi^2(x_2)\phi^2(x_3)u_{,1,1} + \phi^2(x_1)\phi^2(x_3)u_{,2,2} + \phi^2(x_1)\phi^2(x_2)u_{,3,3}\right]\,, \quad u|_{\partial\tau_0} = 0\,. \tag{3.7}$$

Indeed, let us introduce the uniform cubic mesh of the size $\hbar = 2/p = 1/N$ and the notations $\phi_i = \phi(-1+i\hbar)$ and $\mathbf{u} = (u_{\mathbf{i}})_{i_1,i_2,i_3=1}^{p-1}$, assuming the vector $\mathbf{u}$ expanded by zero in the boundary nodes. Then one can write

$$\mathbf{\Lambda}_{I,\mathrm{sp}}\mathbf{u}|_{\mathbf{i}} = -\frac{1}{\hbar^2}\sum_{k=1,2,3}\phi_{i_l}^2\phi_{i_j}^2[u_{\mathbf{i}-\mathbf{e}_k} - 2u_{\mathbf{i}} + u_{\mathbf{i}+\mathbf{e}_k}]\,, \qquad 1 \le i_1, i_2, i_3 \le (p-1)\,, \tag{3.8}$$

where $\mathbf{i} = (i_1, i_2, i_3)$, all numbers $k, l, j \in (1, 2, 3)$ are different, $\mathbf{e}_k = (\delta_{k,l})_{l=1}^3$ – the unite vector, which components are the Kronecker's symbols. For $d = 2$ expressions (3.7),(3.8) simplify:

$$\begin{aligned} L_{\mathrm{sp}}u &= -\left[\phi^2(x_2)u_{,1,1} + \phi^2(x_1)u_{,2,2}\right]\,, \quad u|_{\partial\tau_0} = 0\,, \\ \mathbf{\Lambda}_{I,\mathrm{sp}}\mathbf{u}|_{\mathbf{i}} &= -\textstyle\sum_{k=1,2}\phi_{i_{3-k}}^2[u_{\mathbf{i}-\mathbf{e}_k} - 2u_{\mathbf{i}} + u_{\mathbf{i}+\mathbf{e}_k}]\,, \quad \mathbf{i} = (i_1, i_2)\,. \end{aligned} \tag{3.9}$$

Now we can look for common features of the matrices $\mathbf{\Lambda}_e$, $\mathbf{\Lambda}_{I,\mathrm{sp}}$. At $d = 2$, they and the related differential operators L, L_{sp} are similar. Namely, in each quarter of the square τ_0, the differential expression of the operator L_{sp} is the same as of the operator L up to the constant multiplier (and rotation and translation of the axes). The same is true for the finite-difference operators $\mathbf{\Lambda}_e$, $\mathbf{\Lambda}_{I,\mathrm{sp}}$ and FE matrices $\mathbf{\Lambda}_{\mathrm{e,fem}}, \mathbf{B}_{I,\mathrm{sp}}$, see the definition of the last below. At $d = 3$, the differential and finite-difference operators, related to the preconditioners for hierarchical and spectral elements, are different even in the order: L is the differential operator of the 4-th order, whereas L_{sp} of the 2-nd. However, the multipliers $\mathbf{\Delta}$, $\boldsymbol{\mathcal{D}}$ and respectively $\boldsymbol{\mathcal{D}}_{\mathrm{sp}}$, $\mathbf{\Delta}_{\mathrm{sp}}$ in the representations (2.12),(3.2) of the matrices $\mathbf{\Lambda}_e$, $\mathbf{\Lambda}_{I,\mathrm{sp}}$ by the sums of the Kroneckers products are similar. Due to this, all known fast solvers for the systems with the matrices $\mathbf{\Lambda}_e$ (see, *e.g.*,, [22, 23, 31, 8, 9]) can be adapted to the systems with the matrices $\mathbf{\Lambda}_{I,\mathrm{sp}}$ with the same asymptotic arithmetic cost.

As it was noted earlier, analysis of fast algorithms simplifies, if the preconditioner is a FE matrix. Instead of $\mathbf{\Lambda}_{I,\mathrm{sp}}$, $\widetilde{\mathbf{\Lambda}}_{I,\mathrm{sp}}$ it is possible to use the spectrally equivalent FE matrices, generated with the use of the 1-st order elements. Let $d = 2$. We divide each square nest of the size $\hbar$ on τ_0 in two triangles. On such a triangulation of τ_0, we introduce the space $\overset{\circ}{\mathcal{V}}_{\vartriangle}(\tau_0)$ of the continuous piece wise linear functions vanishing on $\partial\tau_0$. The FE preconditioner $\mathbf{B}_{I,\mathrm{sp}}$ may be defined as the matrix of the bilinear form

$$b_{\tau_0}(u,v) = \sum_{k=1}^{3} \int_{\tau_0} \phi_{3-k}^2 u_{,k} v_{,k}\, d\mathbf{x} \tag{3.10}$$

on the space $\overset{\circ}{\mathcal{V}}_{\triangle}(\tau_0)$. At $d=3$ the operator L_{sp} (3.7) is associated with the bilinear form

$$b_{\tau_0}(u,v) = \sum_{k=1}^{3} \int_{\tau_0} \phi_{k+1}^2 \phi_{k+2}^2 u_{,k} v_{,k}\, d\mathbf{x} \tag{3.11}$$

with the indices $k+1$, $k+2$ understood modulo 3. This form is defined on the space $\overset{\circ}{\mathcal{V}}(\tau_0)$ of continuous functions, which are trilinear on each cubic nest of the mesh and vanish on $\partial\tau_0$. Then $\mathbf{B}_{I,\mathrm{sp}}$ can be the matrix of the bilinear form (3.11) on this space. In a similar to the used in Lemma 2.3 way, it may be shown that

$$\mathbf{B}_{I,\mathrm{sp}} \asymp \hbar^{4-d} \mathbf{\Lambda}_{I,\mathrm{sp}}\,. \tag{3.12}$$

Let us note that instead of the functions $\varphi(x) = x^2$, $x \in [0,1]$, $\phi(x) = \min(x+1, x-1)$, $s \in [-1,1]$ the functions $\varphi(x) = \max(\hbar^2, x^2)$, and $\phi(x) = \max(\hbar, \min(x+1, x-1))$ may be used.

3.2 Multiresolution wavelet solver for Dirichlet problems on 3-d spectral elements

According to Section 3.1, in order to obtain a fast preconditioners-solver for the internal stiffness matrices $\mathbf{A}_{I,\mathrm{sp}}$ of spectral elements, it is sufficient to design a fast preconditioners-solver for the preconditioner $\mathbf{\Lambda}_{I,\mathrm{sp}}$ or $\widetilde{\mathbf{\Lambda}}_{I,\mathrm{sp}}$. In this section, we consider the multilevel preconditioner-solver for the matrix $\mathbf{\Lambda}_{I,\mathrm{sp}}$, based on the multiresolution wavelet analysis, which is similar to the solver initially designed for the preconditioners $\mathbf{\Lambda}_{\mathrm{e,fem}}$ of the stiffness matrices of the hierarchical p elements by Beuchler/Schneider/Schwab [9].

Taking into account the interpretation of $\mathbf{A}_{I,\mathrm{sp}}$ as the finite-difference operator (3.8), we consider the cube τ_0 subdivided by the square mesh of the size $\hbar = 1/p$. Only for convenience and without loss of generality, it is assumed $p = 2N$, $N = 2^{\ell_0 - 1}$. For each $l = 1, 2, ..., \ell_0$, one can introduce the uniform mesh x_i^l, $i = 0, 1, 2, .., 2N_l$, $N_l = 2^{l-1}$, $x_0 = -1$, $x_{2N_l} = 1$ of the size $\hbar_l = 2^{1-l}$ and the space $\mathcal{V}_l(-1,1)$ of the continuous on (-1,1) piece wise linear functions, vanishing at the ends of this interval. The dimension of $\mathcal{V}_l(-1,1)$ is $\mathcal{N}_l = p_l - 1 = 2^l - 1$ with $p_{\ell_0} = p$. Let $\phi_i^l \in \mathcal{V}_l(-1,1)$ be the the nodal basis function for the node x_i^l, so that $\phi_i^l(x_j^l) = \delta_{i,j}$ and $\mathcal{V}_l(-1,1) = \mathrm{span}\left(\phi_i^l\right)_{i=1}^{p_l-1}$. This basis induces the Gram matrices

$$\mathbf{\Delta}_l = \hbar_l \left(\langle (\phi_i^l)', (\phi_j^l)' \rangle_{\omega=1}\right)_{i,j=1}^{p_l-1}, \quad \boldsymbol{\mathcal{M}}_l = \hbar_l^{-1} \left(\langle \phi_i^l, \phi_j^l \rangle_{\omega=\phi}\right)_{i,j=1}^{p_l-1}, \qquad \langle v, u \rangle_\omega := \int_{-1}^{1} \omega^2 v\, u\, d\mathbf{x}\,, \tag{3.13}$$

where ϕ is the function introduced in Section 3.1. Obviously,

$$\boldsymbol{\Delta}_{\ell_0} = \boldsymbol{\Delta}_{\rm sp}\,, \qquad \boldsymbol{\mathcal{M}} := \boldsymbol{\mathcal{M}}_{\ell_0} \asymp \boldsymbol{\mathcal{D}}_{\rm sp}\,. \tag{3.14}$$

The representation of each $\mathcal{V}_l$ by the direct sum $\mathcal{V}_l = \mathcal{V}_{l-1} \oplus \mathcal{W}_l$ results in the decomposition

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus ... \oplus \mathcal{W}_{\ell_0}$$

with the notations $\mathcal{V} = \mathcal{V}_{\ell_0}$ and $\mathcal{W}_1 = \mathcal{V}_1$. Let $(\psi_k^l)_{k,l=1}^{p_{l-1},\ell_0}$ denote the *multiscale wavelet basis*, composed of some *single scale bases* $(\psi_k^l)_{k=1}^{p_{l-1}}$ in the spaces $\mathcal{W}_l$, so that $\mathcal{W}_l = {\rm span}\,[\psi_k^l]_{k=1}^{p_{l-1}}$. The multiscale wavelet basis in $\mathcal{V}$, if it is stable in the norms induced by the scalar products (3.13), allows to define 1-d multilevel preconditioners, which in turn lead to the multidimentional tensor product multilevel preconditioner. Before formulating the result, we introduce additional notations. The basis $(\psi_k^l)_{k,l=1}^{p_{l-1},\ell_0}$ induces the matrices

$$\begin{aligned} &\boldsymbol{\Delta}_{\rm wlet} = \big(\langle(\psi_i^k)',(\psi_j^l)'\rangle_1\big)_{i,j=1;\,k,l=1}^{p_{l-1};\ell_0}\,, &&\boldsymbol{\mathcal{M}}_{\rm wlet} = \big(\langle\psi_i^k,\psi_j^l\rangle_\phi\big)_{i,j=1;\,k,l=1}^{p_{l-1};\ell_0}\,, \\ &\mathbb{D}_1 = {\rm diag}\,[\langle(\psi_i^l)',(\psi_i^l)',\rangle_1]_{i,l=1}^{p_{l-1},\ell_0}\,, &&\mathbb{D}_0 = {\rm diag}\,[\langle\psi_i^l,\psi_i^l,\rangle_\phi]_{i,l=1}^{p_{l-1},\ell_0}\,. \end{aligned} \tag{3.15}$$

By $\mathbf{Q}$ is denoted the transformation matrix from the multiscale wavelet basis to the finite element basis $(\phi_k^{l_0})_{k=1}^{p-1}$. If $\mathbf{v}_{\rm wlet}$ and $\mathbf{v}$ are the vectors of the coefficients of a function from $\mathcal{V}(0,1)$ in these two bases, respectively, then $\mathbf{v} = \mathbf{Q}^\top\,\mathbf{v}_{\rm wlet}$.

Theorem 3.2. *There exist multiscale wavelet bases* $(\psi_k^l)_{k,l=1}^{p_{l-1},\ell_0}$ *such that the matrices* $\boldsymbol{\Delta}_{\ell_0}^{-1}$ *and* $\boldsymbol{\mathcal{M}}_{\ell_0}^{-1}$ *are simultaneously spectrally equivalent to the matrices* $\mathbf{Q}^\top\mathbb{D}_1^{-1}\mathbf{Q}$ *and* $\mathbf{Q}^\top\mathbb{D}_0^{-1}\mathbf{Q}$*, respectively, uniformly in* p*. Besides, the matrix-vector multiplications* $\mathbf{Q}\,\mathbf{v}_{\rm wlet}$ *and* $\mathbf{Q}^\top\,\mathbf{v}$ *require* $\mathcal{O}(p)$ *arithmetic operations.*

Proof. The main step is the proof of the equivalences $\boldsymbol{\Delta}_{\rm wlet} \asymp \mathbb{D}_1$ and $\boldsymbol{\mathcal{M}}_{\rm wlet} \asymp \mathbb{D}_0$, from which Theorem follows by the definition of these matrices and matrices $\boldsymbol{\Delta}_{\ell_0}$, $\boldsymbol{\mathcal{M}}_{\ell_0}$, $\mathbf{Q}$. The proof of these equivalences is simpler than the proof of similar equivalences in Beuchler/Schneider/Schwab [9], because in our case the weight ϕ is symmetric on (-1,1). The sited authors justified existence of multiscale wavelet bases with the required properties in the more difficult case of the space $\mathcal{V}(0,1) := (\,v \in \mathcal{V}(-1,1)\,|\,v(x) = 0 \text{ at } x \notin (0,1)\,)$ and the weight $\phi = x$. □

Theorem 3.3. *Let* $\boldsymbol{\mathcal{B}}_{I,{\rm sp}} = \mathbf{C}\,\mathbb{B}_{I,{\rm sp}}\mathbf{C}$ *and*

$$\mathbb{B}_{I,{\rm sp}}^{-1} = \begin{cases} (\mathbf{Q}^\top \otimes \mathbf{Q}^\top)[\,\mathbb{D}_0 \otimes \mathbb{D}_1 + \mathbb{D}_1 \otimes \mathbb{D}_0\,]^{-1}(\mathbf{Q} \otimes \mathbf{Q}), & d = 2, \\ (\mathbf{Q}^\top \otimes \mathbf{Q}^\top \otimes \mathbf{Q}^\top)[\mathbb{D}_0 \otimes \mathbb{D}_0 \otimes \mathbb{D}_1 + \mathbb{D}_0 \otimes \mathbb{D}_1 \otimes \mathbb{D}_0 + & \\ \qquad \mathbb{D}_0 \otimes \mathbb{D}_0 \otimes \mathbb{D}_1]^{-1}(\mathbf{Q} \otimes \mathbf{Q} \otimes \mathbf{Q}), & d = 3\,, \end{cases} \tag{3.16}$$

then $\boldsymbol{\mathcal{B}}_{I,{\rm sp}} \asymp \mathbf{A}_{I,{\rm sp}}$ *and therefore*

$${\rm cond}\,[\boldsymbol{\mathcal{B}}_{I,{\rm sp}}^{-1}\mathbf{A}_{I,{\rm sp}}] \prec 1\,. \tag{3.17}$$

The arithmetical cost of the operation $\boldsymbol{\mathcal{B}}_{I,{\rm sp}}^{-1}\mathbf{v}$ *for any* $\mathbf{v} \in U_I$ *is* $\mathcal{O}(p^d)$*.*

Proof. In view of Lemma 2.1 and Theorem 3.1, it is sufficient to prove the equivalence

$$\mathrm{cond}\,[\mathbb{B}_{I,\mathrm{sp}}^{-1}\mathbf{\Lambda}_{I,\mathrm{sp}}] \asymp 1\,.$$

The proof of this bound is the consequence of (3.14), Theorem 3.2 and properties of the Kronecker product. □

3.3 Multiresolution wavelet solver for faces

Another important problem in optimization of DD algorithms for spectral discretizations is the development of fast solvers for the internal problems on faces. As it is known, see, *e.g.,* Pavarino/Widlund [37], Korneev/Langer/Xanthis [28], in the wire basket algorithms it is reduced to the preconditioning of the matrix of the quadratic form ${}_{00}|\cdot|^2_{1/2,F_0}$ on the subspace of polynomials $\overset{\circ}{\mathcal{Q}}_{p,\mathbf{x}}$ of two variables $\mathbf{x}=(x_1,x_2)$, vanishing on the boundary of F_0. This quadratic form is the square of the norm in the space $H_{00}^{1/2}(F_0)$, whereas $F_0=(-1,1)\times(-1,1)$ represents a typical face of the 3-d reference cube. As shown in Grisvard [19] and Lions/Magenes [33], one of the characterizations of this norm is

$${}_{00}|\,v\,|^2_{1/2,F_0} = |\,v\,|^2_{1/2,F_0} + \int_{F_0}\frac{|v(\mathbf{x})|^2}{\mathrm{dist}\,[\mathbf{x},\partial F_0]}\,d\mathbf{x}$$

Theorem 3.4. *Let $d_{0,i}, d_{1,i}$ be diagonal entries of matrices $\mathbb{D}_0,\mathbb{D}_1$, respectively, and $\mathbb{D}_{1/2}$ be the diagonal $(p-1)^2\times(p-1)^2$ matrix with the entries on the main diagonal*

$$d_{i,j}^{(1/2)} = d_{0,i}d_{0,j}\sqrt{\frac{d_{1,i}}{d_{0,i}}+\frac{d_{1,j}}{d_{0,j}}}\,.$$

Let also

$$\boldsymbol{\mathcal{S}}_0 = \mathbf{C}\,\mathbb{S}_0\,\mathbf{C}\,, \qquad \mathbb{S}_0^{-1} = (\mathbf{Q}^\top\otimes\mathbf{Q}^\top)\,\mathbb{D}_{1/2}^{-1}\,(\mathbf{Q}\otimes\mathbf{Q})\,.$$

Then for all $v\in\overset{\circ}{\mathcal{Q}}_{p,\mathbf{x}}$ and vectors $\mathbf{v}$, representing v in the basis $\overset{\circ}{\mathcal{M}}_{2,p}$, the norms ${}_{00}|\,v\,|_{1/2,F_0}$ and $||\mathbf{v}||_{\boldsymbol{\mathcal{S}}_0}$ are equivalent uniformly in p.

Proof. The proof is alike the proof of a similar statement for the case of the hierarchical p element $\mathcal{E}_{\mathrm{hi}}$, see Korneev/Langer/Xanthis [28, Theorem 4.4]. First of all, we note that another characterization of the norm ${}_{00}|\cdot|_{1/2,F_0}$ can be obtained by the Peetre's K-method of interpolation between the norms in spaces $L_2(F_0)$ and $\overset{\circ}{H}{}^1(F_0)$, see, *e.g.,* Lions/Magenes [33, pp. 66-69, 98-99]. At the same time from Th. 3.2 and preceding it considerations, it follows that the matrices

$$\mathbf{D}_0 = \mathbb{D}_0\otimes\mathbb{D}_0\,, \qquad \mathbf{D}_1 = \mathbb{D}_1\otimes\mathbb{D}_1 + \mathbb{D}_0\otimes\mathbb{D}_0$$

are spectrally equivalent to the internal mass and the stiffness matrices of the 2-d spectral reference element in the special basis, which is the modified according to the transformation matrix $\mathbf{C}$ multiscale wavelet basis. Indeed, if we introduce matrices

$$\boldsymbol{\mathcal{W}} = \mathbf{C}^{-1}(\mathbf{Q}^\top\times\mathbf{Q}^\top)\,, \qquad \mathbf{M}_{\mathrm{wlet}} = \boldsymbol{\mathcal{W}}^\top\mathbf{M}_{\mathrm{sp}}\boldsymbol{\mathcal{W}}\,, \qquad \mathbf{A}_{\mathrm{wlet}} = \boldsymbol{\mathcal{W}}^\top\mathbf{A}_{\mathrm{sp}}\boldsymbol{\mathcal{W}}\,,$$

then by the chain of equivalences reflected in Lemma 2.1, Theorems 3.1, 3.2 and (3.14), we have

$$\mathbf{M}_{\rm wlet} \asymp \mathbf{D}_0\,, \qquad \mathbf{A}_{\rm wlet} \asymp \mathbf{D}_1\,.$$

Let $v \in \mathcal{Q}_{p,\mathbf{x}}$, $\mathbf{v}$ be the vector representation of v in the basis polinomials of the reference element, and $\mathbf{v} = \boldsymbol{\mathcal{W}}\mathbf{v}_{\rm wlet}$. The above relationships may be also rewritten in the form

$$||\mathbf{v}_{\rm wlet}||_{\,\mathbf{D}_0} \asymp ||v||_{0,\tau_0}\,, \qquad ||\mathbf{v}_{\rm wlet}||_{\,\mathbf{D}_1} \asymp |v|_{1,\tau_0}\,, \tag{3.18}$$

and by the Peetre's K-method of interpolation we have

$$||\mathbf{v}_{\rm wlet}||_{\,\mathbf{D}_{1/2}} \asymp {}_{00}|v|_{1/2,\tau_0}\,. \tag{3.19}$$

In view of the relation between $\mathbf{v}$ and $\mathbf{v}_{\rm wlet}$ this completes the proof, *i.e.,*

$$||\mathbf{v}_F||_{\boldsymbol{\mathcal{S}}_0} \asymp {}_{00}|\,v\,|_{1/2,F_0}\,, \tag{3.20}$$

where now the vector $\mathbf{v}_F \leftrightarrow v$ represents 2-d polynomials $v \in \overset{\circ}{\mathcal{Q}}_{p,\mathbf{x}}$ in the basis $L_{\boldsymbol{\alpha}}(\mathbf{x})$, $2 \le \alpha_1, \alpha_2 \le p$, on the representative face F_0 of the spectral reference element.

□

Presented in Sections 3.2 and 3.3 fast solvers for the internal and face problems can be easily generalized on the "orthotropic" spectral elements with the shape polynomials, having different orders along different axes.

4 Domain decomposition algorithm for discretizations by spectral elements

Now we describe main components of the DD algorithm and formulate the result on its arithmetic cost.

As a model, we consider Dirichlet problem: find $u \in \overset{\circ}{H}{}^1(\Omega)$ satisfying the identity

$$a_\Omega(u,v) := \int_\Omega \varrho(x)\nabla u\cdot\nabla v\,d\mathbf{x} = (f,v)_\Omega\,, \qquad \forall\, v \in \overset{\circ}{H}{}^1(\Omega)\,, \tag{4.1}$$

where $\overset{\circ}{H}{}^1(\Omega)$ is the subspace of functions from $H^1(\Omega)$, vanishing on the boundary $\partial\Omega$. For simplicity, it is assumed that Ω coincides with the computational domain, *i.e.,* it is the domain of an assemblage of geometrically compatible and in general curvilinear finite elements occupying domains τ_r, *i.e.,*

$$\overline{\Omega} = \cup_{r=1}^{\mathcal{R}}\overline{\tau}_r\,.$$

Finite elements and their domains τ_r are specified by nondegenerate mappings $\mathbf{x} = \mathcal{X}^{(r)}(\mathbf{y}) : \overline{\tau}_0 \to \overline{\tau}_r$ with positive Jacobian's, and it is required that these mappings satisfy the conditions, called the *generalized conditions of the angular (shape) quasiuniformity.* If the mappings are trilinear, *i.e.,* elements have straight edges, these conditions are equivalent to the well known conditions of shape regularity, see, *e.g.,* Ciarlet [17]. In a more general case, they are equivalent to the following ones, see [21]. Suppose, each mapping is represented as a superposition of two nondegenerate mappings $\mathcal{X}^{(r)}(\mathbf{y}) = \widetilde{\mathcal{X}}^{(r)}(\mathcal{Z}^{(r)}(\mathbf{y}))$, where $x = \widetilde{\mathcal{X}}^{(r)}(\mathbf{z}) : \bar{\tau}'_r \to \bar{\tau}_r$ is a nonlinear and $\mathbf{z} = \mathcal{Z}^{(r)}(\mathbf{y}) : \bar{\tau}_0 \to \bar{\tau}'_r$ is an affine or trilinear mapping (*e.g.,* with coinciding vertices of τ'_r and τ_r). Then τ'_r must be shape regular, and for the nonlinear mappings and their inverses the Jacobians and their components must be uniformly bounded.

The coefficient ϱ is accepted to be piece wise constant and such that $\varrho(\mathbf{x}) = \varrho_r = \text{const}$ for $\mathbf{x} \in \tau_r$.

The assemblage of spectral finite elements, associated with a single reference element $\mathcal{E}_{\text{sp}}$ by mappings $\mathcal{X}^{(r)}$, defines the FE space

$$\mathbb{V}(\Omega) = \big(v : v \in C(\overline{\Omega}),\ \ v(\mathcal{X}^{(r)}(\mathbf{y}))|_{\mathbf{y}\in\tau_0} \in \mathcal{Q}_{p,\mathbf{x}}\ \text{ for } r = 1, 2, ..., \mathcal{R}\big)\ , \qquad \mathbb{V}(\Omega) \subset H^1(\Omega)\,,$$

and its subspace $\overset{\circ}{\mathbb{V}}(\Omega) = \mathbb{V}(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$. We write the system of FE algebraic equations for the problem (4.1), obtained by means of the subspace $\overset{\circ}{\mathbb{V}}(\Omega)$, in the form

$$\mathbf{K}\mathbf{u} = \mathbf{f}\,. \tag{4.2}$$

At designing the *DD* solver for (4.2), each *p*-element is treated as a subdomain of decomposition, typically for many other papers as well.

It is natural to distinguish *internal, face, edge* and *vertex* degrees of freedom in the FE assemblage and respectively decompose the vector space V of the unknowns as

$$V = V_I \oplus V_F \oplus V_E \oplus V_V\,.$$

DD solvers or their parts are also often based on the decompositions

$$V = V_I \oplus V_F \oplus V_W\,, \qquad V = V_I \oplus V_B\,,$$

where $V_B = V_F \oplus V_E \oplus V_V$ and $V_W = V_E \oplus V_V$ are the subspaces of the *interelement boundary* and *wire basket* degrees of freedom. According to these subspaces, the finite element stiffness matrix may be represented in the block forms

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_B \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IW} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FW} \\ \mathbf{K}_{WI} & \mathbf{K}_{WF} & \mathbf{K}_{WW} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IE} & \mathbf{K}_{IV} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FE} & \mathbf{K}_{FV} \\ \mathbf{K}_{EI} & \mathbf{K}_{EF} & \mathbf{K}_E & \mathbf{K}_{EV} \\ \mathbf{K}_{VI} & \mathbf{K}_{VF} & \mathbf{K}_{VE} & \mathbf{K}_V \end{pmatrix}\,. \tag{4.3}$$

For the corresponding spaces of the FE functions, we use similar notations with V replaced by $\mathbb{V}$.

The restrictions of the introduced above spaces to the finite elements τ_r are supplied with an additional upper index r, *e.g.,* $\mathbb{V}_B^{(r)}$ denotes the subspace spanned by the boundary coordinate functions of a finite element τ_r with $r = 0$ reserved for the reference cube. Similarly, $\mathbf{K}_F^{(r)}$ is the block of the stiffness matrix of an element τ_r, generated by the face coordinate functions. The spaces $V^{(r)}$ and $\mathbb{V}^{(r)}$ for the reference element will be denoted U and $\mathcal{U} = \mathcal{Q}_{p,\mathbf{x}}$, respectively, with the same indexation for subspaces.

We will consider the DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ for the matrix $\mathbf{K}$ of the form

$$\begin{aligned} \boldsymbol{\mathcal{K}}^{-1} &= \overline{\boldsymbol{\mathcal{K}}}_I^{\,+} + \mathbf{P}_{V_B\to V}\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{P}_{V_B\to V}^\top\,, \\ \boldsymbol{\mathcal{S}}_B^{-1} &= \overline{\boldsymbol{\mathcal{S}}}_F^{\,+} + \mathbf{P}_{V_W\to V_B}(\boldsymbol{\mathcal{S}}_W^B)^{-1}\mathbf{P}_{V_W\to V_B}^\top\,, \end{aligned} \tag{4.4}$$

defined by means of the three preconditioners-solvers and two prolongation matrices. First we describe the former three.

i) The block diagonal preconditioner-solver for the internal Dirichlet problems on finite elements has the form

$$\overline{\boldsymbol{\mathcal{K}}}_I^{\,+} := \begin{pmatrix} \boldsymbol{\mathcal{K}}_I^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\mathcal{K}}_I = \operatorname{diag}\left[h_1\varrho_1\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}, h_2\varrho_2\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}, \ldots, h_{\mathcal{R}}\varrho_{\mathcal{R}}\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}\right]$$

and $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}$ is the multiresolution wavelet preconditioner-solver, figuring in Theorem 3.3. The value h_r is the characteristic size of an element, figuring in the generalized conditions of the angular quasiuniformity. It can be set equal to the arithmetic mean of the inscribed and circumscribed spheres for τ_r'. Each block $h_r\varrho_r\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}$ corresponds to one block $\mathbf{K}_I^{(r)}$ in the block $\mathbf{K}_I$ for internal unknowns

$$\mathbf{K}_I = \operatorname{diag}\left[\mathbf{K}_I^{(1)}, \mathbf{K}_I^{(2)}, ..., \mathbf{K}_I^{(\mathcal{R})}\right]$$

of the FE stiffness matrix $\mathbf{K}$.

ii) The block diagonal preconditioner-solver for the internal problems on faces of finite elements

$$\overline{\boldsymbol{\mathcal{S}}}_F^{\,+} = \begin{pmatrix} \boldsymbol{\mathcal{S}}_F^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\mathcal{S}}_F = \operatorname{diag}\left[\kappa_1\boldsymbol{\mathcal{S}}_0, \kappa_2\boldsymbol{\mathcal{S}}_0, \ldots, \kappa_Q\boldsymbol{\mathcal{S}}_0\right], \tag{4.5}$$

Q is the number of different faces $F_k \in \Omega$ of the FE discretizations, κ_k are multipliers, $\boldsymbol{\mathcal{S}}_0^{-1}$ is the multiresolution wavelet preconditioner-solver for one face, as defined in Theorem 3.4. Let for a face F_k of the discretization, $r_1(k)$ and $r_2(k)$ are the numbers of two elements $\overline{\tau}_{r_1(k)}$ and $\overline{\tau}_{r_2(k)}$, sharing the face F_k. Then

$$\kappa_k = \left(h_{r_1(k)}\varrho_{r_1(k)} + h_{r_2(k)}\varrho_{r_2(k)}\right). \tag{4.6}$$

The matrix $\boldsymbol{\mathcal{S}}_F$ is the preconditioner for the $Q(p-1)^2 \times Q(p-1)^2$ block $\mathbf{S}_F$ of the Schur complement

$$\mathbf{S}_B = \mathbf{K}_B - \mathbf{K}_{BI}\mathbf{K}_I^{-1}\mathbf{K}_{IB}\,, \qquad \mathbf{S}_B = \begin{pmatrix} \mathbf{S}_F & \mathbf{S}_{FW} \\ \mathbf{S}_{WF} & \mathbf{S}_W \end{pmatrix},$$

which for the reference element is denoted $\mathbb{S}_B$. Although $\mathbf{S}_F$ is not a block diagonal matrix, obviously, it can be represented in the block form with each $(p-1)^2 \times (p-1)^2$ block on the diagonal related to one face.

iii) The preconditioner-solver $\boldsymbol{\mathcal{S}}_W^B$ related to the wire basket subproblem. We borrow it, as well as the prolongation $\mathbf{P}_{V_W\to V_B}$, from Casarin [16] and Pavarino/Widlund [37] and briefly describe below.

The prolongation operations include

iv) the prolongation $\mathbf{P}_{V_B\to V}$ from the interelement boundary on the whole computational domain $\overline{\Omega}$,

v) the prolongation $\mathbf{P}_{V_W\to V_B}$ from the wire basket on the interelement boundary.

Since **i**) and **ii**) have been completely defined, it is left to define **iii**)–**v**), among which the prolongation $\mathbf{P}_{V_B\to V}$ brings usually the major contribution to the computational cost. For the reason that we have an efficient preconditioner-solver for the internal problems on finite elements, this prolongation can be efficiently completed by means of inexact iterative procedures applied in parallel element wise. We present the simplest variant of such procedure, which adds an extra factor $\log p$ in the estimate of computational work.

The prolongation matrix $\mathbf{P}_{V_B\to V}$ is defined in such a way, that its restriction to each element is $\mathbf{P}^{(r)}_{V_B^{(r)}\to V^{(r)}} = \mathbb{P}_{U_B\to U}$ and $\mathbb{P}_{U_B\to U}$ is the master prolongation matrix, which is used for the prolongation inside any finite element of the discretization. The master prolongation matrix is obtained for the reference element in the following way. Let $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}_{\rm sp}$ or $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}_{\rm p/s}$ and $\mathbb{B}_I$ is another preconditioner for the internal block $\mathbf{A}_I$ of the reference element stiffness matrix, possessing a fast solver. To any $\mathbf{v}_B \in U_B$, one can relate the vector $\overline{\mathbf{v}}_B \in U_B$, which entries are equal to the mean value of the corresponding finite element function $v_B \leftrightarrow \mathbf{v}_B$ on the boundary $\partial\tau_0$, and the vector $\widetilde{\mathbf{v}}_B := \mathbf{v}_B - \overline{\mathbf{v}}_B$. By $\overline{\mathbf{v}} \in U$ is denoted the prolongation of $\overline{\mathbf{v}}_B$ by the constant. The prolongation $\mathbf{u} = \mathbb{P}_{U_B\to U}\mathbf{v}_B$ is the sum of two vectors

$$\mathbf{u} = \overline{\mathbf{v}} + \widetilde{\mathbf{u}}\,, \qquad \text{where} \qquad \widetilde{\mathbf{u}} = (\widetilde{\mathbf{u}}_I^\top, \widetilde{\mathbf{v}}_B^\top)^\top\,,$$

and the subvector $\widetilde{\mathbf{u}}_I = \mathbf{w}_I^{k_0}$ is produced for some fixed number k_0 of the iterations

$$\mathbf{w}_I^{k+1} = \mathbf{w}_I^k - \sigma_{k+1}\mathbb{B}_I^{-1}(\boldsymbol{\mathcal{A}}_I\mathbf{w}_I^k - \boldsymbol{\mathcal{A}}_{IB}\widetilde{\mathbf{v}}_B)\,, \qquad \mathbf{w}_I^0 = \mathbf{0}\,, \tag{4.7}$$

with Chebyshev iteration parameters σ_k. Obviously, the order of k_0 will not increase, if to replace $\boldsymbol{\mathcal{A}}_I, \boldsymbol{\mathcal{A}}_{IB}$ by the respective blocks of the reference element stiffness matrix $\mathbf{A}$ or the preconditioner $\mathbb{A}_\hbar$. However, in general case the multiplications by $\mathbf{A}_I, \mathbf{A}_{IB}$ can be much more expensive, than for two other choices. Note, that $\overline{\mathbf{v}}_B$ can be calculated by means of the quadratures, related to the reference element.

Lemma 4.1. *Suppose* $\underline{\gamma}_I\mathbb{B}_I \le \boldsymbol{\mathcal{A}}_I \le \overline{\gamma}_I\mathbb{B}_I$ *with positive* γ_k. *Then at* $k_0 \ge c(1+\log p)/(\log\rho^{-1})$, *where* $\rho = (1-\theta)/(1+\theta)$, $\theta = \sqrt{\underline{\gamma}_I/\overline{\gamma}_I}$, *the inequality*

$$\|\mathbb{P}_{U_B\to U}\mathbf{v}_B\|_{\mathbf{A}} \le c_{\mathbb{P},0}\|\mathbf{v}_B\|_{\mathbb{S}_B} \tag{4.8}$$

holds with the constant $c_{\mathbb{P}}$ *independent of* p *and* $\mathbb{S}_B = \mathbf{A}_B - \mathbf{A}_{B,I}\mathbf{A}_I^{-1}\mathbf{A}_{I,B}$.

Let us note that the inequality (4.8) is equivalent to

$$|u|_{1,\tau_0} \prec c_{\mathbb{P},0}|v_B|_{1/2,\partial\tau_0} \tag{4.9}$$

where $u \leftrightarrow \mathbf{u}$ and $v_B \leftrightarrow \mathbf{v}_B$. If $\mathbf{v}_B$ is a constant vector, then $\widetilde{\mathbf{v}}_B = \mathbf{0}$, $\mathbf{A}_{IB}\widetilde{\mathbf{v}}_B = \mathbf{0}$, and, therefore, $\mathbf{w}_I^k = \mathbf{0}$ for $k \ge 1$, $\widetilde{\mathbf{u}} = \mathbf{0}$, and $\mathbb{P}_{U_B\to U}\mathbf{v}_B = \overline{\mathbf{v}}$.

Proof. Since inequalities (4.8),(4.9) do not change for all $\mathbf{v}_B = \overline{\mathbf{v}}_B + \widetilde{\mathbf{v}}_B$ with the same $\widetilde{\mathbf{v}}_B$, it is sufficient to prove lemma for the vectors $\mathbf{v}_B = \widetilde{\mathbf{v}}_B$. Under this condition, the proof follows the lines of the proof of Proposition 5.1 in Korneev/Langer/Xanthis [28]. Let the vector $\boldsymbol{\varphi}_I$ be the solution of the system $\boldsymbol{\mathcal{A}}_I \boldsymbol{\varphi}_I = \boldsymbol{\mathcal{A}}_{IB}\mathbf{v}_B$ and $\boldsymbol{\varphi}$ has for the subvectors $\boldsymbol{\varphi}_I$ and $\mathbf{v}_B$. We have the convergence estimate

$$\|\mathbf{u}_I^k - \boldsymbol{\varphi}_I\|_{\boldsymbol{\mathcal{A}}_I} \le \rho^k \|\boldsymbol{\varphi}_I\|_{\boldsymbol{\mathcal{A}}_I}\,, \tag{4.10}$$

from which for $\mathbf{u}^k = \mathbf{u}_I^k + \mathbf{v}_B$ by the spectral equivalence of the matrices $\boldsymbol{\mathcal{A}}$ and $\mathbf{A}$, see Lemma 2.1, and the triangular inequality follows

$$\|\mathbf{u}^k - \boldsymbol{\varphi}\|_{\mathbf{A}} \le \rho^k (\|\boldsymbol{\varphi}\|_{\mathbf{A}} + \|\mathbf{v}_B\|_{\mathbf{A}_B})\,. \tag{4.11}$$

In (4.11), the vector $\mathbf{v}_B$ is considered as an element of U with zero entries for the internal degrees of freedom. To estimate second summand in the right part, we introduce the polynomial $v_B \in \mathcal{U}_B(\tau_0)$, $v_B \leftrightarrow \mathbf{v}_B$, and the piece wise bilinear function $\hat{v}_B \in \mathcal{H}(\tau_0)$, coinciding with v_B at the nodes. The squares of the norms $||\cdot||_{k,\partial\tau_0}, |\cdot|_{k,\partial\tau_0}$, $k = 1, 2$, are understood below as the sums of the squares of the same norms for faces. Equivalence of these norms for the functions v_B and $\hat{v}_B$, as follows from Lemma 2.1, and direct estimation of $|\hat{v}_B|_{1,\tau_0}$ lead to the estimates

$$\|\mathbf{v}_B\|^2_{\mathbf{A}_B} \equiv |v_B|^2_{1,\tau_0} \prec |\hat{v}_B|^2_{1,\tau_0} \prec p^2 ||\hat{v}_B||^2_{1,\partial\tau_0} \prec p^2 ||v_B||^2_{1,\partial\tau_0}\,. \tag{4.12}$$

Then Markov's inequality applied to each face, the definition of the norm $||v_B||_{1/2,\partial\tau_0}$, and the factor space argument allow us to obtain

$$\|\mathbf{v}_B\|_{\mathbf{A}_B} \prec p^2 |v_B|_{1/2,\partial\tau_0}\,. \tag{4.13}$$

Finally, by means of the trace and continuation theorems in polynomial spaces, equipped with Sobolev's norms, see [4] and [5], and the definition of $\mathbb{S}_B$, we come to

$$\|\mathbf{v}_B\|_{\mathbf{A}_B} \prec p^2 \|\mathbf{v}_B\|_{\mathbb{S}_B} \tag{4.14}$$

and

$$\|\boldsymbol{\varphi}\|_{\mathbf{A}} \equiv \|\mathbf{v}_B\|_{\mathbb{S}_B} \prec |v_B|_{1/2,\partial\tau_0} \tag{4.15}$$

Combining (4.11),(4.14) and (4.15), we get

$$\|\mathbf{u}^k\|_{\mathbf{A}} \prec (1 + \rho^k (1 + p^2)) \|\mathbf{v}_B\|_{\mathbb{S}_B}\,,$$

from where the statement of Lemma immediately follows. □

According to Lemma 2.1 and Theorem 3.3, the value of ρ is an absolute constant. Therefore, taking additionally to Lemma 4.1 the conditions of the shape quasiuniformity, we come to the following conclusion.

Corollary 4.1. *If $\mathbb{B}_I = \boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}$, and $k_0 \asymp (1 + \log p)$. Then*

$$\|\mathbf{P}_{V_B \to V} \mathbf{v}_B\|_{\mathbf{K}} \le c_{\mathbb{P}} \|\mathbf{v}_B\|_{\mathbf{S}_B}\,, \qquad c_{\mathbb{P}} = c c_{\mathbb{P},0}\,, \tag{4.16}$$

with the constant c, depending only on the conditions of the shape regularity.

The wire basket preconditioner $\boldsymbol{\mathcal{S}}_W^B$ and the prolongation matrix $\mathbf{P}_{V_W\to V_B}$ figuring in **iii**) and **v**), respectively, were discussed in many papers, see, *e.g.,* Smith [39], Dryja/Smith/Widlund [18], Casarin [16], Pavarino/Widlund [37]. We borrow them without changes from Casarin [16].

Matrices $\boldsymbol{\mathcal{S}}_W^B$, $\mathbf{P}_{V_W\to V_B}$ are assembled of the scaled standard matrices defined for the reference element. For more definiteness we assume that the nodes $\boldsymbol{\eta}_{\boldsymbol{\alpha}}$ of the reference element are the GLL nodes. We denote by $\omega_W \in \omega$ the subset of $\boldsymbol{\alpha}$, corresponding to the nodes on the wire basket W_0 of τ_0, by $\varkappa_{\boldsymbol{\alpha}}$ – the weights of the quadrature, which is assembled of GLL quadratures applied to each edge, and by $\mathbb{S}_{W_0}$ – the matrix of the quadratic form

$$\mathbf{v}_W^\top \mathbb{S}_{W_0}\mathbf{v}_W = \inf_c \sum_{\boldsymbol{\alpha}\in\omega_W} \varkappa_{\boldsymbol{\alpha}}(v_{\boldsymbol{\alpha}} - c)^2 \,, \tag{4.17}$$

where $v_{\boldsymbol{\alpha}}$ are the entries of $\mathbf{v}_W$. If $\mathbf{D}_0$ is the diagonal matrix of the quadrature in (4.17) and $\mathbf{z}_1$ contains unity for all entries, then

$$\mathbb{S}_{W_0} = \mathbf{D}_0 - \frac{\mathbf{D}_0\mathbf{z}_1(\mathbf{D}_0\mathbf{z}_1)^\top}{\mathbf{z}_1^\top\mathbf{D}_0\mathbf{z}_1} = \mathbf{D}_0 - \frac{1}{24}\mathbf{D}_0\mathbf{z}_1(\mathbf{D}_0\mathbf{z}_1)^\top \tag{4.18}$$

and $\boldsymbol{\mathcal{S}}_W^B$ is assembled of the matrices

$$\boldsymbol{\mathcal{S}}_W^{(r)} = h_r\,(1+\log p)\,\varrho_r\,\mathbb{S}_{W_0}\,.$$

For details of the solving procedure for the systems with the preconditioner $\boldsymbol{\mathcal{S}}_W^B$ of the dimension $\mathcal{O}(\mathcal{R}p)\times\mathcal{O}(\mathcal{R}p)$, we refer to Pavarino/Widlund [37] and Casarin [16], where it is described up to solver for the $\mathcal{O}(\mathcal{R})\times\mathcal{O}(\mathcal{R})$ subsystem. Each unknown in this subsystem corresponds to one element and is coupled only with the next neighboring elements. If the number of elements $\mathcal{R}$ is not fixed, it is sufficient for our purposes to assume that the arithmetical cost of the pointed out procedure does not exceed $\mathcal{O}(\mathcal{R}p^3)$.

Let F_0 be the representative face of the reference element and $\mathbf{v}_{\partial F_0}$ be the vector with the entries related to the nodes on ∂F_0. By definition, the vector $\mathbf{1}_{F_0}$ contains 1-s for all internal nodes of the face F_0 and 0-s for all nodes of its boundary, whereas vector $\mathbf{v}_{F_0}$ is the continuation of $\mathbf{v}_{\partial F_0}$ by zero entries to all internal nodes of the face $\boldsymbol{\eta}_{\boldsymbol{\alpha}} \in \overline{F}_0$. Let also $\overline{v}$ be the mean value on ∂F_0 of the finite element function $v_{\partial F_0} \leftrightarrow \mathbf{v}_{\partial F_0}$, which, *e.g.,* can be calculated by quadratures. Then the standard matrix $\mathbb{P}_{\partial F_0\to\overline{F}_0}$ for the prolongation from the boundary ∂F_0 on the whole face $\overline{F}_0$ is defined as

$$\mathbb{P}_{\partial F_0\to\overline{F}_0}\mathbf{v}_{\partial F_0} = \mathbf{v}_{F_0} + \overline{v}\,\mathbf{1}_{F_0}\,. \tag{4.19}$$

A slightly different prolongation is obtained, if $\overline{v}$ is the mean value on ∂F_0 of the piece wise linear function with the entries of $\mathbf{v}_{\partial F_0}$ for the nodal values. The prolongation matrix $\mathbf{P}_{V_W\to V_B}$ is defined in such a way that its restriction to each face $F_k \in \Omega$ is $\mathbb{P}_{\partial F_0\to\overline{F}_0}$.

We have completely defined the DD preconditioner-solver (4.4).

Theorem 4.1. *Under the stated assumptions, DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ provides the condition number* $\mathrm{cond}\,[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}] \le c(1+\log p)^2$, *whereas the arithmetical cost of the operation $\boldsymbol{\mathcal{K}}^{-1}\mathbf{f}$ for any $\mathbf{f}$ is $\mathcal{O}(p^3(1+\log p)\mathcal{R})$.*

Proof. By Theorem 3.3 and conditions of the generalized shape regularity

$$\underline{\beta}_I \boldsymbol{\mathcal{K}}_I \le \mathbf{K}_I \le \overline{\beta}_I \boldsymbol{\mathcal{K}}_I \tag{4.20}$$

with positive constants $\underline{\beta}_I, \overline{\beta}_I$, depending only on the latter conditions. Therefore, taking into account Corollary 4.1 and general results on the condition of DD preconditioners, we reduce the estimate of cond $[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}]$ to deriving the bounds for $\underline{\gamma}_B, \overline{\gamma}_B > 0$ in the inequalities

$$\underline{\gamma}_B \boldsymbol{\mathcal{S}}_B \le \mathbf{S}_B \le \overline{\gamma}_B \boldsymbol{\mathcal{S}}_B \,. \tag{4.21}$$

Indeed, according to (4.20),(4.21) Corollary 4.1 and, *e.g.*, Theorem 8 of Korneev/Langer [26], we obtain

$$\underline{\beta} \boldsymbol{\mathcal{K}} \le \mathbf{K}_I \le \overline{\beta} \boldsymbol{\mathcal{K}} \,,$$

with

$$\begin{aligned} \underline{\beta} &= \min\{\underline{\beta}_I, \underline{\gamma}_B\}(1 - \sqrt{1 - c_{\mathbb{P}}^{-2}}\,) > c_{\mathbb{P}}^{-2} \min\{\underline{\beta}_I, \underline{\gamma}_B\}\,, \\ \overline{\beta} &= \max\{\overline{\beta}_I, \overline{\gamma}_B\}(1 + \sqrt{1 - c_{\mathbb{P}}^{-2}}\,) < 2 \max\{\overline{\beta}_I, \overline{\gamma}_B\}\,. \end{aligned} \tag{4.22}$$

The proof is completed by the use of the bounds for $\underline{\gamma}_B, \overline{\gamma}_B$ given in Theorem 4.2 below, which allow to write

$$\underline{\beta} \ge c_{\mathbb{P}}^{-2} \underline{c} \frac{1}{(1 + \log p)^2}\,, \qquad \overline{\beta} \le 2 \max\{\overline{\beta}_I, \overline{c}\}\,. \tag{4.23}$$

□

Theorem 4.2. *Let the preconditioner-solver $\boldsymbol{\mathcal{S}}_B$ be defined according to (4.4)–(4.6) with $\boldsymbol{\mathcal{S}}_0$ defined as in Theorem 3.4. Then the inequalities (4.21) hold with*

$$\underline{\gamma}_B \ge \underline{c} \frac{1}{(1 + \log p)^2}\,, \qquad \overline{\gamma}_B \le \overline{c} \tag{4.24}$$

and positive constants $\underline{c}, \overline{c}$ depending only on the generalized shape regularity conditions.

Proof. The proof of the bounds follows by, *e.g.,* Theorem 2 of Casarin [16], if to take additionally into account Theorem 3.4 of this paper. The same result may be found in Pavarino/Widlund [37], where it was proved differently from [16]. For completeness, we remind basic inequalities, approving (4.24), and give necessary references. The prolongation $\mathbb{P}_{\partial F_0 \to \overline{F}_0}$ uniquely defines the prolongation $\mathbb{P}_{W_0 \to \partial\tau_0}$ from the wire basket of the reference element on its boundary. If $P_{W_0 \to \partial\tau_0}$ defines the corresponding prolongations from the space of traces on W_0 into the space of traces on $\partial\tau_0$, then

$$\begin{aligned} &|\, P_{W_0 \to \partial\tau_0} v_W \,|^2_{1/2,\partial\tau_0} \le \overline{\beta}_W \inf_c ||\, v_W - c \,||^2_{0,W_0}\,, \\ &\underline{\beta}_W \inf_c ||\, v_W - c \,||^2_{0,W_0} \le (1 + \log p) |\, v \,|^2_{1,\tau_0}\,, \qquad v_W = v|_W\,, \quad \forall\, v \in \mathcal{Q}_{p,x}\,, \\ &\underline{\beta}_F \tfrac{1}{(1+\log p)^2} \textstyle\sum_1^6 {}_{00}|\, v \,|^2_{1/2,F_k} \le |\, v \,|^2_{1/2,\partial\tau_0} \le \overline{\beta}_F \sum_1^6 {}_{00}|\, v \,|^2_{1/2,F_k}\,, \qquad \forall\, v \in \mathcal{U}_F\,, \end{aligned} \tag{4.25}$$

where all β's are positive absolute constants. Due to the spectral equivalence of the reference element stiffness and mass matrices to the FE matrices $\boldsymbol{\mathcal{A}}_{\rm sp}, \boldsymbol{\mathcal{A}}_{\rm p/s}$ and $\boldsymbol{\mathcal{M}}_{\rm sp}, \boldsymbol{\mathcal{M}}_{\rm p/s}$, respectively, see Lemma 2.1, these bounds can be deduced from similar bounds for the h-version on the quasiuniform meshes. It is well illustrated by the proofs of the two first estimates. Indeed, one can introduce the quasiuniform orthogonal mesh on τ_0 with the mesh parameter $h = p^{-2}$, which is embedded in the mesh of nodes of the reference element, and assume by $\mathcal{H}_h(\tau_0)$ the space of continuous piece wise trilinear functions on this mesh. For $\widehat{v} \in \mathcal{H}_h(\tau_0)$, the bound

$$\underline{\mu}_W || \widehat{v} ||^2_{0,W_0} \le (1 + \log h^{-1}) || \widehat{v} ||^2_{1,\tau_0} , \qquad \underline{\mu} = \text{const} , \tag{4.26}$$

was proved in Bramble/Xu [14, Lemma 2.4], from which the second bound (4.25) follows by the inclusion $\mathcal{H}(\tau_0) \subset \mathcal{H}_h(\tau_0)$, factor space argument and Lemma 2.1. Let now $\widehat{v} \in \mathcal{H}(\tau_0)$ has nonzero nodal values only on the wire basket and the polynomial v coincides with $\widehat{v}$ at the nodes. The inequality

$$| \widehat{v} |^2_{1,\tau_0} \le \overline{\mu}_W || \widehat{v} ||^2_{0,W_0} , \qquad \overline{\mu} = \text{const} , \tag{4.27}$$

due to Lemma 2.1 equivalent to

$$| v |^2_{1,\tau_0} \prec \overline{\mu}_W || v ||^2_{0,W_0} , \tag{4.28}$$

is proved by direct calculation. Indeed, the function $\widehat{v}$ is distinct from zero only in the *edge nests*, by which are implied those having at least two vertices on the wire basket. We consider one representative nest $\delta_{i,k,l}$ of the size $\hbar_i \times \hbar_k \times \hbar_j$, $i, k = 0, p$, $1 < j \le p-1$, having one edge on the edge parallel to the axis x_3. Obviously, $\hbar_i = \hbar_k = \hbar_1 \le \hbar_j$ and

$$\int_{\delta_{i,k,l}} (\frac{\partial \widehat{v}}{\partial x_k})^2 d\mathbf{x} \le 3 \int_{\eta_l}^{\eta_{l+1}} \widehat{v}_E^2 dx_3 , \quad k = 1, 2, \quad \int_{\delta_{i,k,l}} (\frac{\partial \widehat{v}}{\partial x_3})^2 d\mathbf{x} \le 6 \frac{\hbar_1}{\hbar_j} \int_{\eta_l}^{\eta_{l+1}} \widehat{v}_E^2 dx_3 ,$$

where $\widehat{v}_E = \widehat{v}_E(x_3)$ is the trace of $\widehat{v}$ on the edge. It is also obvious that, if $j = 0, p$, and, therefore, the nest has a vertex, coinciding with one vertex of τ_0, then $\hbar_i = \hbar_k = \hbar_j = \hbar_1$ and

$$\int_{\delta_{i,k,l}} \nabla \widehat{v} \cdot \nabla \widehat{v} \, d\mathbf{x} \prec \int_{W_\delta} \widehat{v}^2 \, ds , \qquad W_\delta = W_0 \cap \overline{\delta}_{i,k,l} .$$

From these inequalities, the inequality (4.27) follows by summation over all edge nests.

According to (4.19), $v_B = P_{W_0 \to \partial\tau_0} v_W$ has the form $v_B = \tilde{v}_B + \overline{v}_B$ with the summands corresponding to those in (4.19). We consider the 3-d polynomial $u = \tilde{u} + \overline{u}$, for which the restriction to the boundary is v_B, the values at the internal nodes of τ_0 are zeroes, and $\widetilde{u}, \overline{u}$ correspond to $\tilde{v}_B, \overline{v}_B$, respectively. Due to the trace theorem, the factor space argument, and (4.28) applied to $\tilde{u}$, we get

$$| \tilde{v}_B |^2_{1/2,\partial\tau_0} \prec \overline{\mu}_W \inf_c || v_W - c ||^2_{0,W_0} . \tag{4.29}$$

Bound (4.28) is applicable to the contribution of the first term in (4.19). The contribution of the second summand in (4.19) is estimated similarly. Therefore, the bound (4.28) is valid for $v = P_{W_0 \to \partial\tau_0} v_W$ Now, (4.25) is obtained by the factor space argument.

For obtaining the right inequality in the 3-rd line, it is necessary to apply Cauchy inequality and take into account that for any $v \in \mathcal{U}_{F_k}$ the norm ${}_{00}| v |_{1/2,F_k}$ is equivalent to $| v |_{1/2,\partial\tau_0}$, see, *e.g.*, Ben Belgacem [3]. Proofs of the left bound in the 3-rd line may be found in Pavarino/Widlund

[37] and Casarin [16]. Due to Lemma 2.1, again the proof can be essentially based on the similar result for FE first order discretizations on the shape regular meshes.

Let $\mathbf{S}_W^B = \mathbf{S}_W - \mathbf{S}_{WF}\mathbf{S}_F^{-1}\mathbf{S}_{FW}$. From (4.25), we come to the bounds

$$
\begin{aligned}
&\tfrac{1}{(1+\log p)^2}\,\underline{c}_W\underline{\beta}_W||\,\mathbf{v}_W\,||^2_{\boldsymbol{\mathcal{S}}_W^B} \le ||\,\mathbf{P}_{V_W\to V_B}\mathbf{v}_W\,||^2_{\mathbf{S}_B} \le \tfrac{1}{(1+\log p)}\,\overline{c}_W\overline{\beta}_W||\,\mathbf{v}_W\,||^2_{\boldsymbol{\mathcal{S}}_W^B}\,,\\
&||\,\mathbf{P}_{V_W\to V_B}\mathbf{v}_W\,||^{\,2}_{\mathbf{S}_B} \le \overline{c}_W\frac{\overline{\beta}_W}{\underline{\beta}_W}||\,\mathbf{v}_W\,||^{\,2}_{\mathbf{S}_W^B}\,, \qquad (4.30)\\
&\underline{c}_F\,\underline{\beta}_F\,\tfrac{1}{(1+\log p)^2}\,\boldsymbol{\mathcal{S}}_F \le \mathbf{S}_F \le \overline{c}_F\,\overline{\beta}_F\,\boldsymbol{\mathcal{S}}_F\,.
\end{aligned}
$$

At first we obtain them for the reference element. In particular, the left one of the first line is the direct consequence of the second line bound (4.25), equivalence of the norms $|\,v_B\,|_{1/2,\partial\tau_0}$ and $||\,\mathbf{v}_B\,||_{\mathbb{S}_B}$ for the corresponding polynomial and its vector representation and the definition of $\mathbb{S}_{W_0}$ in (4.18). The second line bound (4.30) follows from two first lines of (4.25) and the definition of $\mathbf{S}_W^B$ as the matrix resulting after elimination of the internal and face unknowns. Additionally to the last line of (4.25) and the mentioned above equivalence of the norms $|\,v_B\,|_{1/2,\partial\tau_0}$ and $||\,\mathbf{v}_B\,||_{\mathbb{S}_B}$, the last line of (4.30) requires Theorem 3.4, see (3.20). Having obtained the bounds for the reference element, we proceed further to (4.30) by the use of the generalized conditions of shape regularity and subassembling.

The proof of the estimate of the condition is completed with

$$
\underline{c} = \min\,(\underline{c}_W\underline{\beta}_W, \underline{c}_F\,\underline{\beta}_F)\,, \qquad \overline{c} = \min\,(\overline{c}_W\overline{\beta}_W, \overline{c}_F\,\overline{\beta}_F)\,,
$$

if one takes into account (4.30) and the estimate of the condition for DD preconditioners as it is formulated for a similar case in Korneev/Langer [26, Theorem 8].

The operation $\boldsymbol{\mathcal{K}}^{-1}\mathbf{f}$ involves the operations with the preconditioners and prolongations **i**)-**v**) with the following arithmetic costs:

i) – $\mathcal{O}(p^3\mathcal{R})$ according to Theorem 3.3 and the definition of $\boldsymbol{\mathcal{K}}_I$,
ii) – $\mathcal{O}(p^2\mathcal{R})$ according to Theorem 3.4 and (4.5),
iii) – $\mathcal{O}(p^3\mathcal{R})$ according to the above assumption,
iv) – $\mathcal{O}(p^3(1+\log p)\mathcal{R})$ according to Theorem 3.3 and Corollary 4.1,
v) – $\mathcal{O}(p^2\mathcal{R})$ according to the definition of the prolongation $\mathbf{P}_{V_W\to V_B}$ by means of (4.19).

Therefore, the estimate of the arithmetic cost, given in the theorem, holds as well. □

Relatively to hierarchical elements, spectral elements are less flexible in adaptive computations. With the help of the presented in the paper subproblem solvers, DD preconditioners-solvers for a range of admissible adaptive spectral discretizations can be designed without losses of efficiency. The adjustment of the analysis of their condition and computational cost can be completed along the lines of Korneev/Langer/Xanthis [28].

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