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DIVIDEND MAXIMIZATION UNDER CONSIDERATION OF THE TIME VALUE OF RUIN*

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Abstract

In the Cramér-Lundberg model and its diffusion approximation, it is a classical problem to find the optimal dividend payment strategy that maximizes the expected value of the discounted dividend payments until ruin. One often raised disadvantage of this approach is the fact that such a strategy does not take the life time of the controlled process into account. In this paper we introduce a value function which considers both expected dividends and the time value of ruin. For both the diffusion model and the Cramér-Lundberg model with exponential claim sizes, the problem is solved and in either case the optimal strategy is identified, which for unbounded dividend intensity is a barrier strategy and for bounded dividend intensity is of threshold type.

1 Introduction

The classical optimal dividend problem looks for the strategy that maximizes the expected discounted dividend payments until ruin in an insurance portfolio. For the compound Poisson model, this problem was solved by Gerber [9], identifying so-called *band strategies* as the optimal ones. For exponentially distributed claim sizes this strategy simplifies to a *barrier strategy*, i.e. whenever the surplus exceeds some barrier level b, all the income is paid out as dividends and no dividends are paid out below that surplus level. In [9], the result is first obtained for a discrete version of the model and then obtained for the continuous model by a limiting procedure. Recently, the optimal dividend problem in the compound Poisson model was taken up again by Azcue and Muler [2], who used stochastic optimal control techniques and viscosity solutions.
The corresponding problem in the case of a diffusion risk process was solved in Asmussen & Taksar [1]. Taksar [23] gives an extensive picture over the above and related maximisation problems, where also additional possibilities of control such as reinsurance are treated. Gerber & Shiu [11] showed that in case the admissible dividend payment intensity is bounded above by some constant $M < c$ (where c is the premium intensity of the surplus process), for exponential claim sizes a so-called *threshold strategy* maximizes the expected discounted dividend payments (i.e. whenever the surplus is below a certain threshold, no dividends are paid out and above that level the maximal allowed amount is paid). In a diffusion setting, a corresponding result was already established in [1].

However, all the strategies outlined above lead to ruin with probability one and in many circumstances this is not desirable. On the other hand, there has also been a lot of research activity on using optimal control to minimize the ruin probability. For instance, for the diffusion approximation, Browne [5] considered the case where the insurer is allowed to invest in a risky asset which follows a geometric Brownian motion and identified the optimal investment strategy that minimizes the ruin probability of the resulting risk process. For extensions to the Cramér-Lundberg model, see e.g. Hipp & Plum [13], Gaier & Grandits [8]. The problem of choosing optimal dynamic proportional reinsurance to minimize ruin probabilities was investigated by Schmidli [19] and optimal excess-of-loss reinsurance strategies were considered in Hipp & Vogt [14]. Combinations of both investment and reinsurance are considered in Schmidli [20], see Schmidli [22] for a nice recent survey on this subject.

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In this paper we return to the problem of optimal dividend payments, but add a component to the objective function that penalizes early ruin of the controlled risk process. In particular, this additional term can be interpreted as a continuous payment of a (discounted) constant intensity during the lifetime of the controlled process. It will turn out that this choice of objective function leads to a particularly tractable extension of the corresponding available results for pure dividend maximization (in particular Asmussen & Taksar [1] and Hojgaard & Taksar [15]), and hence considerable parts of the proofs are along the lines of the above papers, however keeping track of the consequences of the additional term in the objective function. The approach should be seen as a first tractable step towards more refined optimization criteria in the corresponding optimal control problems.
The paper is organized as follows. In Section 2, the Cramér-Lundberg model and its diffusion approximation are shortly discussed and the value function underlying our approach is introduced. Section 3 deals with the case of a diffusion risk process and the optimal control problem is solved explicitly, both for bounded and unbounded dividend intensity and the effect of the time value of ruin on the optimal strategy is investigated. It is also shown that if in addition to dividend payouts there is a possibility for dynamic proportional reinsurance, then the optimal strategy from Hojgaard & Taksar [15] is also optimal in our case, just adding a constant term in the value function. Section 4 deals with the above optimal control problem for the classical Cramér-Lundberg process. For exponential claim amounts the explicit solution is obtained, which extends the results of Gerber [9] and Gerber & Shiu [11] for unbounded and bounded dividend intensity, respectively. In each section numerical examples are given that illustrate the modification of the optimal strategy with the additional term in the objective function.

After this manuscript was finished, the authors found an unpublished manuscript of Boguslavskaya [4], who in a financial context used a similar objective function in the diffusion setting and solved it using the theory of free boundary problems. However, the approach in Section 3 provides a somewhat more intuitive way of proof, using classical stochastic optimal control techniques, which also allows us to extend the results to the Cramér-Lundberg model in Section 4.

Finally, we would like to point out that in a recent paper, Gerber et al. [10] conjecture that in case of unbounded dividend intensity, horizontal barrier strategies are optimal for the maximization of the difference of the expected discounted dividends and the deficit at ruin. The results in this paper establish optimality of horizontal barrier strategies for the inclusion of another safety criterion, namely the life-time of the controlled risk process.

2 Model and Value function

Let $(\Omega, \mathcal{F}, P)$ be an underlying complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ that models the flow of information. Let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion with respect to the given filtration. In this paper two models for the collective risk process are considered. In a first approach the risk process $R = (R_t)_{t\geq 0}$ is described by a diffusion process. Apart from the fact that this assumption simplifies the analysis and leads to structural results, it can also be motivated by an approximation argument towards a compound Poisson model (see [12], [16], [21] or [3]). We denote the drift term by $\mu > 0$ and the standard deviation by σ, then the process with initial capital x is defined via

$$dR_t = \mu dt + \sigma dW_t, \quad R_0 = x.$$

Alternatively, we will also work in the Cramér-Lundberg model, where the risk reserve process $R = (R_t)_{t\geq 0}$ with initial capital x is defined by

$$R_t = x + ct - \sum_{i=0}^{N_t} Y_i, \quad t \geq 0. \tag{1}$$

Here $c > 0$ is the constant premium intensity and the claim amounts are an independent and identically distributed sequence $\{Y_i\}_{i\in\mathbb{N}}$ of positive random variables with distribution function $F_Y(y)$. The claim number process $N = (N_t)_{t\geq 0}$ is assumed to be Poisson with intensity $\lambda > 0$, which is independent of $\{Y_i\}_{i\in\mathbb{N}}$.

In the following the insurer is allowed to pay dividends. The cumulated dividends are described by a process $L = (L_t)_{t\geq 0}$, which is called *admissible*, if it is a positive increasing càdlàg process, adapted to $(\mathcal{F}_t)_{t\geq 0}$. L_t represents the total dividends up to time t and the resulting controlled risk process is given by

$$R_t^L = R_t - L_t.$$

The time of ruin for this process is defined by $\tau := \tau^L = \inf\{t \geq 0 \mid R_t^L < 0\}$. Let furthermore $\tau' = \inf\{t \geq 0 \mid R_t^L = 0\}$, then for a pure diffusion process $\tau = \tau'$. We can write

$$L_t = \int_0^t e^{-\beta s} l_s \, ds,$$

where $(l_s)_{s\geq 0}$ is the dividend intensity. Furthermore we require that paying dividends can not cause ruin, $L_t - L_{t-} \leq R_t^L$ and also $L_{0-} = 0$. Moreover no dividends can be paid after ruin, i.e. $L_t = L_\tau$ for all $t > \tau$.

In this paper we aim to identify the dividend payment strategy $L = (L_t)_{t\geq 0}$ that maximizes

$$V(x, L) = \mathbb{E}\left(\int_0^\tau e^{-\beta t} \, dL_t + \int_0^\tau e^{-\beta t} \Lambda \, dt \,\Big|\, R_0^L = x \right) \tag{2}$$

for some $\Lambda > 0$, i.e. we are looking for the value function

$$V(x) = \sup_L V(x, L), \tag{3}$$

where the supremum is taken over all admissible strategies.

Note that compared to the classical value function, which maximizes the expected discounted dividend payments, there is an additional term depending on the time of ruin. $e^{-\beta t}\,\Lambda$ can be interpreted as the present value of an amount which the insurer earns as long as the company is alive. In this way the lifetime of the portfolio becomes part of the value function and is weighted according to the choice of Λ. Another interpretation is that in this way the Laplace transform of the ruin time is part of the value function.

3 Optimal strategy for the diffusion case

Let us distinguish the two cases of bounded and unbounded dividend intensity l_s.

3.1 Bounded Dividend Intensity

Let $0 \leq l_t \leq M$ for $t \geq 0$. Then the value function (3) is given by

$$\begin{aligned} V(x) &= \sup_{0\leq l\leq M} \mathbb{E}\left(\int_0^\tau e^{-\beta t} \, (l_t + \Lambda) dt \,\Big|\, R_0^L = x \right), \\ V(0) &= 0. \end{aligned}$$

Clearly $V(x)$ is bounded by $(M+\Lambda)/\beta$. Standard arguments, see [7], formally yield the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = -\beta V(x) + \sup_{0\leq l\leq M} \left\{ (\mu - l)V'(x) + \frac{\sigma^2}{2} V''(x) + l + \Lambda \right\},$$

which can be rewritten as

$$0 = -\beta V(x) + \mu V'(x) + \frac{\sigma^2}{2} V''(x) + \Lambda + \sup_{0\leq l\leq M} \{(1 - V'(x))l\}. \tag{4}$$

Let us first assume that V is a strictly concave function, $V' > 0$ and $V'' < 0$. Then there exists some point x_0 with the following properties:

$$\begin{aligned} x < x_0 : &\qquad V'(x) > 1, \\ x \geq x_0 : &\qquad V'(x) \leq 1. \end{aligned}$$

In the sequel it will be seen that this working assumption indeed leads to the optimal strategy. Based on the linearity of the control l in (4) we get that the optimal control $l^*(x)$ has to fulfill

$$l^*(x) = \begin{cases} 0 & x < x_0, \\ M & x \geq x_0. \end{cases}$$

Therefore (4) translates into

$$\begin{aligned} 0 &= -\beta V(x) + \mu V'(x) + \frac{\sigma^2}{2} V''(x) + \Lambda, \quad x < x_0, && (5) \\ 0 &= -\beta V(x) + (\mu - M) V'(x) + \frac{\sigma^2}{2} V''(x) + \Lambda + M, \quad x \geq x_0, && (6) \\ 0 &= V(0), \end{aligned}$$

and the crucial point x_0 has to be determined by the method of *smooth fit*. Let V_l denote the solution of (5) and V_r the solution of (6). Since in (5) and (6) there are derivatives of the value function up to order 2, we have to look for a twice differentiable solution. This leads to the following pasting conditions at x_0:

$$\begin{aligned} V_l(x_0) &= V_r(x_0), && (7) \\ V_l'(x_0) &= V_r'(x_0) = 1, && (8) \\ V_l''(x_0) &= V_r''(x_0), && (9) \end{aligned}$$

A general solution of (5) is of the form

$$V_l(x) = \frac{\Lambda}{\beta} + A_1\, e^{R_1 x} + A_2\, e^{R_2 x},$$

with

$$R_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\beta}{\sigma^2}}.$$

Note that $R_1 > 0$ and $R_2 < 0$. The condition $V(0) = 0$ gives $A_2 = -(\Lambda/\beta + A_1)$. Similarly,

$$V_r(x) = \frac{M + \Lambda}{\beta} + B_1\, e^{S_1 x} + B_2\, e^{S_2 x},$$

with

$$S_{1,2} = -\frac{(\mu - M)}{\sigma^2} \pm \sqrt{\frac{(\mu - M)^2}{\sigma^4} + \frac{2\beta}{\sigma^2}}.$$

From the boundedness of the value function we know that if any of the exponents is positive, the corresponding coefficient has to be zero. Hence, from $S_1 > 0$,

$$V_r(x) = \frac{M + \Lambda}{\beta} + B_2\, e^{S_2 x},$$

where $B_2 < 0$ is a constant. Now we use (7)-(9) to determine x_0 and the remaining coefficients A_1 and B_2 (which are functions of x_0). We have

$$\begin{aligned} \frac{\Lambda}{\beta} + A_1\, e^{R_1 x_0} - (A_1 + \frac{\Lambda}{\beta})\, e^{R_2 x_0} &= \frac{M + \Lambda}{\beta} + B_2\, e^{S_2 x_0}, && (10) \\ A_1\, R_1\, e^{R_1 x_0} - (A_1 + \frac{\Lambda}{\beta})\, R_2\, e^{R_2 x_0} &= B_2\, S_2\, e^{S_2 x_0} = 1, && (11) \\ A_1\, R_1^2\, e^{R_1 x_0} - (A_1 + \frac{\Lambda}{\beta})\, R_2^2\, e^{R_2 x_0} &= B_2\, S_2^2\, e^{S_2 x_0}. && (12) \end{aligned}$$

If we use the right equality of (11) in (10), we get with $\delta(M) := M/\beta + 1/S_2$

$$x_0 = \frac{1}{R_1 - R_2} \log\left(\frac{A_1(x_0) + \frac{\Lambda}{\beta}}{A_1(x_0)}\, \frac{1 - \delta(M)\, R_2}{1 - \delta(M)\, R_1} \right). \quad (13)$$

and correspondingly from (11)

$$B_2 = \frac{1}{S_2}\left(\frac{A_1(x_0)+\frac{\Lambda}{\beta}}{A_1(x_0)}\;\frac{1-\delta(M)\,R_2}{1-\delta(M)\,R_1}\right)^{\frac{-S_2}{R_1-R_2}}. \tag{14}$$

After substitution of (13) in (11), A_1 is obtained as a solution of a nonlinear equation, see (15) below.

Lemma 1. *For all $M > 0$*

$$\frac{1}{R_2} < \delta(M) < \frac{1}{R_1}.$$

Proof. From

$$\frac{1}{R_1} - \delta = \frac{1}{2\beta}\left(-M + \sqrt{\mu^2+2\beta\sigma^2} + \sqrt{(\mu-M)^2+2\beta\sigma^2}\right)$$

the right inequality holds if

$$\sqrt{\mu^2+2\beta\sigma^2} > M - \sqrt{(\mu-M)^2+2\beta\sigma^2} := G(M).$$

Indeed, since

$$G'(M) = 1 - \frac{M-\mu}{\sqrt{(\mu-M)^2+2\beta\sigma^2}} > 0$$

and

$$\lim_{M\to\infty} G(M) = \lim_{M\to\infty}\frac{M^2-M^2+2M\mu-\mu^2-2\beta\sigma^2}{M+\sqrt{M^2-2M\mu+\mu^2+2\beta\sigma^2}} = \lim_{M\to\infty}\frac{2\mu-\frac{\mu^2+2\beta\sigma^2}{M}}{1+\sqrt{1-\frac{2\mu}{M}+\frac{\mu^2+2\beta\sigma^2}{M^2}}} = \mu,$$

$G(M)$ is a monotone increasing function with $\lim_{M\to\infty} G(M) = \mu < \sqrt{\mu^2+2\beta\sigma^2}$.
The second inequality follows from

$$\delta - \frac{1}{R_2} = \frac{M\left(1-\frac{M-2\mu}{\sqrt{(\mu-M)^2+2\beta\sigma^2}+\sqrt{\mu^2+2\beta\sigma^2}}\right)}{2\beta}$$

and the fact that $\sqrt{(\mu-M)^2+2\beta\sigma^2} > M-\mu$ and $\sqrt{\mu^2+2\beta\sigma^2} > \mu > 0$. $\square$

Define

$$F(H) := \frac{1+\frac{\Lambda}{\beta}R_2\left(\frac{H+\frac{\Lambda}{\beta}}{H}\,\frac{1-\delta R_2}{1-\delta R_1}\right)^{\frac{R_2}{R_1-R_2}}}{R_1\left(\frac{H+\frac{\Lambda}{\beta}}{H}\,\frac{1-\delta R_2}{1-\delta R_1}\right)^{\frac{R_1}{R_1-R_2}} - R_2\left(\frac{H+\frac{\Lambda}{\beta}}{H}\,\frac{1-\delta R_2}{1-\delta R_1}\right)^{\frac{R_2}{R_1-R_2}}} - H. \tag{15}$$

Note that the denominator of (15) is strictly positive.

Lemma 2. *If $\frac{M+\Lambda}{\beta} + \frac{1}{S_2} \le 0$ then*

$$V^*(x) = \frac{M+\Lambda}{\beta}\left(1-e^{S_2 x}\right),$$

is a twice continuously differentiable strictly concave solution of the HJB equation (4).
If $\frac{M+\Lambda}{\beta} + \frac{1}{S_2} > 0$, then $x_0 > 0$ and

$$V^*(x) = \begin{cases} \frac{\Lambda}{\beta} + A_1\,e^{R_1 x} - (A_1+\frac{\Lambda}{\beta})\,e^{R_2 x} & x < x_0,\\ \frac{M+\Lambda}{\beta} - C_2\,e^{S_2 x} & x \ge x_0,\end{cases}$$

is a twice differentiable strictly concave solution of the HJB equation (4). The coefficient B_2 and x_0 are calculated from (14) and (13), while A_1 is a positive root of $F(H)$ as defined in (15).

Proof. First we look at the case $\frac{M+\Lambda}{\beta}+\frac{1}{S_2}>0$. From Lemma 1 we know that $1-\delta R_2$ and $1-\delta R_1$ are positive. Hence we have to ensure $\frac{A_1+\frac{\Lambda}{\beta}}{A_1}>0$, as otherwise x_0 in (13) is not a real number. This implies $A_1>0$, as the alternative $A_1<-\frac{\Lambda}{\beta}$ would lead to a decreasing function $V^*(x)$ for $x<x_0$. So we are looking for a positive root A_1 of $F(H)$ as defined in (15), which can be rewritten as

$$F(H)=\frac{\frac{\Lambda}{\beta}R_2+\left(\frac{H+\frac{\Lambda}{\beta}}{H}\frac{1-\delta R_2}{1-\delta R_1}\right)^{\frac{-R_2}{R_1-R_2}}}{R_1\left(\frac{H+\frac{\Lambda}{\beta}}{H}\frac{1-\delta R_2}{1-\delta R_1}\right)-R_2}-H.$$

From $0<\frac{-R_2}{R_1-R_2}<1$ we see that $\lim_{H\to 0}F(H)=0$ and $\lim_{H\to\infty}F(H)=-\infty$.
Further we have for sufficiently small $H>0$ that

$$\frac{\Lambda}{\beta}R_2+\left(\frac{H+\frac{\Lambda}{\beta}}{H}\frac{1-\delta R_2}{1-\delta R_1}\right)^{\frac{-R_2}{R_1-R_2}}>-R_2\,H+R_1\left(H+\frac{\Lambda}{\beta}\right)\left(\frac{1-\delta R_2}{1-\delta R_1}\right)>0.$$

The continuity of $F(H)$ thus establishes the existence of a strictly positive root A_1, which is the desired coefficient. In view of (13), $x_0>0$ if

$$\frac{A_1+\frac{\Lambda}{\beta}}{A_1}\frac{1-\delta R_2}{1-\delta R_1}>1,$$

which is equivalent to

$$A_1\delta(R_1-R_2)+\frac{\Lambda}{\beta}(1-\delta R_2)>0,$$

which always holds for $\delta\geq 0$. For $\delta=\frac{M}{\beta}+\frac{1}{S_2}<0$ we need $A_1<-\frac{\Lambda}{\beta}\frac{1-\delta R_2}{\delta(R_1-R_2)}$, which due to

$$F\left(-\frac{\Lambda}{\beta}\frac{1-\delta R_2}{\delta(R_1-R_2)}\right)=\frac{\beta+(M+\Lambda)S_2}{2(\beta+MS_2)\sqrt{\frac{\mu^2}{\sigma^4}+\frac{2\beta}{\sigma^2}}}<0$$

is guaranteed under the assumption $\frac{M+\Lambda}{\beta}+\frac{1}{S_2}>0$. So in this case indeed $x_0>0$.
$V^*(x)$ is clearly differentiable on $\mathbb{R}^+$ and particularly in x_0. Because V_l solves (5) in x_0 and V_r solves (6) in x_0 we get by substitution of $V_l(x_0)=V_r(x_0)$ and $V_l'(x_0)=V_r'(x_0)=1$ in (5) and (6) directly that $V_l''(x_0)=V_r''(x_0)$ also holds.
Next we show that $V^*(x)$ is strictly concave for $x<x_0$. Recall that $A_1>0$ and $A_1+\frac{\Lambda}{\beta}>0$. We have

$$\begin{aligned}V^{*''}(x)&=A_1\,R_1^2\,e^{R_1x}-\left(A_1+\frac{\Lambda}{\beta}\right)\,R_2^2\,e^{R_2x},\\ V^{*'''}(x)&=A_1\,R_1^3\,e^{R_1x}-\left(A_1+\frac{\Lambda}{\beta}\right)\,R_2^3\,e^{R_2x}>0,\end{aligned}$$

so that $V^{*''}$ is strictly increasing. From

$$V^{*''}(0)=A_1\,R_1^2-\left(A_1+\frac{\Lambda}{\beta}\right)\,R_2^2<0$$

and $V^{*''}(x_0)=S_2<0$ we deduce that $V^{*''}(x)<0$ for all $x\in[0,x_0]$. Furthermore, $V^{*'}>0$ and therefore $V^{*'}(x_0)=1$ is a strict lower bound for the derivative in $[0,x_0)$.
On the other hand, it is easy to see that for $x\geq x_0$

$$\begin{aligned}V^{*'}(x)&=B_2\,S_2\,e^{S_2x}>0,\\ V^{*''}(x)&=B_2\,S_2^2\,e^{S_2x}<0,\end{aligned}$$

and hence $V^{*'}(x_0)=1$ is a strict upper bound for $V^{*'}(x)$ in (x_0,∞).

Finally, the case $\frac{M+\Lambda}{\beta} + \frac{1}{S_2} < 0$ is verified by direct calculation. From $S_2 < 0$ we have

$$V^{*'}(x) = -\frac{M+\Lambda}{\beta} S_2\, e^{S_2 x} \leq -\frac{M+\Lambda}{\beta} S_2 < 1$$

and $V^{*'}(x) > 0$ for all $x \geq 0$. Furthermore

$$V^{*''}(x) \quad = \quad -\frac{M+\Lambda}{\beta} S_2^2\, e^{S_2 x} \leq 0,$$

so that $V^*(x)$ is indeed a strictly increasing concave function. □

Remark 1. Equations (10), (11) and (13) generalize equations (2.19), (2.20), (2.21) and (2.26) from [1], where the case $\Lambda = 0$ was treated. Note that in contrast to [1] the unknown x_0 depends on the coefficient A_1 and therefore the equation for A_1 becomes nonlinear (whereas in [1, (2.26)] the independence of x_0 and A_1 led to linear equations). The height of the barrier x_0 raises for increasing Λ, reflecting the reduced risk one is willing to take in case the lifetime of the controlled process is taken into acccount.

Finally we need a verification theorem proving that the value function obtained in Lemma 2 is indeed optimal:

Proposition 3. *Let L be an admissible dividend strategy then for $V^*(x)$ given in Lemma 2, $V^*(x) \geq V(x, L)$ and $V^*(x) = V(x, L^*)$. The strategy L^* is given by*

$$L_t^* = \int_0^t l_s^*\, e^{-\beta s}\, ds,$$

and

$$l^*(x) = \begin{cases} 0 & x < x_0 \\ M & x \geq x_0. \end{cases}$$

Proof. Let L be an admissible strategy with bounded intensity $(l_t)_{t\geq 0}$. From the Itô-formula we obtain

$$e^{-\beta(T\wedge\tau)} V^*(R^L_{T\wedge\tau}) - V^*(x) =$$
$$\int_0^{T\wedge\tau} \left(\frac{1}{2}\sigma^2 V^{*''}(R_t^L) + (\mu - l_t) V^{*'}(R_t^L) - \beta V^*(R_t^L)\right) e^{-\beta t}\, dt + \int_0^{T\wedge\tau} e^{-\beta t} V^{*'}(R_t^L)\sigma\, dW_t. \quad (16)$$

We know that $V^{*'}(x)$ is a monotone decreasing function and therefore bounded by $V^{*'}(0)$, so the stochastic integral in (16) is a square integrable martingale with expectation zero. From the HJB equation (4) we know that the integrand of the first integral is bounded by $-(l_t + \Lambda)\, e^{-\beta t}$, so we get

$$\mathbb{E}\left(e^{-\beta(T\wedge\tau)} V^*(R^L_{T\wedge\tau}) \,\Big|\, R_0^L = x\right) + \mathbb{E}\left(\int_0^{T\wedge\tau} (l_t + \Lambda) e^{-\beta t}\, dt \,\Big|\, R_0^L = x\right) \leq V^*(x) \quad (17)$$

The integrand in the second expectation is bounded by $(M + \Lambda)/\beta$ which is also a bound for $V^*(x)$. We let $T \to \infty$ and use dominated convergence to get

$$\mathbb{E}\left(\int_0^{\tau} (l_t + \Lambda) e^{-\beta t}\, dt \,\Big|\, R_0^L = x\right) = V(x, L) \leq V^*(x).$$

If we use the strategy L^* we get equality in (17). The same bounds hold as before and therefore $V(x, L^*) = V^*(x)$. □

Figure 1 depicts the optimal dividend payout as a function of initial capital x for various values of Λ and Figure 2 shows the optimal threshold level as a function of Λ.

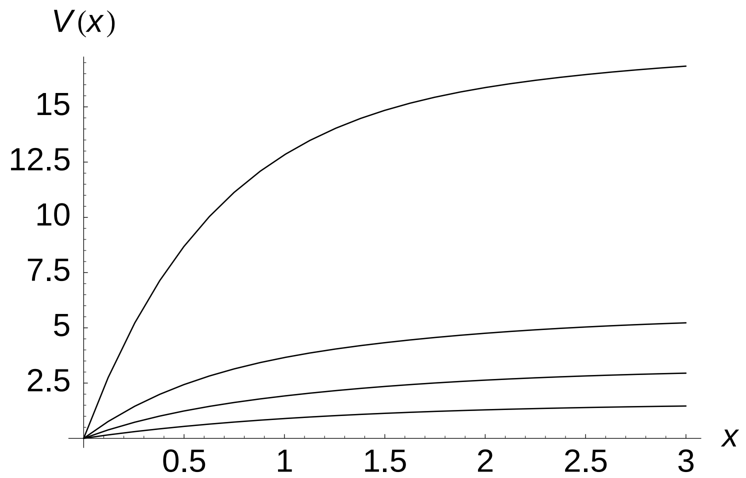

Figure 1: Value function for $\Lambda = 0, 0.08, 0.2, 0.8$

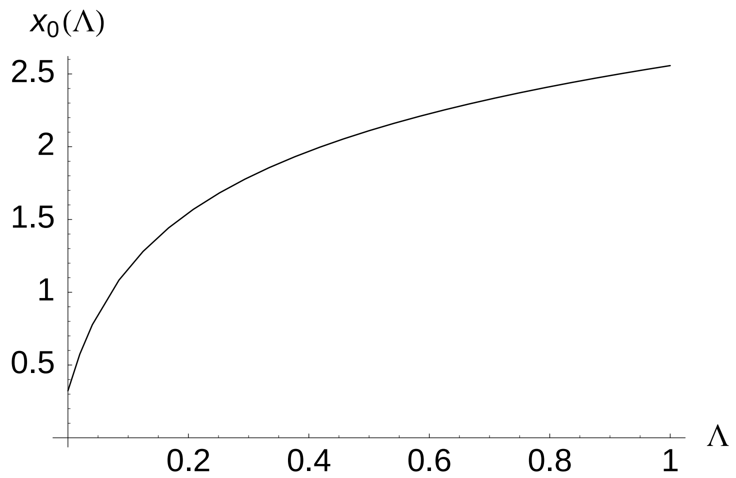

Figure 2: Barrier as function of Λ

3.2 Unbounded Dividend Intensity

Here the cumulated dividends are not absolutely continuous and we have to use tools from *singular control* (see for instance [7]). The amount of dividends associated with an admissible dividend strategy $L = (L_t)_{t\geq 0}$ is given by

$$V(x, L) = \mathbb{E}\left(\int_0^\tau e^{-\beta t}\, dL_t + \int_0^\tau e^{-\beta t}\Lambda\, dt \;\Big|\; R_0^L = x\right)$$

The value function of the optimization problem is

$$V(x) = \sup_L V(x, L),$$

where the supremum is taken over all admissible strategies. The classical variational inequalities (see [7]) deliver the HJB equation of this problem, namely

$$\begin{aligned} 0 &= \max\left\{\mu V'(x) + \frac{\sigma^2}{2} V''(x) - \beta V(x) + \Lambda,\, 1 - V'(x)\right\}, \qquad (18)\\ 0 &= V(0). \end{aligned}$$

At first we again assume that $V(x)$ is strictly concave and that a crucial point x_0 with $V'(x) > 1$ for $x < x_0$, $V'(x_0) = 1$ and $V'(x) < 1$ for $x > x_0$ exists (x_0 will play the role of a classical dividend barrier). This gives

$$0 = \mu V'(x) + \frac{\sigma^2}{2} V''(x) - \beta V(x) + \Lambda, \quad x < x_0, \tag{19}$$

$$0 = 1 - V'(x), \quad x \geq x_0. \tag{20}$$

As in the bounded case, due to the principle of *smooth fit*, the value function has to fulfill

$$V_l(x_0) = V_r(x_0), \tag{21}$$

$$V_l'(x_0) = V_r'(x_0) = 1, \tag{22}$$

$$V_l''(x_0) = V_r''(x_0) = 0, \tag{23}$$

where again $V_l(x)$ and $V_r(x)$ denote the function $V(x)$ for $x < x_0$ and $x \geq x_0$, respectively. Hence

$$V_l(x) = \frac{\Lambda}{\beta} + A_1\, e^{R_1 x} - \left(A_1 + \frac{\Lambda}{\beta}\right) e^{R_2 x},$$

with

$$R_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + 2\frac{\beta}{\sigma^2}}.$$

Note that $R_2 < 0 < R_1$ with $|R_2| > R_1$. The solution for the right part $x \geq x_0$ is a straight line given by

$$V_r(x) = B_1 + x.$$

In terms of these two functions the conditions (21)-(23) read as follows:

$$\begin{aligned} \frac{\Lambda}{\beta} + A_1\, e^{R_1 x_0} - \left(A_1 + \frac{\Lambda}{\beta}\right) e^{R_2 x_0} &= B_1 + x_0, \\ A_1\, R_1\, e^{R_1 x_0} - \left(A_1 + \frac{\Lambda}{\beta}\right) R_2\, e^{R_2 x_0} &= 1, \qquad (24)\\ A_1\, R_1^2\, e^{R_1 x_0} - \left(A_1 + \frac{\Lambda}{\beta}\right) R_2^2\, e^{R_2 x_0} &= 0. \end{aligned}$$

For x_0 we get

$$x_0 = \frac{1}{R_1 - R_2} \log\left(\frac{A_1(x_0) + \frac{\Lambda}{\beta}}{A_1(x_0)} \frac{R_2^2}{R_1^2}\right). \tag{25}$$

The constant B_1 is determined by $B_1 = V_l(x_0) - x_0$ and the coefficient A_1 is a root of the function

$$F(H) := H\, R_1\, e^{R_1 x_0} - \left(H + \frac{\Lambda}{\beta}\right) R_2\, e^{R_2 x_0} - 1.$$

Remark 2. Note that equations (24) and (25) reduce to equations (3.20) and (3.19) from [1] for $\Lambda = 0$. With (25), $F(H)$ can be expressed as

$$F(H) = H^{\frac{-R_2}{R_1-R_2}} \left(H + \frac{\Lambda}{\beta}\right)^{\frac{R_1}{R_1-R_2}} \left(R_1 \left(\frac{R_2^2}{R_1^2}\right)^{\frac{R_1}{R_1-R_2}} - R_2 \left(\frac{R_2^2}{R_1^2}\right)^{\frac{R_2}{R_1-R_2}}\right) - 1. \tag{26}$$

We again need to show that $x_0 > 0$, which is certainly fulfilled if $A_1 > 0$, i.e. $F(H)$ has to have a positive root which, due to the continuity of $F(H)$ together with $F(0) = -1$ and $\lim_{H\to\infty} F(H) = \infty$ is indeed the case. Moreover, the uniqueness of A_1 follows from

$$F'(H) = \frac{H^{\frac{R_1}{R_2-R_1}} \left(H + \frac{\Lambda}{\beta}\right)^{\frac{R_1}{R_1-R_2}} (H\beta R_1 - R_2(H\beta + \Lambda))}{(R_1 - R_2)(H\beta + \Lambda)} \left(R_1 \left(\frac{R_2^2}{R_1^2}\right)^{\frac{R_1}{R_1-R_2}} - R_2 \left(\frac{R_2^2}{R_1^2}\right)^{\frac{R_2}{R_1-R_2}}\right) > 0.$$

Lemma 4. *The function*

$$V^*(x) = \begin{cases} \frac{\Lambda}{\beta} + A_1\, e^{R_1 x} - (A_1 + \frac{\Lambda}{\beta})\, e^{R_2 x} & x < x_0, \\ x - x_0 + \frac{\Lambda}{\beta} + A_1\, e^{R_1 x_0} - (A_1 + \frac{\Lambda}{\beta})\, e^{R_2 x_0} & x \geq x_0, \end{cases}$$

is a twice differentiable and (strictly for $x < x_0$) concave solution to the HJB equation (18).

Proof. It only remains to show that

$$V_l(x) = \frac{\Lambda}{\beta} + A_1\, e^{R_1 x} - (A_1 + \frac{\Lambda}{\beta})\, e^{R_2 x}$$

is strictly concave. Clearly $V_l'(x) = A_1\, R_1\, e^{R_1 x} - (A_1 + \frac{\Lambda}{\beta})\, R_2\, e^{R_2 x} > 0$. To see that $V_l''(x) < 0$ for $x < x_0$, observe that

$$\begin{aligned} V_l''(0) &= A_1\, R_1^2 - R_2^2 \left(A_1 + \frac{\Lambda}{\beta}\right) < 0, \\ V_l''(x_0) &= 0, \\ V_l'''(x) &= A_1\, R_1^3\, e^{R_1 x} - \left(A_1 + \frac{\Lambda}{\beta}\right)\, R_2^3\, e^{R_2 x} > 0. \end{aligned}$$

Finally, $V^{*'}(x) > 1$ for $x < x_0$. □

Proposition 5. *For every admissible dividend strategy L, the function $V^*(x)$ of Lemma 4 dominates $V(x, L)$, $V^*(x) \geq V(x, L)$. Let L^* be the barrier strategy given by the barrier x_0. Then $V^*(x) = V(x, L^*)$.*

Proof. Let $L = (L_t)_{t\geq 0}$ be any admissible strategy. From Dynkin's formula, see [18], we know that

$$e^{-\beta t\wedge\tau} V^*(R^L_{t\wedge\tau}) - V(x) - \int_0^{t\wedge\tau} e^{-\beta s} \mathbf{A} V^*(R^L_s) ds$$

is a martingale with expectation zero, where $\mathbf{A}V^*(R^L_s)$ denotes the infinitesimal generator of the process $R^L_t = x + \mu\, t + \int_0^t \sigma\, dW_s - L_t$. To get the generator of the jump part of the process (which in this case can only originate from dividend payments) we use a generalized Itô formula from [6],

$$\begin{aligned} V^*(R^L_t) - V^*(R^L_0) &= \int_0^t V^{*'}(R^L_s)\, dR^{L,c}_s + \int_0^t \frac{\sigma^2}{2} V^{*''}(R^L_s)\, ds \\ &\quad + \sum_{\Delta R^L_s \neq 0,\ 0\leq s\leq t} (V^*(R^L_{s-} + \Delta R^L_s) - V^*(R^L_{s-})) \\ &= \int_0^t \mu V^{*'}(R^L_s)\, ds + \int_0^t \sigma V^{*'}(R^L_s)\, dW_s - \int_0^t V^*(R^L_s) dL^c_s \\ &\quad + \int_0^t \frac{\sigma^2}{2} V^{*''}(R^L_s)\, ds + \sum_{\Delta R^L_s \neq 0,\ 0\leq s\leq t} (V^*(R^L_{s-} + \Delta R^L_s) - V^*(R^L_{s-})), \end{aligned}$$

where the superscript c refers to the continuous component of the process. Note that the sum is negative because $R^L_{s-} + \Delta R^L_s \leq R^L_{s-}$. Concretely, we have

$$\mathbb{E}\left(e^{-\beta(t\wedge\tau)}V^*(R^L_{t\wedge\tau})\right) = V^*(x) + \mathbb{E}\left(\int_0^{t\wedge\tau} e^{-\beta s}\left(\mu V^{*'}(R^L_s) + \frac{\sigma^2}{2}V^{*''}(R^L_s) - \beta V^*(R^L_s)\right) ds \right.$$
$$\left. - \int_0^{t\wedge\tau} e^{-\beta s}V^{*'}(R^L_s)\, dL^c_s + \sum_{\Delta R^L_s \neq 0, 0\leq s\leq t\wedge\tau} e^{-\beta s}\left(V^*(R^L_{s-} + \Delta R^L_s) - V^*(R^L_{s-})\right)\right).$$

From the HJB equation (18) we get that the first integrand on the right side is smaller than $-e^{-\beta s}\,\Lambda$. Furthermore $V^{*'} \geq 1$. In addition, we have to find a bound for the left hand side and the sum. Because $V^*(x)$ is concave, it can be bounded by a straight line of the form $kx + d$ and so the left hand side is bounded by $e^{-\beta t}(d + k|\mu t + \sigma W_t|)$ (note that for $\tau < t$ we can use $V^*(0) = 0$) and this term converges to zero for $t \to \infty$.
Jumps of the reserve occur if and only if jumps of the dividends occur, so $R^L_s - R^L_{s-} = L_{s-} - L_s$. Together with the concavity we get $V^*(R^L_{s-} + \Delta R^L_s) - V^*(R^L_{s-}) \leq L_{s-} - L_s$. Now we are allowed to let $t \to \infty$ and together with the bounds above we get

$$\mathbb{E}\left(\int_0^{\tau} e^{-\beta s}\Lambda\, ds + \int_0^{\tau} e^{-\beta s}\, dL^c_s + \sum_{\Delta L_s \neq 0, 0\leq s\leq \tau} e^{-\beta s}(L_s - L_{s-})\right)$$
$$= \mathbb{E}\left(\int_0^{\tau} e^{-\beta s}\Lambda\, ds + \int_0^{\tau} e^{-\beta s}\, dL_s\right) \leq V^*(x)$$

Therefore $V^*(x) \geq V(x, L)$ holds.
Now look at the barrier strategy L^* derived from x_0. We have that $R^{L^*}_t \leq x_0$ for all $t \geq 0$ and dividends are only paid at times at which $R^{L^*}_t = x_0$, note that $V^{*'}(x)I_{x=x_0} = I_{x=x_0}$. Because V^* fulfills the HJB equation (18) for $x \leq x_0$ we get from Dynkin's formula,

$$\begin{aligned}\mathbb{E}\left(e^{-\beta t\wedge\tau}V^*(R^{L^*}_{t\wedge\tau})\right) &= V^*(x) - \mathbb{E}\left(\int_0^{t\wedge\tau} e^{-\beta s}\Lambda\, ds\right. \\ &\left. - \int_0^{t\wedge\tau} e^{-\beta s}I_{R^{L^*}_s = x_0}\, dL^{*c}_s + \sum_{\Delta R^{L^*}_s \neq 0, 0\leq s\leq t\wedge\tau} e^{-\beta s}\left(V^*(R^{L^*}_{s-} + \Delta R^{L^*}_s) - V^*(R^{L^*}_{s-})\right)\right)\end{aligned}$$

From the construction of V^* and L^* jumps can only happen when $R^{L^*}_s > x_0$ and

$$V^*(R^{L^*}_{s-} + \Delta R^{L^*}_s) - V^*(R^{L^*}_{s-}) = V^*(x_0) - (R^{L^*}_s - x_0 + V^*(x_0)) = -R^{L^*}_s + x_0 = L^*_{s-} - L^*_s.$$

Therefore

$$\mathbb{E}\left(\int_0^{t\wedge\tau} e^{-\beta s}I_{R^{L^*}_s = x_0}\, dL^{*c}_s - \sum_{\Delta R^{L^*}_s \neq 0, 0\leq s\leq t\wedge\tau} e^{-\beta s}\left(V^*(R^{L^*}_{s-} + \Delta R^{L^*}_s) - V^*(R^{L^*}_{s-})\right)\right) = \mathbb{E}\left(\int_0^{t\wedge\tau} e^{-\beta s}\, dL^*_s\right)$$

As before all relevant terms are bounded and for $t \to \infty$ we get the result

$$V^*(x) = \mathbb{E}\left(\int_0^{\tau} e^{-\beta t}\Lambda\, dt + \int_0^{\tau} e^{-\beta t}\, dL^*_t\right).$$

□

Figure 3 depicts the optimal dividend payout with unbounded intensity as a function of initial capital x for various values of Λ and Figure 4 shows the corresponding optimal barrier levels as a function of Λ.

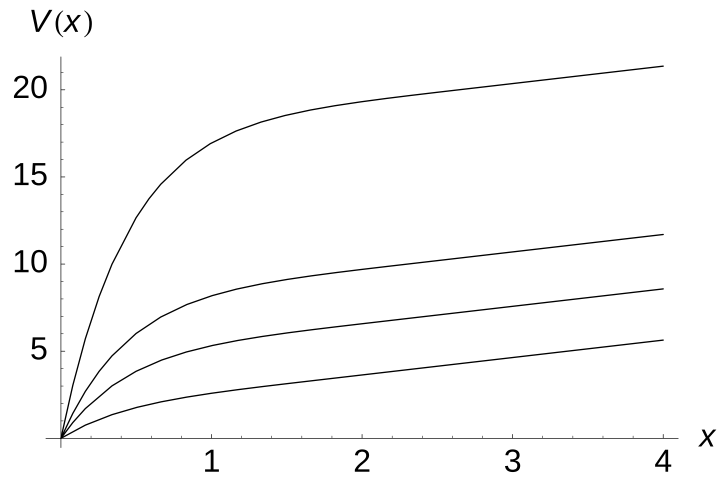

Figure 3: Value function for $\Lambda = 0,\ 0.1,\ 0.2\ 0.5$

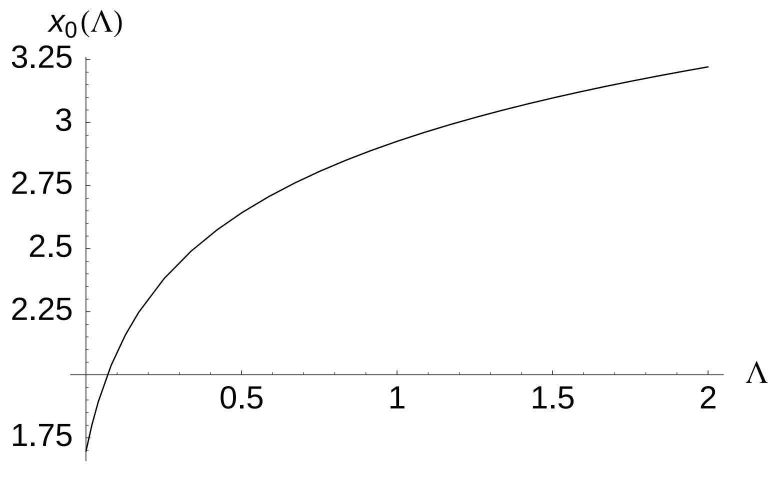

Figure 4: Barrier as function of Λ

3.3 Optimal Dividends and Proportional Reinsurance

In the literature for the diffusion model, optimal control problems were also extended to maximize expected dividend payments with additionally being able to take dynamic proportional reinsurance (see e.g. Hojgaard & Taksar [15]), where the insurer passes on some fraction $0 \leq 1 - A_t \leq 1$ of the premiums (in the diffusion model of the drift μ), and correspondingly proportionally reduces the risk (in the diffusion model the volaitility σ). This leads to the modified risk process

$$dR_t^A = A_t \, \mu dt + A_t \, \sigma dW_t$$

for the dynamic reinsurance strategy $A = (A_t)_{t \geq 0}$. A strategy is admissible if it is an adapted process and $0 \leq A_t \leq 1$ for all $t \geq 0$. It is natural to ask for the optimal combination of dividend and reinsurance strategy maximizing

$$V(x, A, L) = \mathbb{E}\left(\int_0^\tau e^{-\beta t} \, dL_t + \int_0^\tau e^{-\beta t} \Lambda \, dt \,\Big|\, R_0^{A,L} = x \right)$$

among all admissible strategies A and L. However, since for $\Lambda = 0$ the optimal reinsurance strategy is to pass on all the risk ($A^*(0) = 0$) and stay at zero forever, this means that ruin can not occur for this controlled process and hence we always obtain the maximal reward $\frac{\Lambda}{\beta}$ from the second summand of our value function. Consequently, the optimal strategy is not influenced by this additional term and $V^*(x)$ is always given by the value for $\Lambda = 0$ (already determined in [15]) plus $\frac{\Lambda}{\beta}$. If one formulates and solves the HJB equation for $\Lambda > 0$, the above conclusion is reflected by the fact that the initial condition $V^*(0) = \frac{\Lambda}{\beta}$ neutralizes the additional factor Λ in the differential equations arising from the HJB equation.

4 Optimal strategy for the Cramér-Lundberg model

In this section we will investigate the impact of the term Λ for the optimal dividend payout scheme for the Cramér-Lundberg model (1), where in addition we assume exponentially distributed claim amounts. In the Cramér-Lundberg model the value function does not satisfy the boundary condition $V(0) = 0$ (since being in 0 does now not necessarily imply ruin) and hence we have to look for another condition.

4.1 Bounded dividend intensity

Let us start again with the case of a bounded dividend intensity $0 \leq l_t \leq M$ for a bound $0 \leq M < c$. The generator of the controlled risk reserve process is given by

$$\mathbf{A}g(x) = (c - l)g'(x) + \lambda \int_0^x g(x - y) - g(x) \, dF_Y(y). \tag{27}$$

Now $g(x)$ is not continuous in 0. Such a case can be handled by introducing the concept of a stopped risk reserve process (by considering an additional dimension with two states, reflecting "stopped" or "unstopped"). For details of this technique in the framework of *Piecewise Deterministic Markov Processes* see Rolski et al. [18]. The HJB equation in the bounded case reads as follows

$$0 = \sup_{0 \leq l \leq M} \left\{ \Lambda + l + (c - l)V'(x) + \lambda \int_0^x V(x - y) dF_Y(y) - (\beta + \lambda)V(x) \right\}. \tag{28}$$

From now on we specify $F_Y(y) = 1 - e^{-\alpha y}$ and assume the existence of a strictly increasing concave solution of (28). Because of the linearity in the control l we get a crucial point x_0 with $V'(x) > 1$ for $x < x_0$, $V'(x_0) = 1$ and $V'(x) < 1$ for $x > x_0$. As in Section 3, it is possible that $x_0 = 0$. Under these assumptions the HJB equation (28) is equal to

$$0 = \Lambda + cV'(x) + \lambda \int_0^x V(x - y) \alpha e^{-\alpha y} dy - (\beta + \lambda)V(x), \quad x \leq x_0, \tag{29}$$

$$0 = \Lambda + M + (c - M)V'(x) + \lambda \int_0^x V(x - y) \alpha e^{-\alpha y} dy - (\beta + \lambda)V(x), \quad x > x_0. \tag{30}$$

Equation (29) can be rewritten as

$$cV''(x) + (\alpha c - (\beta + \alpha))V'(x) - \alpha\beta V(x) + \alpha\Lambda = 0,$$

with a general solution of the form

$$V_l(x) = \frac{\Lambda}{\beta} + A_1\, e^{R_1 x} + A_2\, e^{R_2 x},$$

where

$$R_{1,2} = -\frac{(\alpha c - (\beta + \lambda))}{2c} \pm \sqrt{\frac{(\alpha c - (\beta + \lambda))^2}{4c^2} + \frac{\alpha\beta}{c}}.$$

Clearly $R_2 < 0 < R_1$ and $|R_1| < |R_2|$. Correspondingly, (30) has a solution of the form

$$V_r(x) = \frac{\Lambda + M}{\beta} + B_1\, e^{S_1 x} + B_2\, e^{S_2 x}$$

with

$$S_{1,2} = -\frac{(\alpha(c - M) - (\beta + \lambda))}{2(c - M)} \pm \sqrt{\frac{(\alpha(c - M) - (\beta + \lambda))^2}{4(c - M)^2} + \frac{\alpha\beta}{(c - M)}},$$

and $S_2 < 0 < S_1$, $R_2 < S_2$. The value function is bounded by $\frac{\Lambda+M}{\beta}$, so that $B_1 = 0$ and $B_2 < 0$.
If $x_0 = 0$ (i.e. it is optimal to pay dividends at rate M for any initial capital $x \geq 0$), then the value function has to fulfill (30) for all $x \geq 0$. Putting $V_r(x)$ into (30) gives

$$V_r(x) = \frac{\Lambda + M}{\beta}\left(1 - \frac{\alpha + S_2}{\alpha}\, e^{S_2 x}\right).$$

This function is increasing and concave, because $\alpha + S_2 > 0$. It is indeed the optimal solution if $V_r'(0) \leq 1$, which happens if $-\frac{(\alpha+S_2)}{\alpha\beta} S_2(\Lambda + M) \leq 1$.
From now on we consider the opposite case $-\frac{(\alpha+S_2)}{\alpha\beta} S_2(\Lambda + M) > 1$. Since we need a differentiable solution of (29) and (30), the following three equations have to hold in x_0:

$$\frac{\Lambda}{\beta} + A_1\, e^{R_1 x_0} + A_2\, e^{R_2 x_0} = \frac{\Lambda + M}{\beta} + B_2\, e^{S_2 x_0}, \tag{31}$$

$$A_1\, R_1\, e^{R_1 x_0} + A_2\, R_2\, e^{R_2 x_0} = B_2\, S_2\, e^{S_2 x_0} = 1. \tag{32}$$

With the notation $\delta(M) := \frac{M}{\beta} + \frac{1}{S_2}$, we obtain from (31)

$$x_0 = \frac{1}{R_1 - R_2} \log\left(-\frac{A_2(x_0)}{A_1(x_0)} \frac{1 - \delta(M)\, R_2}{1 - \delta(M)\, R_1}\right). \tag{33}$$

Again we have highlighted the dependence of the coefficients A_i on x_0 to see that the equation is not explicit in x_0 (opposed to the case $\Lambda = 0$). Given x_0, (31) together with substitution of V_l and V_r in (29) and (30) imply that the coefficients are the solution of the linear system of equations

$$\begin{pmatrix} e^{R_1 x_0} & e^{R_2 x_0} & -e^{S_2 x_0} \\ \frac{\alpha}{\alpha+R_1} & \frac{\alpha}{\alpha+R_2} & 0 \\ \frac{\alpha\, e^{R_1 x_0}}{\alpha+R_1} & \frac{\alpha\, e^{R_2 x_0}}{\alpha+R_2} & -\frac{\alpha\, e^{S_2 x_0}}{\alpha+S_2} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{M}{\beta} \\ -\frac{\Lambda}{\beta} \\ \frac{M}{\beta} \end{pmatrix}.$$

From the right equality (32) we directly get $B_2 = \frac{e^{-S_2 x_0}}{S_2} < 0$ (which coincides with the solution from the system only if $V'(x_0)$ exists and equals 1). Then

$$A_1 = \frac{(\alpha + R_1)(e^{R_2 x_0}\Lambda(R_2 - S_2) - MS_2)}{\alpha\beta(e^{R_1 x_0}(R_1 - S_2) + e^{R_2 x_0}(S_2 - R_2))},$$

$$A_2 = \frac{(\alpha + R_2)(e^{R_1 x_0}\Lambda(R_1 - S_2) - MS_2)}{\alpha\beta(e^{R_2 x_0}(R_2 - S_2) + e^{R_1 x_0}(S_2 - R_1))}. \tag{34}$$

Here $A_2 < 0$ for any value of x_0, whereas $A_1 > 0$ if $x_0 > \frac{1}{R_2} \log\left(\frac{MS_2}{\Lambda(R_2 - S_2)}\right)$.

Lemma 6. *For $M < c$ we have*

$$\frac{1}{R_2} < \delta(M) < \frac{1}{R_1}.$$

Proof. Define $H(M) := \delta(M) - \frac{1}{R_1(M)}$. Then $H(0) = \frac{1}{R_2} - \frac{1}{R_1} < 0$. From

$$\begin{aligned}\lim_{M\to c^-} S_2 &= \lim_{M\to c^-} -\frac{\alpha}{2} + \frac{(\beta+\lambda)}{2(c-M)} - \frac{1}{(c-M)}\sqrt{\frac{\alpha^2(c-M)^2 + (\beta+\lambda)^2 - 2\alpha(\beta+\lambda)(c-M)}{4} + \alpha\beta(c-M)} \\ &= \lim_{M\to c^-} -\frac{\alpha}{2} + \frac{-\frac{\alpha^2(c-M)}{4} + \frac{\alpha(\beta+\lambda)}{2} - \alpha\beta}{\frac{(\beta+\lambda)}{2} + \sqrt{\frac{\alpha^2(c-M)^2+(\beta+\lambda)^2-2\alpha(\beta+\lambda)(c-M)}{4} + \alpha\beta(c-M)}} \\ &= -\frac{\alpha\beta}{(\beta+\lambda)} < 0\end{aligned}$$

we get

$$\lim_{M\to c^-} H(M) = \frac{\alpha c - (\beta+\lambda)}{\alpha\beta} - \frac{\frac{(\alpha c-(\beta+\lambda))}{2} + \sqrt{\frac{(\alpha c-(\beta+\lambda))^2}{4} + \alpha\beta c}}{\alpha\beta} < 0.$$

The monotonicity of $H(M)$ in $\in [0, c)$ follows from

$$H'(M) = \frac{2\lambda}{\alpha^2(c-M)^2 + (\beta+\lambda)^2 + 2\alpha(\beta-\lambda)(c-M) - (\alpha c+\beta-\lambda)\sqrt{\alpha^2(c-M)^2 + (\beta+\lambda)^2 + 2\alpha(\beta-\lambda)(c-M)}}.$$

Indeed, for $(\alpha c + \beta - \lambda) < 0$ we immediately have $H'(M) > 0$. In the opposite case $(\alpha c + \beta - \lambda) > 0$ one observes

$$\begin{aligned}(\alpha c+\beta-\lambda) < \sqrt{\alpha^2(c-M)^2 + (\beta+\lambda)^2 + 2\alpha(\beta-\lambda)(c-M)} \Leftrightarrow \\ \alpha^2(c-M)^2 + (\beta-\lambda)^2 + 2\alpha(\beta-\lambda)(c-M) < \alpha^2(c-M)^2 + (\beta+\lambda)^2 + 2\alpha(\beta-\lambda)(c-M) \Leftrightarrow \\ 0 < 4\beta\lambda,\end{aligned}$$

also implying $H'(M) > 0$. Hence $H(M) < 0$ for $M \in [0, c)$.
For the second inequality, define $H_1(M) := \delta - \frac{1}{R_2} > 0$. $H_1'(M) = \frac{\partial}{\partial M}\left(\frac{M}{\beta} + \frac{1}{S_2}\right) = H'(M) > 0$ and hence $H_1(M)$ is strictly increasing for $M \in [0, c)$. Furthermore, for $M = \frac{\beta c}{\alpha c-\lambda}$ we have $\delta = 0$. Therefore $H_1(0) = 0$ and $H_1\left(\frac{\beta c}{\alpha c-\lambda}\right) = -\frac{1}{R_2} > 0$ and finally $H_1(M) > 0$ for $M \in (0, \frac{\beta c}{\alpha c-\lambda})$. For $M \geq \frac{\beta c}{\alpha c-\lambda}$, the term δ is positive and so $(1 - \delta R_2) > 0$ also holds for this case. □

From (33) and (34) one sees that x_0 is the solution of the nonlinear equation

$$x = \frac{1}{R_1 - R_2}\log\left(\frac{\alpha+R_2}{\alpha+R_1}\frac{(e^{R_1 x}\Lambda(R_1-S_2) - MS_2)}{(e^{R_2 x}\Lambda(R_2-S_2) - MS_2)}\frac{(\beta(R_2-S_2) + MR_2S_2)}{(\beta(R_1-S_2) + MR_1S_2)}\right). \tag{35}$$

Remark 3. Note that for $\Lambda = 0$ equation (35) reduces to equation (9.15) of Gerber & Shiu [11].

Lemma 7. *For $-\frac{(\alpha+S_2)}{\alpha\beta}S_2(\Lambda + M) > 1$ equation (35) has an unique positive solution x_0.*

Proof. To simplify notation, define

$$G(x) := \frac{1}{R_1 - R_2}\log\left(\frac{\alpha+R_2}{\alpha+R_1}\frac{(e^{R_1 x}\Lambda(R_1-S_2) - MS_2)}{(e^{R_2 x}\Lambda(R_2-S_2) - MS_2)}\frac{(\beta(R_2-S_2) + MR_2S_2)}{(\beta(R_1-S_2) + MR_1S_2)}\right).$$

Note that

$$\frac{\beta(R_2-S_2) + MR_2S_2}{\beta(R_1-S_2) + MR_1S_2} = \frac{1-\delta R_2}{1-\delta R_1}$$

and observe

$$\lim_{x\to\infty} G'(x) = \frac{R_1}{R_1 - R_2} < 1.$$

This means that for large x, $G(x)$ tends to a linear function with slope less than one. Furthermore

$$G''(x) = \frac{e^{(R1+2R_2)x}\,(-\Lambda^3 M R_1^2 S_2 (R_1-S_2)(R_2-S_2)^2) + e^{(2R_1+R_2)x}(\Lambda^3 M R_2^2 S_2 (R_1-S_2)^2 (R_2-S_2))}{(R_1-R_2)\left(e^{R_1x}\Lambda(R_1-S_2)-MS_2\right)^2\left(e^{R_2x}\Lambda(R_2-S_2)-MS_2\right)^2}$$
$$+\frac{e^{(R_1+R_2)x}(2\Lambda^2 M^2 S_2^2 (R_1-R_2)(R_1+R_2)(R_1-S_2)(R_2-S_2)) + e^{R_1x}(\Lambda M^3 R_1^2 S_2^3 (S_2-R_1))}{(R_1-R_2)\left(e^{R_1x}\Lambda(R_1-S_2)-MS_2\right)^2\left(e^{R_2x}\Lambda(R_2-S_2)-MS_2\right)^2}$$
$$+\frac{e^{R_2x}(\Lambda M^3 R_2^2 S_2^3 (R_2-S_2)}{(R_1-R_2)\left(e^{R_1x}\Lambda(R_1-S_2)-MS_2\right)^2\left(e^{R_2x}\Lambda(R_2-S_2)-MS_2\right)^2} > 0,$$

since all coefficients of the exponential terms are positive. Hence $G(x)$ is convex. It has a pole at $\hat{x} = \frac{1}{R_2}\log\left(\frac{MS_2}{\Lambda(R_2-S_2)}\right)$ with

$$\lim_{x\to\hat{x}^+} G(x) = \infty.$$

If $\hat{x} \geq 0$ (which holds for $S_2 \geq \frac{R_2\Lambda}{(\Lambda+M)}$) this implies the existence of a unique positive root x_0 of $x = G(x)$. If $\hat{x} < 0$ (i.e. $S_2 < \frac{\Lambda R_2}{(\Lambda+M)}$), one can consider

$$G(0) = \frac{1}{(R_1-R_2)}\log\left(\frac{(\alpha+R_2)}{(\alpha+R_1)}\frac{(1-\delta R_2)}{(1-\delta R_1)}\frac{\Lambda R_1-(\Lambda+M)S_2}{\Lambda R_2-(\Lambda+M)S_2}\right).$$

Here all terms of the denominator are positive and $G(0) > 0$ is equivalent to

$$(\alpha+R_2)(1-\delta R_2)(\Lambda R_1-(\Lambda+M)S_2) - (\alpha+R_1)(1-\delta R_1)(\Lambda R_2-(\Lambda+M)S_2) > 0.$$

From $\alpha+R_1 > \alpha+R_2 > 0$ we have that the expression above is greater than

$$M(R_1-R_2)(\alpha+R_2)S_2(\beta+(\Lambda+M)S_2),$$

which is positive because

$$\beta < \frac{\alpha\beta}{(\alpha+S_2)} < -S_2(\Lambda+M).$$

Hence $G(0) > 0$ and again the existence of a unique root x_0 of $x = G(x)$ follows. □

Lemma 8. *If* $-\frac{(\alpha+S_2)}{\alpha\beta}S_2(\Lambda+M) \leq 1$ *then*

$$V^*(x) = \frac{\Lambda+M}{\beta}\left(1-\frac{(\alpha+S_2)}{\alpha}\,e^{S_2x}\right)$$

is a differentiable, increasing and concave solution of the HJB equation (28).
If $-\frac{(\alpha+S_2)}{\alpha\beta}S_2(\Lambda+M) > 1$*, then*

$$V^*(x) = \begin{cases} \frac{\Lambda}{\beta} + A_1\,e^{R_1x} + A_2\,e^{R_2x} & x \leq x_0, \\ \frac{\Lambda+M}{\beta} + B_2\,e^{S_2x} & x_0 < x \end{cases}$$

with x_0 *the unique solution of (35) and* A_1, A_2, B_2 *determined by the equations given above, is a differentiable increasing and concave solution to (28).*

Proof. It only remains to show that for $-\frac{(\alpha+S_2)}{\alpha\beta}S_2(\Lambda+M) > 1$ one indeed has $V^{*'}(x) > 1$ for $x < x_0$, $0 < V^{*'}(x) < 1$ for $x_0 < x$ and $V^{*''}(x) < 0$ for $x \geq 0$.
Due to $A_1 > 0$, $A_2 < 0$ and $B_2 < 0$, together with $V^{*'}(x_0) = 1$ this holds for $x > x_0$. For $x < x_0$, $V^{*''}(x) < 0$ follows if

$$x_0 \leq \frac{1}{R_1-R_2}\log\left(\frac{R_2^2}{R_1^2}\frac{(\alpha+R_2)}{(\alpha+R_1)}\frac{(e^{R_1x_0}\,\Lambda(R_1-S_2)-MS_2)}{(e^{R_2x_0}\,\Lambda(R_2-S_2)-MS_2)}\right) \iff R_2^2(1-\delta R_1) > R_1^2(1-\delta R_2).$$

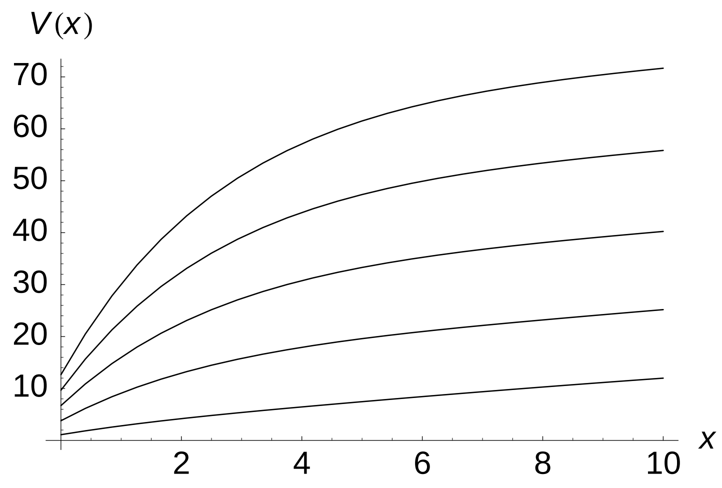

Figure 5: Value function for $\Lambda = 0,\ 0.5,\ 1,\ 1.5,\ 2$

However, the last inequality holds because

$$R_2^2(1-\delta R_2) - R_1^2(1-\delta R_2) =$$
$$-\frac{\sqrt{\beta^2+(\alpha c-\lambda)^2+2\beta(\alpha c+\lambda)}\left(\beta+\lambda-\alpha(c-M)-\sqrt{(\alpha(c-M)-(\beta+\lambda))^2+4\alpha\beta(c-M)}\right)}{2c^2},$$

and

$$\left(\beta+\lambda-\alpha(c-M)-\sqrt{(\alpha(c-M)-(\beta+\lambda))^2+4\alpha\beta(c-M)}\right) < 0.$$

□

Proposition 9. *The function $V^*(x)$ given in Lemma 8 fulfills $V^*(x) = V(x)$ and the threshold strategy with threshold level x_0 is optimal among all admissible strategies with bounded density in the case of $Exp(\alpha)$ distributed claim amounts.*

Proof. Use the generator of the controlled process given in (27) and proceed in exactly the same way as in Proposition 3. □

Figure 5 depicts the value function for initial capital x for several values of Λ and Figure 6 shows the threshold level as a function of Λ for the parameter set $\alpha = 2$, $\lambda = 3$, $\beta = 0.03$, $c = 1.75$ and $M = 1$. One can again observe that x_0 is increasing in Λ.

4.2 Unbounded dividend intensity

If the dividend intensity is not bounded, the associated HJB equation reads as follows

$$\max\left\{\Lambda + cV'(x) + \lambda\int_0^x V(x-y)dF_Y(y) - (\beta+\lambda)V(x), 1 - V'(x)\right\} = 0. \tag{36}$$

We again consider the special case of $Exp(\alpha)$ distributed claim amounts and first assume the existence of a concave differentiable solution to (36). The crucial point where the first derivative of the value function becomes smaller than one is again denoted by x_0. For $x > x_0$ we then have $1 - V'(x) = 0$, which immediately gives $V(x) = x + B_2$ for some constant B_2. For $x \le x_0$, we have to solve

$$\Lambda + cV'(x) + \lambda\int_0^x V(x-y)\alpha\, e^{-\alpha y}dy - (\beta+\lambda)V(x) = 0, \tag{37}$$

which can be rewritten as

$$cV''(x) + (\alpha c - (\beta+\lambda))V'(x) - \alpha\beta V(x) + \alpha\Lambda = 0,$$

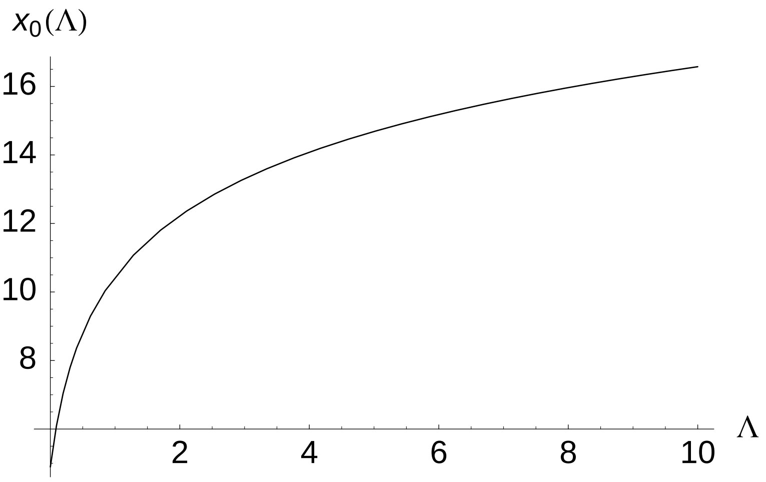

Figure 6: Barrier as function of Λ

with general solution

$$V_l(x) = \frac{\Lambda}{\beta} + A_1\, e^{R_1 x} + A_2\, e^{R_2 x},$$

where the exponents $\{R_1, R_2\}$ are again the roots of the polynomial

$$P(R) = cR^2 + (\alpha c - (\beta + \lambda))R - \alpha\beta.$$

Substitution in (37) then leads to

$$A_2 = -\frac{\alpha + R_2}{\alpha}\left(\frac{\alpha}{\alpha + R_1} A_1 + \frac{\Lambda}{\beta}\right).$$

We now need to find a differentiable solution. The corresponding pasting conditions at x_0 give $B_2 = -x_0 + V_l(x_0)$ and $V_l'(x_0) = 1$, yielding

$$A_1 = \frac{(\alpha + R_1)(\alpha\beta + e^{R_2 x_0}\Lambda R_2(\alpha + R_2))}{\alpha\beta(e^{R_1 x_0} R_1(\alpha + R_1) - e^{R_2 x_0} R_2(\alpha + R_2))}.$$

We are still short of an additional condition to determine x_0. In the diffusion case of Section 3.2, this additional condition was the request of a twice differentiable solution, implying $V''(x_0) = 0$. Here we do not have second derivatives in the equations. Nevertheless, $V''(x_0) = 0$ also turns out to be the appropriate condition in this case: Above we have seen that the coefficients $\{A_1, A_2\}$ are functions of the barrier x_0. Denote this barrier by b for a moment. Some calculations show that the optimal barrier height is then determined by setting

$$\frac{\partial}{\partial b}\left(\frac{\Lambda}{\beta} + A_1(b)\, e^{R_1 x} + A_2(b)\, e^{R_2 x}\right) = -\frac{(\alpha + R_1)\, e^{R_1 x} - (\alpha + R_2)\, e^{R_2 x}}{R_1(\alpha + R_1)\, e^{R_1 b} - R_2(\alpha + R_2)\, e^{R_2 b}}\, V_b''(b) = 0, \tag{38}$$

where $V_b(x)$ is the value function belonging to a specified barrier b, given by

$$V_b(x) = \begin{cases} \frac{\Lambda}{\beta} + A_1(b)\, e^{R_1 x} + A_2(b)\, e^{R_2 x} & x \le b, \\ x - b + V_b(b) & x > b. \end{cases}$$

The denominator of the right-hand side of (38) is strictly positive, therefore we have to find a root of $V_b''(b)$ to obtain the optimal barrier. In the following we will show that $V_b''(b)$ has an unique root giving the optimal value function and therefore the optimal barrier. $V_b''(b) = 0$ is equivalent to

$$A_1\, R_1^2\, e^{R_1 x_0} + A_2\, R_2^2\, e^{R_2 x_0} = 0,$$

which leads to

$$x_0 = \frac{1}{R_1 - R_2}\log\left(\frac{R_2^2}{R_1^2}\frac{(\alpha + R_2)}{(\alpha + R_1)}\frac{(\alpha\beta + e^{R_1 x_0}(\alpha + R_1)\Lambda R_1)}{(\alpha\beta + e^{R_2 x_0}(\alpha + R_2)\Lambda R_2)}\right). \tag{39}$$

Remark 4. For $\Lambda = 0$, equation (39) again reduces to equation (5.2.6) of Gerber [9].

Lemma 10. *If* $\alpha\lambda(c+\Lambda) - (\beta+\lambda)^2 \leq 0$ *then*

$$V^*(x) = x + \frac{c+\Lambda}{\beta+\lambda},$$

is a solution of (36).
If $\alpha\lambda(c+\Lambda) - (\beta+\lambda)^2 > 0$, *then the function*

$$V^*(x) = \begin{cases} \frac{\Lambda}{\beta} + A_1(x_0)\, e^{R_1 x} + A_2(x_0)\, e^{R_2 x} & x \leq x_0, \\ x - x_0 + V^*(x_0) & x > x_0 \end{cases}$$

is a concave solution to (36), where x_0 is the unique root of equation (39).

Proof. We start with looking at the case $\alpha\lambda(c+\Lambda) - (\beta+\lambda)^2 \leq 0$. It is obvious that $V^*(x) = x + \frac{c+\Lambda}{\beta+\lambda}$ solves $1 - V^{*'}(x) = 0$. Thus we have to show that the first part of the left hand side of (36) is not positive. Substituting $V^*(x) = x + \frac{c+\Lambda}{\beta+\lambda}$ into (36) we thus have to show that

$$\left(e^{-\alpha x} - 1\right) \frac{\beta\lambda + \lambda^2 - \alpha\lambda(c+\Lambda)}{\alpha(\beta+\lambda)} - \beta x \,\leq\, 0,$$

which clearly holds in zero. With $\beta\lambda + \lambda^2 - \alpha\lambda(c+\Lambda) \leq -\beta(\beta+\lambda)$ we indeed have

$$\left(e^{-\alpha x} - 1\right) \frac{\beta\lambda + \lambda^2 - \alpha\lambda(c+\Lambda)}{\alpha(\beta+\lambda)} \leq \frac{\beta}{\alpha}\left(1 - e^{-\alpha x}\right) \leq \beta x.$$

Note that the second inequality is strict for $x > 0$. On the barrier (which is equal to zero), both terms of (36) are equal to zero, while for $x > 0$ the first one is strictly negative.
From now on we deal with the case $\alpha\lambda(c+\Lambda) - (\beta+\lambda)^2 > 0$. Here $x + \frac{(c+\Lambda)}{(\beta+\lambda)}$ is not a solution to (36): The first part of the maximum in (36) gives

$$\left(1 - e^{-\alpha x}\right) \frac{\alpha\lambda(c+\Lambda) - (\beta+\lambda)\lambda}{\alpha(\beta+\lambda)} - \beta x, \tag{40}$$

which is zero for $x = 0$, but for

$$0 \leq x < \frac{1}{\alpha}\, \log\left(\frac{\alpha\lambda(c-\Lambda) - (\beta+\lambda)\lambda}{\beta(\beta+\lambda)}\right),$$

the first derivative of (40) is strictly positive and therefore the first part in the maximum of (36) is positive, so that the line $x + \frac{(c+\Lambda)}{(\beta+\lambda)}$ does not solve (36).
Next we show that (39) has a unique positive solution denoted by x_0. Let

$$F(x) := \frac{1}{R_1 - R_2} \log\left(\frac{R_2^2}{R_1^2} \frac{(\alpha+R_2)}{(\alpha+R_1)} \frac{(\alpha\beta + e^{R_1 x}(\alpha+R_1)\Lambda R_1)}{(\alpha\beta + e^{R_2 x}(\alpha+R_2)\Lambda R_2)}\right).$$

We have

$$\lim_{x\to\infty} F'(x) = \frac{\Lambda^2 R_2(\alpha+R_2)(R_1-R_2) + \alpha\beta R_1}{\alpha\beta(R_1-R_2)} < 1.$$

Consequently, for large values of x the right hand side of (39) grows linearly with smaller slope than the left hand side, so that a desired solution exists if $F(x) > x$, for some $x > 0$.
The numerator in the logarithm above is positive. Furthermore, the denominator is positive for $x > \hat{x}$ where

$$\hat{x} = \frac{1}{R_2} \log\left(\frac{-\alpha\beta}{\lambda R_2(\alpha+R_2)}\right)$$

and

$$\lim_{x\to\hat{x}^+} F(x) = \infty.$$

As result we get that if $\hat{x} \geq 0$, then a positive root x_0 of (39) exists. For $\hat{x} < 0$ (i.e. $\Lambda R_2(\alpha+R_2)+\alpha\beta > 0$), consider

$$F(0) = \frac{1}{R_1 - R_2} \log\left(\frac{(\alpha\beta + \Lambda R_1(\alpha + R_1))R_2^2(\alpha + R_2)}{(\alpha\beta + \Lambda R_2(\alpha + R_2))R_1^2(\alpha + R_1)}\right)$$

and $F(0) > 0$ if

$$(\alpha\beta + \Lambda R_1(\alpha + R_1))R_2^2(\alpha + R_2) - (\alpha\beta + \Lambda R_2(\alpha + R_2))R_1^2(\alpha + R_1) =$$
$$\frac{\alpha\beta\sqrt{\frac{\beta^2+2\beta(\alpha c+\lambda)+(\lambda-\alpha c)^2}{c^2}}(\alpha\lambda(c + \Lambda) - (\beta + \lambda)^2)}{c^2} > 0,$$

which holds under the condition $\alpha\lambda(c + \Lambda) - (\beta + \lambda)^2 > 0$ of the lemma. Finally, the uniqueness of x_0 follows by verifying that $F''(x) > 0$ and hence convexity of $F(x)$, which can be done by some algebraic manipulations.
For $x \geq x_0$, the concavity of $V(x)$ is obvious. For $x < x_0$ the first and second derivatives are

$$\begin{aligned}
V'(x) &= \frac{e^{R_1 x}\, R_1(\alpha + R_1)(\alpha\beta + e^{R_2 x_0}\, \Lambda R_2(\alpha + R_2) - e^{R_2 x}\, R_2(\alpha + R_2)(\alpha\beta + e^{R_1 x_0}\, \Lambda R_1(\alpha + R_1))}{\alpha\beta(e^{R_1 x_0}\, R_1(\alpha + R_1) - e^{R_2 x_0}\, R_2(\alpha + R_2))}, \\
V''(x) &= \frac{e^{R_1 x}\, R_1^2(\alpha + R_1)(\alpha\beta + e^{R_2 x_0}\, \Lambda R_2(\alpha + R_2) - e^{R_2 x}\, R_2^2(\alpha + R_2)(\alpha\beta + e^{R_1 x_0}\, \Lambda R_1(\alpha + R_1))}{\alpha\beta(e^{R_1 x_0}\, R_1(\alpha + R_1) - e^{R_2 x_0}\, R_2(\alpha + R_2))}
\end{aligned}$$

and $V'(x) > 0$ follows. Also, the definition of x_0 implies $V''(x) < 0$ for $x < x_0$ and, as mentioned before, we get that $V''(x) > 0$ for $x > x_0$. Therefore x_0 is the only root of $V''(x)$. □

Proposition 11. *For $Exp(\alpha)$ distributed claim amounts a barrier strategy characterized by x_0 is optimal among all admissible strategies and the function $V^*(x)$ defined in Lemma 10 is the value function $V(x)$.*

Proof. Because under the assumptions of the proposition a unique solution of the HJB equation (36) exists, the result can be proved in exactly the same way as in Proposition 5. □

Figure 7 shows $V(x)$ as a function of initial capital x for several values of Λ, whereas Figure 8 depicts the optimal barrier level as a function of Λ for the same set of parameters underlying Figures 5 and Figure 6. Finally, Figure 9 shows the corresponding expected discounted dividends for three values of Λ (where $\Lambda = 0$ again refers to the case of pure dividend maximization). A formula for the expected ruin time under a horizontal barrier strategy and exponential claim amounts can for instance be found in Lin et al. [17] and an application of that formula for the respective values of $x_0(\Lambda)$ gives the functions depicted in Figure 10. A comparison of Figures 9 and 10 reveals that for the used set of parameters the increase of the barrier x_0 due to a positive value of Λ leads to a much larger expected ruin time whereas the dividend reduction is moderate.

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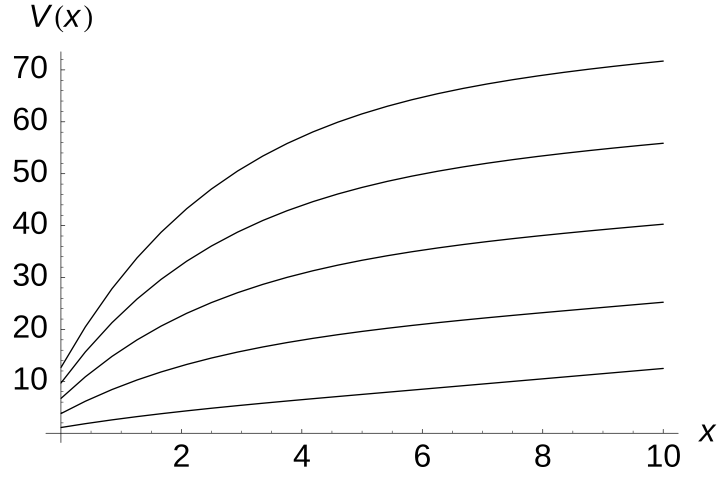

Figure 7: Value function for $\Lambda = 0,\ 0.5,\ 1,\ 1.5,\ 2$

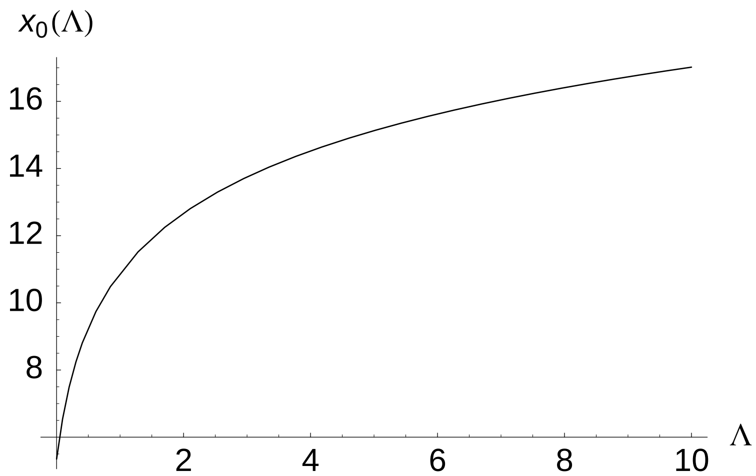

Figure 8: Barrier as function of Λ

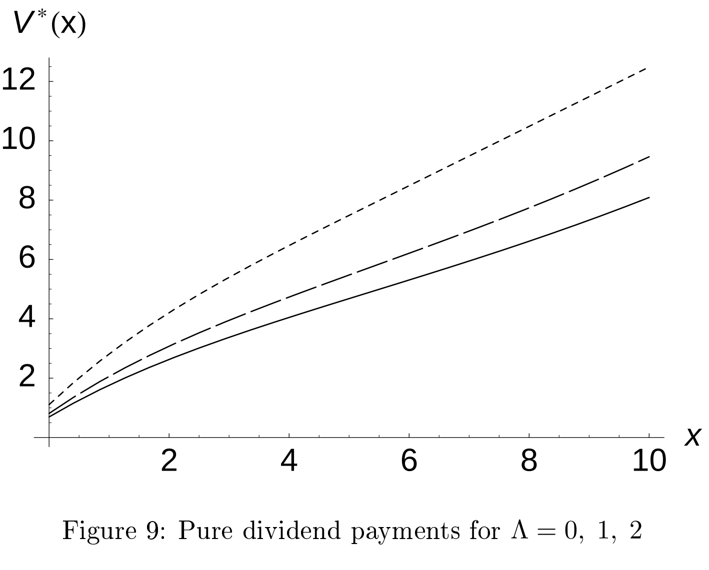

Figure 9: Pure dividend payments for $\Lambda = 0,\ 1,\ 2$

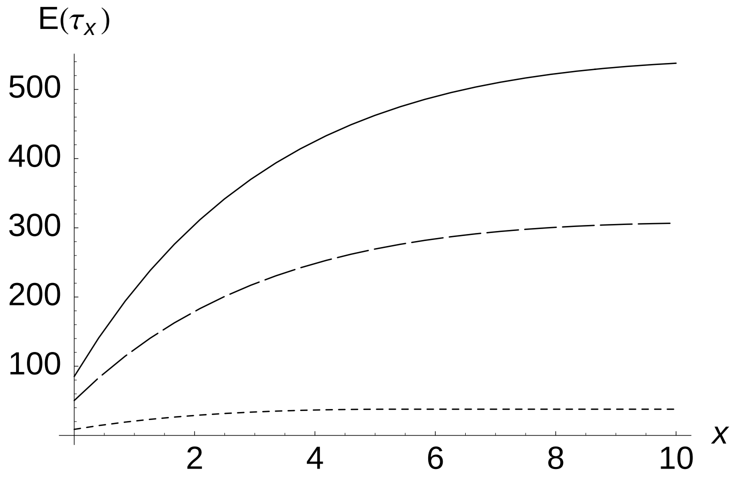

Figure 10: Expected time of ruin for $\Lambda = 0,\ 1,\ 2$

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