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Abstract

Many real phenomena may be modelled as random closed sets in $\mathbb{R}^d$, of different Hausdorff dimensions. The authors have recently revisited the concept of *mean geometric densities* of random closed sets Θ_n with Hausdorff dimension $n \leq d$ with respect to the standard Lebesgue measure on $\mathbb{R}^d$, in terms of expected values of a suitable class of linear functionals (Delta functions *à la Dirac*). In many real applications such as fiber processes, n-facets of random tessellations of dimension $n \leq d$ in spaces of dimension $d \geq 1$, several problems are related to the estimation of such mean densities; in order to face such problems in the general setting of spatially inhomogeneous processes, we suggest and analyze here an approximation of mean densities for sufficiently regular random closed sets. We will show how some known results in literature follow as particular cases. A series of examples throughout the paper are provided to exemplify various relevant situations.

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1 Introduction

Many real phenomena may be modelled as random closed sets in $\mathbb{R}^d$ of different Hausdorff dimensions (see for example [4, 6, 19, 21, 22, 23]).
We remind that a *random closed set* Ξ in $\mathbb{R}^d$ is a measurable map

$$\Xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $\mathbb{F}$ denotes the class of the closed subsets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the so called *hit-or-miss topology* (see [17]).
Let $\Xi = \Theta_n$ be almost surely a set of locally finite Hausdorff n-dimensional measure, and denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure on $\mathbb{R}^d$. The set Θ_n induces a random measure μ_{Θ_n} defined by

$$\mu_{\Theta_n}(A) := \mathcal{H}^n(\Theta_n \cap A), \qquad A \in \mathcal{B}_{\mathbb{R}^d}$$

(for a discussion of the delicate issue of measurability of the random variables $\mathcal{H}^n(\Theta_n \cap A)$, we refer to [3, 16, 24]).

Under suitable regularity assumptions on a random closed set $\Theta_n \subset \mathbb{R}^d$ with Hausdorff dimension $n \leq d$, in [10] the concept of *mean geometric density*, i.e. the mean density of the expected measure

$$\mathbb{E}[\mu_{\Theta_n}](A) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)], \qquad A \in \mathcal{B}_{\mathbb{R}^d}$$

with respect to the standard Lebesgue measure ν^d on $\mathbb{R}^d$, has been revisited in terms of expected values of a suitable class of linear functionals (Delta functions *à la Dirac*)[10] .

It is clear that, if $n < d$ and $\mu_{\Theta_n(\omega)}$ is a Radon measure for almost every $\omega \in \Omega$, then it is singular with respect to the d-dimensional Lebesgue measure ν^d. On the other hand, in dependence of the probability law of Θ_n, the expected measure may be either singular or absolutely continuous with respect to ν^d.
Thus, it is of interest to distinguish between random closed sets which induce a singular expected measure, and random closed sets which induce an absolutely continuous one; in this latter case we say that a random closed set is *absolutely continuous in mean*, and its mean density is the classical Radon-Nikodym derivative of $\mathbb{E}[\mu_{\Theta_n}]$ with respect to ν^d.

In many real applications such as fiber processes, n-facets of random tessellations of dimension $n \leq d$ in spaces of dimension $d \geq 1$, several problems are related to the estimation of their mean densities (see e.g. [4, 19, 23]).
In order to face such problems in the general setting of spatially inhomogeneous processes, we suggest and analyze here an approximation of mean densities for sufficiently regular random closed sets Θ_n in $\mathbb{R}^d$. We will show how some known results in literature follow as particular cases.

Since points and lines are ν^2-negligible, it is natural to make use of a 2-D box approximation of points of them. As a matter of fact, a computer graphics representation of them is anyway provided in terms of pixels, which can only offer a 2-D box approximation of points in $\mathbb{R}^2$.
This is the motivation of this paper, which tends to suggest unbiased estimators for densities of random sets of lower dimensions in a given d-dimensional space, by means of their approximation in terms of their d-dimensional enlargement by Minkowski addition.

A series of examples throughout the paper are provided to exemplify various relevant situations.

2 Preliminaries and notations

In this section we collect some basic facts and terminology that will be useful in the sequel.

We will call *Radon measure* in $\mathbb{R}^d$ any nonnegative and σ-additive set function μ defined on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ which is finite on bounded sets.
We know that every Radon measure μ on $\mathbb{R}^d$ can be represented in the form

$$\mu = \mu_{\ll} + \mu_{\perp},$$

where $\mu_{\ll}$ and $\mu_{\perp}$ are the absolutely continuous part of μ with respect to ν^d, and the singular part of μ, respectively. As a consequence of the Besicovitch Derivation Theorem (see [1], p.54), we have that the limit

$$\delta_\mu(x) := \lim_{r\to 0} \frac{\mu(B_r(x))}{\nu^d(B_r(x))}$$

exists in $\mathbb{R}$ for ν^d-a.e. $x \in \mathbb{R}^d$, and it is a version of the Radon-Nikodym derivative of $\mu_{\ll}$, while $\mu_{\perp}$ is the restriction of μ to the ν^d-negligible set $\{x \in \mathbb{R}^d : \delta_\mu(x) = \infty\}$.

According to Riesz theorem, Radon measures can be canonically identified with linear and order preserving functionals on $C_c(\mathbb{R}^d)$, the space of continuous functions with compact support in $\mathbb{R}^d$. The identification is provided by the integral operator, i.e.

$$(\mu, f) := \int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x) \qquad \forall f \in C_c(\mathbb{R}^d).$$

If $\mu \ll \nu^d$, it admits, as Radon-Nikodym density, a classical function δ_μ defined almost everywhere in $\mathbb{R}^d$, so that

$$(\mu, f) = \int_{\mathbb{R}^d} f(x)\delta_\mu(x)\mathrm{d}x \qquad \forall f \in C_c(\mathbb{R}^d)$$

in the usual sense of Lebesgue integral.

If $\mu \perp \nu^d$, we may speak of a density δ_μ only in the sense of distributions (it is almost everywhere trivial, but it is ∞ on a set of ν^d-measure zero), according to the duality beween measures and smooth (or $C_c(\mathbb{R}^d)$) functions. In this case the symbol

$$\int_{\mathbb{R}^d} f(x)\delta_\mu(x)\mathrm{d}x := (\mu, f)$$

can still be adopted, provided the integral on the left hand side is understood in a generalized sense, and not as a Lebesgue integral.

In either cases, from now on, we will denote by (δ_μ, f) the quantity (μ, f).

Accordingly, we say that a sequence of measures μ_n *weakly* converges* to a Radon measure μ if (δ_{μ_n}, f) converges to (δ_μ, f) for any $f \in C_c(\mathbb{R}^d)$. A classical criterion (see for instance [13] or [1]) states that μ_n weakly* converge to μ if and only if $\mu_n(A) \to \mu(A)$ for any bounded open set A with $\mu(\partial A) = 0$.

Given an integer $n \leq d$, we say that a set $C \subset \mathbb{R}^d$ is *countably* $\mathcal{H}^n$-rectifiable if there exist countably many Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\left(C \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n)\right) = 0.$$

Rectifiable sets include piecewise C^1 sets, and still have nice properties from the measure-theoretic viewpoint (for instance, one can define a n-dimensional tangent space to them, in an approximate sense): we refer to [1] for the basic properties of this class of sets.

The n-dimensional *Minkowski content* of a closed set $S \subset \mathbb{R}^d$ is defined by

$$\lim_{r\to 0} \frac{\nu^d(S_{\oplus r})}{b_{d-n} r^{d-n}}$$

whenever the limit exists. Here $S_{\oplus r}$ is the closed r-neighborhood of S, i.e.

$$S_{\oplus r} := \left\{x \in \mathbb{R}^d : \exists y \in S \text{ with } |x-y| \le r\right\},$$

known also as Minkowski addition of S with the closed ball $B_r(0)$.
We quote the following result from [1], p.110:

Theorem 1 *Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that for all $x \in S$*

$$\eta(B_r(x)) \ge \gamma r^n \quad \forall r \in (0,1) \tag{1}$$

holds for some $\gamma > 0$ and some Radon measure η in $\mathbb{R}^d$ absolutely continuous with respect to $\mathcal{H}^n$. Then

$$\lim_{r\to 0} \frac{\nu^d(S_{\oplus r})}{b_{d-n} r^{d-n}} = \mathcal{H}^n(S).$$

3 Generalized densities

In the sequel we will consider a class of sufficiently regular random closed sets in the Euclidean space $\mathbb{R}^d$, of integer dimension $n \le d$. We start with the deterministic case.

Definition 2 (n-regular sets) *Given an integer $n \in [0,d]$, we say that a closed subset S of $\mathbb{R}^d$ is* n-regular*, if it satisfies the following conditions:*

(i) $\mathcal{H}^n(S \cap B_R(0)) < \infty$ *for any* $R > 0$*;*

(ii) $\lim_{r\to 0} \frac{\mathcal{H}^n(S \cap B_r(x))}{b_n r^n} = 1$ *for* $\mathcal{H}^n$*-a.e.* $x \in S$*.*

Here b_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 3 *Note that condition (ii) is related to a characterization of the countable $\mathcal{H}^n$-rectifiability of the set S ([14], p.256, 267, [1], p.83).*

To any n-regular closed subset $S \subset \mathbb{R}^d$ we associate the Radon measure

$$\mu_{S_n}(B) := \mathcal{H}^n(S_n \cap B) \qquad B \in \mathbb{B}_{\mathbb{R}^d}.$$

Under the regularity assumption on S, we have

$$\lim_{r\to 0} \frac{\mu_{S_n}(B_r(x))}{b_n r^n} = \begin{cases} 1 & \mathcal{H}^n\text{-a.e. } x \in S_n, \\ 0 & \forall x \notin S_n. \end{cases}$$

As a consequence (by assuming $0 \cdot \infty = 0$), for $0 \le n < d$ we have

$$\lim_{r\to 0} \frac{\mu_{S_n}(B_r(x))}{b_d r^d} = \lim_{r\to 0} \frac{\mathcal{H}^n(S_n \cap B_r(x))}{b_n r^n} \frac{b_n r^n}{b_d r^d} = \begin{cases} \infty & \mathcal{H}^n\text{-a.e. } x \in S_n, \\ 0 & \forall x \notin S_n. \end{cases}$$

Therefore, setting

$$\delta_{S_n}(x) := \lim_{r\to 0} \frac{\mathcal{H}^n(S_n \cap B_r(x))}{b_d r^d},$$

we have that this density takes only the values 0 and ∞, and so provides almost no information of practical use on S_n, or even on μ_{S_n}. This is not the case for some natural approximations at the scale r of δ_{S_n}, defined below.

Definition 4 (Density at the scale r) *Let $r > 0$ and let S_n be n-regular. We set*

$$\delta_{S_n}^{(r)}(x) := \frac{\mu_{S_n}(B_r(x))}{b_d r^d} = \frac{\mathcal{H}^n(S_n \cap B_r(x))}{b_d r^d}$$

and, correspondingly, the associated measures $\mu_{S_n}^{(r)} = \delta_{S_n}^{(r)} \nu^d$:

$$\mu_{S_n}^{(r)}(B) := \int_B \delta_{S_n}^{(r)}(x)\mathrm{d}x, \qquad B \in \mathcal{B}_{\mathbb{R}^d}.$$

Identifying, as usual, measures with linear functionals on $C_c(\mathbb{R}^d)$, according to the notations introduced in the previous section, we may consider the linear functionals associated with the measures $\mu_{S_n}^{(r)}$ and μ_{S_n} as follows:

$$(\delta_{S_n}^{(r)}, f) := \int_{\mathbb{R}^d} f(x)\mu_{S_n}^{(r)}(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\delta_{S_n}^{(r)}(x)\mathrm{d}x, \quad (\delta_{S_n}, f) := \int_{\mathbb{R}^d} f(x)\mu_{S_n}(\mathrm{d}x),$$

for any $f \in C_c(\mathbb{R})$.
It can be proved (see for instance [10]) that the measures $\mu_{S_n}^{(r)}$ weakly* converge to the measure μ_{S_n} as $r \to 0$; this convergence result can also be understood noticing that $\delta_{S_n}^{(r)}(x)$ is the convolution of the measure μ_{S_n} with the kernels (here $\mathbf{1}_E$ stands for the characteristic function of E)

$$\rho_r(y) := \frac{1}{b_d r^d}\mathbf{1}_{B_r(0)}(y).$$

In the case $n = d$, δ_{S_d} is a classical function, and we will also use the following well known fact:

$$\delta_{S_d}^{(r)}(x) \to \delta_{S_d}(x) \text{ as } r \to 0 \text{ for } \nu^d\text{-a.e. } x \in \mathbb{R}^d. \tag{2}$$

Remark 5 *By definition, $\delta_{S_n} = \lim_{r\to 0} \delta_{S_n}^{(r)}$ and it can be interpreted as the* generalized density *(or the* generalized Radon-Nikodym derivative*) of the measure μ_{S_n} with respect to the d-dimensional Lebesgue measure ν^d, so that, with the adopted formal integral notations,*

$$\int_{\mathbb{R}^d} f(x)\delta_{S_n}(x)\mathrm{d}x := (\delta_{S_n}, f);$$

by the weak convergence of $\delta_{S_n}^{(r)}$ to δ_{S_n}, we have the formal exchange between limit and integral*

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(x)\delta_{S_n}^{(r)}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\delta_{S_n}(x)\,\mathrm{d}x.$$

We consider now random closed sets.

Definition 6 (Random n-regular sets) *Given an integer n, with $0 \le n \le d$, we say that a random closed set Θ_n in $\mathbb{R}^d$ is* n-regular*, if it satisfies the following conditions:*

(i) for almost all $\omega \in \Omega$, $\Theta_n(\omega)$ is an n-regular closed set in $\mathbb{R}^d$;

(ii) $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_R(0))] < \infty$ *for any* $R > 0$.

If Θ_n is a random n-regular closed set in $\mathbb{R}^d$, by condition (ii) the random measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot)$$

is almost surely a Radon measure, and now $\delta_{\Theta_n}(x)$ is a random distribution; by the equivalence between measures and linear functionals on $C_c(\mathbb{R}^d)$, μ_{Θ_n} can be viewed as a *random linear functional* (i.e. (δ_{Θ_n}, f) is a real random variable for any test function $f \in C_c(\mathbb{R}^d)$).

Definition 7 (Expected linear functionals and measures) *By extending the definition of expected value of a random operator* à la Pettis *(or Gelfand-Pettis, [2, 5]), we may define the* expected linear functional $\mathbb{E}[\delta_{\Theta_n}]$ *associated with* δ_{Θ_n} *as follows:*

$$(\mathbb{E}[\delta_{\Theta_n}], f) := \mathbb{E}[(\delta_{\Theta_n}, f)], \quad f \in C_c(\mathbb{R}^d).$$

By the Riesz duality between continuous functions and Radon measures, the linear functional $\mathbb{E}[\delta_{\Theta_n}]$ *corresponds to a Radon measure, that we denote by* $\mathbb{E}[\mu_{\Theta_n}]$*, and call* expected measure*. It satisfies*

$$\mathbb{E}[(\delta_{\Theta_n}, f)] = \int_{\mathbb{R}^d} f \mathbb{E}[\mu_{\Theta_n}](\mathrm{d}x) \qquad \forall f \in C_c(\mathbb{R}^d). \tag{3}$$

Remark 8 *By approximating characteristic functions of bounded open sets by* C_c *functions, from* (3)*, we get*

$$\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] = \mathbb{E}[\mu_{\Theta_n}](A)$$

for any bounded open set A*. A simple application of Dynkin's lemma then gives that the identity above holds for all bounded Borel sets* A*, and provides an alternative possible definition of the expected measure.*

For any lower dimensional random n-regular closed set Θ_n in $\mathbb{R}^d$, while it is clear that $\mu_{\Theta_n(\omega)}$ is a singular measure, it may well happen (e.g. when the process Θ_n is stationary) that the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is absolutely continuous with respect to ν^d, and so the Radon-Nikodym theorem ensures the existence of a density of this measure with respect to ν^d. In this case it is interesting to try to find *explicit* formulas for the computation of the "mean density" $\lambda_{\Theta_n} := \mathbb{E}[\delta_{\Theta_n}]$.

Definition 9 (Absolutely continuous processes and mean densities) *Let* Θ_n *be a random* n*-regular closed set in* $\mathbb{R}^d$*. We say that* Θ_n *is* absolutely continuous in mean *if the expected measure* $\mathbb{E}[\mu_{\Theta_n}]$ *is absolutely continuous with respect to* ν^d*. In this case we call mean density of* Θ_n*, and denote by* λ_{Θ_n}*, the Radon-Nikodym derivative of* $\mathbb{E}[\mu_{\Theta_n}]$ *with respect to* ν^d*.*

It is easy to check that the definition above is consistent with the case when Θ is a real random variable or a random point in $\mathbb{R}^d$, corresponding to $n = 0$:

Remark 10 (The 0-dimensional and d-dimensional cases) *If $n = 0$ and $\Theta_0(\omega) = \{X(\omega)\}$ is a random point, then $\mathbb{E}[\mathcal{H}^0(\Theta_0 \cap \cdot\,)] = \mathbb{P}(X \in \cdot\,)$. Therefore Θ_0 is absolutely continuous in mean if and only if the law of X is absolutely continuous, and λ_{Θ_0} coincides with the pdf of X.*

On the other hand, any random d-regular set is absolutely continuous: it suffices to apply Fubini's theorem (in $\Omega \times \mathbb{R}^d$, with the product measure $\mathbb{P} \times \nu^d$) to obtain

$$\mathbb{E}[\mu_{\Theta_d}(B)] = \mathbb{E}\left[\int_B \delta_{\Theta_d}(x)\mathrm{d}x\right] = \int_B \mathbb{E}[\delta_{\Theta_d}(x)]\mathrm{d}x = \int_B \mathbb{P}(x \in \Theta_d)\mathrm{d}x \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

Therefore

$$\lambda_{\Theta_d}(x) = \mathbb{E}[\delta_{\Theta_d}(x)] = \mathbb{P}(x \in \Theta_d) \qquad \text{for } \nu^d\text{-a.e. } x \in \mathbb{R}^d. \tag{4}$$

Remark 11 *We have seen that any random d-dimensional closed set is absolutely continuous in mean. However, a stronger definition of absolute continuity in mean of a random closed set has been given in [9], in terms of the expected measure of its boundary.*

We can now provide approximations of mean densities at the scale r, as in the deterministic case. We define

$$\mathbb{E}[\delta^{(r)}_{\Theta_n}](x) := \frac{\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]}{b_d r^d}, \tag{5}$$

and, using Fubini's theorem, we find:

$$\int_{\mathbb{R}^d} \mathbb{E}[\delta^{(r)}_{\Theta_n}](x) f(x)\ \mathrm{d}x = \mathbb{E}\left[\int_{\mathbb{R}^d} \delta^{(r)}_{\Theta_n}(x) f(x)\mathrm{d}x\right] \qquad \forall f \in C_c(\mathbb{R}^d). \tag{6}$$

Now, the deterministic case tells us that for all $f \in C_c(\mathbb{R}^d)$ we have $(\delta^{(r)}_{\Theta_n}, f) \to (\delta_{\Theta_n}, f)$ as $r \to 0$ almost surely (whenever μ_{Θ_n} is a Radon measure), and this convergence is easily seen to be dominated, by condition (ii) in Definition 6. Therefore (6) gives

$$\lim_{r\to 0}(\mathbb{E}[\delta^{(r)}_{\Theta_n}], f) = \lim_{r\to 0}\mathbb{E}[(\delta^{(r)}_{\Theta_n}, f)] = \mathbb{E}[\lim_{r\to 0}(\delta^{(r)}_{\Theta_n}, f)] = \mathbb{E}[(\delta_{\Theta_n}, f)] = (\mathbb{E}[\delta_{\Theta_n}], f)$$

for any $f \in C_c(\mathbb{R}^d)$. Therefore $\mathbb{E}[\delta^{(r)}_{\Theta_n}]\nu^d$ weakly* converge to $\mathbb{E}[\mu_{\Theta_n}]$ as $r \to 0$.

Remark 12 (The d-dimensional case) *In the d-dimensional case a pointwise convergence result holds, namely $\mathbb{E}[\delta^{(r)}_{\Theta_d}]$ converge to λ_{Θ_d} ν^d-a.e. (and in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, by dominated convergence). Indeed, by (2), we have*

$$\delta^{(r)}_{\Theta_d(\omega)}(x) \to \delta_{\Theta_d(\omega)}(x) \text{ for } \nu^d\text{-a.e. } x \in \mathbb{R}^d$$

for any ω (because $\mu_{\Theta_d(\omega)}$ is a Radon measure). By Fubini's theorem, we can find a ν^d-negligible set $N \subset \mathbb{R}^d$ such that

$$\delta^{(r)}_{\Theta_d(\omega)}(x) \to \delta_{\Theta_d(\omega)}(x) \text{ for } \mathbb{P}\text{-almost every } \omega$$

for all $x \in \mathbb{R}^d \setminus N$. As these functions are less than 1, we can take expectations in both sides and use (4) to obtain the convergence of $\mathbb{E}[\delta^{(r)}_{\Theta_d}](x)$ to $\lambda_{\Theta_d}(x)$ for all $x \in \mathbb{R}^d \setminus N$.

4 Approximation of mean densities

Let us notice that if $n = 0$, so that $\Theta_0 = \{X\}$ is a random point, then $B_r(X)$ is the "enlargement" of the random point Θ_0 by Minkowski addition, so that

$$\frac{\mathcal{H}^0(\Theta_0 \cap B_r(x))}{b_d r^d} = \frac{\mathbf{1}_{B_r(x)}(X)}{b_d r^d} = \frac{\mathbf{1}_{x_{\oplus r}}(X)}{b_d r^d} = \frac{\mathbf{1}_{\Theta_{0_{\oplus r}}}(x)}{b_d r^d}.$$

In the case $d = 1$ we have in particular that

$$\mathbb{E}[\delta^{(r)}_{\Theta_0}](x) = \frac{\mathbb{P}(X \in [x - r, x + r])}{2r}; \tag{7}$$

if X is a random variable with absolutely continuous law and pdf p_X, we have (see Remark 10) that $\mathbb{E}[\delta_{\Theta_0}] = p_X$, so that (7) leads to the usual histogram estimation of probability densities (see [20], § VII.13).
Given an n-regular random closed set Θ_n, even if a natural sequence of approximating functions of the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is given by $\mathbb{E}[\delta^{(r)}_{\Theta_n}]$ defined by (5), problems might arise in the estimation of $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap B_r(x))]$, as the computation of the Hausdorff measure is typically non-trivial even in the deterministic case. Therefore we are led to consider a new approximation, based on the Lebesgue measure (much more robust and computable) of the enlargement of the random set. This procedure is obviously consistent with (7). A crucial result is given in the following proposition.

Proposition 13 *Let Θ_n be a random n-regular set, and let $A \in \mathcal{B}$. If*

$$\lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta_{n_{\oplus r}} \cap A)]}{b_{d-n} r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)], \tag{8}$$

then

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)]. \tag{9}$$

Proof. For a random closed set Ξ in $\mathbb{R}^d$, Fubini's theorem gives

$$\mathbb{E}[\nu^d(\Xi \cap A)] = \int_A \mathbb{P}(x \in \Xi)\mathrm{d}x.$$

Therefore, the following chain of equalities holds:

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x = \lim_{r\to 0} \frac{\mathbb{E}[\nu^d(\Theta_{n_{\oplus r}} \cap A)]}{b_{d-n} r^{d-n}} \stackrel{(8)}{=} \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)].$$

□

Motivated by the previous proposition, we define

$$\delta_n^{\oplus r}(x) := \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}}$$

and, accordingly, the absolutely continuous Radon measures $\mu^{\oplus r} = \delta_n^{\oplus r} \nu^d$, i.e.

$$\mu^{\oplus r}(B) := \int_B \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x \qquad \forall B \in \mathcal{B}_{\mathbb{R}^d}.$$

We may notice that

$$\mathbb{P}(x \in \Theta_{n_{\oplus r}}) = \mathbb{P}(\Theta_n \cap B_r(x) \neq \emptyset) = T_{\Theta_n}(B_r(x)),$$

thus making explicit the reference to T_{Θ_n}, the capacity functional characterizing the probability law of the random set Θ_n [17].

Corollary 14 *Let Θ_n be a random n-regular set, and assume that* (8) *holds for any bounded open set A such that $\mathbb{E}[\mu_{\Theta_n}](\partial A) = 0$. Then the measures $\mu^{\oplus r}$ weakly* converge to the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ as $r \to 0$.*

Note that, if Θ_n is absolutely continuous, we have

$$\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] = \int_A \lambda_{\Theta_n}(x)\mathrm{d}x,$$

where λ_{Θ_n} is the density of the expected measure $\mathbb{E}[\mu_{\Theta_n}]$. So, in this case, we can rephrase (9) as

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x = \int_A \lambda_{\Theta_n}(x)\mathrm{d}x. \tag{10}$$

In particular, if Θ_n is a stationary random closed set, then $\delta_n^{\oplus r}(x)$ is independent of x and the expected measure $\mathbb{E}[\mu_{\Theta_n}]$ is motion invariant, i.e. $\lambda_{\Theta_n}(x) = L \in \mathbb{R}_+$ for ν^d-a.e. $x \in \mathbb{R}^d$. It follows that

$$\lim_{r\to 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \mathrm{d}x = \lim_{r\to 0} \frac{\mathbb{P}(x_0 \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} \nu^d(A)$$

for any $x_0 \in \mathbb{R}^d$, and so by (10) we infer

$$\lim_{r\to 0} \frac{\mathbb{P}(x_0 \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} = L \qquad \forall x_0 \in \mathbb{R}^d. \tag{11}$$

Recall that all these conclusions hold under the assumption, made in Proposition 13, that (8) holds. So, the main *problem* is to find conditions on Θ_n ensuring that this condition holds. If Θ_n is such that almost every realization $\Theta_n(\omega)$ has Minkowski content equal to the Hausdorff measure, i.e.

$$\lim_{r\to 0} \frac{\nu^d(\Theta_{n_{\oplus r}}(\omega))}{b_{d-n} r^{d-n}} = \mathcal{H}^n(\Theta_n(\omega)), \tag{12}$$

then it is clear that, taking the expected values on both sides, (8) is strictly related to the possibility of exchanging limit and expectation. So we ask whether (12) implies a similar result when we consider the intersection of $\Theta_{n_{\oplus r}}(\omega)$ with an open set A in $\mathbb{R}^d$, and for which kind of random closed sets the convergence above is dominated, so that exchanging limit and expectation is allowed.

The following result is a local version of Theorem 1.

Lemma 15 *Let S be a compact subset of $\mathbb{R}^d$ satisfying the hypotheses of Theorem 1. Then, for any $A \in \mathcal{B}_{\mathbb{R}^d}$ such that*

$$\mathcal{H}^n(S \cap \partial A) = 0, \tag{13}$$

the following holds

$$\lim_{r\to 0}\frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}}=\mathcal{H}^n(S\cap A). \tag{14}$$

Proof. If $n=d$, then equality (14) is easily verified. Thus, let $n<d$.
We may notice that, by the definition of rectifiability, if $C\subset\mathbb{R}^d$ is closed, then the compact set $S\cap C$ is still countably $\mathcal{H}^n$-rectifiable; besides (1) holds for all point $x\in S\cap C$ (since it holds for any point $x\in S$). As a consequence, by Theorem 1, we may claim that for any closed subset C of $\mathbb{R}^d$, the following holds

$$\lim_{r\to 0}\frac{\nu^d((S\cap C)_{\oplus r})}{b_{d-n}r^{d-n}}=\mathcal{H}^n(S\cap C). \tag{15}$$

Let A be as in the assumption.

- Let $\varepsilon>0$ be fixed. We may observe that the following holds:

$$S_{\oplus r}\cap A\subset(S\cap \mathrm{clos}A)_{\oplus r}\cup(S\cap \mathrm{clos}A_{\oplus\varepsilon}\setminus \mathrm{int}A)_{\oplus r}\qquad \forall r<\varepsilon.$$

Indeed, if $x\in S_{\oplus r}\cap A$ then there exists $y\in S$ with $|x-y|\leq r$, and $y\in \mathrm{clos}A_{\oplus\varepsilon}$. Then, if $x\notin(S\cap\mathrm{clos}A)_{\oplus r}$, we must have $y\in S\setminus\mathrm{clos}A$, hence $y\in S\cap\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{clos}\mathrm{A}$.

By (15), since $\mathrm{clos}A$ and $\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{int}A$ are closed, we have

$$\lim_{r\to 0}\frac{\nu^d((S\cap\mathrm{clos}A)_{\oplus r})}{b_{d-n}r^{d-n}}=\mathcal{H}^n(S\cap\mathrm{clos}A)\overset{(13)}{=}\mathcal{H}^n(S\cap A), \tag{16}$$

$$\lim_{r\to 0}\frac{\nu^d(S\cap\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{int}A)_{\oplus r}}{b_{d-n}r^{d-n}}=\mathcal{H}^n(S\cap\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{int}A). \tag{17}$$

Thus,

$$\begin{aligned}
&\limsup_{r\to 0}\frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}}\\
\leq\quad&\limsup_{r\to 0}\frac{\nu^d((S\cap\mathrm{clos}A)_{\oplus r}\cup(S\cap\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{int}A)_{\oplus r})}{b_{d-n}r^{d-n}}\\
\leq\quad&\limsup_{r\to 0}\frac{\nu^d((S\cap\mathrm{clos}A)_{\oplus r})+\nu^d((S\cap\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{int}A)_{\oplus r})}{b_{d-n}r^{d-n}}\\
\overset{(16),(17)}{=}\quad&\mathcal{H}^n(S\cap A)+\mathcal{H}^n(S\cap\mathrm{clos}A_{\oplus\varepsilon}\setminus\mathrm{int}A);
\end{aligned}$$

by taking the limit as ε tends to 0 we obtain

$$\limsup_{r\to 0}\frac{\nu^d(I_r(S)\cap A)}{b_{d-n}r^{d-n}}\leq\mathcal{H}^n(S\cap A)+\underbrace{\mathcal{H}^n(S\cap\partial A)}_{=0}=\mathcal{H}^n(S\cap A).$$

- Now, let B be a closed set well contained in A, i.e. $\mathrm{dist}(A,B)>0$. Then there exists $\tilde{r}>0$ such that $B_{\oplus r}\subset A$, $\forall r<\tilde{r}$. So,

$$\begin{aligned}
\mathcal{H}^n(S\cap B)\quad&\overset{(15)}{=}\quad\liminf_{r\to 0}\frac{\nu^d((S\cap B)_{\oplus r})}{b_{d-n}r^{d-n}}\\
&\leq\quad\liminf_{r\to 0}\frac{\nu^d(S_{\oplus r}\cap B_{\oplus r})}{b_{d-n}r^{d-n}}\\
&\leq\quad\liminf_{r\to 0}\frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}}.
\end{aligned}$$

Let us consider an increasing sequence of closed sets $\{B_n\}_{n\in\mathbb{N}}$ well contained in A such that $B_n \nearrow \text{int}A$. By taking the limit as n tends to ∞, we obtain that

$$\liminf_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \geq \lim_{n\to\infty} \mathcal{H}^n(S\cap B_n) = \mathcal{H}^n(S\cap \text{int}A) \stackrel{(13)}{=} \mathcal{H}^n(S\cap A).$$

We summarize,

$$\mathcal{H}^n(S\cap A) \leq \liminf_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \leq \limsup_{r\to 0} \frac{\nu^d(S_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \leq \mathcal{H}^n(S\cap A),$$

and so the thesis follows. $\square$

If we consider the sequence of random variables $\frac{\nu^d(\Theta_{n_{\oplus r}}\cap A)}{b_{d-n}r^{d-n}}$, for r going to 0, we ask which conditions have to be satisfied by a random set Θ_n, so that they are dominated by an integrable random variable. In this way we could apply the Dominated Convergence Theorem in order to exchange limit and expectation in (14).

Lemma 16 *Let K be a compact subset of $\mathbb{R}^d$ and assume that for all $x\in K$*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall r\in(0,1) \tag{18}$$

holds for some $\gamma>0$ and some probability measure η in $\mathbb{R}^d$.
Then, for all $r<2$,

$$\frac{\nu^d(K_{\oplus r})}{b_{d-n}r^{d-n}} \leq \frac{1}{\gamma}2^n 4^d \frac{b_d}{b_{d-n}}.$$

Proof. Since $K_{\oplus r}$ is compact, then it is possible to cover it with a finite number p of closed balls $B_{3r}(x_i)$, with $x_i\in K_{\oplus r}$, such that

$$|x_i - x_j| > 3r \qquad i\neq j. \tag{19}$$

(A first ball is taken off $K_{\oplus r}$, and so on until the empty set is obtained).
As a consequence, there exist $y_1,\dots y_p$ such that

- $y_i\in K$, $i=1,\dots,p$,
- $|y_i-y_j|>r$, $i\neq j$,
- $K_{\oplus r}\subseteq \bigcup_{i=1}^p B_{4r}(y_i)$.

In fact, if $x_i\in K$, then we choose $y_i=x_i$; if $x_i\in K_{\oplus r}\setminus K$, then we choose $y_i\in B_r(x_i)\cap K$. As a consequence, $|y_i-x_i|\leq r$ and $B_{4r}(y_i)\supseteq B_{3r}(x_i)$ for any $i=1,\dots,p$. So

$$\bigcup_{i=1}^p B_{4r}(y_i) \supseteq \bigcup_{i=1}^p B_{3r}(x_i) \supseteq K_{\oplus r},$$

and

$$3r \leq |x_i-x_j| \leq |x_i-y_i| + |y_i-y_j| + |y_j-x_j| \leq 2r + |y_i-y_j| \qquad i\neq j.$$

For $r < 2$, $B_{r/2}(y_i) \cap B_{r/2}(y_j) = \emptyset$. Since by hypothesis η is a probability measure satisfying (18), we have that

$$1 \geq \eta\left(\bigcup_{i=1}^{p} B_{r/2}(y_i)\right) = \sum_{i=1}^{p} \eta(B_{r/2}(y_i)) \overset{(18)}{\geq} p\gamma\left(\frac{r}{2}\right)^n,$$

and so

$$p \leq \frac{1}{\gamma}\frac{2^n}{r^n}.$$

In conclusion,

$$\frac{\nu^d(K_{\oplus r})}{b_{d-n}r^{d-n}} \leq \frac{\nu^d(\bigcup_{i=1}^{p} B_{4r}(y_i))}{b_{d-n}r^{d-n}} \leq \frac{pb_d(4r)^d}{b_{d-n}r^{d-n}} \leq \frac{1}{\gamma}2^n 4^d \frac{b_d}{b_{d-n}}.$$

□

In the following theorem we consider $n \in \{0, 1, \dots, d-1\}$, since the particular case $n = d$ is trivial.

Theorem 17 *Let Θ_n be a countably $\mathcal{H}^n$-rectifiable random closed set in $\mathbb{R}^d$ (i.e., for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\Theta_n(\omega) \subseteq \mathbb{R}^d$ is a countably $\mathcal{H}^n$-rectifiable closed set), such that $\mathbb{E}[\mu_{\Theta_n}]$ is a Radon measure. Let $W \subset \mathbb{R}^d$ be a compact set and let $\Gamma_W : \Omega \longrightarrow \mathbb{R}$ be the function so defined:*

$$\Gamma_W(\omega) := \max\big\{\gamma \geq 0 \,:\, \exists \text{ a probability measure } \eta \ll \mathcal{H}^n \text{ such that } \eta(B_r(x)) \geq \gamma r^n \quad \forall x \in \Theta_n(\omega) \cap W_{\oplus 1},\ r \in (0,1)\big\}.$$

If there exists a random variable Y with $\mathbb{E}[Y] < \infty$, such that $1/\Gamma_W(\omega) \leq Y(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, then, for all $A \in \mathcal{B}_{\mathbb{R}^d}$ such that

$$A \subset \operatorname{int} W_{\oplus 1} \quad \text{and} \quad \mathbb{E}[\mathcal{H}^n(\Theta_n \cap \partial A)] = 0, \tag{20}$$

we have

$$\lim_{r \to 0} \frac{\mathbb{E}[\nu^d(\Theta_{n_{\oplus r}} \cap A)]}{b_{d-n}r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)].$$

Proof. Since $\mathbb{E}[Y] < \infty$, then $Y(\omega) < \infty$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.
Let $A \in \mathcal{B}_{\mathbb{R}^d}$ be satisfying (20). Let us define

$\Omega_A := \{\omega \in \Omega \,:\, \mathcal{H}^n(\Theta_n(\omega) \cap \partial A) = 0\}$,

$\Omega_T := \{\omega \in \Omega \,:\, \Theta_n(\omega)$ is countably $\mathcal{H}^n$-rectifiable and closed$\}$,

$\Omega_Y := \{\omega \in \Omega \,:\, Y(\omega) < \infty\}$,

$\Omega_\Gamma := \{\omega \in \Omega \,:\, \frac{1}{\Gamma_W(\omega)} \leq Y(\omega)\}$;

by hypothesis $\mathbb{P}(\Omega_A) = \mathbb{P}(\Omega_T) = \mathbb{P}(\Omega_Y) = \mathbb{P}(\Omega_\Gamma) = 1$.
Thus, if $\Omega' := \Omega_A \cap \Omega_T \cap \Omega_Y \cap \Omega_\Gamma$, it follows that $\mathbb{P}(\Omega') = 1$.

Let $\omega \in \Omega'$ be fixed. Then

- $\Gamma_W(\omega) > 0$, i.e. a probability measure $\eta \ll \mathcal{H}^k$ exists such that

$$\eta(B_r(x)) \geq \Gamma_W(\omega) r^n \quad \forall x \in \Theta_n(\omega) \cap W_{\oplus 1},\ r \in (0,1),$$

- $\mathcal{H}^n(\Theta_n(\omega) \cap \partial A) = 0$, so, by applying Lemma 15 to $\Theta_n \cap W_{\oplus 1}$, we get

$$\lim_{r\to 0} \frac{\nu^d(\Theta_{n_{\oplus r}}(\omega)\cap A)}{b_{d-n}r^{d-n}} = \mathcal{H}^n(\Theta_n(\omega)\cap A);$$

i.e. we may claim that

$$\lim_{r\to 0} \frac{\nu^d(\Theta_{n_{\oplus r}}\cap A)}{b_{d-n}r^{d-n}} = \mathcal{H}^n(\Theta_n\cap A) \quad \text{almost surely.}$$

Further, for all $\omega \in \Omega'$, $\Theta_n(\omega)\cap W_{\oplus 1}$ satisfies the hypotheses of Lemma 16, and so

$$\begin{aligned} \frac{\nu^d(\Theta_{n_{\oplus r}}(\omega)\cap A)}{b_{d-n}r^{d-n}} &= \frac{\nu^d((\Theta_n(\omega)\cap W_{\oplus 1})_{\oplus r}\cap A)}{b_{d-n}r^{d-n}} \leq \frac{\nu^d((\Theta_n(\omega)\cap W_{\oplus 1})_{\oplus r})}{b_{d-n}r^{d-n}} \\ &\leq \frac{1}{\Gamma_W(\omega)}2^n 4^d \frac{b_d}{b_{d-n}} \leq Y(\omega)2^n4^d\frac{b_d}{b_{d-n}} \in \mathbb{R}. \end{aligned}$$

Let Z be the random variable so defined:

$$Z(\omega) := Y(\omega)2^n4^d\frac{b_d}{b_{d-n}}, \qquad \omega\in\Omega'.$$

By assumption $\mathbb{E}[Z]<\infty$, so that the Dominated Convergence Theorem gives

$$\lim_{r\to 0}\mathbb{E}\left[\frac{\nu^d(\Theta_{n_{\oplus r}}\cap A)}{b_{d-n}r^{d-n}}\right] = \mathbb{E}[\mathcal{H}^n(\Theta_n\cap A)].$$

□

Notice that in the statement of Theorem 17 we introduced the auxiliary function $Y(\omega)$ in order to avoid the non-trivial issue of the measurability of $\Gamma_W(\omega)$; as a matter of fact, in all examples, one can estimate $1/\Gamma_W(\omega)$ from above in a measurable way.

Notice also that if Θ_n satisfies the assumption of the theorem for some closed W, then it satisfies the assumption for all closed $W'\subset W$; analogously, any random n-regular closed set Θ'_n contained almost surely in Θ_n still satisfies the assumption of the theorem.

Remark 18 (The case $n=0$) *Let us consider the well-known and studied case of a random point $\{X\}$ in $\mathbb{R}^d$, and see how it is consistent with our framework (of course, the special cases $n=0$ and $n=d$ can be handled with much more elementary tools). Thus, we are in the particular case in which $n=0$ and $\Theta_0(\omega)=\{X(\omega)\}$.*
First of all it is immediate to check that X satisfies the hypotheses of Theorem 17 with $\eta := \mathcal{H}^0(\Theta_0(\omega)\cap\cdot)$ (i.e. the unit Dirac mass on $X(\omega)$) and $\Gamma(\omega)=1$ for all $\omega\in\Omega$. In particular, for any bounded Borel set A such that $\mathbb{E}[\mu_{\Theta_0}](\partial A)=0$ we have

$$\lim_{r\to 0}\frac{\mathbb{E}[\nu^d(\Theta_{0_{\oplus r}}\cap A)]}{b_d r^d} = \mathbb{E}[\mathcal{H}^0(\Theta_0\cap A)].$$

Theorem 19 (Main result) *Let Θ_n be a random n-regular closed set in $\mathbb{R}^d$ and let $\mathbb{E}[\mu_{\Theta_n}]$ be its expected measure. Assume that Θ_n satisfies the density lower bound assumption of Theorem 17 for any compact set $W \subset \mathbb{R}^d$. Then*

$$\lim_{r\to 0}\int_A \frac{\mathbb{P}(x\in\Theta_{n\oplus r})}{b_{d-n}r^{d-n}}\mathrm{d}x = \mathbb{E}[\mu_{\Theta_n}](A)$$

for any bounded Borel set $A \subset \mathbb{R}^d$ such that

$$\mathbb{E}[\mu_{\Theta_n}](\partial A) = 0. \tag{21}$$

In particular, if Θ_n is absolutely continuous in mean, we have

$$\lim_{r\to 0}\int_A \frac{\mathbb{P}(x\in\Theta_{n\oplus r})}{b_{d-n}r^{d-n}}\mathrm{d}x = \int_A \lambda_{\Theta_n}(x)\mathrm{d}x \tag{22}$$

for any bounded Borel set $A \subset \mathbb{R}^d$ with $\nu^d(\partial A) = 0$, where λ_{Θ_n} is the mean density of Θ_n. Finally, if Θ_n is stationary we have

$$\lim_{r\to 0}\frac{\mathbb{P}(x_0\in\Theta_{n\oplus r})}{b_{d-n}r^{d-n}} = \lambda_{\Theta_n} \qquad \forall x_0 \in \mathbb{R}^d. \tag{23}$$

Proof. The first statement follows by (9) in Proposition 13: indeed, the assumption (8) of that proposition is fulfilled, thanks to Theorem 17. The second statement is a direct consequence of the first one. Finally, in the stationary case (23) follows directly by (22), as explained after Corollary 14. □

Finally, notice that condition (21), when restricted to bounded open sets A, is "generically satisfied" in the following sense: given any family of bounded open sets $\{A_t\}_{t\in\mathbb{R}}$ with $\mathrm{clos}A_s \subseteq A_t$ for $s<t$, the set

$$T := \{t\in\mathbb{R}:\ \mathbb{E}[\mu_{\Theta_n}](\partial A_t) > 0\}$$

is at most countable. This is due to the fact that the sets $\{\partial A_t\}_{t\in T}$ are pairwise disjoint, and all with strictly positive $\mathbb{E}[\mu_{\Theta_n}]$-measure.

Remark 20 (Mean density as a pointwise limit) *It is tempting to try to exchange limit and integral in* (22)*, to obtain*

$$\lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta_{n\oplus r})}{b_{d-n}r^{d-n}} = \lambda_{\Theta_n}(x), \tag{24}$$

at least for ν^d-a.e. $x\in\mathbb{R}^d$. The proof of the validity of this formula for absolutely continuous (in mean) processes seems to be a quite delicate problem, with the only exception of stationary processes. However, in the extreme cases $n=d$ and $n=0$ it is not hard to prove it.

In the case $n=d$ we know from Remark 10 that $\lambda_{\Theta_d}(x) = \mathbb{P}(x\in\Theta_d)$ for ν^d-a.e. x, and obviously $\mathbb{P}(x\in\Theta_{d\oplus r})$ converges to $\mathbb{P}(x\in\Theta_d)$ for all x.
In the case $n=0$, let $\Theta_0=\{X\}$, with X absolutely continuous random point in $\mathbb{R}^d$ with pdf p_X, and notice that

$$\frac{\mathbb{P}(x\in\Theta_{0\oplus r})}{b_d r^d} = \frac{\mathbb{P}(X\in B_r(x))}{b_d r^d} = \frac{1}{b_d r^d}\int_{B_r(x)} p_X(y)\mathrm{d}y.$$

Therefore (24) *with $\lambda_{\Theta_0} = p_X$ holds at any Lebesgue point of p_X, and therefore for ν^d-a.e. $x\in\mathbb{R}^d$.*

5 Applications

In many real applications Θ_n is given by a random collection of geometrical objects, so that it may be described as the union of a family of n-regular random closed sets E_i in $\mathbb{R}^d$:

$$\Theta_n = \bigcup_i E_i. \tag{25}$$

Here we don't make any specific assumption regarding the stochastic dependence among the E_i. In fact, if Θ_n is known to be absolutely continuous in mean, a problem of interest is to determine its mean density λ_{Θ_n}, and Theorem 17 seems to require sufficient regularity of the E_i's, rather than stringent assumptions about their probability law. As a simple example, consider the case in which, for any i, E_i is a random segment in $\mathbb{R}^d$ such that the mean number of segments which hit a bounded region is finite; then we will show in the sequel that Θ_1 satisfies Proposition 13, without any other assumption on the probability law of the E_i's (e.g. the law of the point process associated with the centers of the segments).

Note also that geometric processes like segment-, line-, or surface- processes may be described by the so called *union set of a particle process* (see, for example, [4, 23]):

$$\Theta_n = \bigcup_{K \in \Psi} K,$$

where Ψ is a point process on the state space of $\mathcal{H}^n$-rectifiable closed sets. It is known that the stationarity of Θ_n depends on the stationarity of the point process Ψ.

Other random closed sets represented by unions, as in (25), are given by

$$\Theta_n = \bigcup_{i=1}^{\Phi} E_i, \tag{26}$$

where Φ is a positive integer valued random variable, representing the random number of geometrical objects E_i.
This kind of representation may be used to model a class of time dependent geometric processes, too. For example, at any fixed time $t \in \mathbb{R}_+$, let Θ^t be given by

$$\Theta^t = \bigcup_{i=1}^{\Phi_t} E_i,$$

where Φ_t is a counting process in $\mathbb{R}_+$; e.g., if Φ_t is a Poisson counting process with intensity λ, then at any time t the number of random objects E_i is given by a random variable distributed as $\mathrm{Po}(\lambda t)$; in this case the process $\{\Theta^t\}$ is additionally determined by a marked point process $\tilde{\Phi}$ in $\mathbb{R}_+$, with marks in a suitable space: the marginal process is given by the counting process Φ_t, while the marks are given by a family of random closed sets E_i.

In literature, many geometric processes like this are investigated, as point-, line-, segment-, or plane processes, random mosaics, grain processes,...
Note that Θ_n may be unbounded, given by an infinite union of random sets E_i

(e.g. as in (25) with E_i random line). In such a case, when we consider the restriction of Θ_n to a bounded window $W \subset \mathbb{R}^d$, by the usual assumption that the mean number of E_i's hitting a bounded region is finite, we may represent

$$\Theta_n \cap W = \bigcup_{i=1}^{\Phi} E_i^W,$$

with $E_i^W = E_i \cap W$, the union above being finite almost surely.

We give now some significant simple examples of random sets of this kind, to which the results of the previous sections apply.

Example 1. A class of random sets satisfying hypotheses of Theorem 17 is given by all sets Θ_n which are random union of random closed sets of dimension $n < d$ in $\mathbb{R}^d$ as in (26), such that

(i) $\mathbb{E}[\Phi] < \infty$,

(ii) $E_1, E_2, \ldots$ are IID as E and independent of Φ,

(iii) $\mathbb{E}[\mathcal{H}^n(E)] = C < \infty$ and $\exists \gamma > 0$ such that for any $\omega \in \Omega$,

$$\mathcal{H}^n(E(\omega) \cap B_r(x)) \geq \gamma r^n \qquad \forall x \in E(\omega),\ r \in (0,1). \tag{27}$$

We can choose $\eta(\cdot) := \dfrac{\mathcal{H}^n(\Theta_n(\omega) \cap \cdot)}{\mathcal{H}^n(\Theta_n(\omega))}$ for any fixed $\omega \in \Omega$. As a consequence, η is a probability measure absolutely continuous with respect to $\mathcal{H}^n$, and such that

$$\eta(B_r(x)) \geq \frac{\gamma}{\mathcal{H}^n(\Theta(\omega))} r^n \qquad \forall x \in \Theta_n(\omega),\ r \in (0,1).$$

In fact, if $x \in \Theta_n(\omega)$, then there exists an $\bar{\imath}$ such that $x \in E_{\bar{\imath}}(\omega)$; since $\Theta_n(\omega) = \bigcup_{i=1}^{\Phi(\omega)} E_i(\omega)$, we have

$$\eta(B_r(x)) = \frac{\mathcal{H}^n(\Theta_n(\omega) \cap B_r(x))}{\mathcal{H}^n(\Theta_n(\omega))} \geq \frac{\mathcal{H}^n(E_{\bar{\imath}}(\omega) \cap B_r(x))}{\mathcal{H}^n(\Theta_n(\omega))} \geq \frac{\gamma}{\mathcal{H}^n(\Theta(\omega))} r^n.$$

As a result, the function Γ defined as in Theorem 17 is such that

$$\frac{1}{\Gamma(\omega)} \leq \frac{\mathcal{H}^n(\Theta_n(\omega))}{\gamma} =: Y(\omega),$$

and so it remains to verify only that $\mathbb{E}[\mathcal{H}^n(\Theta_n)] < \infty$:

$$\begin{aligned}
\mathbb{E}[\mathcal{H}^n(\Theta_n)] &= \mathbb{E}[\mathbb{E}[\mathcal{H}^n(\Theta_n) \mid \Phi]] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[\mathcal{H}^n(\bigcup_{i=1}^{k} E_i) \mid \Phi = k] \mathbb{P}(\Phi = k) \\
&\overset{\text{(ii)}}{\leq} \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}[\mathcal{H}^n(E_i)] \mathbb{P}(\Phi = k) \\
&\overset{\text{(iii)}}{=} \sum_{k=1}^{\infty} Ck \mathbb{P}(\Phi = k) \\
&= C\mathbb{E}[\Phi] \overset{\text{(i)}}{<} \infty.
\end{aligned}$$

Note that we have not made any particular assumption on the probability laws of Φ and E. Further, it is clear that the same proof holds even in the case in which the E_i's are not IID, provided that $\mathbb{E}[\mathcal{H}^n(E_i)] \leq C$, $\forall i$, and (27) is true for any E_i (with γ independent of ω and i).

By keeping the general assumption that Φ is an integrable positive, integer valued random variable, we may write the probability that a point x belongs to the set $\Theta_{n_{\oplus r}}$ in terms of the mean number of E_i which intersect the ball $B_r(x)$. We prove the following proposition.

Proposition 21 *Let $n < d$, let Φ be a positive integer valued random variable with $\mathbb{E}[\Phi] < \infty$, and let $\{E_i\}$ be a collection of random closed sets with dimension n. Let Θ_n be the random closed set so defined:*

$$\Theta_n = \bigcup_{i=1}^{\Phi} E_i.$$

If $E_1, E_2, \ldots$ are IID as E and independent of Φ, then, for any $x \in \mathbb{R}^d$ such that $\mathbb{P}(x \in E) = 0$,

$$\lim_{r\to 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} = \lim_{r\to 0} \frac{\mathbb{E}[\#\{E_i \, : \, x \in \mathbb{E}_{i_{\oplus r}}\}]}{b_{d-n} r^{d-n}},$$

provided that at least one of the two limits exist.

Proof. The following chain of equalities holds:

$$\begin{aligned}
\mathbb{P}(x \in \Theta_{n_{\oplus r}}) &= \mathbb{P}(x \in \bigcup_{i=1}^{\Phi} E_{i_{\oplus r}}) \\
&= 1 - \mathbb{P}(x \notin \bigcup_{i=1}^{\Phi} E_{i_{\oplus r}}) \\
&= 1 - \mathbb{P}(\bigcap_{i=1}^{\Phi} \{x \notin E_{i_{\oplus r}}\}) \\
&= 1 - \sum_{k=1}^{\infty} \mathbb{P}(\bigcap_{i=1}^{k} \{x \notin E_{i_{\oplus r}}\}) \,|\, \Phi = k) \mathbb{P}(\Phi = k);
\end{aligned}$$

since the E_i's are IID and independent of Φ,

$$\begin{aligned}
&= 1 - \sum_{k=1}^{\infty} [\mathbb{P}(x \notin E_{\oplus r})]^k \mathbb{P}(\Phi = k) \\
&= 1 - \mathbb{E}[(\mathbb{P}(x \notin E_{\oplus r}))^{\Phi}] \\
&= 1 - \mathrm{G}(\mathbb{P}(x \notin E_{\oplus r})), \qquad (28)
\end{aligned}$$

where G is the probability generating function of the random variable Φ.

Now, let us observe that

$$\begin{aligned}
\mathbb{E}[\#\{E_i\,:\,x\in\mathbb{E}_{i_{\oplus r}}\}] &= \sum_{k=1}^{\infty}\mathbb{E}[\sum_{i=0}^{k}\mathbf{1}_{E_{i_{\oplus r}}}(x)\,|\,\Phi=k]\mathbb{P}(\Phi=k)\\
&= \sum_{k=1}^{\infty}k\mathbb{P}(x\in E_{\oplus r})\mathbb{P}(\Phi=k)\\
&= \mathbb{P}(x\in E_{\oplus r})\sum_{k=1}^{\infty}k\mathbb{P}(\Phi=k)\\
&= \mathbb{E}[\Phi]\mathbb{P}(x\in E_{\oplus r}). \qquad (29)
\end{aligned}$$

We remind that $\mathbb{E}[\Phi]=\mathrm{G}'(1)$ and $1=\mathrm{G}(1)$.
In order to simplify the notation, let $s(r):=\mathbb{P}(x\in E_{\oplus r})$. By hypothesis we know that $s(r)\to 0$ as $r\to 0$; thus by (28) and (29) we have

$$\begin{aligned}
\lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta_{n_{\oplus r}})}{\mathbb{E}[\#\{E_i\,:\,x\in E_{i_{\oplus r}}\}]} &= \lim_{r\to 0}\frac{\mathrm{G}(1)-\mathrm{G}(1-s(r))}{\mathrm{G}'(1)s(r)}\\
&= \frac{1}{\mathrm{G}'(1)}\lim_{r\to 0}\frac{\mathrm{G}(1-s(r))-\mathrm{G}(1)}{-s(r)}\\
&= \frac{1}{\mathrm{G}'(1)}\mathrm{G}'(1)=1. \qquad (30)
\end{aligned}$$

In conclusion we obtain

$$\begin{aligned}
\lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}} &= \lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta_{n_{\oplus r}})}{\mathbb{E}[\#\{E_i\,:\,x\in E_{i_{\oplus r}}\}]}\frac{\mathbb{E}[\#\{E_i\,:\,x\in E_{i_{\oplus r}}\}]}{b_{d-n}r^{d-n}}\\
&\overset{(30)}{=} \lim_{r\to 0}\frac{\mathbb{E}[\#\{E_i\,:\,x\in E_{i_{\oplus r}}\}]}{b_{d-n}r^{d-n}}. \qquad (31)
\end{aligned}$$

□

Corollary 22 *Under the same assumptions of Proposition 21,* (29) *and* (31) *yield*

$$\lim_{r\to 0}\frac{\mathbb{P}(x\in\Theta_{n_{\oplus r}})}{b_{d-n}r^{d-n}}=\mathbb{E}[\Phi]\lim_{r\to 0}\frac{\mathbb{P}(x\in E_{\oplus r})}{b_{d-n}r^{d-n}} \qquad (32)$$

for any $x\in\mathbb{R}^d$ where at least one of the two limits exists.

Remark 23 *1. Whenever it is possible to exchange limit and integral in (10), we can use the fact that A is arbitrary to obtain*

$$\lambda_{\Theta_n}(x)=\mathbb{E}[\Phi]\lim_{r\to 0}\frac{\mathbb{P}(x\in E_{\oplus r})}{b_{d-n}r^{d-n}}=\mathbb{E}[\Phi]\lambda_E(x),$$

for ν^d-a.e. $x\in R^d$, where λ_{Θ_n} and λ_E are the mean densities of μ_{Θ_n} and μ_E, respectively. In particular, when E is stationary (which implies Θ_n stationary as well), $\lambda_{\Theta_n}(x)\equiv L_{\Theta_n}\in\mathbb{R}^+$ and $\lambda_E(x)\equiv L_E\in\mathbb{R}^+$, so that

$$L_{\Theta_n}=\mathbb{E}[\Phi]\lim_{r\to 0}\frac{\mathbb{P}(x_0\in E_{\oplus r})}{b_{d-n}r^{d-n}}=\mathbb{E}[\Phi]L_E.$$

2. Let Θ_n be a random closed set as in Proposition 21. By (32) *we infer that the probability that a point x belongs to the intersection of two or more enlarged sets E_i is an infinitesimal faster than r^{d-n}. In fact, denoting by $F_r(x)$ this event and by $\chi_G : \Omega \to \{0,1\}$ the characteristic function of an event G, we have*

$$\chi_{F_r(x)} \leq \sum_i \chi_{\{x \in E_{i\oplus r}\}} - \chi_{\{x \in \Theta_{n\oplus r}\}},$$

so that, as (29) *gives*

$$\mathbb{E}[\sum_i \chi_{\{x \in E_{i\oplus r}\}}] = \mathbb{E}[\Phi]\mathbb{P}(x \in E_{\oplus r}),$$

taking expectations in both sides and dividing by r^{d-n} we get that $\mathbb{P}(F_r(x))/r^{d-n} \to 0$ as $r \to 0$.

Example 2. (Poisson line process.)
Now, we recall the definition of a Poisson line process, given in [23] as a simple example of applicability of the above arguments. We shall obtain the same result for the mean density of the random length measure.
Line processes are the simplest examples of fibre processes. Such random patterns can be treated directly as random sets; however, they can also be considered as point processes with constituent "points" lying not in the Euclidean space, but in the space of lines in the plane, which can be parameterized as a cylinder $\mathbf{C}^*$ in $\mathbb{R}^3$ (see [23], Ch. 8):

$$\mathbf{C}^* = \{(\cos\alpha,\ \sin\alpha,\ p)\ :\ p \in \mathbb{R},\ \alpha \in (0,\pi]\},$$

where p is the signed perpendicular distance of the line l from the origin 0 (the sign is positive if 0 lies to the left of l and negative if it lies to the right), and α is the angle between l and the x-axis, measured in an anti-clockwise direction.
"A *line process* is a random collection of lines in the plane which is locally finite, i.e. only finitely many lines hit each compact planar set. Formally it is defined as a random subset of the representation space $\mathbf{C}^*$. The process is locally finite exactly when the representing random subset is a random locally finite subset, hence a point process, on $\mathbf{C}^*$. Such point processes are particular cases of point processes on $\mathbb{R}^2$, because, as suggested by the parametrization (p, α), the cylinder can be cut and embedded as the subset $\mathbb{R} \times (0, 2\pi]$ of $\mathbb{R}^2$" ([23], p.248).
A line process $\Theta_1 = \{l_1, l_2, \ldots\}$, when regarded as a point process on $\mathbf{C}^*$, yields an *intensity measure* Λ on $\mathbf{C}^*$:

$$\Lambda(A) = \mathbb{E}[\#\{l :\ l \in \Theta_1 \cap A\}]$$

for each Borel subset A of $\mathbf{C}^*$.
A *Poisson line process* Ξ is the line process produced by a Poisson process on $\mathbf{C}^*$. Consequently it is characterized completely by its intensity measure Λ. Under the assumption of stationarity of Ξ, it follows that there exists a constant $L_\Xi > 0$ such that the intensity measure Λ of Ξ is given by

$$\Lambda(\mathrm{d}(p,\alpha)) = L_\Xi \cdot \mathrm{d}p \cdot \frac{\mathrm{d}\alpha}{2\pi};$$

besides, it is clear that the measure $\mathbb{E}[\mathcal{H}^1(\Xi\cap\cdot)]$ is motion invariant on $\mathbb{R}^2$, and so there exists a constant c such that

$$\mathbb{E}[\mathcal{H}^1(\Xi\cap A)] = c\nu^2(A)$$

for any $A\in\mathcal{B}_{\mathbb{R}^2}$.
Such a constant c can be calculated using the cylinder representation of Ξ; it is shown that $c = L_\Xi$ (see [23], p.249).

We show now that the same statement can be obtained as a consequence of (9).
As a matter of fact, by stationarity we know that

$$c\nu^2(A) = \lim_{r\to 0}\frac{\mathbb{P}(0\in\Xi_{\oplus r})}{2r}\nu^2(A) \tag{33}$$

holds for any Borel set A which satisfies condition (8), so that it is sufficient to prove that

$$\lim_{r\to 0}\frac{\mathbb{E}[\nu^2(\Xi_{\oplus r}\cap A)]}{2r} = \mathbb{E}[\mathcal{H}^1(\Xi\cap A)]$$

holds for a particular fixed A. Let us choose a closed square W in $\mathbb{R}^2$ with edges P_1, P_2, P_3, P_4, and side length h.
Note that $\Theta_1 := \Xi\cap W_{\oplus 1}$ is a countably $\mathcal{H}^1$-rectifiable and compact random set and, by the absolute continuity of the expected measure, $\mathbb{E}[\mathcal{H}^1(\Xi\cap\partial W] = 0$ (so that $\mathbb{P}(\mathcal{H}^1(\Xi\cap\partial W) > 0) = 0$).
For any $\omega\in\Omega$ let us define

$$\eta(\cdot) := \frac{\mathcal{H}^1(\Theta_1(\omega)\cap\cdot)}{\mathcal{H}^1(\Theta_1(\omega))}.$$

Then η is a probability measure absolutely continuous with respect to $\mathcal{H}^1$ such that

$$\eta(B_r(x)) \geq \frac{1}{\mathcal{H}^1(\Theta_1)}r \qquad \forall x\in\Theta_1(\omega),\ r\in(0,1),$$

and we may notice that $\mathcal{H}^1(l_i(\omega)\cap W_{\oplus 1}) \leq (h+2)\sqrt{2}$ for any $\omega\in\Omega$, for any i.
Let $I := \{i : l_i\cap W_{\oplus 1}\neq\emptyset\}$, and $\Phi^W := \mathrm{card}(I)$; we know that $\mathbb{E}[\Phi^W] < \infty$, so

$$\mathbb{E}[\mathcal{H}^1(\Theta_1)] = \mathbb{E}[\sum_{i\in I}\mathcal{H}^1(l_i\cap W_{\oplus 1})] \leq (h+2)\sqrt{2}\mathbb{E}[\Phi_W] < \infty.$$

The hypotheses of Theorem 17 are satisfied with $A = W$ and $Y = \mathcal{H}^1(\Theta_1)$, thus we obtain

$$\lim_{r\to 0}\frac{\mathbb{E}[\nu^2(\Xi_{\oplus r}\cap W)]}{2r} = \mathbb{E}[\mathcal{H}^1(\Xi\cap W)].$$

In conclusion, remembering that the number N_r of lines of Ξ hitting the ball $B_r(0)$ is a Poisson random variable with mean $2rL_\Xi$ ([23], p.250), by (33) we obtain

$$\begin{aligned} c &= \lim_{r\to 0}\frac{\mathbb{P}(0\in\Xi_{\oplus r})}{2r} = \lim_{r\to 0}\frac{\mathbb{P}(\Xi\cap B_r(0)\neq\emptyset)}{2r} \\ &= \lim_{r\to 0}\frac{\mathbb{P}(N_r\geq 1)}{2r} = \lim_{r\to 0}\frac{1-e^{-2rL_\Xi}}{2r} = L_\Xi. \end{aligned}$$

Remark 24 *In the above example we have chosen the measure η as the restriction of the Hausdorff measure $\mathcal{H}^1$ to a suitable set containing $\Xi \cap W$. As a matter of fact, due to the stochasticity of the relevant random closed set Ξ, problems may arise in identifying a measure η needed for the application of Theorem 17. A proper choice of η can be made by referring to another suitable random set containing Ξ. We further clarify such procedure by the following example.*

Example 3. (Segment processes.)
Let Θ_1 be a random closed set in $\mathbb{R}^2$ such that

$$\Theta_1 := \bigcup_{i=1}^{\Phi} S_i,$$

where Φ is a counting process (i.e. a positive integer valued random variable) with $\mathbb{E}[\Phi] < \infty$, and $S_1, S_2, \ldots$, are random segments independent of Φ, randomly distributed in the plane with random lengths $\mathcal{H}^1(S_i)$ in $[0, M]$.
Let us consider a realization $\Theta_1(\omega)$ and define $\eta(\cdot) := \dfrac{\mathcal{H}^1(\Theta_1(\omega) \cap \cdot)}{\mathcal{H}^1(\Theta_1(\omega))}$.
Let $x \in \Theta_1(\omega)$; then an $\bar{\imath}$ exists such that $x \in S_{\bar{\imath}}(\omega)$, and so

$$\eta(B_r(x)) = \frac{\mathcal{H}^1(\Theta_1(\omega) \cap B_r(x))}{\mathcal{H}^1(\Theta_1(\omega))} \geq \frac{\mathcal{H}^1(S_{\bar{\imath}}(\omega) \cap B_r(x))}{\mathcal{H}^1(\Theta_1(\omega))}.$$

Fixed $r \in (0,1)$, observe that, if $S_{\bar{\imath}}(\omega) \cap \partial B_r(x) \neq \emptyset$, then $\mathcal{H}^1(S_{\bar{\imath}}(\omega) \cap B_r(x)) \geq r$, while if $S_{\bar{\imath}}(\omega) \subseteq B_r(x)$, then $\mathcal{H}^1(S_{\bar{\imath}}(\omega) \cap B_r(x)) = \mathcal{H}^1(S_{\bar{\imath}}(\omega)) \geq \mathcal{H}^1(S_{\bar{\imath}}(\omega))r$. Suppose that $\Phi(\omega) = n$ and define

$$L(\omega) := \min_{i=1,\ldots,n} \{\mathcal{H}^1(S_i(\omega))\}.$$

We have that

$$\eta(B_r(x)) \geq \frac{\min\{1, L(\omega)\}}{\mathcal{H}^1(\Theta_1(\omega))} r, \qquad \forall x \in \Theta_1(\omega),\ r \in (0,1).$$

Thus, $\Theta_1(\omega)$ satisfies the hypotheses of Theorem 1.
If we want to apply Theorem 17, the above is not a good choice for η. In fact,

$$\frac{1}{\Gamma(\omega)} \leq \max\{\mathcal{H}^1(\Theta_1(\omega)), \frac{\mathcal{H}^1(\Theta_1(\omega))}{L(\omega)}\} =: Y(\omega),$$

and we may well have $\mathbb{E}[Y] \not< \infty$. In this case, a possible solution to the problem is to extend all the segments with length less than 2 (the extension can be done omothetically from the center of the segment, so that measurability of the process is preserved). In particular, for any $\omega \in \Omega$, let

$$\tilde{S}_i(\omega) = \begin{cases} S_i(\omega) & \text{if } \mathcal{H}^1(S_i(\omega)) \geq 2, \\ S_i(\omega) \text{ extended to length 2} & \text{if } \mathcal{H}^1(S_i(\omega)) < 2; \end{cases}$$

and

$$\tilde{\Theta}_1(\omega) := \bigcup_{i=1}^{\Phi(\omega)} \tilde{S}_i(\omega).$$

In this way, for every $x \in \Theta_1(\omega)$, there exists an $\bar{\imath}$ such that $x \in \tilde{S}_{\bar{\imath}}(\omega)$ with $\tilde{S}_{\bar{\imath}}(\omega) \cap \partial B_r(x) \neq \emptyset$ for any $r \in (0,1)$. If we define

$$\eta(\cdot) := \frac{\mathcal{H}^1(\tilde{\Theta}_1(\omega) \cap \cdot)}{\mathcal{H}^1(\tilde{\Theta}_1(\omega))},$$

then

$$\eta(B_r(x)) \geq \frac{1}{\mathcal{H}^1(\tilde{\Theta}_1(\omega))} r \qquad \forall x \in \Theta_1(\omega),\ r \in (0,1),$$

and so in this case we have $Y = \mathcal{H}^1(\tilde{\Theta}_1)$, and

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathcal{H}^1(\bigcup_{i=1}^{\Phi} \tilde{S}_i)] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{\Phi} \mathcal{H}^1(\tilde{S}_i) \,|\, \Phi]] \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \mathbb{E}[\mathcal{H}^1(\tilde{S}_i) \,|\, \Phi = n] \mathbb{P}(\Phi = n) \leq \sum_{n=1}^{\infty} n(M+2)\mathbb{P}(\Phi = n) \\
&= (M+2)\mathbb{E}[\Phi] < \infty.
\end{aligned}$$

(The same holds whenever the S_i's are IID as S with $\mathbb{E}[\mathcal{H}^1(S)] < \infty$.)
Note the peculiar role played by the geometrical properties of the random set.

A particular segment process is the well known *stationary Poisson segment process* in $\mathbb{R}^d$ [4, 23]. In this case each segment S_i is determined by its reference point c_i, length and orientation. The c_i's are given by a stationary Poisson point process Ψ with intensity $\alpha > 0$, while length, say R, and orientation are supposed to be random and independent, with $\mathbb{E}[R] < \infty$. Then the measure $\mathbb{E}[\mu_{\Theta_1}]$ induced by the segment process is stationary, and it can be proved (see e.g. [4], p. 42, [23]) that its density is given by $\lambda(x) = \alpha\mathbb{E}[R] =: L$, for any $x \in \mathbb{R}^d$.
Clearly, the resulting random closed set is not compact and the mean number of segments which intersect a fixed bounded region is finite. Using Theorem 19, and because of stationarity, we know that

$$L = \lim_{r \to 0} \frac{\mathbb{P}(0 \in \Theta_{1_{\oplus r}})}{b_{d-1} r^{d-1}}.$$

Let us first consider for simplicity the particular case in which orientation and length R are both fixed. Then, denoted by K the subset of $\mathbb{R}^d$ such that a segment with reference point in K hits the ball $B_r(0)$, then it is easy to see that $\nu^d(K) = b_{d-1} r^{d-1} R + b_d r^d$, and

$$\mathbb{P}(0 \in \Theta_{1_{\oplus r}}) = \mathbb{P}(\Psi(K) > 0) = 1 - e^{-\alpha \nu^d(K)},$$

so that we obtain

$$\lim_{r \to 0} \frac{1 - e^{-\alpha(b_{d-1} r^{d-1} R + b_d r^d)}}{b_{d-1} r^{d-1}} = \alpha R. \tag{34}$$

Note that this does not depend on orientation, so, if now we return to consider the case in which length and orientation are random, it easily follows that

$$L = \alpha \mathbb{E}[R],$$

as we expected.

Geometric processes of great interest in applications are the so called **fibre processes**. A fibre process Θ_1 is a random collection of rectifiable curves. A relevant real system which can be modelled as a fibre process is the system of vessels in tumor driven angiogenesis. Estimation of the mean length intensity of such system are useful for suggesting important methods of diagnosis and of dose response in clinical treatments [7, 11, 12].

It is clear that, as in Example 3, Theorem 17 can be applied also to this kind of $\mathcal{H}^1$-rectifiable random closed sets, going to consider as $\tilde{\Theta}_1$ the random closed set given by the union of suitably extended fibres. As a consequence, Proposition 13 holds for fibre processes, and we may obtain information about the measure $\mathbb{E}[\mu_{\Theta_1}]$, also under hypotheses of inhomogeneity of the process.

Example 4. (Boolean models.)
Another geometric process, well known in literature, is given by a inhomogeneous Boolean model of spheres (see [23]); i.e. Θ_{d-1} turns out to be a random union of spheres in $\mathbb{R}^d$, and so it may be represented as follows

$$\Theta_{d-1}(\omega) := \bigcup_i \partial B_{R_i(\omega)}(Y_i(\omega)),$$

where Y_i is a random point in $\mathbb{R}^d$, given by a Poisson point process in $\mathbb{R}^d$, and R_i is a positive random variable (e.g. $R \sim U[0, M]$). As a consequence, the mean number of balls which intersect any compact set K is finite. In order to claim that (8) holds, we proceed in an analogous way as in the previous example for a stationary Poisson segment process; it is clear that if Theorem 17 holds for a random closed set

$$\Xi_{d-1} := \bigcup_{i=1}^{\Phi} \partial B_{R_i(\omega)}(Y_i(\omega)), \tag{35}$$

where Φ is a positive integer valued random variable with finite expected value, and Y_i is a random point in $\mathbb{R}^d$, then the thesis follows.
Since in general $1/\mathcal{H}^{d-1}(\partial B_{R_i}(Y_i))$ has not a finite expected value, using the same approach as in the previous example, we are going to consider a suitable random set $\tilde{\Xi}_{d-1}$ containing Ξ_{d-1}.
Let $d = 2$; the case $d > 2$ follows similarly. For any $\omega \in \Omega$, let

$$B_i(\omega) = \begin{cases} B_{R_i(\omega)}(Y_i(\omega)) & \text{if } R_i(\omega) \geq \frac{1}{2}, \\ B_{R_i(\omega)}(Y_i(\omega)) \cup l_{Y_i(\omega)} & \text{if } R_i(\omega) < \frac{1}{2}, \end{cases}$$

where $l_{Y_i(\omega)}$ is a segment centered in $Y_i(\omega)$ with length 3, and

$$\tilde{\Xi}_1(\omega) := \bigcup_{i=1}^{\Phi(\omega)} \partial B_i(\omega).$$

In this way, for every $x \in \Xi_1(\omega)$, there exists an $\bar{\imath}$ such that $x \in \partial B_{\bar{\imath}}(\omega)$ with

$\partial B_{\bar{1}}(\omega) \cap \partial B_r(x) \neq \emptyset$ for any $r \in (0,1)$. We define

$$\eta(\cdot) := \frac{\mathcal{H}^1(\tilde{\Xi}_1(\omega) \cap \cdot)}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))}.$$

Let $r \in (0,1)$ be fixed and observe that, if $R_{\bar{1}}(\omega) \geq \frac{r}{2}$, then $\partial B_r(x) \cap \partial B_{R_{\bar{1}}(\omega)}(Y_{\bar{1}}(\omega)) \neq \emptyset$, and so

$$\eta(B_r(x)) \geq \frac{2r}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))}.$$

On the other hand, $R_{\bar{1}}(\omega) < \frac{r}{2} < 1/2$, let $s := \mathrm{dist}(x, l_{Y_{\bar{1}}(\omega)})$, and $m := \mathcal{H}^1(l_{Y_{\bar{1}}(\omega)} \cap B_r(x))$; then $s^2 \leq R_{\bar{1}}(\omega)^2 \leq \frac{r^2}{4}$ and

$$\eta(B_r(x)) \geq \frac{m}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))} = \frac{2\sqrt{r^2 - s^2}}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))} \geq \frac{\sqrt{3}r}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))}.$$

Hence, summarizing, we have

$$\eta(B_r(x)) \geq \frac{\sqrt{3}}{\mathcal{H}^1(\tilde{\Xi}_1(\omega))} r \qquad \forall x \in \Xi_1(\omega),\ \forall r \in (0,1),$$

and it is clear that $\mathbb{E}[\mathcal{H}^1(\tilde{\Xi}_1)] < \infty$; thus Theorem 17 holds for $\tilde{\Xi}_1$, with $W = \mathbb{R}^d$ and $Y := \mathcal{H}^1(\tilde{\Xi}_1)/\sqrt{3}$.

Consider now the random closed set Ξ_1 defined by (35), where Φ, R_i and X_i are chosen as before. It is clear that Theorem 17 holds for Ξ_1 as well, since $\Xi_1 \subseteq \tilde{\Xi}_1$.

Example 5. (Birth-and-growth processes.)
Let $\{\Theta^t\}$ be a *birth-and-growth process* [8] with a nucleation process defined by a random measure N on $\mathbb{R}_+ \times \mathbb{R}^d$, and a constant growth rate $G > 0$.

The random measure N, defining the nucleation process, is given by

$$N = \sum_{n=1}^{\infty} \varepsilon_{T_n, X_n},$$

where

- T_n is an $\mathbb{R}_+$-valued random variable representing the time of birth of the n-th nucleus,
- X_n is random point in $\mathbb{R}^d$ representing the spatial location of the nucleus born at time T_n,
- $\varepsilon_{t,x}$ is the Dirac mass concentrated at (t,x).

For any fixed time t, Θ^t is the union of a finite and random number of random balls in $\mathbb{R}^d$:

$$\Theta^t = \bigcup_{i:T_i \leq t} B_{G(t-T_i)}(X_i).$$

As a consequence of Theorem 17 we have that Proposition 13 applies, so that

$$\lim_{r\to 0}\frac{\mathbb{E}[\mathcal{H}^d((\partial\Theta^t)_{\oplus r}\cap A))]}{2r}=\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t\cap A)].$$

If the random set $\partial\Theta^t$ turns out to be absolutely continuous in mean, then we have

$$\lim_{r\to 0}\int_A\frac{\mathbb{P}(x\in\partial\Theta^t_{\oplus r})}{b_{d-n}r^{d-n}}\mathrm{d}x=\int_A S_V(t,x)\mathrm{d}x,$$

where S_V is the so called *mean surface density* S_V associated to the birth and growth process.

As a simple example in which $\partial\Theta^t$ is absolutely continuous, but not stationary, let us consider a nucleation process N given by an inhomogeneous Poisson point process, with intensity $\alpha(t,x)$. We may prove this as follows:

By absurd, let $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t\cap\cdot)]$ be not absolutely continuous with respect to ν^d; then there exists $A\subset\mathbb{R}^d$ with $\nu^d(A)=0$ such that $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t\cap A)]>0$. It is clear that

$$\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t\cap A)]>0\;\Rightarrow\;\mathbb{P}(\mathcal{H}^{d-1}(\partial\Theta^t\cap A)>0)>0,$$

and

$$\mathbb{P}(\mathcal{H}^{d-1}(\partial\Theta^t\cap A)>0)\le\mathbb{P}(\exists(T_j,X_j)\,:\,\mathcal{H}^{d-1}(\partial B_{G(t-T_j)}(X_j)\cap A)>0).$$

As a consequence, we have

$$\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t\cap A)]>0\;\Rightarrow\;\mathbb{P}(\Phi(\mathcal{A})\neq 0)>0,$$

where

$$\mathcal{A}:=\{(s,y)\in[0,t]\times\mathbb{R}^d\,:\,\mathcal{H}^{d-1}(\partial B_{G(t-s)}(y)\cap A)>0)\}.$$

Denoting by $\mathcal{A}_s:=\{y\in\mathbb{R}^d\,:\,(s,y)\in\mathcal{A}\}$ the section of $\mathcal{A}$ at time s, and by $\mathcal{A}^y:=\{s>0\,:\,(s,y)\in\mathcal{A}\}$ the section of $\mathcal{A}$ at y, we notice that $\nu^1(\mathcal{A}^y)=0$ for all y, because $\nu^d(A)=0$ (it suffices to use spherical coordinates centered at y to obtain that ν^1-a.e. ball with radius s centered at y intersects A in a $\mathcal{H}^{d-1}$-negligible set). Therefore we may apply Fubini's theorem to get

$$\int_0^\infty\nu^d(\mathcal{A}_s)\,\mathrm{d}s=\int_0^\infty\int_{\mathbb{R}^d}\chi_A\,\mathrm{d}y\,\mathrm{d}s=\int_{\mathbb{R}^d}\int_0^\infty\chi_A\,\mathrm{d}s\,\mathrm{d}y=\int_{\mathbb{R}^d}\nu^1(\mathcal{A}^y)\,\mathrm{d}y=0.$$

It follows that $\nu^d(\mathcal{A}_s)=0$ for ν^1-almost every $s\in[0,t]$, and so

$$\mathbb{E}[\Phi(\mathcal{A})]=\int_{\mathcal{A}}\alpha(s,y)\,\mathrm{d}s\mathrm{d}y=\int_0^t\int_{\mathcal{A}_s}\alpha(s,y)\,\mathrm{d}y\,\mathrm{d}s=0.$$

But this is an absurd, since

$$\mathbb{P}(\Phi(\mathcal{A})\neq 0)>0\;\Rightarrow\;\mathbb{E}[\Phi(\mathcal{A})]>0.$$

Example 6. (Random Johnson-Mehl tessellations.)

In a birth-and-growth process as in the previous example, where N is a time inhomogeneous Poisson marked point process, one may consider the associated random Johnson-Mehl tessellation generated by the impingement of two grains which stop their growth at points of contact (see [18]); briefly, the system of n-facets of a Johnson-Mehl tessellation at time $t > 0$ is a random finite union of a system of random n-regular sets $F_i^{(n)}(t)$, $0 \leq n \leq d$:

$$\Xi_n^t := \bigcup_i F_i^{(n)}(t).$$

Again, it can be shown that Proposition 13 applies, so that we may approximate mean n-facet densities, for all $0 \leq n \leq d$.

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