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EXACT SOLUTIONS FOR DIVIDEND STRATEGIES OF THRESHOLD AND LINEAR BARRIER TYPE IN A SPARRE ANDERSEN MODEL*

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Abstract

For the classical risk model, in order to allow for a positive probability of survival, a threshold dividend strategy has recently been introduced in the literature to improve upon the horizontal barrier strategy at the expense of some profit in terms of dividend payments. In this paper we both extend several of these results to a Sparre Andersen model with generalized Erlang(n)-distributed interclaim times and compare the performance of the threshold strategy to a linear dividend barrier model. In particular, (partial) integro-differential equations for the corresponding ruin probabilities and expected discounted dividend payments are provided for both models and explicitly solved for $n = 2$ and exponentially distributed claim amounts. Finally, the explicit solutions are used to identify parameter sets for which one strategy outperforms the other and vice versa.

Keywords: Sparre Andersen model; dividend payments, piece-wise deterministic Markov processes, ruin probability

1 Introduction

In collective risk theory, the Sparre Andersen model to describe the surplus process of an insurance portfolio has a long history, starting with the original paper [24]. In this model, the claim counting process $(N_t)_{t\geq 0}$ for time t is assumed to be an ordinary renewal process, which can be written as

$$N_t = \min\{k \ : \ T_1 + \ldots + T_{k+1} > t\}, \quad t \geq 0,$$

where $(T_i)_{i\in\mathbb{N}}$ is the sequence of independent interarrival times.
This renewal assumption allows for more flexibility than the classical risk process (where N_t constitutes a homogeneous Poisson process) and enables to some extent contagion between claim occurrences. As usual, the premium inflow is assumed to be continuous over time with constant intensity c, and the claim amounts $(Y_i)_{i\in\mathbb{N}}$ are independent and

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identically distributed positive random variables with distribution function F_Y and mean $\mu < \infty$. Then, for initial capital u, the risk reserve process $(R_t)_{t\geq 0}$ of the insurance portfolio at time t is given by

$$R_t = x + c\,t - S_t.$$

where $S_t = \sum_{i=1}^{N_t} Y_i$ denotes the aggregate claim amount at time t. The net profit condition in this model is given by $c > \mu/\mathbb{E}(T_i)$. Typical quantities studied in this context are the time of ruin

$$\tau(u) = \inf\{t > 0 \mid R_t < 0\ ,\ R_0 = u\},$$

and the probability of ruin

$$\psi(u) = P\{\tau(u) < \infty \mid R_0 = u\}.$$

In this paper we will assume that the interarrival times (T_i) follow a generalized Erlang(n) distribution, in which case each T_i is a convolution of n independent exponentially distributed random variables with parameters $\lambda_1, \ldots, \lambda_n$. This specific property allows to extend the Markovian character of the claim number process in the classical model together with the analytically strong tools to study properties of the risk process to this renewal setting. This approach was recently exploited by Gerber & Shiu [15] and Li & Garrido [18], see also Dickson [9], Dickson & Hipp [10, 11] and Cheng & Tang [6].

The analytical tractability of this renewal setup has also been used to include a horizontal dividend barrier strategy in the model (see Li & Garrido [19] for properties of the resulting risk process (generalizing results of Lin et al. [21] for $n = 1$) and, on the other hand, Albrecher et al. [1] for the calculation of moments of dividend payments resulting from this strategy). However, one considerable disadvantage of the horizontal barrier strategy is that the corresponding risk process will lead to ruin with probability 1, which typically makes this strategy inappropriate in practice. More than that, even if one is just interested in maximizing expected discounted dividend payouts, while for the compound Poisson model there are some optimality properties for horizontal barriers (see e.g. [12]), it can be shown that horizontal barrier strategies are not optimal in the Sparre Andersen model (partly because then R_t is not a Lévy process, see [2]). These two issues motivate to look at alternative dividend payment strategies that allow for a positive probability of survival, still have a satisfying level of dividend payouts and allow for analytical expressions so that the parameters of the strategy can be tuned towards a given target.

Among these, there is the so-called *threshold* dividend strategy (which for the classical compound Poisson model was discussed in Asmussen [5] and recently studied in detail by Lin & Pavlova [20] and Gerber & Shiu [17]; for a diffusion setup, see also [16]). Following this strategy, one fixes a level $b > 0$ and no dividends are paid out if the surplus level is below b. Whenever the surplus is above b, dividends are paid with intensity a, $0 < a < c$ (and the surplus increases with intensity $c - a$) until the surplus falls again below b due to the occurrence of a claim. Finally, the dividend payments are stopped at the time of ruin. So the dynamics of the modified risk process R^{thr} are given by

$$\begin{aligned} dR_t^{thr} &= c\,dt - dS_t, \quad 0 \leq R_t^{thr} < b, \\ dR_t^{thr} &= (c-a)\,dt - dS_t, \quad R_t^{thr} \geq b, \\ dD_t^{thr} &= a\,dt, \qquad R_t^{thr} \geq b, \end{aligned}$$

where $(D_t^{thr})_{t\geq 0}$ denotes the accumulated dividend payments at time t . The expected discounted dividends of such a strategy are given by

$$W(u,b) = \mathbb{E}\left(\int_0^{\tau(u)} a\ I_{\{R_t \geq b\}}\ e^{-\delta t}\ dt \ \Big|\ R_0 = u \right),$$

where $\delta \geq 0$ is the discounting factor. Observe that $\psi(u) = 1$ if the process above b does not fulfill the net profit condition $(c-a)\mathbb{E}(T_i) > \mu$, see [5].

The second strategy we will focus on is the so-called *linear barrier* dividend strategy, where the barrier $b_t = b + at$ $(b > 0, 0 < a < c)$ grows linearly in time and dividends are paid out with intensity $(c-a)$ whenever R_t reaches b_t, while the reserve increases with intensity a. On the other hand, nothing is paid out when the surplus is below the barrier. Dividend payments again stop at the event of ruin. The dynamics of the modified risk process R^{lin} are thus given by

$$\begin{aligned} dR_t^{lin} &= c\,dt - dS_t, \quad 0 \leq R_t^{lin} < b_t, \\ dR_t^{lin} &= a\ dt - dS_t, \quad R_t^{lin} = b_t, \\ dD_t^{lin} &= (c-a)\ dt, \qquad R_t^{lin} = b_t. \end{aligned}$$

The expected discounted dividend payments are in this case

$$W(u,b) = \mathbb{E}\left(\int_0^{\tau(u)} (c-a)\ I_{(R_t = b_t)} e^{-\delta t}\ dt \ \Big|\ R_0 = u,\ b_0 = b \right).$$

The linearity of the barrier enables several classical techniques for the computation of quantities of interest in this more general model (such as martingale techniques (see e.g. Gerber [13])). Historically, together with the positivity of the corresponding survival probability, this was one of the reasons to consider linear barriers. Explicit formulae for $W(u,b)$ and $\psi(u,b)$ for light-tailed claim sizes were derived in [14, 23]. Recently, the discounted penalty function and higher moments of discounted dividend payments for the linear barrier strategy were investigated in [3].
Figure 1 depicts the two strategies for a sample path that leads to ruin in both cases.
An objection sometimes raised against the linear barrier model is the fact that the strategy depends on the point in time, i.e. the payment strategy is different for each t for an otherwise identical situation. However, if one is forced to fix a dividend strategy at time 0 and is interested in both maximizing $W(u,b)$ and keeping $\psi(u,b)$ below a specified level, it is intuitively clear that for small t one will try to deduct a high dividend amount possibly involving some higher risk, whereas for larger t the main focus will be on securing the survival, since, due to the discount factor δ, the dividend contributions at this later stage will only be marginal. Indeed, as will be illustrated in Section 4, for higher values of δ the linear barrier model often outperforms the threshold model in terms of finding a compromise between the values of $W(u,b)$ and $\psi(u,b)$.

In this paper, we will derive (partial) integro-differential equations (PIDE's) for the ruin probability and moments of dividend payments for both the threshold and the linear barrier strategy in the Sparre Andersen model with generalized Erlang(n) interclaim times.

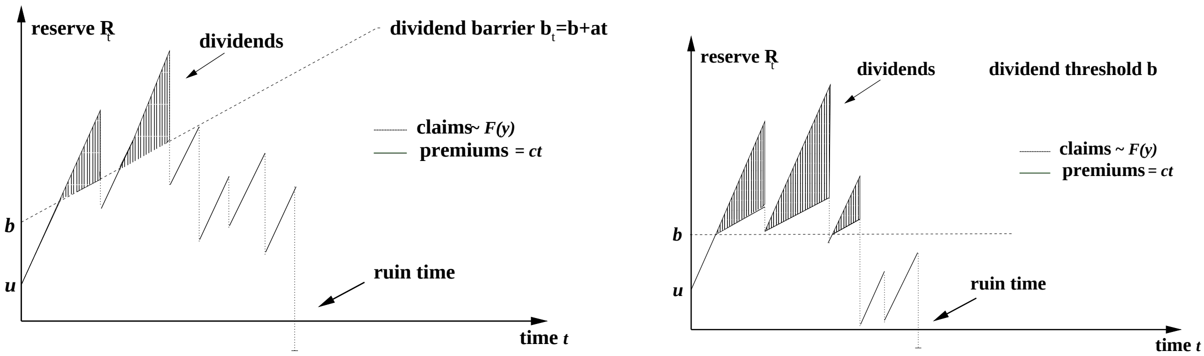

Figure 1: The linear barrier- and the threshold strategy for a sample path of R_t

These equations can in principle be solved explicitly whenever the claim size distribution is itself of generalized Erlangian type. We will demonstrate this solution procedure for the case of Erlang(2) interclaim times and exponential claim amounts. The explicit formulae pave the way for a fast numerical assessment of the performance of these dividend strategies for a given set of parameters.

In Section 2, we use the differential approach to establish a PIDE for the moment-generating function of discounted dividend payments for the threshold strategy in the Sparre Andersen model with Erlang(n) interclaim times. Subsequently, the corresponding IDE's for arbitrary moments of the discounted dividend payments and for the ruin probability are derived. These equations are explicitly solved for $n = 2$ and exponential claim amounts in Section 2.4. Section 3 provides PIDE's for the survival probability as well as the expected discounted dividend payments in the linear barrier model with Erlang(n) interclaim times. As an alternative to the differential approach, these are derived using the framework of piece-wise deterministic Markov processes. Again, an explicit solution is provided for $n = 2$ and exponential claim amounts and the exact results are compared with simulation in Section 3.2.4. Finally, in Section 4 the analytical results of the previous sections are used to compare the performance of the threshold and linear barrier model for various sets of parameters.

2 The Threshold Dividend Strategy

2.1 A system of PIDEs for the moment-generating function

Let us decompose every inter-occurrence time with generalized Erlang(n)-distribution into the independent sum of n exponential random variables with parameters $\lambda_1, \ldots, \lambda_n$, each causing a "sub-claim" of size 0 and at the time of the n-th sub-claim an actual claim with distribution function F_Y occurs. This is done by defining n states for the risk process (see e.g. [1]). Starting at time 0 in state 1, every sub-claim causes a transition to the next state and at the time of occurrence of the n-th sub-claim, an actual claim with distribution function F_Y occurs and the risk process jumps into state 1 again. Let

$$M^{(j)}(u, y, b) = \mathbb{E}\left(\exp\left(y \int_0^{\tau(u)} e^{-\delta t} a I_{\{R_t > b\}} \, \mathrm{d}t \right) \middle| R_0 = u, \text{state} = j \right)$$

denote the moment-generating function of the discounted dividend payments, given that the risk process starts in state j ($j = 1, \ldots, n$).
Furthermore we split up the moment-generating functions in two regions below and above the barrier,

$$M^{(j)}(u, y, b) = M_1^{(j)}(u, y, b) I_{\{u<b\}} + M_2^{(j)}(u, y, b) I_{\{u \geq b\}}.$$

For $j = 1, \ldots, n-1$, we condition on the occurrence of a claim within an infinitesimal time interval, which gives

$$M_1^{(j)}(u, y, b) = (1 - \lambda_j dt) M_1^{(j)}(u + cdt, ye^{-\delta dt}, b) + \lambda_j dt \; M_1^{(j+1)}(u + cdt, ye^{-\delta dt}, b) + o(dt),$$

and

$$\begin{aligned} M_2^{(j)}(u, y, b) &= (1 - \lambda_j dt) e^{yadt} M_2^{(j)}(u + (c-a)dt, ye^{-\delta dt}, b) \\ &\quad + \lambda_j dt \, e^{yadt} \, M_2^{(j+1)}(u + (c-a)dt, ye^{-\delta dt}, b) + o(dt). \end{aligned}$$

The analogous equations for $j = n$ are

$$\begin{aligned} M_1^{(n)}(u, y, b) &= (1 - \lambda_n dt) M_1^{(n)}(u + cdt, ye^{-\delta dt}, b) \\ &\quad + \lambda_n dt \int_0^{u+cdt} M_1^{(1)}(u + cdt - z, ye^{-\delta dt}, b) \, \mathrm{d}F_Y(z) \\ &\quad + \lambda_n dt \int_{u+cdt}^{\infty} \mathrm{d}F_Y(z) + o(dt), \\ M_2^{(n)}(u, y, b) &= (1 - \lambda_n dt) e^{yadt} M_2^{(n)}(u + (c-a)dt, ye^{-\delta dt}, b) \\ &\quad + \lambda_n dt e^{yadt} \int_0^{u+cdt} M^{(1)}(u + (c-a)dt - z, ye^{-\delta dt}, b) \, \mathrm{d}F_Y(z) \\ &\quad + \lambda_n dt e^{yadt} \int_{u+cdt}^{\infty} \mathrm{d}F_Y(z) + o(dt). \end{aligned}$$

Taylor expansion and collection of suitable terms leads to the following partial (integro-) differential equations ($j = 1, \ldots, n-1$):

$$\begin{aligned} M_1^{(j+1)}(u, y, b) &= \left(\frac{-c\frac{\partial}{\partial u} + \lambda_j + \delta y \frac{\partial}{\partial y}}{\lambda_j} \right) M_1^{(j)}(u, y, b), \\ M_2^{(j+1)}(u, y, b) &= \left(\frac{-(c-a)\frac{\partial}{\partial u} + (\lambda_j - ya) + \delta y \frac{\partial}{\partial y}}{\lambda_j} \right) M_2^{(j)}(u, y, b), \end{aligned}$$

and

$$\left(\frac{-c\frac{\partial}{\partial u} + \lambda_n + \delta y \frac{\partial}{\partial y}}{\lambda_n} \right) M_1^{(n)}(u, y, b) - (1 - F_Y(u)) - \int_0^u M_1^{(1)}(u - z, y, b) \, \mathrm{d}F_Y(z) = 0,$$

$$\left(\frac{-(c-a)\frac{\partial}{\partial u}+(\lambda_n-ya)+\delta y\frac{\partial}{\partial y}}{\lambda_n}\right) M_2^{(n)}(u,y,b)$$
$$-(1-F_Y(u))-\int_0^u M^{(1)}(u-z,y,b)\,\mathrm{d}F_Y(z)=0.$$

The quantity of eventual interest is $M^{(1)}(u,y,b) := M(u,y,b)$, which from the above equations is seen to be the solution of the following system of partial integro-differential equations:

$$\begin{aligned} 0 &= \left(\prod_{j=1}^{n}\frac{-c\frac{\partial}{\partial u}+\lambda_j+\delta y\frac{\partial}{\partial y}}{\lambda_j}\right) M_1(u,y,b)-(1-F_Y(u)) \\ &\quad -\int_0^u M_1(u-z,y,b)\,\mathrm{d}F_Y(z), \qquad (1)\\ 0 &= \left(\prod_{j=1}^{n}\frac{-(c-a)\frac{\partial}{\partial u}+(\lambda_j-ya)+\delta y\frac{\partial}{\partial y}}{\lambda_j}\right) M_2(u,y,b)-(1-F_Y(u)) \\ &\quad -\int_0^u M(u-z,y,b)\,\mathrm{d}F_Y(z) \qquad (2)\end{aligned}$$

(note that the product $y\frac{\partial}{\partial y}$ in the above operator is not commutative). Boundary conditions are given by

$$\begin{aligned}\lim_{b\to\infty} M_1(u,y,b) &= 1,\\ \lim_{u\to\infty} M_2(u,y,b) &= e^{ya/\delta}.\end{aligned}$$

Moreover, at $u=b$, by continuity we have to have

$$\lim_{u\to b^+} M_2^{(j)}(u,y,b)=\lim_{u\to b^-} M_1^{(j)}(u,y,b)$$

for all states $j=1,\ldots,n$, which translates into

$$\left((c-a)\frac{\partial^+}{\partial u}-\delta y\frac{\partial^+}{\partial y}+ya\right)^{j-1} M_2\Bigg|_{u=b}=\left(c\frac{\partial^-}{\partial u}-\delta y\frac{\partial^-}{\partial y}\right)^{j-1} M_1\Bigg|_{u=b}, \qquad (3)$$

where the derivatives are assumed to be one-sided.

2.2 The moments of the discounted dividends

The results of the previous subsection can be used to derive an integro-differential equation for the mth moment $W_m(u,b)$ of the discounted sum of dividend payments ($m\in\mathbb{N}$). Again, we write

$$W_m(u,b)=W_{m,1}(u,b)\,I_{\{u<b\}}+W_{m,2}(u,b)\,I_{\{u\geq b\}}.$$

With the representation

$$M(u,y,b) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!} W_m(u,b),$$

and the equations (1)-(3), a comparison of coefficients of y^m gives

$$\left(\prod_{j=1}^{n} \frac{-c\frac{\partial}{\partial u} + \lambda_j + \delta\bar{\Delta}}{\lambda_j}\right) W_{m,1}(u,y,b) - \int_0^u W_{m,1}(u-z,y,b)\, \mathrm{d}F_Y(z) = 0,$$

$$\left(\prod_{j=1}^{n} \frac{-(c-a)\frac{\partial}{\partial u} + (\lambda_j - a\Delta) + \delta\bar{\Delta}}{\lambda_j}\right) W_{m,2}(u,y,b) - \int_0^u W_m(u-z,y,b)\, \mathrm{d}F_Y(z) = 0,$$

with the operators $\Delta W_m := mW_{m-1}$, $\bar{\Delta} W_m := mW_m$. Moreover, $W_0 = 1, W_{-i} = 0$ ($i \in \mathbb{N}$). Here, the product $\Delta\bar{\Delta}$ of operators is not commutative and is given by $(\Delta\bar{\Delta})W_m = \bar{\Delta}(\Delta W_m) = m(m-1)W_{m-1}$ and $(\bar{\Delta}\Delta)W_m = \Delta(\bar{\Delta} W_m) = m^2 W_{m-1}$.
We have the boundary conditions:

$$\lim_{b\to\infty} W_{m,1}(u,y,b) = 0, \tag{4}$$

$$\lim_{u\to\infty} W_{m,2}(u,y,b) = \left(\frac{a}{\delta}\right)^m. \tag{5}$$

Moreover, by assuming that all moments are continuous we have

$$\left(c\frac{\partial^-}{\partial u}\right)^{j-1} W_{m,1}\Bigg|_{u=b} = \left((c-a)\frac{\partial^+}{\partial u} + a\Delta\right)^{j-1} W_{m,2}\Bigg|_{u=b}, \quad (j = 1,\ldots,n).$$

Remark 2.1. Note that the above formulas extend equations (5.1)-(5.3) of Gerber & Shiu [17], who studied the case $m = 1, n = 1$.

2.3 Probability of ruin

The probability of ruin is defined through

$$\psi(u,b) = \mathbb{E}\left(I_{\{\tau<\infty\}} | R_0 = u\right).$$

Let us again split the function in two regions below and above the barrier b,

$$\psi(u,b) = \psi_1(u,b) I_{\{u<b\}} + \psi_2(u,b) I_{\{u\geq b\}}.$$

Analogously to Section 2.1, one can now decompose the process into n states and subsequently apply the differential approach to obtain

$$\left(\prod_{j=1}^{n} \frac{\lambda_j - c\frac{\partial}{\partial u}}{\lambda_j}\right) \psi_1(u,b) - (1 - F_Y(u)) - \int_0^u \psi_1(u-z,b)\, \mathrm{d}F_Y(z) = 0, \tag{6}$$

$$\left(\prod_{j=1}^{n} \frac{\lambda_j - (c-a)\frac{\partial}{\partial u}}{\lambda_j}\right) \psi_2(u,b) - (1 - F_Y(u)) - \int_0^u \psi(u-z,b)\, \mathrm{d}F_Y(z) = 0. \tag{7}$$

The natural boundary conditions are

$$\lim_{u\to\infty} \psi_2(u,b) = 0 \tag{8}$$

and

$$\lim_{b\to\infty} \psi_1(u,b) = \bar{\psi}(u), \tag{9}$$

where $\bar{\psi}(u)$ denotes the ruin probability without dividend payments. Moreover, from the continuity assumptions,

$$\left((c-a)\frac{\partial^+}{\partial u}\right)^{j-1} \psi_2(u,b)\Big|_{u=b} = \lim_{u\to b^-} \left(c\frac{\partial^-}{\partial u}\right)^{j-1} \psi_1(u,b)\Big|_{u=b}, \quad (j=1,\dots,n). \tag{10}$$

Remark 2.2. For the compound Poisson model $n=1$, (6) and (7) appear implicitly in [5, 20].

2.4 Erlang(2) interarrivals and exponential claims

In principle, the above equations can be explicitly solved for Erlang distributed claim sizes. In the following we will illustrate the solution procedure for the specific case of Erlang$(2,\lambda)$ distributed interclaim times and Exp(α) distributed claim amounts. From Section 2.2, we then obtain

$$\left(\lambda + \delta m - c\frac{\partial}{\partial u}\right)^2 W_{m,1} - \lambda^2 \int_0^u W_{m,1}(u-z,b)\,\mathrm{d}F_Y(z) = 0$$

and

$$\begin{aligned}\left(\lambda + \delta m - (c-a)\frac{\partial}{\partial u}\right)^2 W_{m,2} - \lambda^2 \int_0^u W_m(u-z,b)\,\mathrm{d}F_Y(z)\\ = a\,m\left(-2(c-a)\frac{\partial}{\partial u} + 2\lambda + (2m-1)\delta\right) W_{m-1,2} - a^2(m-1)mW_{m-2,2}.\end{aligned}$$

together with the boundary conditions (4),(5),

$$W_{m,1}(b,b) = W_{m,2}(b,b)$$

and

$$c\frac{\partial}{\partial u}W_{m,1}\Big|_{u=b} = (c-a)\frac{\partial}{\partial u}W_{m,2}\Big|_{u=b} + a\,mW_{m-1,2}(b,b).$$

Let us consider the case $m=1$. We then have

$$c^2W''_{1,1}(u,b) - 2c(\delta+\lambda)W'_{1,1}(u,b) + (\delta+\lambda)^2W_{1,1}(u,b) - \lambda^2\alpha\, e^{-\alpha u}\int_0^u W_{1,1}(v,b)\,e^{\alpha v}dv = 0 \tag{11}$$

and

$$\begin{aligned}(c-a)^2W''_{1,2}(u,b) - 2(c-a)(\delta+\lambda)W'_{1,2}(u,b) + (\delta+\lambda)^2W_{1,2}(u,b) - a(2\lambda+\delta)\\ - \lambda^2\alpha\, e^{-\alpha u}\int_0^u W_1(v,b)\,e^{\alpha v}dv = 0\end{aligned} \tag{12}$$

together with

$$W_{1,1}(b-,b) = W_{1,2}(b+,b), \tag{13}$$

and

$$c\frac{\partial^- W_{1,1}}{\partial u}(u,b)\Big|_{u=b} = (c-a)\frac{\partial^+ W_{1,2}}{\partial u}(u,b) + a\Big|_{u=b}. \tag{14}$$

Applying the operator $\left(\frac{\partial\cdot}{\partial u} + \alpha\right)$ to (11) and (12) yields the differential equations

$$\begin{aligned} 0 &= c^2 W_{1,1}'''(u,b) + (\alpha c^2 - 2c(\delta+\lambda))W_{1,1}''(u,b) + ((\lambda+\delta)^2 - 2c\alpha(\delta+\lambda))W_{1,1}'(u,b) \\ &\quad +(\alpha(\delta+\lambda)^2 - \alpha\lambda^2)W_{1,1}(u,b), \end{aligned} \tag{15}$$

$$\begin{aligned} 0 &= (c-a)^2 W_{1,2}'''(u,b) + (\alpha(c-a)^2 - 2(c-a)(\delta+\lambda))W_{1,2}''(u,b) \\ &\quad +((\delta+\lambda)^2 - 2\alpha(c-a)(\delta+\lambda))W_{1,2}'(u,b) + (\alpha(\delta+\lambda)^2 - \alpha\lambda^2)W_{1,2}(u,b) - a\alpha(2\lambda+\delta). \end{aligned} \tag{16}$$

The solution of (15) is of the form

$$W_{1,1}(u,b) = \sum_{i=1}^{3} A_1^{(i)}(b) e^{R_1^{(i)}u}, \tag{17}$$

where $R_1^{(1)}, R_1^{(2)} > 0$ and $R_1^{(3)} < 0$ are the three roots of

$$(\delta + \lambda - c\,R)^2(R+\alpha) - \alpha\lambda^2 = 0.$$

Substitution in (11) then gives the condition

$$\sum_{i=1}^{3} \frac{A_1^{(i)}(b)}{R_1^{(i)} + \eta} = 0$$

On the other hand, equation (16) has a solution of the form

$$W_{1,2}(u,b) = \frac{a}{\delta} + A_2^{(1)}(b)\, e^{R_2^{(1)}u}, \tag{18}$$

where $\frac{a}{\delta}$ is a particular solution of (16) and $R_2^{(1)}$ is the negative root of

$$(\delta + \lambda - (c-a)R)^2(R+\alpha) - \alpha\lambda^2 = 0$$

(that this equation has indeed exactly one negative root follows by a Rouché-type argument, see e.g. Gerber & Shiu [15]). The coefficients of the positive roots have to be zero according to (4), hence these terms do not appear in (18).
Substituting (17) and (18) in (12), a comparison of coefficients gives

$$\frac{A_2^{(1)}\, e^{R_2^{(1)}b}}{R_2^{(1)} + \alpha} - \frac{A_1^{(1)}\, e^{R_1^{(1)}b}}{R_1^{(1)} + \alpha} - \frac{A_1^{(2)}\, e^{R_1^{(2)}b}}{R_1^{(2)} + \alpha} - \frac{A_1^{(3)}\, e^{R_1^{(3)}b}}{R_1^{(3)} + \alpha} + \frac{a}{\alpha\delta} = 0$$

(note that the integral on the right hand side of (12) has to be written as $\int_0^u W_1(v,b)\, e^{\alpha v}\, dv = \int_0^b W_{1,1}(v,b)\, e^{\alpha v}\, dv + \int_b^u W_{1,2}(v,b)\, e^{\alpha v}\, dv$). Together with conditions (13) and (14), we hence obtain the explicit solution (17) and (18), where the coefficients are determined by the system of linear equations

$$\begin{pmatrix} 0 & \frac{1}{R_1^{(1)}+\alpha} & \frac{1}{R_1^{(2)}+\alpha} & \frac{1}{R_1^{(3)}+\alpha} \\ -\frac{\alpha}{R_2^{(1)}+\alpha}e^{R_2^{(1)}b} & \frac{\alpha}{R_1^{(1)}+\alpha}e^{R_1^{(1)}b} & \frac{\alpha}{R_1^{(2)}+\alpha}e^{R_1^{(2)}b} & \frac{\alpha}{R_1^{(3)}+\alpha}e^{R_1^{(3)}b} \\ -e^{R_2^{(1)}b} & e^{R_1^{(1)}b} & e^{R_1^{(2)}b} & e^{R_1^{(3)}b} \\ -(c-a)R_2^{(1)}e^{R_2^{(1)}b} & cR_1^{(1)}e^{R_1^{(1)}b} & cR_1^{(2)}e^{R_1^{(2)}b} & cR_1^{(3)}e^{R_1^{(3)}b} \end{pmatrix} \begin{pmatrix} A_2^{(1)} \\ A_1^{(1)} \\ A_1^{(2)} \\ A_1^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{a}{\delta} \\ \frac{a}{\delta} \\ a \end{pmatrix}.$$

For the ruin probability, one has to apply the operator $(\alpha + \frac{\partial}{\partial u})$ on (6) and (7) to obtain the differential equations

$$0 = \left(\left(\alpha + \frac{\partial}{\partial u}\right)\left(\prod_{j=1}^{2} \frac{-c\frac{\partial}{\partial u} + \lambda_j}{\lambda_j}\right) - \alpha\right) \psi_1(u,b),$$

and

$$0 = \left(\left(\alpha + \frac{\partial}{\partial u}\right)\left(\prod_{j=1}^{2} \frac{-(c-a)\frac{\partial}{\partial u} + \lambda_j}{\lambda_j}\right) - \alpha\right) \psi_2(u,b).$$

Thus for $i = 1, 2$, we have solutions of the form

$$\psi_i(u,b) = A_i^{(3)} + \sum_{j=1}^{2} A_i^{(j)} e^{R_i^{(j)} u}.$$

Consider the corresponding Lundberg equations

$$0 = (\alpha + R)\left(\prod_{j=1}^{2} \frac{-cR + \lambda_j}{\lambda_j}\right) - \alpha \tag{19}$$

and

$$0 = (\alpha + R)\left(\prod_{j=1}^{2} \frac{-(c-a)R + \lambda_j}{\lambda_j}\right) - \alpha.$$

For each of the two, one solution is 0 and both equations have exactly one negative solution (see again [15]), which are denoted by $R_i^{(1)}$, $i = 1, 2$. The remaining positive solutions are called $R_i^{(2)}$. Thus for $i = 1, 2$, one has

$$\psi_i(u,b) = A_i^{(3)} + \sum_{j=1}^{2} A_i^{(j)} e^{R_i^{(j)} u},$$

where the $A_{i,j}$ depend on the choice of b. Condition (9) gives

$$\lim_{b\to\infty} \psi_1(u,b) = \frac{\alpha + R_1^{(1)}}{\alpha} e^{R_1^{(1)} u} \tag{20}$$

(cf. [15]) and (8) translates into

$$\psi_2(u,b) = A_2^{(1)} e^{R_2^{(1)} u}.$$

Thus, we now have four unknown constants. Two equations are obtained by using (10) for $j = 1, 2$ and the remaining two are found by a comparison of coefficients of the solutions in the IDEs (6) and (7). Altogether, the resulting system of equations is given by

$$\begin{pmatrix} e^{R_1^{(1)} b} & e^{R_1^{(2)} b} & e^{R_1^{(3)} b} & -e^{R_2^{(1)} b} \\ cR_1^{(1)} e^{R_1^{(1)} b} & cR_1^{(2)} e^{R_1^{(2)} b} & cR_1^{(3)} e^{R_1^{(3)} b} & -(c-a)R_2^{(1)} e^{R_2^{(1)} b} \\ \frac{\alpha}{\alpha+R_1^{(1)}} & \frac{\alpha}{\alpha+R_1^{(2)}} & \frac{\alpha}{\alpha+R_1^{(3)}} & 0 \\ \frac{1}{\alpha+R_1^{(1)}} e^{R_1^{(1)} b} & \frac{1}{\alpha+R_1^{(2)}} e^{R_1^{(2)} b} & \frac{1}{\alpha+R_1^{(3)}} e^{R_1^{(3)} b} & -\frac{1}{\alpha+R_2^{(1)}} e^{R_2^{(1)} b} \end{pmatrix} \begin{pmatrix} A_1^{(1)} \\ A_1^{(2)} \\ A_1^{(3)} \\ A_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

3 The Linear Dividend Barrier Strategy

Consider a linear barrier of the form $b_t = b + at$. With this strategy, dividends are paid out with intensity $(c-a)$ whenever R_t reaches b_t, while the reserve increases with intensity a until the next claim occurs. On the other hand, nothing is paid if the surplus is below the barrier.

3.1 Integro-differential equation for $U(u, b)$

For convenience, let us think of the risk process with linear dividend barrier as a piecewise deterministic Markov process (PDMP) with n external states (see e.g. Davis [8] or Rolski et al. [22]), where the transition from state i to state $i+1$ is generated by an $\mathrm{Exp}(\lambda_i)$ random variable. This can again be interpreted as a decomposition of the interclaim time T_i into n exponential distributed summands, see [1]. For $i = 1, \ldots, n-1$ the process only changes the state, for $i = n$ the state moves to state 1 and a claim with distribution function F_Y occurs. The generator $\mathbf{A}$ for a suitable function g (depending on the state i, the risk process and the barrier b) is given by

$$c\frac{\partial g^{(i)}}{\partial u}(u,b) + a\frac{\partial g^{(i)}}{\partial b}(u,b) + \lambda_i \left(g^{(i+1)}(u,b) - g^{(i)}(u,b) \right), \quad (0 \le u < b,\ i = 1, \ldots, n-1),$$
$$c\frac{\partial g^{(n)}}{\partial u}(u,b) + a\frac{\partial g^{(n)}}{\partial b}(u,b) + \lambda_n \left(\int_0^u g^{(1)}(u-y,b)dF_Y(y) - g^{(n)}(u,b) \right), \quad (0 \le u < b).$$

and at the barrier

$$a\frac{\partial g^{(i)}}{\partial b}(u,b) + \lambda_i \left(g^{(i+1)}(u,b) - g^{(i)}(u,b) \right), \quad (u = b,\ i = 1, \ldots, n-1),$$
$$a\frac{\partial g^{(n)}}{\partial b}(u,b) + \lambda_n \left(\int_0^u g^{(1)}(u-y,b)dF_Y(y) - g^{(n)}(u,b) \right), \quad (u = b).$$

To get candidates for the survival probability $U^{(i)}(u) := 1 - \psi^{(i)}(u)$ (for initial state i) we have to solve the equations $\mathbf{A}U^{(i)}(u,b) = 0$ together with the boundary conditions.

From the equations above we get the following integro-differential equation for the survival probability in state 1 (which is the one we are in fact interested in):

$$\prod_{i=1}^{n}\left(\lambda_i - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right) U^{(1)}(u,b) = \prod_{i=1}^{n}\lambda_i \int_0^u U^{(1)}(u-y,b)dF_Y. \tag{21}$$

By continuity at $u=b$, the boundary conditions are

$$\prod_{j=1}^{i-1}\left(\frac{\lambda_j - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}}{\lambda_j}\right)\frac{\partial U^{(1)}}{\partial u}(u,b)\Big|_{u=b} = 0, \quad i=1,\ldots,n,$$

where $\prod_{j=1}^{0}\cdot = 1$. Moreover,

$$\lim_{b\to\infty} U^{(1)}(u,b) = \bar{U}(u)$$

(with $\bar{U}(u)$ denoting the survival probability in the renewal model without dividend payments) and $\lim_{u,b\to\infty} U^{(1)}(u,b) = 1$ if u and b go to infinity uniformly (cf. [14]).

3.1.1 Erlang(2) interarrivals and exponential claims

In this section we look for an explicit solution for the survival probability in case of Erlang$(2,\lambda)$-distributed interclaim times and Exp(α) claim amounts. Write $U(u,b) := U^{(1)}(u,b)$. The integro-differential equation then reads

$$\left(\lambda - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right)^2 U(u,b) = \lambda^2\int_0^u U(u-y,b)\alpha e^{-\alpha y}dy \tag{22}$$

together with the boundary conditions

$$\frac{\partial U}{\partial u}(u,b)\Big|_{u=b} = 0 \tag{23}$$

$$c\frac{\partial^2 U}{\partial u^2}(u,b)\Big|_{u=b} + a\frac{\partial^2 U}{\partial u\,\partial b}(u,b)\Big|_{u=b} = 0, \tag{24}$$

$$\lim_{b\to\infty} U(u,b) = \bar{U}(u), \tag{25}$$

$$\lim_{u,b\to\infty} U(u,b) = 1. \tag{26}$$

Analogous to Section 2, equation (22) can be transformed into a partial differential equation with constant coefficients:

$$\left(\lambda - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right)^2\frac{\partial U}{\partial u}(u,b) + \alpha\left(\lambda - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right)^2 U(u,b) - \alpha\lambda^2 U(u,b) = 0. \tag{27}$$

From (20) we have

$$\bar{U}(u) = 1 - \frac{\alpha + R_{1,1}^{(1)}}{\alpha}e^{R_{1,1}^{(1)}u},$$

where $R_{1,1}^{(1)}$ is the unique negative solution of (19).

We will construct an explicit solution to the above problem. Using fixed-point arguments it follows that there is a unique solution to the problem, so we have actually solved the problem completely (see Albrecher & Kainhofer [4] for details on the uniqueness argument in a related model and Cohen & Down [7] in a queueing framework). The solution of (27) will be of the form

$$\sum_{k=0}^{\infty} e^{S^{(k)}b} \left(A_1^{(k)} \, e^{R_1^{(k)}u} + A_2^{(k)} \, e^{R_2^{(k)}u} + A_3^{(k)} \, e^{R_3^{(k)}u} \right), \tag{28}$$

where for each $k \geq 0$ the pairs $(S^{(k)}, R_j^{(k)})$, $j = 1, 2, 3$, are zeroes of the polynomial

$$P(R, S) = (R + \alpha) \; (\lambda - cR - aS)^2 - \alpha\lambda^2.$$

In the spirit of [23], the initial step $k = 0$ is chosen in order to satisfy (25), i.e.

$$S^{(0)} = 0, \; R_1^{(0)} = 0, \; A_1^{(0)} = 1, A_2^{(0)} = -\frac{R_{1,1}^{(1)} + \alpha}{\alpha}, \; A_3^{(0)} = 0.$$

If $S^{(k)} < 0$ and all $S^{(k)} + R_j^{(k)} < 0$ for all $k \geq 1, j = 1, 2, 3$, then the two conditions (25) and (26) are fulfilled. Thus, if for some k and j, $S^{(k)} + R_j^{(k)} \geq 0$, then necessarily $A_j^{(k)} = 0$. It turns out that one can choose $A_3^{(k)} = 0$ for all $k \geq 0$. Plugging (28) into the original equation (21), one obtains for each k

$$A_2^{(k)} = -\frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} \, A_1^{(k)}.$$

Boundary condition (23) reduces to

$$\sum_{k=0}^{\infty} A_1^{(k)} \left(R_1^{(k)} e^{(S^{(k)} + R_1^{(k)})b} - \frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} \, R_2^{(k)} e^{(S^{(k)} + R_2^{(k)})b} \right) = 0$$

and condition (24) can be rewritten as

$$\sum_{k=0}^{\infty} A_1^{(k)} \left(R_1^{(k)} \, (cR_1^{(k)} + aS^{(k)}) \, e^{(S^{(k)} + R_1^{(k)})b} - \frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} \, R_2^{(k)} \, (cR_2^{(k)} + aS^{(k)}) \, e^{(S^{(k)} + R_2^{(k)})b} \right) = 0$$

For each k, the summand in (28) solves the integro-differential equation (27). For $k = 0$, $S^{(0)} = R_1^{(0)} = 0$, so only the second term produces an error concerning the two boundary conditions above (note that in [14, 23], only one boundary condition had to be satisfied this way). For each of the two conditions, the first summand of a larger index k in (28) will be used to correct for it. However, the second summand will again provide a mismatch with respect to this boundary condition and will itself be corrected by a first summand of higher index etc. It will turn out that these correction terms converge to zero and thus in the limit we have found the exact solution. In fact, the convergence is fast and with only a few terms of the series (28) the approximation to the exact value is satisfying.

The deletion algorithm

So, for a general step k, fix two new steps $\hat{k} > k$ and $k' > k$ such that

$$S^{(k)} + R_2^{(k)} = S^{(k')} + R_1^{(k')} = S^{(\hat{k})} + R_1^{(\hat{k})}. \tag{29}$$

The coefficients $A_1^{(k')}$ and $A_1^{(\hat{k})}$ of the new steps have to solve the linear equations

$$\begin{aligned} \left(\frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha}\right) A_1^{(k)} R_2^{(k)} &= A_1^{(k')} R_1^{(k')} + A_1^{(\hat{k})} R_1^{(\hat{k})}, \qquad (30) \\ \left(\frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha}\right) A_1^{(k)} R_2^{(k)}(cR_2^{(k)} + aS^{(k)}) &= A_1^{(k')} R_1^{(k')}(cR_1^{(k')} + aS^{(k')}) + A_1^{(\hat{k})} R_1^{(\hat{k})}(cR_1^{(\hat{k})} + aS^{(\hat{k})}). \end{aligned}$$

To that end, it is essential that $S^{(k')} \neq S^{(\hat{k})}$ and $R_1^{(k')} \neq R_1^{(\hat{k})}$. In the following it is shown that it is always possible to find two distinct roots $R_1^{(k')}$ and $R_1^{(\hat{k})}$ such that $S^{(k')} + R_1^{(k')} < 0$ and $S^{(\hat{k})} + R_1^{(\hat{k})} < 0$ and at the same time $S^{(k')} + R_2^{(k')}$, $S^{(\hat{k})} + R_2^{(\hat{k})} < 0$ holds.
For each fixed $S < 0$, $P(R, S)$ has three real roots in R which satisfy

$$r_1(S) < 0 < r_2(S) < \frac{\lambda - a\,S}{c} < r_3(S) \tag{31}$$

so that

$$r_2(S) + S < \frac{\lambda}{c} - \left(\frac{a}{c} - 1\right) S;$$

whence $r_2(S) + S < 0$ if

$$S < -\frac{\lambda}{c - a}.$$

Later on we will see from the construction of $S^{(k)}$ that this is guaranteed. Moreover, it turns out that $\lim_{S \to \infty} r_1(S) = -\alpha$, whereas $r_2(S)$ and $r_3(S)$ do not have a finite limit. On the other hand, if we fix R, then $P(R, S) = 0$ has two solutions in S given by

$$s_{1,2}(R) = \frac{a(\alpha + R)(\lambda - c\,R) \pm \sqrt{\alpha\, a^2 \lambda^2 (\alpha + R)}}{a^2(\alpha + R)}.$$

We get that $s_2(R) < 0$ for $R \in (-\infty, r_1(0)) \cup (\min\{\frac{\lambda}{c}, r_2(0)\}, \min\{\frac{\lambda}{c}, r_3(0)\}) \cup (\frac{\lambda}{c}, \infty)$. On the other hand, $s_1(R) > 0$ if $R \in (\max\{\frac{\lambda}{c}, r_1(0)\}, \max\{\frac{\lambda}{c}, r_2(0)\}) \cup (\max\{\frac{\lambda}{c}, r_3(0)\}, \infty)$.

Let us now turn to the determination of $R_1^{(k')}$ and $R_1^{(\hat{k})}$ for a given step k so as to match (29). For that purpose, consider the polynomial

$$P(R, S^{(k)} + R_2^{(k)} - R),$$

which has three real roots $\{\tilde{R}_1, \tilde{R}_2, \tilde{R}_3\}$ in R. A closer look at its behavior reveals that we again have $\tilde{R}_1, \tilde{R}_2 > 0$ and $\tilde{R}_3 < 0$.

In the following it is shown that $S^{(k)} + R_2^{(k)} < 0$, therefore the following choice is possible:

$$\begin{aligned} R_1^{(k')} &:= \tilde{R}_1, \qquad & S^{(k')} &= S^{(k)} + R_2^{(k)} - R_1^{(k')}, \\ R_1^{(\hat{k})} &:= \tilde{R}_2, \qquad & S^{(\hat{k})} &= S^{(k)} + R_2^{(k)} - R_1^{(\hat{k})}. \end{aligned} \tag{32}$$

If $S^{(k)} + R_2^{(k)} < 0$, then clearly both $S^{(k')} + R_2^{(k')} < 0$ and $S^{(\hat{k})} + R_2^{(\hat{k})} < 0$. Moreover, from $R_1^{(k')}, R_1^{(\hat{k})} > 0$ it follows that $S^{(k')}, S^{(\hat{k})} < 0$, as required. Consequently, due to (31) it is always possible to choose $R_2^{(k')}, R_2^{(\hat{k})} < 0$ as the negative solutions of $P(R, S^{(k')}) = 0$.

Summarizing, starting with $S^{(0)} = 0$, choose $R_2^{(0)}$ as the negative zero of $P(R, 0)$, so that

$$S^{(0)} + R_2^{(0)} < 0.$$

Then, the coefficients of two next steps k' and $\hat{k}$ are chosen according to (32) and (30). Subsequently, the same procedure is applied to each of the two steps and so on. By the above considerations and induction, $S^{(k)} + R_2^{(k)} < 0$ holds for all $k \geq 0$. In addition,

$$R_2^{(k')} + S^{(k')} = R_2^{(k)} + S^{(k)} - R_1^{(k')} + R_2^{(k')} < R_2^{(k)} + S^{(k)},$$

since $R_1^{(k')} > 0$ and $R_2^{(k')} < 0$. So this sum decreases in every step of the algorithm and, moreover, $S^{(k')} < S^{(k)}$ (the same argument holds with k' replaced by $\hat{k}$).
A numerical illustration of this solution algorithm will be given in Section 3.2.4.

3.2 Integro-Differential equation for $W(u, b)$

Since one of the boundary conditions will involve the expected discounted dividends of the linear barrier strategy in case the payments are continued after the event of ruin, we will first discuss this variant of the model.

3.2.1 Dividend payments continue after ruin

It is well-known that in this case it suffices to look at the process $z_t = b_t - R_t$ (see e.g. [14]). Dividends are then paid whenever $z_t = 0$. The value function with discounting factor $\delta \geq 0$ is

$$\begin{aligned} V(z) &= \mathbb{E}\left(\int_0^\infty (c-a)\, I_{(z_t=0)} e^{-\delta t}\, dt \;\middle|\; z_0 = z \right), \\ \lim_{z\to\infty} V(z) &= 0, \end{aligned}$$

which is bounded by

$$\int_0^\infty (c-a) e^{-\delta t} dt = (c-a)/\delta. \tag{33}$$

As in Section 3.1 we think of $z = (z_t)_{t\geq 0}$ as a PDMP with n external states. The generator $\mathbf{A}$ for a suitable function g depending on the state i and the process z is given by

$$-(c-a)\frac{\partial g^{(i)}}{\partial z}(z) + \lambda_i \left(g^{(i+1)}(z) - g^{(i)}(z)\right), \quad (z>0,\ i=1,\dots,n-1),$$

$$-(c-a)\frac{\partial g^{(n)}}{\partial z}(z) + \lambda_n \left(\int_0^\infty g^{(1)}(z+y)dF_Y(y) - g^{(n)}(z)\right), \quad (z>0),$$

and at the boundary $z=0$ we get

$$\lambda_n \left(\int_0^\infty g^{(1)}(y)dF_Y(y) - g^{(n)}(0)\right) \quad \text{and} \quad \lambda_i \left(g^{(i+1)}(0) - g^{(i)}(0)\right), \quad (i=1,\dots,n-1).$$

Here, a function g is *suitable*, if for all states i it is absolutely continuous on $(0,\infty)$ and

$$\mathbb{E}\left(\sum_{j,\sigma_j<t} \left|g^{(i)}(z_{\sigma_j}) - g^{(i)}(z_{\sigma_j}-)\right|\right) < \infty \quad \forall t>0,$$

where $\{\sigma_i\}_{i\geq 1}$ denote the claim occurrence times (this condition is certainly fulfilled if g is bounded). From [22, Thm. 11.2.3] we know that for a suitable function f which fulfills

$$\mathbf{A}f(z) - \delta f(z) + \gamma(z) = 0,$$

the relation

$$f(z_0) = \mathbb{E}\left(\int_0^{t_0} \gamma(z_t)e^{-\delta t}dt + e^{-\delta t_0} f_{ter}(z_{t_0})\right),$$

holds for any $t_0>0$ (and for a bounded function f_{ter} the second summand vanishes for $t_0\to\infty$).
Let $V^{(i)}(z)$ denote the value function for initial state i and set $\gamma(z)=(c-a)I_{(z=0)}$. Then we can write

$$V^{(i)}(z,t_0) = \mathbb{E}\left(\int_0^{t_0} \gamma(z_t)e^{-\delta t}dt + e^{-\delta t_0} V_{ter}(i,z_{t_0})\right).$$

From the upper bound (33) it is clear that V_{ter} is bounded and hence neglible in the limit $t_0\to\infty$ in the above expression. Hence $V^{(i)}(z)=V^{(i)}(z,\infty)$ is given as the solution of

$$-(c-a)\frac{\partial V^{(i)}}{\partial z}(z) + \lambda_i \left(V^{(i+1)}(z) - V^{(i)}(z)\right) - \delta V^{(i)}(z) = 0, \quad (i=1,\dots,n-1,\ z>0) \tag{34}$$

and

$$-(c-a)\frac{\partial V^{(n)}}{\partial z}(z) + \lambda_n \left(\int_0^\infty V^{(1)}(z+y)dF_Y(y) - V^{(n)}(z)\right) - \delta V^{(n)}(z) = 0, \quad (z>0). \tag{35}$$

For $z=0$ we get the boundary conditions

$$\lambda_i \left(V^{(i+1)}(0) - V^{(i)}(0)\right) - \delta V^{(i)}(0) + (c-a) = 0,\ i = 1, \dots, n-1,$$
$$\lambda_n \left(\int_0^\infty V^{(1)}(y) dF_Y(y) - V^{(n)}(0)\right) - \delta V^{(n)}(0) + (c-a) = 0.$$

Moreover, continuity of $V^{(i)}(z)$ implies $\frac{\partial V^{(i)}}{\partial z}(0) = -1$ for $i = 1, \dots, n$. Eventually, we are interested in the quantity $V(z) = V^{(1)}(z)$. From (34) we have

$$V^{(i+1)}(z) = \prod_{j=1}^{i} \left(\frac{\delta + \lambda_j + (c-a)\frac{\partial \cdot}{\partial z}}{\lambda_j} \right) V^{(1)}(z)$$

and together with (35) we arrive at

$$\prod_{i=1}^{n} \lambda_i \int_0^\infty V^{(1)}(z+y) dF_Y(y) = \prod_{i=1}^{n} \left(\delta + \lambda_i + (c-a)\frac{\partial \cdot}{\partial z} \right) V^{(1)}(z), \quad (z > 0) \tag{36}$$

and for the boundary $z = 0$

$$\prod_{j=1}^{i-1} \left(\frac{\delta + \lambda_j + (c-a)\frac{\partial \cdot}{\partial z}}{\lambda_j} \right) V^{(1)}(0) = -1, \quad (i = 1, \dots, n). \tag{37}$$

3.2.2 Dividend payments stop at ruin

If the dividend payments stop at the event of ruin, the value function of the expected discounted dividend payments is given by

$$W(u,b) = \mathbb{E}\left(\int_0^{\tau(u)} (c-a)\ I_{(R_t = b_t)} e^{-\delta t}\ dt\ \middle|\ R_0 = u,\ b_0 = b \right).$$

The PDMP approach analogous to Section 3.2.1 leads to the partial integro-differential equation

$$\prod_{i=1}^{n} \lambda_i \int_0^u W(u-v,b) dF_Y(v) = \prod_{i=1}^{n} \left((\delta + \lambda_i) - c\frac{\partial \cdot}{\partial u} - a\frac{\partial \cdot}{\partial b} \right) W(u,b),$$

and boundary conditions are

$$\begin{aligned} V(z) &= \lim_{u \to \infty} W(u, u+z), \\ 1 &= \prod_{k=1}^{j-1} \left(\frac{(\delta + \lambda_k) - c\frac{\partial \cdot}{\partial u} - a\frac{\partial \cdot}{\partial b}}{\lambda_k} \right) \frac{\partial W(u,b)}{\partial u}\bigg|_{u=b},\ j = 1, \dots, n. \end{aligned}$$

3.2.3 Erlang(2) interarrivals and exponential claims

In the special case of Erlang$(2, \lambda)$ distributed interclaim times and Exp(α) distributed claim amounts, the integro-differential equation (36) can (similarly to the previous sections) be transformed into the ordinary linear differential equation with constant coefficients

$$(c-a)^2 V'''(z) + (2(\delta+\lambda)(c-a) - \alpha(c-a)^2)V''(z) \\ + ((\delta+\lambda)^2 - 2\alpha(\delta+\lambda)(c-a))V'(z) + (\alpha\lambda^2 - \alpha(\delta+\lambda)^2)V(z) = 0,$$

with a solution of the form

$$V(z) = \hat{A}_1\, e^{\hat{R}_1 z} + \hat{A}_2\, e^{\hat{R}_2 z} + \hat{A}_3\, e^{\hat{R}_3 z},$$

where $\{\hat{R}_1, \hat{R}_2, \hat{R}_3\}$ denote the roots of the polynomial

$$P_1(R) = (c-a)^2R^3 + (2(\delta+\lambda)(c-a) - \alpha(c-a)^2)R^2 + ((\delta+\lambda)^2 - 2\alpha(\delta+\lambda)(c-a))R + (\alpha\lambda^2 - \alpha(\delta+\lambda)^2).$$

It is easy to see that $P_1(R)$ has three real roots, two of which are negative. The condition $\lim_{z\to\infty} V(z) = 0$ implies that if $\hat{R}_3$ refers the positive root, $\hat{A}_3 = 0$.

Under the assumption of exponential claim amounts, The boundary conditions (37) can be rewritten as

$$V'(0) = -1 \quad \text{and} \quad V''(0) = \frac{\delta}{c-a}.$$

Altogether this leads to the explicit solution

$$V(z) = \hat{A}_1\, e^{\hat{R}_1 z} + \hat{A}_2\, e^{\hat{R}_2 z}. \tag{38}$$

with

$$\hat{A}_1 = \frac{\hat{R}_2 + \frac{\delta}{c-a}}{\hat{R}_1\hat{R}_2 - \hat{R}_1^2} \quad \text{and} \quad \hat{A}_2 = \frac{\hat{R}_1 + \frac{\delta}{c-a}}{\hat{R}_2^2 - \hat{R}_1\hat{R}_2}.$$

The function $V(z)$ is differentiable and bounded. Thus it fulfills the conditions of [22, Thm.11.2.3] and is indeed the solution to the problem.

For $W(u, b)$, one has to solve

$$\left(\delta + \lambda - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right)^2 W(u, b) = \lambda^2 \int_0^u W(u-v, b)\alpha e^{-\alpha v} dv, \tag{39}$$

which by applying the operator $\left(\frac{\partial\cdot}{\partial u} + \alpha\right)$ leads to the PDE

$$\left(\delta + \lambda - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right)^2 \frac{\partial W(u,b)}{\partial u} + \alpha\left(\delta + \lambda - c\frac{\partial\cdot}{\partial u} - a\frac{\partial\cdot}{\partial b}\right)^2 W(u, b) - \alpha\lambda^2 W(u, b) = 0. \tag{40}$$

The boundary conditions simplify to

$$\lim_{b\to\infty} W(u, b) = 0, \tag{41}$$

$$\frac{\partial W(u,b)}{\partial u}\Big|_{u=b} = 1, \tag{42}$$

$$c\frac{\partial^2 W(u,b)}{\partial u^2}\Big|_{u=b} + a\frac{\partial^2 W(u,b)}{\partial u \partial b}\Big|_{u=b} = \delta, \tag{43}$$

$$\lim_{u\to\infty} W(u, u+z) = V(z), \tag{44}$$

where $V(z)$ is given by (38).

(40) is a homogeneous differential equation of third order with constant coefficients and we will construct a solution of the form

$$W(u,b) = \sum_{k=0}^{\infty} e^{S^{(k)}b}(A_1^{(k)}\, e^{R_1^{(k)}u} + A_2^{(k)}\, e^{R_2^{(k)}u} + A_3^{(k)}\, e^{R_3^{(k)}u}).$$

where $R_i^{(k)}(S^{(k)})$, $(i = 1, 2, 3)$, denote the roots of the polynomial

$$P(R, S^{(k)}) = R\left((\delta+\lambda) - cR - aS^{(k)}\right)^2 + \alpha\left((\delta+\lambda) - cR - aS^{(k)}\right)^2 - \alpha\lambda^2$$

for a given value of $S^{(k)}$.
The main idea is again that each of the above summands solves (40) and the combination of such solutions is used to match all the necessary boundary conditions. Substitution of each term in the original integro-differential equation (39) gives

$$\frac{A_1^{(k)}}{R_1^{(k)} + \alpha} + \frac{A_2^{(k)}}{R_2^{(k)} + \alpha} + \frac{A_3^{(k)}}{R_3^{(k)} + \alpha} = 0$$

As in the case of the survival probability, the choice $A_3^{(k)} = 0$ for all $k \geq 0$ turns out to be feasible and hence

$$A_2^{(k)} = -\frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} A_1^{(k)}.$$

So we actually look for a solution of the form

$$W(u,b) = \sum_{k=0}^{\infty} A_1^{(k)}\, e^{S^{(k)}b}\left(e^{R_1^{(k)}u} - \frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha}\, e^{R_2^{(k)}u}\right).$$

Condition (41) is automatically satisfied as long as $S^{(k)} < 0$ for $k \geq 0$. In view of (38) and (44), define for $k = 0$

$$\begin{aligned} A_1^{(0)} &:= \hat{A}_1, & S^{(0)} &:= \hat{R}_1 & R_1^{(0)} &:= -\hat{R}_1, \\ A_1^{(1)} &:= \hat{A}_2, & S^{(1)} &:= \hat{R}_2 & R_1^{(1)} &:= -\hat{R}_2. \end{aligned}$$

At the same time this choice already determines the values of $R_2^{(0)}$ and $R_2^{(1)}$. Note also that the combination of $R_1^{(0)}$ and $R_1^{(1)}$ is possible, since $P(-R, R) = -P_1(R)$. By construction, $R_1^{(0)}$ and $R_1^{(1)}$ are positive and for $S^{(i)} = \hat{R}_i$ the polynomial $P(R, S^{(i)})$ also has a negative root, the value of which is assigned to $R_2^{(i)}$ $(i = 0, 1)$. If both $S^{(0)} + R_2^{(0)}$ and $S^{(1)} + R_2^{(1)}$ and all remaining sums $S^{(k)} + R_i^{(k)}$, $k \geq 2$, $i \in \{1, 2\}$, are negative, condition (44) will be

fulfilled.
Let us now turn the attention to (42). Inserting the above choice of the first terms, this condition reads

$$- \hat{A}_1 \hat{R}_1 - \hat{A}_2 \hat{R}_2 - \hat{A}_1 \frac{R_2^{(0)} + \alpha}{\alpha - \hat{R}_1} R_2^{(0)} e^{(R_2^{(0)} + \hat{R}_1)b} - \hat{A}_2 \frac{R_2^{(1)} + \alpha}{\alpha - \hat{R}_2} R_2^{(1)} e^{(R_2^{(1)} + \hat{R}_2)b} + \sum_{k=2}^{\infty} A_1^{(k)} \left(R_1^{(k)} e^{(R_1^{(k)} + S^{(k)})b} - \frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} R_2^{(k)} e^{(R_2^{(k)} + S^{(k)})b} \right) = 1.$$

From $V'(0) = -1$ we know that $\hat{A}_1 \hat{R}_1 + \hat{A}_2 \hat{R}_2 = -1$, so the above equation can be simplified to

$$- \hat{A}_1 \frac{R_2^{(0)} + \alpha}{\alpha - \hat{R}_1} R_2^{(0)} e^{(R_2^{(0)} + \hat{R}_1)b} - \hat{A}_2 \frac{R_2^{(1)} + \alpha}{\alpha - \hat{R}_2} R_2^{(1)} e^{(R_2^{(1)} + \hat{R}_2)b} + \sum_{k=2}^{\infty} A_1^{(k)} \left(R_1^{(k)} e^{(R_1^{(k)} + S^{(k)})b} - \frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} R_2^{(k)} e^{(R_2^{(k)} + S^{(k)})b} \right) = 0,$$

which will be fulfilled by an appropiate definition of the coefficients $A_1^{(k)}$. However, in addition we have to satisfy the boundary condition (43). Inserting all the chosen initial values and using the identity $\hat{A}_1 \hat{R}_1^2 + \hat{A}_2 \hat{R}_2^2 = \delta/(c-a)$, (43) can be written as

$$- \hat{A}_1 R_2^{(0)} \frac{R_2^{(0)} + \alpha}{\alpha - \hat{R}_1} (c R_2^{(0)} + a \hat{R}_1) e^{(R_2^{(0)} + \hat{R}_1)b} - \hat{A}_2 R_2^{(1)} \frac{R_2^{(1)} + \alpha}{\alpha - \hat{R}_2} (c R_2^{(1)} + a \hat{R}_2) e^{(R_2^{(1)} + \hat{R}_2)b}$$
$$+ \sum_{k=2}^{\infty} \left(A_1^{(k)} R_1^{(k)} (c R_1^{(k)} + a S^{(k)}) e^{(R_1^{(k)} + S^{(k)})b} - A_1^{(k)} R_2^{(k)} \frac{R_2^{(k)} + \alpha}{R_1^{(k)} + \alpha} (c R_2^{(k)} + a S^{(k)}) e^{(R_2^{(k)} + S^{(k)})b} \right) = 0.$$

The procedure needed now to create the correction terms for these two remaining boundary conditions is analogous to the case of the survival probability in Section 3.1.1 and will therefore not be given in detail. The additional factor δ in all the polynomial equations does not cause any harm. A different feature of the present case as compared to Section 3.1.1 is that in order to satisfy condition (44), here coefficients for both the steps $k = 0$ and $k = 1$ have to be assigned, so we start with two (instead of one) terms to be deleted and the algorithm of Section 3.1.1 has to be applied to each of the two separately; the sequence $(S^{(k)})_{k \in \mathbb{N}}$ then consists of strictly decreasing subsequences and tends to $-\infty$ again.

3.2.4 Numerical Illustration

In the following, the exact solutions derived in the previous sections are approximated by truncating the series after 18 terms. The accuracy of this approximation is already striking. One should note that these values are obtained virtually instantaneously, whereas Monte Carlo simulation (including variance reduction procedures) takes several minutes to achieve a comparable accuracy. Tables 1 and 2 show exact and simulated values of the survival probability $U^{lin}(u,b)$ and the expected discounted dividends $W^{lin}(u,b)$ in the linear barrier model with Erlang(2) interclaim times and exponential claim sizes for two different parameter sets. In these tables, the Monte Carlo estimates are based on $N = 10000$ iterations. As an illustration, for $N = 20000$ iterations, one obtains the simulation estimates $U^{lin}(1.1, 2) = 0.51925$ and $W^{lin}(1.1, 2) = 0.00046$ in Table 2, which is still not fully satisfying. One should note in this context that opposed to horizontal barrier models, here one has positive probability of survival of the trajectories, which increases the simulation time and effort.

4 Comparing the two dividend models

The availability of exact solutions provides a quick way to compare the two dividend models investigated in this paper. Note that with the linear barrier strategy, dividends are paid with intensity $c - a$ at the time-dependent barrier, while the process increases with intensity a. On the other hand, with the threshold strategy in its usual notation, dividends are paid with intensity a_{thr} and the surplus process increases with intensity $c - a_{thr}$ (note that one needs $(c - a_{thr}) > \frac{\lambda}{2\alpha}$ to ensure a positive survival probability).
One motivation for the introduction of the threshold dividend model was the positive survival probability, while the expected discounted dividend payments are still of reasonable size. Let us assume that we consider the survival probabilities U and the expected sums of discounted dividend payments W as the only quantities of interest and that we compare the two proposed dividend strategies at time 0 on that basis. Then, at least for larger values of the discounting factor, dividends earned at a rather late stage do not provide a substantial contribution to the overall sum of discounted dividend payments and hence it seems preferable to focus on securing survival once time has evolved. At the same time, dividends paid out at an earlier stage contribute significantly to the overall value of the discounted sum of dividend payments. For large δ, this aspect is perhaps better captured by linear dividend barrier models (where the barrier departs from the ruin level as time evolves) than by threshold models (where the payment strategy is not "safer" at later times).
The numerical values worked out in this section are intended to give an impression on how the performance of the dividend strategy differs in various regions of the parameter space.

In Table 3, for a given set parameters including the value b of the linear barrier model, the threshold b_{thr} is calibrated in such a way that the expected dividend payments of the two strategies are of comparable size. It turns out that in this case also the survival probabilities are comparable. However, if instead b_{thr} is more than halved, then the survival probabilities are not affected, whereas the expected dividends are much higher (see the two columns on the right of Table 3).
Table 4 depicts a situation where a_{thr} and b_{thr} are chosen so that the ruin probabilities of the two strategies are comparable, but where then the linear barrier strategy outperforms the layer strategy in terms of expected dividend payments. Table 5 illustrates the importance of the discount factor δ in comparing the performance of the two strategies.
Table 6 shows a parameter choice with comparable survival probabilities, where the linear barrier strategy is preferable although the discount factor is of moderate size ($\delta = 0.03$). For still smaller values of δ, the situation is reversed again (see the two columns on the right of Table 6).

This indicates that for every parameter setting and initial surplus u a critical value $\delta^*(u)$ exists such that for $\delta > \delta^*(u)$ the linear barrier strategy performs better and for $\delta < \delta^*(u)$ the threshold strategy is to be preferred (while the level of survival probability is not affected by the choice of δ).
Finally, in Table 7 we indicate a combination of parameters, for which much more risk must be taken with the threshold strategy to achieve expected dividends of the order of the linear barrier model.

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	$U^{lin}(u,3)$		$W^{lin}(u,3)$	
u	Exact	Simulation	Exact	Simulation
2.1	0.733224	0.7380	1.46862	1.46972
2.2	0.739212	0.7333	1.54505	1.53293
2.3	0.744364	0.7462	1.62477	1.65721
2.4	0.748668	0.7506	1.70782	1.72777
2.5	0.752118	0.7574	1.79422	1.80451
2.6	0.754721	0.7543	1.88392	1.89729
2.7	0.756511	0.7542	1.97677	1.95437
2.8	0.757559	0.7615	2.07247	2.04798
2.9	0.758001	0.7540	2.17051	2.18002
3.0	0.758073	0.7554	2.27010	2.27564

Table 1: Exact and simulated values for the survival probability and the expected sum of discounted dividend payments for $\lambda = n = 2$, $\alpha = 1$, $\delta = 0.03$, $c = 1.5$, $a = 0.7$

	$U^{lin}(u,2)$		$W^{lin}(u,2)$	
u	Exact	Simulation	Exact	Simulation
1.1	0.518345	0.5226	0.000442681	0.00060890
1.2	0.536764	0.5355	0.000897554	0.00112367
1.3	0.554457	0.5453	0.001819520	0.00169736
1.4	0.571422	0.5630	0.003687090	0.00398161
1.5	0.587612	0.5858	0.007464610	0.00782931
1.6	0.602876	0.6070	0.015079200	0.01474820
1.7	0.616821	0.6167	0.030302600	0.02912370
1.8	0.628532	0.6319	0.060134400	0.06046390
1.9	0.636225	0.6347	0.115667000	0.11660100
2.0	0.638223	0.6319	0.204578000	0.20514300

Table 2: Exact and simulated values for the survival probability and the expected sum of discounted dividend payments for $\lambda = 4$, $n = 2$, $\alpha = 1.5$, $\delta = 0.03$, $c = 5/3$, $a = 4/3$

u	$U^{lin}(u,2)$	$U^{thr}(u,35)$	$W^{lin}(u,2)$	$W^{thr}(u,35)$	$U^{thr}(u,15)$	$W^{thr}(u,15)$
1.0	0.910725	0.912509	2.47362	2.94955	0.912509	7.89945
1.1	0.921141	0.923443	2.56674	2.99669	0.923443	8.02571
1.2	0.930043	0.933011	2.66011	3.03995	0.933011	8.14155
1.3	0.937560	0.941383	2.75400	3.07981	0.941383	8.24831
1.4	0.943794	0.948709	2.84862	3.11672	0.948709	8.34716
1.5	0.948818	0.955119	2.94414	3.15106	0.955119	8.43914
1.6	0.952685	0.960728	3.04064	3.18318	0.960728	8.52515
1.7	0.955437	0.965636	3.13812	3.21335	0.965636	8.60597
1.8	0.957132	0.969931	3.23650	3.24185	0.969931	8.68228
1.9	0.957896	0.973689	3.33565	3.26888	0.973689	8.75468
2.0	0.958029	0.976977	3.43538	3.29465	0.976977	8.82371

Table 3: Comparison for $\alpha = 2$, $\lambda = 2$, $\delta = 0.03$, $c = 1.1$, $a = 0.55$, $a_{thr} = 0.55$

u	$U^{lin}(u,20)$	$U^{thr}(u,25)$	$W^{lin}(u,20)$	$W^{thr}(u,25)$
10	0.270068	0.221158	0.972399	0.147233
11	0.285057	0.235981	1.171740	0.166242
12	0.298725	0.250344	1.410030	0.187142
13	0.311005	0.264260	1.695140	0.210175
14	0.321820	0.277744	2.036520	0.235608
15	0.331085	0.290809	2.445470	0.263735
16	0.338698	0.303468	2.935500	0.294881
17	0.344551	0.315733	3.522600	0.329407
18	0.348533	0.327617	4.225038	0.367710
19	0.350586	0.339132	5.059620	0.410232
20	0.351000	0.350288	6.019980	0.457460

Table 4: Comparison for $\alpha = 0.5$, $\lambda = 4$, $c = 4.2$, $\delta = 0.08$, $a = 0.6$, $a_{thr} = 0.1$

			$\delta = 0.03$		$\delta = 0.1$	
u	$U^{lin}(u,1.5)$	$U^{thr}(u,=2.5)$	$W^{lin}(u,1.5)$	$W^{thr}(u,=2.5)$	$W^{lin}(u,1.5)$	$W^{thr}(u,=2.5)$
0.5	0.598238	0.522446	2.84655	4.13162	0.98854	0.919823
0.6	0.619711	0.545412	2.99965	4.33540	1.06783	0.976708
0.7	0.637969	0.566298	3.14423	4.52543	1.14843	1.032120
0.8	0.653200	0.585293	3.28100	4.70307	1.23070	1.086340
0.9	0.665577	0.602567	3.41055	4.86955	1.31493	1.139630
1.0	0.675266	0.618277	3.53345	5.02600	1.40141	1.192220
1.1	0.682442	0.632565	3.65021	5.17343	1.49034	1.244320
1.2	0.687309	0.645560	3.76139	5.31275	1.58188	1.296120
1.3	0.690138	0.657380	3.86771	5.44480	1.67609	1.347810
1.4	0.691330	0.668132	3.97017	5.57032	1.77286	1.399530
1.5	0.691525	0.677914	4.07045	5.69000	1.87191	1.451440

Table 5: Comparison for $\alpha = 2$, $\lambda = 2$, $c = 0.8$, $a = 0.2$, $a_{thr} = 0.25$

			$\delta = 0.03$		$\delta = 0.01$	
u	$U^{lin}(u,10)$	$U^{thr}(u,20)$	$W^{lin}(u,10)$	$W^{thr}(u,20)$	$W^{lin}(u,10)$	$W^{thr}(u,20)$
9.0	0.611476	0.622957	8.38890	5.44013	14.2771	22.0554
9.1	0.611840	0.625355	8.48463	5.47752	14.3777	22.1631
9.2	0.612144	0.627722	8.58098	5.51489	14.4781	22.2699
9.3	0.612390	0.630058	8.67791	5.55226	14.5784	22.3761
9.4	0.612584	0.632363	8.77540	5.58962	14.6786	22.4814
9.5	0.612729	0.634639	8.87343	5.62698	14.7788	22.5860
9.6	0.612831	0.636886	8.97194	5.66435	14.8788	22.6900
9.7	0.612895	0.639104	9.07090	5.70172	14.9789	22.7931
9.8	0.612930	0.641292	9.17025	5.73910	15.0788	22.8956
9.9	0.612944	0.643453	9.26991	5.77649	15.1788	22.9974
10.0	0.612946	0.645586	9.36982	5.81389	15.2788	23.0986

Table 6: Comparison for $\alpha = 0.5$, $\lambda = 2$, $c = 2.5$, $a = 0.5$, $a_{thr} = 0.4$.

u	$U^{lin}(u,15)$	$U^{thr}(u,20)$	$W^{lin}(u,15)$	$W^{thr}(u,20)$
14.0	0.210288	0.00589161	3.36797	3.50556
14.1	0.210418	0.00599345	3.45753	3.52634
14.2	0.210526	0.00609513	3.54874	3.54716
14.3	0.210615	0.00619665	3.64152	3.56801
14.4	0.210685	0.00629801	3.73579	3.58889
14.5	0.210737	0.00639921	3.83143	3.60981
14.6	0.210774	0.00650025	3.92832	3.63076
14.7	0.210797	0.00660113	4.02628	3.65174
14.8	0.210809	0.00670185	4.12513	3.67275
14.9	0.210814	0.00680241	4.22464	3.69389
15.0	0.210815	0.00690281	4.32454	3.71488

Table 7: Comparison for $\alpha = 0.25$, $\lambda = 2$, $c = 4.2$, $\delta = 0.02$, $a = 1.9$, $a_{thr} = 0.19$