Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

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R. Becker, B. Vexler

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OPTIMAL CONTROL OF THE CONVECTION-DIFFUSION EQUATION USING STABILIZED FINITE ELEMENT METHODS

R. BECKER[†] AND B. VEXLER[‡]

Abstract. In this paper we analyze discretization of optimal control problems governed by convection-diffusion equations which are subject to pointwise control constraints. We present a stabilization scheme which leads to improved approximate solutions even on corse meshes in the convection dominated case. Moreover, the in general different approaches "optimize-then-discretize" and "discretize-then-optimize" coincide for the proposed discretization scheme. This allows for a symmetric optimality system on the discrete level and optimal order of convergence.

Key words. optimal control, stabilized finite elements, error estimates, pointwise constraints

AMS subject classifications. 49K20, 49M25, 65N15, 65N30

1. Introduction. This paper is concerned with the discretization of optimal control problems using stabilized finite element methods. We consider the following optimal control problem governed by linear convection diffusion reaction equations on a polygonal domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

$$\text{Minimize } J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}, \quad u \in V, q \in Q_{ad} \tag{1.1}$$

subject to

$$-\varepsilon\Delta u + \beta \cdot \nabla u + \sigma u = f + q \quad \text{in } \Omega\,, \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega\,. \tag{1.3}$$

The state variable u is searched for in the space $V = H^1_0(\Omega)$ and the admissible set $Q_{ad} \subset Q = L^2(\Omega)$ is given by pointwise box constraints, i.e.:

$$Q_{ad} = \{q \in L^2(\Omega) \ : \ a \le q \le b \quad \text{a.e. in } \Omega\},$$

where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$.

For the data of the problem we assume: $f, u_d \in L^\infty(\Omega)$, $\sigma \in L^\infty(\Omega)$ with $\sigma \ge \sigma_0 > 0$ and $\alpha > 0$. To simplify the notation we set throughout $\sigma = 1$. Further we assume β to be a constant vector of length $\|\beta\|$.

It is well known that for convection dominated problems standard finite element discretizations applied to the equations (1.2) – (1.3) lead to strongly oscillatory solutions unless the mesh size h is sufficiently small with respect to the ratio between ε and $\|\beta\|$. Several methods are known to improve the approximation properties of the pure Galerkin discretization and to reduce the oscillatory behavior, see e.g. [6, 13, 15, 21, 22].

In [9] the authors apply the SUPG method (streamline upwind Petrov Galerkin method, see e.g. [15]) to the optimal control problem (1.1) – (1.3). They discuss two different approaches to the discretization of the optimal control problem: "optimize-then-discretize" and "discretize-then-optimize". In the "optimize-then-discretize" approach first the necessary optimality conditions are established on the continuous

[†]Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour BP 1155, 64013 PAU Cedex, France, roland.becker@univ-pau.fr

[‡]Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Straße 69, 4040 Linz, Austria, boris.vexler@oeaw.ac.at

level consisting of the state, adjoint and the optimality equations, and then these equations are discretized using a stabilized finite element scheme, e.g. SUPG. In the "discretize-then-optimize" approach the state equation is discretized and then the optimality system for the finite dimensional optimization problem is derived. It is well known that these two approaches lead to the same discretization scheme provided a pure Galerkin discretization is used. However, in the presence of stabilization terms these approaches may differ. In [9] it is shown by numerical computations that for the SUPG discretization the "optimize-then-discretize" approach leads to better asymptotic convergence properties. However, the "discretize-then-optimize" approach has the important advantage of consistency of the state and the adjoint equations on the discrete level which is reflected in the fact that the corresponding optimality system is symmetric. In [18] the authors propose a method with similar behavior for a SUPG-like discretization, based on a non-symetric modification of the Lagrangian.

In this paper we analyze a stabilization method, which leads to symmetric optimality systems and has optimal order of convergence. For the resulting discretization scheme the approaches "optimize-then-discretize" and "discretize-then-optimize" coincide. The presented method uses standard finite element discretization with stabilization based on local projections (called LPS-method), see in [13] for convection diffusion reaction equations and for the Stokes-, Oseen- and Navier-Stokes equations see [2, 4, 5]. The control space is likewise discretized by first-order finite elements.

The main contribution of this paper is the a priori error analysis of the discretization of the optimal control problem (1.1) – (1.3) by stabilized finite elements. For both the unconstrained case ($Q_{ad} = Q$) and the constrained case ($Q_{ad} \neq Q$) we obtain the estimate of order $\mathcal{O}(h^{3/2})$ for the L^2-error in the control, state and the adjoint state.

For the a priori error analysis of pure Galerkin finite element discretizations for optimal control problems with pointwise inequality constraints, we refer e.g. to [1, 10, 14, 20]. In [24] discretization by stabilized finite element for an optimal control problem governed by Stokes equations is analyzed.

Our results are optimal for the following two reasons: First, it is well known that stabilized finite elements leads to optimal order of convergence of $\mathcal{O}(h^{3/2})$ in $L^2(\Omega)$-norm for the convection diffusion reaction equation (1.2) – (1.3) on general quasi-uniform meshes, see e.g. [26]. Second, the presence of control constraints leads to the fact, that the optimal control $\bar{q}$ is in general not in $H^2(\Omega)$ and only $\mathcal{O}(h^{3/2})$ convergence can be expected for the piecewise (bi)linear discretization of the control space, see e.g. [23].

The outline of the paper is as follows: In the next section we discuss the precise formulation of the optimal control problem and the optimality conditions. In Section 3 we describe the discretization of (1.1) – (1.3) and formulate our main results. In Section 4 we prove an error estimate for the optimal control problem without control constraints. The extension to the case with pointwise control constraints is presented in Section 5. In Section 6 we present numerical results.

2. Optimal control problem. In this section we discuss the optimality conditions for the optimal control problem (1.1) – (1.3).

A weak solution $u \in V = H_0^1(\Omega)$ of the state equation (1.2) – (1.3) is determined by

$$a(u, v) = (f + q, v) \quad \forall v \in V$$

using the bilinear form $a\colon V \times V \to \mathbb{R}$ given by:

$$a(u, v) = \varepsilon(\nabla u, \nabla v) + (\beta \cdot \nabla u, v) + (u, v).$$

Throughout the paper we make the following assumption on the domain Ω:

ASSUMPTION 1. *There exists $s \in \mathbb{R}$, $s > d$ such that the weak solution $v \in H_0^1(\Omega)$ of*

$$-\varepsilon\Delta v + \beta \cdot \nabla v + v = g \quad \text{in } \Omega\,, \tag{2.1}$$

$$v = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

satisfies the a priori error estimates:

$$\|v\|_{W^{2,p}(\Omega)} \leq c\,\|g\|_{L^p(\Omega)} \quad \text{for all } g \in L^p(\Omega), \quad 2 \leq p \leq s.$$

REMARK 2.1. *For $d = 2$ a sufficient condition for the above assumption is the convexity of the domain Ω, see e.g. [12]. For $d = 3$ one needs an additional angle condition, see e.g. [16].*

THEOREM 2.2. *Under the above assumption the optimal control problem* (1.1) – (1.3) *admits a unique solution $(\bar{q}, \bar{u}) \in (Q_{ad} \cap W^{1,\infty}(\Omega)) \times (H_0^1(\Omega) \cap W^{2,s}(\Omega))$.*

The proof of this theorem follows standard arguments, see e.g. [11, 19, 25].

REMARK 2.3. *In the case $Q = Q_{ad}$, i.e. in the absence of control constraints, the optimal control $\bar{q}$ has better regularity properties, i.e. $\bar{q} \in W^{2,s}(\Omega)$.*

We denote by $S\colon Q \to H_0^1(\Omega) \cap H^2(\Omega)$ the solution operator of the state equations (1.2) – (1.3) and introduce the reduced cost functional $j\colon Q \to \mathbb{R}$ by:

$$j(q) = J(q, Sq).$$

This allows us to eliminate the state equation and to reformulate the optimization problem as:

$$\text{Minimize} \quad j(q), \quad q \in Q_{ad}.$$

The reduced cost functional j is continously differentiable and its derivatives are given in the following lemma:

LEMMA 2.4. *There holds:*

$$j'(q)(\delta q) = (z, \delta q) + \alpha(q, \delta q),$$

where $z \in H_0^1(\Omega) \cap W^{2,s}(\Omega)$ is the solution of the adjoint equation:

$$-\varepsilon\Delta z - \beta \cdot \nabla z + z = u - u_d \quad \text{in } \Omega\,, \tag{2.3}$$

$$z = 0 \quad \text{on } \partial\Omega\,, \tag{2.4}$$

where $u = Sq$ is the associated state to q; z is called adjoint state.

Proof. Using standard arguments, see e.g. [25], we obtain the above formula for the derivatives of j with $z \in H_0^1(\Omega)$. Due to the fact that $u \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and consequently $u - u_d \in L^\infty(\Omega)$, Assumption 1 implies that $z \in W^{2,s}(\Omega)$. □

Due to the fact that Q_{ad} is convex the necessary and sufficient optimality condition for this problem can be formulated as follows:

$$j'(\bar{q})(\delta q - \bar{q}) \geq 0 \quad \forall \delta q \in Q_{ad}. \tag{2.5}$$

REMARK 2.5. *Using Lemma 2.4 condition* (2.5) *can be rewritten as:*

$$(\bar{z} + \alpha\bar{q}, \delta q - \bar{q}) \geq 0 \quad \forall \delta q \in Q_{ad},$$

where $\bar{z}$ is the associated adjoint state to $\bar{q}$. This condition together with the state equation (1.2) – (1.3) *and the adjoint equation* (2.3) – (2.4) *builds the optimality system for optimal control problem* (1.1) – (1.3)*.*

In the sequel we will need the gradient $\nabla j(q) \in L^2(\Omega)$ of the reduced cost functional j, which is given as the Riesz representative of $j'(q)(\cdot)$ by:

$$\nabla j(q) = z + \alpha q.$$

Obviously, there holds $\nabla j(\bar{q}) \in W^{1,\infty}(\Omega)$.

Since j is quadratic, the second derivative $j''(q)(\delta q, \tau q)$ does not depend on q and there holds:

$$j''(q)(\delta q, \delta q) \geq \alpha \|\delta q\|^2_{L^2(\Omega)} \quad \forall \delta q \in Q. \tag{2.6}$$

3. Discretization and main results. In this section we present the discretization of the optimal control problem (1.1) – (1.3) and formulate our main result concerning the asymptotic convergence properties.

For the discretization of the optimal control problem (1.1) – (1.3) we consider a family $\{\mathcal{T}_h\}_{h>0}$ of two- or three dimensional meshes consisting of (open) cells K which are either triangles, tetrahedra, quadrilaterals, or hexahedra and constitute a non-overlapping covering of the computational domain Ω. The mesh parameter h is defined as a cell-wise constant function by setting $h_{|K} = h_K$ and h_K is the diameter of K. We use the symbol h also for the maximal cell size, i.e.

$$h = \max_{K \in \mathcal{T}_h} h_K. \tag{3.1}$$

The family of meshes $\{\mathcal{T}_h\}_{h>0}$ is assumed to fulfill the standard conditions of quasi-uniformity and shape-regularity, see e.g. [8]. In addition, we require that the mesh $\mathcal{T}_h$ is organized in a patch-wise manner. This means that it results from a coarser regular mesh $\mathcal{T}_{2h}$ by an uniform refinement. By a patch P of elements we denote a group of cells in $\mathcal{T}_h$ which results from a common coarser cell in $\mathcal{T}_{2h}$.

For each node x of the mesh $\mathcal{T}_h$ we denote by $\mathcal{N}_h(x) \subset \Omega$ the “neighborhood” of x, which is the union of all cells K with $x \in \partial K$. For quasi-uniform, shape-regular meshes there exists a constant $c_\mathcal{N}$ such that

$$\operatorname{diam}(\mathcal{N}_h(x)) \leq c_\mathcal{N} h. \tag{3.2}$$

On the mesh $\mathcal{T}_h$ we define finite element spaces $Q_h \subset H^1(\Omega)$ and $V_h \subset H^1_0(\Omega)$ consisting of linear or bilinear shape functions, see e.g. [15]. In addition we define the space of cell-wise constant functions on patches V^{disc}_{2h}.

In the sequel, we will need an L^2-orthogonal projection operator $\pi_h \colon L^2(\Omega) \to V^{disc}_{2h}$. This operator has the following approximation and stability properties for $P \in \mathcal{T}_{2h}$:

$$\|\pi_h v\|_{L^2(P)} \leq c\|v\|_{L^2(P)} \qquad \forall v \in L^2(P), \tag{3.3}$$

$$\|v - \pi_h v\|_{L^2(P)} \leq ch\|\nabla v\|_{L^2(P)} \quad \forall v \in H^1(P). \tag{3.4}$$

For a positive stabilization parameter δ we introduce a stabilization (bilinear, symmetric) form $s^\delta_h \colon V_h \times V_h \to \mathbb{R}$ by:

$$s^\delta_h(u_h, v_h) = \delta\big(\beta \cdot \nabla u_h - \pi_h(\beta \cdot \nabla u_h), \beta \cdot \nabla v_h - \pi_h(\beta \cdot \nabla v_h)\big),$$

and the discrete semilinear form $a_h\colon V_h \times V_h \to \mathbb{R}$ by:

$$a_h(u_h, v_h) = a(u_h, v_h) + s_h^\delta(u_h, v_h). \tag{3.5}$$

Then, the discretized optimal control problem is formulated as follows:

$$\text{Minimize } J(q_h, u_h), \quad q \in Q_{ad,h} = Q_{ad} \cap Q_h,\ u_h \in V_h \tag{3.6}$$

subject to

$$a_h(u_h, v_h) = (f + q_h, v_h) \quad \forall v_h \in V_h. \tag{3.7}$$

This problem possess a unique solution $(\bar{q}_h, \bar{u}_h) \in Q_{ad,h} \times V_h$. Similar to the continuous case we introduce a discrete solution operator $S_h\colon Q \to V_h$ defined by:

$$a_h(S_h q, v_h) = (f + q, v_h) \quad \forall v_h \in V_h$$

and the discrete reduced cost functional

$$j_h(q) = J(q, S_h q).$$

The optimality condition for the discretized problem reads:

$$j_h'(\bar{q}_h)(\delta q_h - \bar{q}_h) \geq 0 \quad \forall \delta q_h \in Q_{ad,h}. \tag{3.8}$$

The derivatives of j_h are given similar to the continuous case in the following lemma:

Lemma 3.1. *There holds:*

$$j_h'(q)(\delta q) = (z_h, \delta q) + \alpha(q, \delta q),$$

where $z_h \in V_h$ is the solution of the adjoint equation:

$$a_h(v_h, z_h) = (u_h - u_d, v_h) \quad \forall v_h \in V_h. \tag{3.9}$$

where $u_h = S_h q$ is the associated discrete state to q. The solution z_h is the associated discrete adjoint state.

Remark 3.2. *We note that we obtain the same discrete adjoint equation as* (3.9) *if we directly discretize the continuous adjoint equation by the same (stabilized) discretization scheme. This property relies on the symmetry of the stabilization term and leads to the fact that the approaches "optimize-then-discretize" and "discretize-then-optimize" coincide in our case.*

We note that similar to the continuous case the second derivative $j_h''(q)$ does depend on q and we have analog to (2.6):

$$j_h''(q)(\delta q, \delta q) \geq \alpha \|\delta q\|^2_{L^2(\Omega)} \quad \forall \delta q \in Q. \tag{3.10}$$

In the following theorem we formulate our main result for the optimal control problem without control constraints.

Theorem 3.3. *Let $(\bar{q}, \bar{u})$ be the solution of the optimal control problem* (1.1) – (1.3) *with $Q_{ad} = Q$ and $(\bar{q}_h, \bar{u}_h)$ be the solution of the discretized problem* (3.6) – (3.7) *with $Q_{ad,h} = Q_h$, then the following estimate holds:*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq \frac{c}{\alpha}\, \tau(\delta)\, h\, (\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)}) + \frac{c}{\alpha}\, h^2\, \|\bar{q}\|_{H^2(\Omega)}\,,$$

where

$$\tau(\delta) = \big(\delta^{1/2}\|\beta\| + \min(\delta^{-1/2}, \|\beta\|\varepsilon^{-1/2})h + h + \varepsilon^{1/2}\big). \tag{3.11}$$

The proof of this theorem will be given in the next section. As a direct consequence of this theorem we give a rule for choosing the stabilization parameter δ. A similar rule is known for the SUPG or LPS discretization of convection-diffusion-reaction equations. The stabilization parameter is chosen in dependence of the Peclet number:

$$\mathrm{Pe} = \frac{h\,\|\beta\|}{\varepsilon}$$

by

$$\delta = \begin{cases} 0, & \text{if} \quad \mathrm{Pe} < 1, \\ \frac{h}{\|\beta\|}, & \text{if} \quad \mathrm{Pe} \geq 1. \end{cases} \tag{3.12}$$

This choice leads to the following corollary

Corollary 3.4. *Let* $(\bar q, \bar u)$ *be the solution of the optimal control problem* (1.1) – (1.3) *with* $Q_{ad} = Q$ *and* $(\bar q_h, \bar u_h)$ *be the solution of the discretized problem* (3.6) – (3.7) *with* $Q_{ad,h} = Q_h$*. Let moreover* δ *be chosen as in* (3.12)*, then the following estimates hold:*

1. *if* $Pe < 1$

$$\|\bar q - \bar q_h\|_{L^2(\Omega)} \leq \frac{c}{\alpha}\,\varepsilon^{1/2}\,h\,(\|\bar u\|_{H^2(\Omega)} + \|\bar z\|_{H^2(\Omega)}) + \frac{c}{\alpha}\,h^2\,\|\bar q\|_{H^2(\Omega)}\,,$$

2. *if* $Pe \geq 1$

$$\|\bar q - \bar q_h\|_{L^2(\Omega)} \leq \frac{c}{\alpha}\,\|\beta\|^{1/2}\,h^{3/2}\,(\|\bar u\|_{H^2(\Omega)} + \|\bar z\|_{H^2(\Omega)}) + \frac{c}{\alpha}\,h^2\,\|\bar q\|_{H^2(\Omega)}\,.$$

Remark 3.5. *In the case that* $\beta \neq Const$ *or in the case of variating cell diameter* h_K *the Peclet number has to be defined locally, i.e. cell-wise. Then, the stabilization parameter* δ *is also chosen locally.*

Remark 3.6. *We note, that our error estimate is proportional to* $\frac{1}{\alpha}$*. This is a typical situation if the coercivity constant of* j *coincide with the regularization parameter* α*, see* (3.10)*. However, this not always the case.*

Let us consider a variant of the optimal control problem (3.6) – (3.7) *with a finite dimensional control space* $Q = \mathbb{R}^n$*:*

$$\textit{Minimize } J(q,u) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_Q, \quad u \in V, q \in Q_{ad}$$

subject to

$$\begin{aligned} -\varepsilon\Delta u + \beta\cdot\nabla u + \sigma u &= f + \sum_{i=1}^n q_i\,g_i \quad && \textit{in } \Omega \\ u &= 0 && \textit{on } \partial\Omega\,, \end{aligned}$$

where $g_i \in L^\infty(\Omega)$ *are linearly independent functions. This problem possess a solution also in the case* $\alpha = 0$*. Moreover, there holds:*

$$j''(\bar q)(\delta q, \delta q) \geq \gamma\|\delta q\|^2_Q \quad \textit{with } \gamma \geq \alpha.$$

For this problem similar error estimates can obtained which are proportional to $\frac{1}{\gamma}$.

In order to formulate our main result for the control constrained case we need an additional assumption. We group the cells of the mesh $\mathcal{T}_h$ in two classes $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ with $\mathcal{T}_h^1 \cap \mathcal{T}_h^2 = \emptyset$ as follows: The cell K belongs to $\mathcal{T}_h^1$ if and only if one of the following conditions is satisfied:

(a) $\bar{q} \equiv a$ on K,

(b) $\bar{q} \equiv b$ on K,

(c) $a < \bar{q} < b$ on $\mathcal{N}_h(x_i)$ for each node $x_i \in \partial K$.

The set $\mathcal{T}_h^2$ is given by $\mathcal{T}_h^2 = \mathcal{T}_h \setminus \mathcal{T}_h^1$ and consists of the cells which lie "close to the free boundary between the active and the inactive sets".

Assumption 2. *We assume that*

$$\Big| \bigcup_{K \in T_h^2} K \Big| \le c\, h.$$

A similar assumption is used in [20] and in [24].

Remark 3.7. *This assumption is valid if the boundary of the level sets*

$$\{x \,:\, \bar{q}(x) = a\} \quad \textit{and} \quad \{x \,:\, \bar{q}(x) = b\}$$

consists of a finite number of rectifiable curves.

In addition we assume $-\infty < a < b < +\infty$ and introduce the inactive set in optimum:

$$\Omega_I = \{x \in \Omega \,:\, a < \bar{q}(x) < b\}.$$

Using this set we define a norm, we need in the sequel:

$$\|q\|_{2,ad} = \Big(\|q\|^2_{W^{1,\infty}(\Omega)} + \|\nabla^2 q\|^2_{L^2(\Omega_I)} \Big)^{1/2}.$$

In the following theorem we formulate our main result for the optimal control problem with control constraints.

Theorem 3.8. *Let* $(\bar{q}, \bar{u})$ *be the solution of the optimal control problem* (1.1) – (1.3) *and* $(\bar{q}_h, \bar{u}_h)$ *be the solution of the discretized problem* (3.6) – (3.7). *Let moreover Assumption 2 be fulfilled. Then the following estimate holds:*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \le \frac{c}{\alpha}\, \tau(\delta)\, h \left(\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)} \right) + \frac{c}{\alpha} h^{3/2} \|\bar{q}\|_{2,ad}$$

with $s > d$ *from Assumption 1 and* $\tau(\delta)$ *defined in* (3.11).

The proof of this theorem is given is Section 5.

For the choice of δ according to (3.12) we obtain the following corollary:

Corollary 3.9. *Let* $(\bar{q}, \bar{u})$ *be the solution of the optimal control problem* (1.1) – (1.3) *and* $(\bar{q}_h, \bar{u}_h)$ *be the solution of the discretized problem* (3.6) – (3.7). *Let moreover Assumption 2 be fulfilled and* δ *be chosen according to* (3.12). *Then the following estimates hold:*

1. *if* $Pe < 1$

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \le \frac{c}{\alpha}\, \varepsilon^{1/2}\, h \left(\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)} \right) + \frac{c}{\alpha} h^{3/2} \|\bar{q}\|_{2,ad}\,,$$

2. *if* $Pe \ge 1$

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \le \frac{c}{\alpha}\, \|\beta\|^{1/2}\, h^{3/2} \left(\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)} \right) + \frac{c}{\alpha} h^{3/2} \|\bar{q}\|_{2,ad}\,.$$

Remark 3.10. *Theorem 3.3 and Theorem 3.8 provide error estimates for the error in the control variable. The corresponding estimates for the state and the adjoint variable have similar structure since there holds:*

$$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)}+\|\bar{z}-\bar{z}_h\|_{L^2(\Omega)}\le ch^2(\|\bar{u}\|_{H^2(\Omega)}+\|\bar{z}\|_{H^2(\Omega)})+c\|\bar{q}-\bar{q}_h\|_{L^2(\Omega)}.$$

4. Error analysis for the unconstrained problem. In this section we consider the unconstrained case, i.e. for $Q_{ad}=Q$ and prove Theorem 3.3. To this end we need a special interpolation operator:

Lemma 4.1. *There is an interpolation operator $I_h\colon V\cap H^2(\Omega)\to V_h$ with the following properties: There holds for all $v\in V\cap H^2(\Omega)$*

$$\text{(i)}\quad (v-I_hv,w_h)=0\quad \forall w_h\in V_{2h}^{disc}, \tag{4.1}$$

$$\text{(ii)}\quad \|v-I_hv\|_{L^2(\Omega)}+h\,\|\nabla(v-I_hv)\|_{L^2(\Omega)}\le ch^2\,\|\nabla^2v\|_{L^2(\Omega)}. \tag{4.2}$$

A constructive proof of this lemma can be found in [4].

In the sequel, we will need an assertion of the term $s_h^\delta(I_hv,I_hv)$. This assertion is given in the following lemma.

Lemma 4.2. *For any $v\in V\cap H^2(\Omega)$ the following estimate holds:*

$$|s_h^\delta(I_hv,I_hv)|\le c\,\delta\,\|\beta\|^2\,h^2\,\|v\|^2_{H^2(\Omega)}.$$

Proof. We start by

$$s_h^\delta(I_hv,I_hv)=s_h^\delta(v+I_hv-v,v+I_h-v)\le 2\big(s_h^\delta(v,v)+s_h^\delta(v-I_hv,v-I_hv)\big)$$

For the first term we use the interpolation property of π_h, see (3.4):

$$s_h^\delta(v,v)=\delta\|\beta\cdot\nabla v-\pi_h(\beta\cdot\nabla v)\|^2_{L^2(\Omega)}\le c\,\delta\,h^2\|\nabla(\beta\cdot\nabla)\|^2_{L^2(\Omega)}\le c\,\delta\,h^2\|\beta\|^2\,\|v\|^2_{H^2(\Omega)}.$$

For the second term we use the L^2-stability of π_h and obtain:

$$\begin{aligned}s_h^\delta(v-I_hv,v-I_hv)&\le c\,\delta\,\|\beta\cdot\nabla(v-I_hv)\|^2_{L^2(\Omega)}\\ &\le c\,\delta\,\|\beta\|^2\|\nabla(v-I_hv)\|^2_{L^2(\Omega)}\le c\,\delta\,\|\beta\|^2\,h^2\,\|v\|^2_{H^2(\Omega)}.\end{aligned}$$

This completes the proof. □

In the following lemma we give an error estimate for the discretization of the state equation with an additional perturbation in the right hand side. To this end we first introduce a norm:

$$|||v|||^2=\|v\|^2_{L^2(\Omega)}+\|\varepsilon^{1/2}\nabla v\|^2_{L^2(\Omega)}+s_h^\delta(v,v).$$

Lemma 4.3. *Let for $q\in Q$, $u=Sq\in V\cap H^2(\Omega)$ be a associated solution of the state equation* (1.2) – (1.3)*, and for $p\in Q$, $w_h=S_hp\in V_h$ be the associated discrete solution, i.e:*

$$a_h(w_h,v_h)=(f+p,v_h)\quad\forall v_h\in V_h.$$

Then the following estimate holds:

$$|||u - w_h||| \le \|q - p\|_{L^2(\Omega)} + c\,\tau(\delta)\,h\,\|u\|_{H^2(\Omega)},$$

where $\tau(\delta)$ *is defined as in* (3.11).

Proof. For the error $u - w_h$ there holds a perturbed Galerkin orthogonality relation:

$$a(u - w_h, v_h) = s_h^\delta(w_h, v_h) + (q - p, v_h) \quad \forall v_h \in V_h.$$

We split the error $u - w_h = \eta + \xi$ with

$$\eta = u - I_h u \quad \xi = I_h u - w_h$$

For the interpolation error η we obtain:

$$\|\eta\|_{L^2(\Omega)} \le c\,h^2\,\|u\|_{H^2(\Omega)} \tag{4.3}$$

and

$$|||\eta||| \le c\,(h + \varepsilon^{1/2} + \delta^{1/2}\|\beta\|)\,h\|u\|_{H^2(\Omega)}. \tag{4.4}$$

For $\xi \in V_h$ we use the perturbed Galerkin orthogonality and obtain:

$$|||\xi|||^2 = a(\xi,\xi) + s_h^\delta(\xi,\xi) = (q - p, \xi) + s_h^\delta(I_h u, \xi) - a(\eta, \xi).$$

For the first term we have:

$$(q - p, \xi) \le \|q - p\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \le \|q - p\|_{L^2(\Omega)}|||\xi|||.$$

For the second term we use Lemma 4.2:

$$s_h^\delta(I_h u, \xi) \le (s_h^\delta(I_h u, I_h u))^{1/2}(s_h^\delta(\xi,\xi))^{1/2} \le \delta^{1/2}\,h\,\|\beta\|\,\|u\|_{H^2(\Omega)}\,|||\xi|||$$

For the third term we obtain:

$$|a(\eta,\xi)| \le |||\eta|||\,|||\xi||| + |(\beta\cdot\nabla\eta, \xi)|.$$

The last term here can be estimated either using the property (4.1) of I_h

$$\begin{aligned}
|(\beta\cdot\nabla\eta,\xi)| &= |(\eta, \beta\cdot\nabla\xi)| = |(\eta, \beta\cdot\nabla\xi - \pi_h(\beta\cdot\nabla\xi))| \\
&\le \delta^{-1/2}\|\eta\|_{L^2(\Omega)}\,\delta^{1/2}\|\beta\cdot\nabla\xi - \pi_h(\beta\cdot\nabla\xi)\|_{L^2(\Omega)} \le \delta^{-1/2}\|\eta\|_{L^2(\Omega)}\,|||\xi|||
\end{aligned}$$

or directly:

$$\begin{aligned}
|(\beta\cdot\nabla\eta,\xi)| = |(\eta, \beta\cdot\nabla\xi)| &\le \varepsilon^{-1/2}\,\|\beta\|\,\|\eta\|_{L^2(\Omega)}\,\|\varepsilon^{1/2}\nabla\xi\|_{L^2(\Omega)} \\
&\le \varepsilon^{-1/2}\,\|\beta\|\,\|\eta\|_{L^2(\Omega)}\,|||\xi|||.
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
|||\xi||| \le \|q - p\|_{L^2(\Omega)} &+ \min(\delta^{-1/2}, \varepsilon^{-1/2}\,\|\beta\|)\,\|\eta\|_{L^2(\Omega)} \\
&+ \delta^{1/2}\,h\,\|\beta\|\,\|u\|_{H^2(\Omega)} + |||\eta|||.
\end{aligned}$$

Now we use the interpolation estimates (4.3) and (4.4) and obtain the desired estimate. □

As a direct consequence of the above lemma we obtain an estimate for the error in the adjoint states:

LEMMA 4.4. *Let for $q \in Q$, $z \in V \cap H^2(\Omega)$ be the associated adjoint solution, i.e the solution of* (2.3) – (2.4), *and for $p \in Q$, $y_h \in V_h$ denote the associated adjoint discrete solution, then the following estimate holds:*

$$|||z - y_h||| \leq \|q - p\|_{L^2(\Omega)} + c\,\tau(\delta)\,h\,(\|u\|_{H^2(\Omega)} + \|z\|_{H^2(\Omega)}),$$

where $\tau(\delta)$ is defined as in (3.11).

The proof of this lemma is obtained similar to the proof of Lemma 4.3.

Now, we come to the proof of Theorem 3.3.

Proof. Let $p_h \in Q_h$ be arbitrary. We obtain from (3.10):

$$\alpha\|p_h - \bar{q}_h\|^2_{L^2(\Omega)} \leq j''_h(\bar{q}_h)(p_h - \bar{q}_h, p_h - \bar{q}_h) = j'_h(p_h)(p_h - \bar{q}_h) - j'_h(\bar{q}_h)(p_h - \bar{q}_h).$$

Due to $Q_{ad} = Q$ and $Q_{ad,h} = Q_h$ we have:

$$j'_h(\bar{q}_h)(p_h - \bar{q}_h) = 0 = j'(\bar{q})(p_h - \bar{q}_h). \tag{4.5}$$

Hence,

$$\alpha\|p_h - \bar{q}_h\|^2_{L^2(\Omega)} \leq j'_h(p_h)(p_h - \bar{q}_h) - j'(\bar{q})(p_h - \bar{q}_h).$$

We use Lemma 2.4 and Lemma 3.1 and obtain:

$$\alpha\|p_h - \bar{q}_h\|^2_{L^2(\Omega)} \leq (y_h, p_h - \bar{q}_h) - (\bar{z}, p_h - \bar{q}_h),$$

where $\bar{z}$ is the associated adjoint state to $\bar{q}$ and y_h is the associated discrete adjoint state to p_h. Using Lemma 4.4 we obtain:

$$\|p_h - \bar{q}_h\|_{L^2(\Omega)} \leq \frac{1}{\alpha}\|\bar{q} - p_h\|_{L^2(\Omega)} + \frac{c}{\alpha}\,\tau(\delta)\,h\,(\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)}),$$

where $\tau(\delta)$ is defined as in (3.11). Due to $Q_{ad} = Q$ we have that

$$\bar{q} = \frac{1}{\alpha}\bar{z} \in V \cap H^2(\Omega).$$

Therefore we can choose $p_h \in Q_h$ as the pointwise interpolation of $\bar{q}$ with

$$\|\bar{q} - p_h\|_{L^2(\Omega)} \leq c\,h^2\|\bar{q}\|_{H^2(\Omega)}.$$

Using this fact we obtain the desired estimate:

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq \frac{c}{\alpha}\,\tau(\delta)\,h\,(\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)}) + \frac{c}{\alpha}\,h^2\,\|\bar{q}\|_{H^2(\Omega)}\,.$$

□

5. Error analysis for the problem with control constraints. In this section we provide the error estimate for the control constrained case. In the case $Q_{ad} \neq Q$ we can not use the same argument as for the proof of Theorem 3.3 since the condition (4.5) does not hold any more. To overcome this difficulty we will construct a special interpolation $p_h \in Q_{ad,h}$ of the solution $\bar{q} \in W^{1,\infty}(\Omega)$ which fulfills the following condition:

$$j'(\bar{q})(r - p_h) \geq 0 \quad \forall r \in Q_{ad}. \tag{5.1}$$

In the following lemma we use this condition in order to get an error estimate.

LEMMA 5.1. *Let* $(\bar{q}, \bar{u})$ *be the solution of the optimal control problem* (1.1) – (1.3) *and* $(\bar{q}_h, \bar{u}_h)$ *be the solution of the discretized problem* (3.6) – (3.7). *Let moreover* $p_h \in Q_{ad,h}$ *fulfill condition* (5.1). *Then the following estimate holds:*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq \frac{1}{\alpha}\|\bar{q} - p_h\|_{L^2(\Omega)} + \frac{c}{\alpha}\,\tau(\delta)\,h\,(\|\bar{u}\|_{H^2(\Omega)} + \|\bar{z}\|_{H^2(\Omega)}),$$

where $\tau(\delta)$ *is defined as in* (3.11).

Proof. As in the unconstrained case there holds:

$$\alpha\|p_h - \bar{q}_h\|^2_{L^2(\Omega)} \leq j''_h(\bar{q}_h)(p_h - \bar{q}_h, p_h - \bar{q}_h) = j'_h(p_h)(p_h - \bar{q}_h) - j'_h(\bar{q}_h)(p_h - \bar{q}_h).$$

We use the discrete optimality condition (3.8) and the condition (5.1) with $r = \bar{q}_h$. We obtain:

$$-j'_h(\bar{q}_h)(p_h - \bar{q}_h) \leq 0 \leq -j'(\bar{q})(p_h - \bar{q}_h).$$

Hence,

$$\alpha\|p_h - \bar{q}_h\|^2 \leq j'_h(p_h)(p_h - \bar{q}_h) - j'(\bar{q})(p_h - \bar{q}_h).$$

Then we proceed as in the proof of Theorem 3.3 and obtain the desired estimate. ∎

The interpolation $p_h \in Q_{ad,h}$ is constructed as follows: We prescribe the value of p_h at each node x of the mesh $\mathcal{T}_h$. To this end we use the "neighborhood" $\mathcal{N}_h(x_i)$ of a node x_i defined in Section 3. We set

$$p_h(x_i) = \begin{cases} a, & \text{if } \min\limits_{x \in \mathcal{N}_h(x_i)} \bar{q}(x) = a \\ b, & \text{if } \max\limits_{x \in \mathcal{N}_h(x_i)} \bar{q}(x) = b \\ \bar{q}(x_i), & \text{else.} \end{cases} \tag{5.2}$$

A similar construction can be found in [7].

In the following lemma we show that p_h is well defined if h is small enough and that p_h fulfills condition (5.1).

LEMMA 5.2. *Let*

$$h < \frac{b - a}{c_{\mathcal{N}}\,\|\bar{q}\|_{W^{1,\infty}(\Omega)}}\,. \tag{5.3}$$

Then p_h *is well defined and fulfills condition* (5.1).

Proof. Due to the fact that $\bar{q} \in W^{1,\infty}(\Omega)$ there holds:

$$\max_{x \in \mathcal{N}_h(x_i)} \bar{q}(x) - \min_{x \in \mathcal{N}_h(x_i)} \bar{q}(x) \leq \|\bar{q}\|_{W^{1,\infty}(\Omega)} \mathrm{diam}(\mathcal{N}_h(x_i)) \leq \|\bar{q}\|_{W^{1,\infty}(\Omega)}\, c_{\mathcal{N}}\, h,$$

with $c_{\mathcal{N}}$ introduced by (3.2).

Therefore, either the first or the second or the third case applies, provided h fulfills the condition (5.3). This proves that p_h is well defined.

In order to prove that p_h fulfills (5.1) we first consider a neighborhood $\mathcal{N}(x_i)$ such that

$$\min_{x \in \mathcal{N}(x_i)} \bar{q}(x) = a.$$

Then there holds:

$$\bar{q}(x) < b \quad \forall x \in \mathcal{N}(x_i)$$

and consequently $\nabla j(\bar{q}) \geq 0$ pointwise on $\mathcal{N}(x_i)$. Therefore we obtain for an arbitrary $r \in Q_{ad}$

$$\nabla j(\bar{q})(r - p_h) = \nabla j(\bar{q})(r - a) \geq 0$$

pointwise on $\mathcal{N}(x_i)$. For the cases

$$\max_{x \in \mathcal{N}(x_i)} \bar{q}(x) = b$$

and

$$\min_{x \in \mathcal{N}(x_i)} \bar{q}(x) > a \text{ and } \max_{x \in \mathcal{N}(x_i)} \bar{q}(x) < b$$

we proceed similarly and obtain that $\nabla j(\bar{q})(r - p_h) \geq 0$ pointwise on Ω for all $r \in Q_{ad}$. This completes the proof. ☐

To complete the proof of Theorem 3.8 we have to provide an estimate for the interpolation error $\|\bar{q} - p_h\|_{L^2(\Omega)}$.

LEMMA 5.3. *Let $p_h \in Q_{ad,h}$ be constructed as in* (5.2) *and Assumption 2 be fulfilled, then the following estimate holds:*

$$\|\bar{q} - p_h\|_{L^2(\Omega)} \leq \frac{c}{\alpha} h^{3/2} \|\bar{q}\|_{2,ad},$$

with $s > d$ from Assumption 1.

Proof. We start with

$$\|\bar{q} - p_h\|^2_{L^2(\Omega)} = \sum_{K \in \mathcal{T}_h} \|\bar{q} - p_h\|^2_{L^2(K)} = \sum_{K \in \mathcal{T}_h^1} \|\bar{q} - p_h\|^2_{L^2(K)} + \sum_{K \in \mathcal{T}_h^2} \|\bar{q} - p_h\|^2_{L^2(K)} \quad (5.4)$$

For the first sum we have:

$$\sum_{K \in \mathcal{T}_h^1} \|\bar{q} - p_h\|^2_{L^2(K)} \leq \sum_{K \in \mathcal{T}_h^1} ch^4 \|\nabla^2 \bar{q}\|^2_{L^2(K)} \leq ch^4 \|\nabla^2 \bar{q}\|^2_{L^2(\Omega_I)}, \quad (5.5)$$

since $q \in H^2(K)$ for each cell $K \in \mathcal{T}_h^1$ and p_h is a usual point interpolation on the cells from $\mathcal{T}_h^1$.

For a cell $K \in T_h^2$ we obtain that $K \subset \mathcal{N}_h(x_i)$ with either

$$(i) \qquad \min_{x \in \mathcal{N}_h(x_i)} \bar{q}(x) = a$$

or

$$(ii) \qquad \max_{x \in \mathcal{N}_h(x_i)} \bar{q}(x) = b.$$

In the first case we obtain:

$$\|\bar{q} - p_h\|^2_{L^2(K)} \leq \|\bar{q} - p_h\|^2_{\mathcal{N}_h(x_i)} \leq \|\bar{q} - a\|^2_{\mathcal{N}_h(x_i)}.$$

Due to the fact that $\bar{q} \in W^{1,\infty}(\Omega)$ we have:

$$\|\bar{q} - a\|^2_{\mathcal{N}_h(x_i)} \leq |\mathcal{N}_h(x_i)| \, \|\bar{q} - a\|^2_{L^\infty(\Omega)} \leq |\mathcal{N}_h(x_i)| \operatorname{diam}(\mathcal{N}_h(x_i))^2 \, \|\bar{q}\|^2_{W^{1,\infty}(\Omega)}.$$

The same estimate is obtained in the case (ii). By summing up and using Assumption 2, we obtain:

$$\sum_{K \in \mathcal{T}_h^2} \|\bar{q} - p_h\|^2_{L^2(K)} \leq ch^2 \, \|\bar{q}\|^2_{W^{1,\infty}(\Omega)} \sum_{K \in \mathcal{T}_h^2} |K| \leq ch^3 \|\bar{q}\|^2_{W^{1,\infty}(\Omega)} \tag{5.6}$$

Inserting (5.5) and (5.6) in (5.4) we obtain the desired estimate. ∎

To complete the proof of Theorem 3.8 we combine the results from Lemma 5.1, Lemma 5.2 and Lemma 5.3.

6. Numerical example. In this section we present a numerical example confirming our results. To this end we consider the optimization problem (1.1) – (1.3) on the unit square, i.e with $\Omega = (0,1)^2$ and with following parameters:

$$\varepsilon = 10^{-3}, \quad \beta = (-1,-2)^t, \quad \sigma = 1,$$

$$f = 1, \quad u_d = 1,$$

and

$$\alpha = 0.1, \quad a = 0.5, \quad b = 10\,.$$

The state and the control variables are discretized by bilinear finite elements as described in Section 3. The discretized control constraint problems are solved by primal dual active set method, see, e.g., [3] and [17].

Figure 6.1 shows the optimal control $\bar{q}_h$ computed on the mesh $\mathcal{T}_h$ with $h = 2^{-6}\sqrt{2}$, i.e with Peclet number $Pe \approx 50$ with and without stabilization. It turns out, that the solution of the stabilized problem has slight oscillations only in the boundary layer, whereas the solution of the discrete problem without stabilization shows strong oscillations almost in the whole domain.

Since the exact solution of the optimal control problem under consideration is not known, we show the evolution of the values of the cost functional $J(\bar{q}_h, \bar{u}_h)$ for a sequence of uniformly refined meshes $\mathcal{T}_h$, for h tending to zero. From this sequence, we compute the approximative order of convergence, see Table 6.1. The mean observed order of convergence is $r \approx 1.73$.

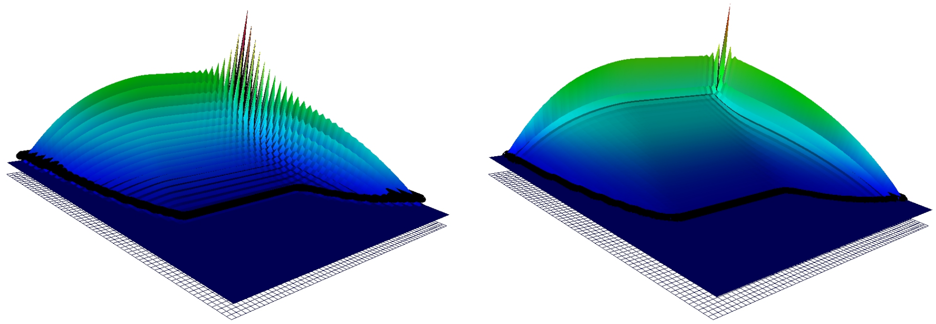

FIG. 6.1. *Optimal control $\bar{q}_h$ (level sets) computed on the mesh $\mathcal{T}_h$ with $h = 2^{-6}\sqrt{2}$, $Pe \approx 50$ without stabilization (left) and with stabilization (right). The active sets are indicated by the black lines.*

TABLE 6.1
Evolution of the values of the cost functional $J(\bar{q}_h, \bar{u}_h)$ for a sequence of uniformly refined meshes $\mathcal{T}_h$, for h tending to zero

$h/\sqrt{2}$	$J(\bar{q}_h, \bar{u}_h)$	$J(\bar{q}_h, \bar{u}_h) - J(\bar{q}_{2h}, \bar{u}_{2h})$	order
2^{-2}	0.293169	–	–
2^{-3}	0.273468	-1.97006e-2	–
2^{-4}	0.265199	-8.26853e-3	1.25254
2^{-5}	0.262427	-2.77226e-3	1.57657
2^{-6}	0.261613	-8.13916e-4	1.76811
2^{-7}	0.261409	-2.04352e-4	1.99383
2^{-8}	0.261360	-4.86628e-5	2.07016
2^{-9}	0.261346	-1.44169e-5	1.75505

7. Conclusion. In this paper we have derived a priori error estimates for a stabilized finite element discretization of optimal control problems governed by advection-diffusion equations subject to pointwise control constraints. The method has optimal convergence behavior and has the commutativity property of discretization and optimization, which is very convenient from an algorithmic point of view. No modification of the Lagrangian is necessary. The presented analysis can be used to analyze related stabilization methods as the one proposed in [6].

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