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A SUCCESSIVE SDP–NSDP APPROACH TO A ROBUST OPTIMIZATION PROBLEM IN FINANCE *

F. LEIBFRITZ AND J. H. MARUHN †

Abstract. The robustification of trading strategies is of particular interest in financial market applications. In this paper we robustify a portfolio strategy recently introduced in the literature against model errors in the sense of a worst case design. As it turns out, the resulting optimization problem can be solved by a sequence of linear and nonlinear semidefinite programs (SDP/NSDP), where the nonlinearity is introduced by the parameters of a parabolic differential equation. The nonlinear semidefinite program naturally arises in the computation of the worst case constraint violation which is equivalent to an eigenvalue minimization problem. Further we prove convergence for the iterates generated by the sequential SDP–NSDP approach.

Key Words. *Robust Optimization, Nonlinear Semidefinite Programming, Static Hedging, Barrier Options, Eigenvalue Minimization.*

AMS subject classification. 45C05, 47A75, 62G35, 78M50, 90C22, 90C30, 91B28.

1. Introduction. During the last few years, robust optimization techniques have been an active area of research in the optimization community. For example, the solution of robustified linear and quadratic programming problems can be carried out efficiently by modern optimization methods like conic or semidefinite programming (see e.g. [2], [12], [30]).

Due to the success of the robust optimization framework, these ideas have also been applied to financial market problems. For example the authors in [12] study the robust counterpart of a portfolio selection problem to make optimal portfolios less sensitive with respect to perturbations in the market data. As in the general theory, the structure of the non–robust optimization problem allows to simplify the robust counterpart significantly which results in a second order cone program (SOCP).

In this paper we also apply the idea of robustness to a specific portfolio optimization problem recently developed by Maruhn and Sachs in [25] and [26]. In this problem a bank seeks to identify the cheapest *hedge portfolio* which produces a payoff greater or equal to the payoff of another financial instrument in every state of the economy. This leads to an infinite number of constraints in which the value of the hedge portfolio enters in a very nonlinear fashion. To robustify the hedge portfolio with respect to future changes of market prices, Maruhn and Sachs include possible changes of implied model parameters as an additional number of infinite constraints in the optimization problem.

However, up to this point the authors did not take model or implementation errors into account. To avoid the associated risk, we incorporate possible deviations from model prices in the real world in the sense of a worst case design. These deviations are mathematically described as ellipsoidal uncertainty sets around the asset prices described by Heston's stochastic volatility model. This results in a linear semi–infinite optimization problem with a typically 15 to 30–dimensional parameter space of the semi–infinite constraints.

Clearly, the high dimension of this parameter space makes the numerical solution of the problem very hard. However, by using equivalent transformations, we can reduce the dimension of the parameter space significantly to six. The price for this reduction is a nonlinearity in the form of second order cone constraints. By employing suitable semi–infinite optimization results, we can prove convergence of an iterative method successively solving second order cone programs and nonlinear programming problems to compute the worst case constraint violation for each iterate.

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In addition we are able to reformulate this iterative procedure as a successive solution of semidefinite programming problems and nonlinear semidefinite programs (NSDPs). In particular our derivations show, that NSPDs naturally arise as subproblems of robustified linear semi–infinite programming problems to compute the constraint violation of the iterates.

Nonlinear semidefinite programs also often appear in the design of static output feedback control laws for linear time–invariant control systems. For example $\mathcal{H}_2$–, $\mathcal{H}_\infty$– or mixed $\mathcal{H}_2/\mathcal{H}_\infty$–design problems for ODE or PDE systems lead to non–convex NSDPs (see e. g. [18], [20], [21], [23] and [24]). Finding a solution to non–convex NSDPs is a difficult task, particularly, if the dimension of the problem is large. This may be one reason, why general "off the shelf solvers" for NSDPs are unavailable. To our knowledge, there only exist some specialized solvers for particular NSDPs (see e. g. SLMSDP [16], PENBMI [17], IPCTR [22], [23], [24], SSDP [8]). On the other hand, for the solution of linear SDPs and SOCPs a lot of solvers are freely available over the internet (e. g. DSDP [1], SeDuMi [28], SDPT3 [29] and many more). Due to this gap, it is necessary to develop, test and analyze solvers for more general NSDPs. During the development of new NSDP solvers it is important to test the code on several benchmark examples. Fortunately Leibfritz has built the publicly available benchmark collection *COMPl$_e$ib* – the *CO*nstrained *M*atrix–optimization *P*roblem *lib*rary [19]. *COMPl$_e$ib* can be used for testing a wide variety of algorithms solving matrix optimization problems, e. g. NSDP solvers like IPCTR or SSDP, bilinear matrix inequality (BMI) codes like PENBMI or linear SDP solvers. The financial market application presented in this paper provides an additional example why the development of general NSDP solvers is urgently needed.

The paper is organized as follows. In Section 2 we briefly describe the financial market application under consideration. Note that a more detailed discussion of the underlying stochastic optimization problem can be found in [26]. Section 3 then adds robustness to the problem taking possible model as well as implementation errors into account. Furthermore, we briefly sketch the numerical solution method employed to solve the problem and derive a convergence result. In Section 4 we finally show how the robust optimization problem can be solved by a sequence of SDPs and NSDPs. In particular, we will prove that a nonlinear second order cone problem is equivalent to the minimization of the minimal eigenvalue of a matrix function, where this matrix depends in a very nonlinear fashion on the parameters. Then, by using duality arguments, the problem of minimizing the minimal eigenvalue of this matrix function will be reformulated as a nonlinear minimization problem with SDP constraints.

Notation. Throughout this paper, $\mathbb{S}^m$ denotes the linear space of real symmetric $m \times m$ matrices. In the space of real $m \times n$ matrices we define the inner product by $\langle M, Z\rangle = Tr(M^T Z)$ for $M, Z \in \mathbb{R}^{m\times n}$, where $Tr(\cdot)$ is the trace operator. For a matrix $M \in \mathbb{S}^m$ we use the notation $M \succ 0$ or $M \succeq 0$ if it is positive definite or positive semidefinite, respectively.

2. Description of the Hedging Problem. Nowadays a large variety of financial products is available on the capital market. Usually, the buyer of such a product pays some fixed amount to the bank at time $t = 0$. In turn, the bank enters the obligation to pay some insecure future payment at time $t = T$ to the customer which is based on the performance of a so–called underlying, for example the stock price, from time $t = 0$ to $t = T$.

Once the bank has sold the product, it is exposed to the risk of the insecure future payment. To reduce this risk, the bank immediately buys a portfolio of alternative financial instruments which replicates the value of the sold product as good as possible in any future state of the economy.

In our case, the bank sells a so–called *barrier option*, more precisely an *up–and–out call* at time $t = 0$. The future amount the bank has to pay to the buyer of this product depends on the value of the stock price S_t from time $t = 0$ to $t = T$. In case the stock price $(S_t)_{0\le t\le T}$ hits the barrier D $(0 < S_0 < D)$ at some time $t \in [0, T]$ (*knock–out*), the option expires worthless. However, in case the stock price never touches the barrier D, the value of the up–and–out call is given by $C_{uo} = \max(S_T - K, 0)$, where $0 < K < D$ denotes the *strike price*. Figure 2.1 illustrates these two cases for two possible future evolvements of the stock price.

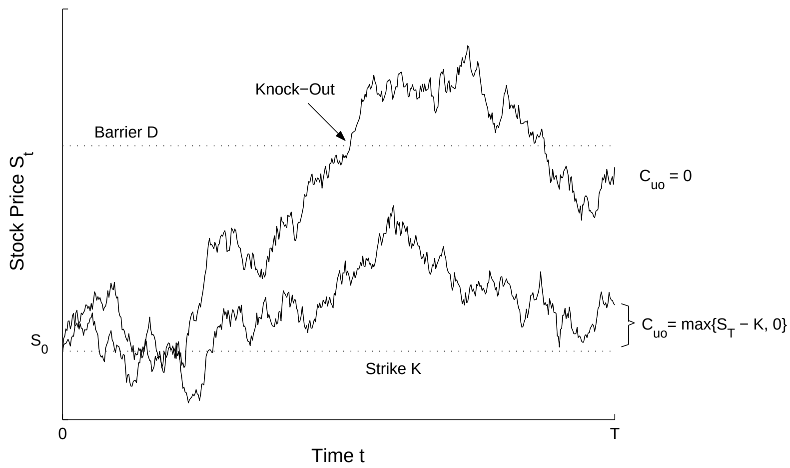

FIG. 2.1. *Payoff of the up–and–out call for two possible stock–price paths*

Regarding these two cases, a natural choice for a hedge portfolio of such an option is as follows. At time $t = 0$ the bank buys a portfolio of alternative financial instruments and holds these instruments constant until either the barrier D is hit or until time T is reached. At these time instances the hedge portfolio is sold immediately and hopefully guarantees a payoff that is at least as high as the amount the bank has to pay to the customer.

If we denote the financial instruments in the hedge portfolio by $C_1, ..., C_n$ with value $C_i(t, S_t)$ at time t and the units invested in product C_i by α_i, the value of the hedge portfolio at time t is given by $\Pi(t, \alpha) = \sum_{i=1}^n \alpha_i C_i(t, S_t)$. Hence the optimization problem of finding the cheapest trading strategy guaranteeing a payoff greater or equal to the payoff of the up-and-out call in all states of the economy is given by

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^n} \Pi(0, \alpha) = \sum_{i=1}^n \alpha_i C_i(0, S_0) \\ \text{s.t.} \quad &\Pi(t, \alpha) = \sum_{i=1}^n \alpha_i C_i(t, D) \geq 0 \text{ for all times t the barrier might be hit.} \\ &\Pi(T, \alpha) = \sum_{i=1}^n \alpha_i C_i(T, S_T) \geq \max(S_T - K, 0) \text{ if the barrier is not hit at all.} \end{aligned} \tag{2.1}$$

Research has shown, that standard calls with payoff $C_i(T_i, S_{T_i}) = \max(S_{T_i} - K_i, 0)$ at maturity $0 < T_i \leq T$ satisfying $K_i \geq D$ if $T_i < T$ are particularly suited for the task of hedging barrier options (see e.g. [3], [6], [7], [10] and [25]). Here $K_i \geq 0$ denotes the strike of the standard call C_i.

Note that, in case the stock price has not hit the barrier until time $t \in (0, T]$, the value of the hedge portfolio at time t is independent of the calls C_i with maturity $T_i < t$. This can easily be seen by observing that $S_s < D \; \forall \; s \in (0, t]$, which implies in particular $S_{T_i} < D \leq K_i$ such that $C_i(T_i, S_{T_i}) = \max(S_{T_i} - K_i, 0) = 0$. Hence these calls can be omitted in the formulation of optimization problem (2.1) and we obtain (for a more detailed discussion see [25]):

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^n} \sum_{i=1}^n \alpha_i C_i(0, S_0) \\ \text{s.t.} \quad &\sum_{i:T_i \geq t} \alpha_i C_i(t, D) \geq 0 \text{ for all times t the barrier might be hit.} \\ &\sum_{i:T_i = T} \alpha_i \max(S_T - K_i) \geq \max(S_T - K, 0) \text{ if the barrier is not hit at all.} \end{aligned} \tag{2.2}$$

The times t at which the barrier can be hit as well as the value $C_i(t, S_t)$ in problem (2.2) will depend on the financial market model and hence the model parameters $p \in \mathbb{R}^k$ under consideration such that $C_i(t, S_t) = C_i(t, S_t, p)$. Clearly, the quality of the solution of the optimization problem depends on the ability of the model to fit the market prices of call options. A well suited model for this task is Heston's stochastic volatility model [14], in which the call option prices C_i are given as the solution of the parabolic differential equation

$$\frac{1}{2}vS^2\frac{\partial^2 C_i}{\partial S^2} + \rho\sigma vS\frac{\partial^2 C_i}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C_i}{\partial v^2} + rS\frac{\partial C_i}{\partial S} + \kappa(\theta - v)\frac{\partial C_i}{\partial v} + \frac{\partial C_i}{\partial t} = rC_i \qquad (2.3)$$
$$(t, S, v) \in (0, T_i) \times (0, \infty) \times (0, \infty)$$

with initial and boundary conditions

$$\begin{gathered} C_i(T_i, S, v) = \max(S - K_i, 0) \\ C_i(t, 0, v) = 0 \\ \frac{\partial C_i}{\partial S}(t, \infty, v) = 1 \\ rS\frac{\partial C_i}{\partial S}(t, S, 0) + \kappa\theta\frac{\partial C_i}{\partial v}(t, S, 0) + \frac{\partial C_i}{\partial t}(t, S, 0) = rC_i(t, S, 0) \\ C_i(t, S, \infty) = S. \end{gathered}$$

Here v denotes the initial variance of the stock price, θ the long term mean of the variance process, κ its mean reversion speed and σ the volatility. ρ denotes the correlation between the stock price and variance process and r specifies the risk–free rate.

Thus the call option prices C_i in Heston's model depend on five model parameters such that $C_i = C_i(t, S_t, v, \kappa, \theta, \sigma, \rho)$. As it is unknown how market prices and hence the associated model parameters will change in the future, one will ask the first constraint in (2.2) to hold for a given set of model parameters $p = (v, \kappa, \theta, \sigma, \rho) \in P \subset \mathbb{R}^5$.

Intuitively it is clear, that the barrier can be hit any time from 0 to T and in case no barrier hit occurs, the stock price S_T can attain any value in $[0, D]$. Combining these facts, the optimization problem of finding the cheapest strategy super–replicating the value of the up–and–out call is given by the following definition (for a more detailed derivation see Maruhn [26]).

Definition 2.1. *Let an up–and–out call with payoff $C_{uo} = \max(S_T - K, 0)1_{\{max_{0\le t\le T} S_t < D\}}$ be given, where $T > 0$ denotes the maturity, $K > 0$ the strike, D the barrier, $0 < K, S_0 < D$ and S_t the stock price at time t. Further let C_i, $i = 1, ..., n$, be standard calls with maturities $0 < T_i \le T$ and strikes $K_i \ge 0$ satisfying $K_i \ge D$ for $T_i < T$, whose prices satisfy the parabolic differential equation (2.3). In addition let $\alpha_i^{lb} \le 0$ and $\alpha_i^{ub} \ge 1$ be lower and upper bounds on the portfolio weights. Then a cost–optimal superhedge is defined as the solution of the optimization problem*

$$\begin{aligned} &\min_{\alpha\in\mathbb{R}^n} \sum_{i=1}^{n} \alpha_i C_i(0, S_0, v_0, \kappa_0, \theta_0, \sigma_0, \rho_0) \\ s.t. \quad &\sum_{i:T_i\ge t} \alpha_i C_i(t, D, v, \kappa, \theta, \sigma, \rho) \ge 0 \ \forall\ t \in [0, T]\ \forall\ p = (v, \kappa, \theta, \sigma, \rho) \in P \qquad (2.4) \\ &\sum_{i:T_i=T} \alpha_i \max(s - K_i) \ge \max(s - K, 0)\ \forall\ s \in [0, D] \\ &\alpha_i^{lb} \le \alpha_i \le \alpha_i^{ub},\ i = 1, ..., n \end{aligned}$$

where $p_0 = (v_0, \kappa_0, \theta_0, \sigma_0, \rho_0)$ denotes the implied model parameters at time $t = 0$ and p are the future implied model parameters expected to vary in the compact uncertainty set $P \subset \mathbb{R}^5$.

It is obvious, that the feasible set of the linear semi–infinite optimization problem (2.4) is closed, convex and due to the box constraints also compact. Furthermore the feasible set is nonempty, if the standard call with maturity $T_i = T$ and strike $K_i = K$ is included as call C_j in the hedge portfolio, because then the strategy $\alpha_j = 1$, $\alpha_i = 0$, $i \neq j$ satisfies all constraints. Hence a solution of the optimization problem exists if this particular call is included in the portfolio.

Maruhn shows in [26], that the hedging strategy solving the optimization problem has very attractive properties. However, although the Heston model fits prices of standard calls quite well, the optimization problem does not yet take model errors into account. In the next section we will incorporate an additional robustness against model errors in the problem definition and discuss possibilities of numerically solving the optimization problem.

3. Robustification and Numerical Solution. As an advanced model in finance, Heston's stochastic volatility model already provides a good fit of market prices of standard calls. However, as is well known in the robust optimization community, even small perturbations of the data of linear programming problems can result in unexpected effects regarding optimality and feasibility of the solution. Of course, this sensitivity of solutions transfers in a similar way to solutions of linear optimization problems with an infinite number of linear constraints like problem (2.4). Thus our goal in this section is to derive a robustified version of this problem taking possible model errors into account. Furthermore we will briefly discuss some aspects of the numerical solution.

Note that optimization problem (2.4) is of the form

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^n} c^T \alpha \\ \text{s.t.} \quad & a_1(t,p)^T \alpha \geq 0 \quad \forall\ (t,p) \in [0,T] \times P \\ & a_2(s)^T \alpha \geq b_2(s) \quad \forall\ s \in [0,D] \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub}, \quad i = 1, ..., n \end{aligned} \tag{P}$$

with suitable vectors c, a_1, a_2 and scalars b_2. The performance of the optimal hedge portfolio crucially depends on the model prices $C_i(t, D, v, \kappa, \theta, \sigma, \rho)$ included in the vector a_1. The second constraint $a_2(s)^T \alpha \geq b_2(s)$ is model–independent and hence can be neglected in the context of robustification.

We now robustify the solution of optimization problem (P) with respect to perturbations of the vector a_1 in the sense of model errors by asking the corresponding inequality to hold in a small ellipsoid around the model prices. These ellipsoids shall be defined by associated matrices $E(t,p)$. This results in the following robust optimization problem.

DEFINITION 3.1. *Consider problem (2.4) of finding the cost–optimal superhedge in the notation of problem (P) with suitable vectors c, a_1, a_2 and scalars b_2. For $(t,p) \in [0,T] \times P$ let $E(t,p) \in \mathbb{R}^{n \times n}$ be positive definite matrices defining ellipsoids around the model prices $a_1(t,p)$. Further assume that the map $E : [0,T] \times P \to \mathbb{R}^{n \times n}$ is continuous. Then the robust counterpart of problem (P) is given by the following optimization problem:*

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^n} c^T \alpha \\ s.t. \quad & (a_1(t,p) + E(t,p)u)^T \alpha \geq 0 \quad \forall\ u \in \mathbb{R}^n : \|u\|_2 \leq 1 \quad \forall\ (t,p) \in [0,T] \times P \\ & a_2(s)^T \alpha \geq b_2(s) \quad \forall\ s \in [0,D] \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub}, \quad i = 1, ..., n \end{aligned} \tag{RP}$$

In the simplest case the matrix $E(t,p)$ might be chosen as a small multiple of the identity matrix, but in a more realistic setting the model error and hence the matrix will depend on time t and the model–parameters p. In the following we assume, that the matrices $E(t,p)$ are chosen in such a way that the feasible set of problem (RP) is nonempty. Due to the compactness of the feasible set we then readily obtain, that a solution of the optimization problem exists.

ASSUMPTION 3.2. *Assume that the feasible set of problem (RP) is nonempty.*

Note that problem (RP) is still a linear semi–infinite optimization problem, but now the complexity of the problem has increased significantly due to the additional variable u varying in the n–dimensional unit ball. Hence it would be very desirable to reduce the dimension of the semi–infinite parameter set by eliminating u from problem (RP). The next theorem shows that this can actually be achieved.

THEOREM 3.3. *The linear semi–infinite optimization problem (RP) is equivalent to the semi–infinite second order cone problem:*

$$\begin{aligned} & \min_{\alpha \in \mathbb{R}^n} c^T \alpha \\ s.t. \quad & a_1(t,p)^T \alpha \geq \|E(t,p)^T \alpha\|_2 \quad \forall\ (t,p) \in [0,T] \times P \\ & a_2(s)^T \alpha \geq b_2(s) \quad \forall\ s \in [0,D] \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub}, \quad i = 1,...,n \end{aligned} \tag{RP–SOCP}$$

Proof. Similar to Ben–Tal and Nemirovski [2], we prove the equivalence by showing that for every $(t,p) \in [0,T] \times P$ the infinite number of constraints

$$(a_1(t,p) + E(t,p)u)^T \alpha \geq 0 \quad \forall\ u \in \mathbb{R}^n : \|u\|_2 \leq 1 \tag{3.1}$$

is equivalent to the single constraint $a_1(t,p)^T \alpha \geq \|E(t,p)^T \alpha\|$. Note that (3.1) can be rewritten as

$$\min_{u \in \mathbb{R}^n : \|u\|_2 \leq 1} (a_1(t,p) + E(t,p)u)^T \alpha = \min_{u \in \mathbb{R}^n : \|u\|_2 \leq 1} \left(a_1(t,p)^T \alpha + (E(t,p)^T \alpha)^T u \right) \geq 0$$

It is easy to prove that the minimum of this linear function on the unit circle is attained for the vector $u_* = -E(t,p)^T \alpha / \|E(t,p)^T \alpha\|_2$ if $E(t,p)^T \alpha \neq 0$. However, as $E(t,p)$ is a positive definite matrix, the case $E(t,p)^T \alpha = 0$ would imply $\alpha = 0$ which is not admissible for problem (RP) because then the second constraint in this problem would imply $a_2(s)^T \alpha = a_2(s)^T 0 = 0 \geq b_2(s) = \max(s - K, 0) > 0$ for $s \in (K, D]$. Hence (3.1) is equivalent to

$$(a_1(t,p) + E(t,p)u_*)^T \alpha = a_1(t,p)^T \alpha - (E(t,p)^T \alpha)^T \frac{E(t,p)^T \alpha}{\|E(t,p)^T \alpha\|_2} \geq 0$$

This expression in turn can be transformed to $a_1(t,p)^T \alpha \geq \|E(t,p)^T \alpha\|_2$ which proves the theorem. □

The new equivalent formulation (RP–SOCP) of (RP) also allows to interprete the robustness added to (P). If the first constraint of (RP–SOCP) is compared to the corresponding constraint of (P), it is clear that the hedge portfolio described by the robust problem offers a safety margin $\|E(t,p)^T \alpha\|_2 > 0$ that protects the portfolio against model errors.

Further note that problem (RP–SOCP) is still a semi–infinite optimization problem, but compared to problem (RP) the parameter set $\{u \in \mathbb{R}^n : \|u\|_2 \leq 1\} \times [0,T] \times P$ has been reduced to $[0,T] \times P$. The price for this reduction is the nonlinearity which now enters the constraints of problem (RP–SOCP) in the form of second order cone constraints.

In Section 4 we will present an alternative problem formulation also reducing the complexity of the parameter space but still preserving the linear structure of the underlying problem (RP). But before we turn to the associated transformations, we will briefly discuss the numerical solution of problems (RP) and (RP–SOCP). As the two optimization problems are equivalent, we will focus on the reduced problem (RP–SOCP).

In general, an algorithm solving problem (RP–SOCP) will replace the infinite number of constraints associated with the parameter sets $[0,T] \times P$ and $[0,D]$ by a discrete set of constraints. Let the discrete approximations of these parameter sets be denoted by $M_1 \subset [0,T]\times P$ and $M_2 \subset [0,D]$, $|M_1|, |M_2| < \infty$. The resulting discrete version of (RP–SOCP) is then given by

$$\begin{aligned} &\min_{\alpha\in\mathbb{R}^n} c^T\alpha \\ \text{s.t.}\quad & a_1(t,p)^T\alpha \geq \|E(t,p)^T\alpha\|_2 \quad \forall\ (t,p)\in M_1 \\ & a_2(s)^T\alpha \geq b_2(s) \quad \forall\ s\in M_2 \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub}, \quad i=1,...,n \end{aligned} \tag{RP–SOCP–DISCR}$$

Clearly, problem (RP–SOCP–DISCR) is a second order cone program where we can assure the existence of solutions for any discrete sets M_1, M_2 due to the compactness of the associated feasible set and Assumption 3.2. Hence problem (RP–SOCP–DISCR) can be solved by standard SOCP– or SDP–solvers. However, the solution of this problem is in general not feasible for the original problem (RP–SOCP). To overcome this difficulty, we successively add the most violating constraints to the sets M_1, M_2. This leads to the following algorithm.

Algorithm 3.4. *Let $M_1 \subset [0,T]\times P$ and $M_2 \subset [0,D]$, $|M_1|, |M_2| < \infty$ be given initial grids. Further let $\epsilon > 0$ be a suitable convergence tolerance and $k = 0$.*

(S1) Calculate an optimal solution α^k of the discretized problem (RP–SOCP–DISCR).

(S2) Determine the constraint violation (CV) of α^k for problem (RP–SOCP) by minimizing the slack–functions at α^k:

$$\begin{aligned} \delta_1 &= \min_{(t,p)\in[0,T]\times P} a_1(t,p)^T\alpha^k - \|E(t,p)^T\alpha^k\|_2 \\ \delta_2 &= \min_{s\in[0,D]} a_2(s)^T\alpha^k - b_2(s) \end{aligned} \tag{CV–SOCP}$$

If $\min(\delta_1,\delta_2) \geq -\epsilon$ then STOP.

(S3) Add the minimizers of the slack functions (the most violating constraints) to M_1, M_2. Set $k \to k+1$ and go to step (S1).

In order to solve problem (RP–SOCP), the algorithm successively solves second order cone programs (RP–SOCP–DISCR) and nonlinear optimization problems in the form of (CV–SOCP). Note that the latter problem is in fact a PDE–constrained optimization problem as the vector a_1 depends on the prices $C_i(t,D,\nu,\kappa,\theta,\sigma,\rho)$ which in turn are the solution of PDE (2.3). The next theorem shows, that these subproblems and hence Algorithm 3.4 is well-defined and that each limit point of the sequence is an optimal solution of problem (RP–SOCP).

Theorem 3.5. *Assume that Assumption 3.2 holds. Then Algorithm 3.4 is well defined. Furthermore, if $\epsilon = 0$, every limit point of the sequence $(\alpha^k)_{k\in\mathbb{N}}$ is an optimal solution of problem (RP–SOCP).*

Proof. As mentioned before, step (S1) is well defined due to Assumption 3.2. Furthermore, the two slack minimizations in step (S2) always have a solution, because in these problems continuous functions are minimized on compact sets. The continuity of a_2 is obvious, the map $E : [0,T]\times P \to \mathbb{R}^{n\times n}$ was assumed to be continuous and for the continuity of a_1 see Maruhn [26].

Note that, due to Theorem 3.3, optimization problem (RP–SOCP) is equivalent to the linear semi–infinite optimization problem (RP). Hence it is sufficient to show, that Algorithm 3.4 in fact produces iterates of an algorithm for the solution of a linear semi–infinite optimization problem and then apply the corresponding convergence theory.

The proof of Theorem 3.3 shows, that the solution of the discretized problem (RP–SOCP–

DISCR) in step (S1) of Algorithm 3.4 is equivalent to the linear semi–infinite programming problem

$$\begin{aligned} &\min_{\alpha\in\mathbb{R}^n} c^T\alpha \\ \text{s.t.}\quad &(a_1(t,p)+E(t,p)u)^T\alpha \geq 0 \quad \forall\ (u,t,p)\in\{u:\|u\|_2\leq 1\}\times M_1 \\ &a_2(s)^T\alpha \geq b_2(s) \quad \forall\ s\in M_2 \\ &\alpha_i^{lb}\leq \alpha_i \leq \alpha_i^{ub}, \quad i=1,...,n \end{aligned} \tag{3.2}$$

Furthermore, due to Theorem 3.3 and its proof, the problem of computing the worst case constraint violation (CV–SOCP) in step (S2) is equivalent to

$$\min_{(u,t,p)\in\{u:\|u\|_2\leq 1\}\times[0,T]\times P} (a_1(t,p)+E(t,p)u)^T\alpha \tag{3.3}$$

because one can eliminate the variable u from the minimization by explicitly computing the optimal $u_*(t,p) := -E(t,p)^T\alpha/\|E(t,p)^T\alpha\|_2$. In particular, if (t_*,p_*) denotes the optimal solution of problem (CV–SOCP), then $(u_*(t_*,p_*),t_*,p_*)$ is also the solution of (3.3) and vice versa.

Adding the point (t_*,p_*) to the set M_1 in step (S3) of Algorithm 3.4 and hence the constraint $a_1(t_*,p_*)^T\alpha \geq \|E(t_*,p_*)^T\alpha\|_2$ to the next SOCP in step (S1), is equivalent to adding the infinite number of constraints

$$(a_1(t_*,p_*)+E(t_*,p_*)u)^T\alpha \geq 0 \quad \forall\ u\in\{u:\|u\|_2\leq 1\}$$

to problem (3.2). Hence the components (u,t_*,p_*), $u\in\{u:\|u\|_2\leq 1\}$, can be interpreted as an infinite number of feasibility cuts for the linear semi–infinite optimization problem which are added to the parameter sets $\{u:\|u\|_2\leq 1\}\times M_1$.

Applying the standard convergence theory of linear semi–infinite optimization (see e.g. Theorem 11.2 in Goberna and Lopez [11]) to a successive solution of problems (3.2), (3.3) and with the mesh update $\{u:\|u\|_2\leq 1\}\times(M_1\cup\{t_*,p_*\})$ mentioned previously then proves the convergence. □

The proof of Theorem 3.5 made heavy use of the equivalence of the linear semi–infinite optimization problem (RP) and the semi–infinite second order cone problem (RP–SOCP). For the numerical implementation, it seems to be advantegous to solve problem (RP–SOCP) instead of (RP), because the dimension of the semi–infinite parameter space $\{u\in\mathbb{R}^n:\|u\|_2\leq 1\}\times[0,T]\times P$ is reduced drastically from typically $15-30$ (depending on the number of financial products n included in the portfolio) to 6, the dimension of $[0,T]\times P$. However, from a computational point of view, the drawback is the additional nonlinearity in the form of the second order cone constraints. In the next section we will see how the additional SOCP–nonlinearity can be eliminated from the optimization problems arising in Algorithm 3.4.

4. Equivalent SDP–NSDP Formulation. In this section we restate the robust second order cone problem (RP–SOCP) as a semi–infinite semidefinite programming problem. As it turns out, this equivalent formulation can be used to derive an iterative procedure similar to Algorithm 3.4 which successively solves SDPs instead of the second order cone problems (RP–SOCP–DISCR) and NSDPs instead of the nonlinear programming problems (CV–SOCP). Based on Theorem 3.5 we can prove convergence for this mixed SDP–NSDP procedure.

It is a well known fact that for fixed $(t,p)\in[0,T]\times P$ a conic quadratic constraint of the form

$$a_1(t,p)^T\alpha \geq \|E(t,p)^T\alpha\|_2$$

can be explicitly converted to an SDP constraint. The following result presents the equivalent SDP formulation of the second order cone problem (RP–SOCP). For completeness of the paper we present a detailed proof.

LEMMA 4.1. *The conic quadratic problem (RP–SOCP) is equivalent to the following semi–infinite optimization problem with an infinite number of SDP constraints*

$$\begin{aligned} &\min_{\alpha \in \mathbb{R}^n} c^T \alpha \\ s.t.\quad & \mathcal{A}(t,p;\alpha) \succeq 0 \quad \forall\ (t,p) \in [0,T] \times P \\ & a_2(s)^T \alpha \geq b_2(s) \quad \forall\ s \in [0,D] \\ & \alpha_i^{lb} \leq \alpha_i \leq \alpha_i^{ub}, \quad i = 1,...,n, \end{aligned} \tag{RP–SDP}$$

where

$$\mathcal{A}(t,p;\alpha) := \begin{pmatrix} a_1(t,p)^T \alpha\ I_n & -E(t,p)^T \alpha \\ -\alpha^T E(t,p) & a_1(t,p)^T \alpha \end{pmatrix} \in \mathbb{S}^{n+1}, \tag{4.1}$$

$I_n \in \mathbb{R}^{n \times n}$ *denotes the identity matrix and* $\mathbb{S}^{n+1}$ *is the space of all real symmetric* $(n+1) \times (n+1)$ *matrices.*

Proof. To show the equivalence of (RP–SOCP) and (RP–SDP) it suffices to observe that

$$\mathcal{A}(t,p;\alpha) \succeq 0 \quad \Longleftrightarrow \quad a_1(t,p)^T \alpha \geq \|E(t,p)^T \alpha\|_2.$$

Suppose $\mathcal{A}(t,p;\alpha) \succeq 0$, then for any $z = (\xi, \tau)^T \in \mathbb{R}^{n+1}$ $(\xi \in \mathbb{R}^n, \tau \in \mathbb{R})$ we know

$$0 \leq z^T \mathcal{A}(t,p;\alpha) z = a_1(t,p)^T \alpha\ ||\xi||_2^2 - 2\tau\ \alpha^T E(t,p)\xi + a_1(t,p)^T \alpha\ \tau^2. \tag{4.2}$$

Assuming $a_1(t,p)^T\alpha < 0$, then for $\xi \in \mathbb{R}^n \backslash \{0\}$ and $\tau = 0$ the relation (4.2) would imply $0 \leq z^T \mathcal{A}(t,p;\alpha) z = a_1(t,p)^T \alpha\ ||\xi||_2^2 < 0$ which is a contradiction. Hence, $\mathcal{A}(t,p;\alpha) \succeq 0$ always yields $a_1(t,p)^T \alpha \geq 0$.

If $a_1(t,p)^T\alpha = 0$, then for $\xi = E(t,p)^T\alpha$ and $\tau > 0$ we deduce from (4.2) that $E(t,p)^T\alpha = 0$, since, in this case, (4.2) yields $0 \leq z^T \mathcal{A}(t,p;\alpha) z = -2\tau ||E(t,p)^T\alpha||_2^2$ which in turn is equivalent to $||E(t,p)^T\alpha||_2 = 0$ ($\Longleftrightarrow$ $\alpha = 0$ since $E(t,p) \succ 0$). Therefore, we get

$$a_1(t,p)^T \alpha \geq 0 = ||E(t,p)^T \alpha||_2$$

in the case of $a_1(t,p)^T\alpha = 0$.

Otherwise, if $a_1(t,p)^T\alpha > 0$, then, using (4.2), the choice $z = (E(t,p)^T\alpha, a_1(t,p)^T\alpha)^T$ yields

$$0 \leq z^T \mathcal{A}(t,p;\alpha) z = a_1(t,p)^T \alpha\ \left(\left(a_1(t,p)^T\alpha\right)^2 - \left\|E(t,p)^T\alpha\right\|_2^2 \right)$$

which also implies

$$a_1(t,p)^T \alpha \geq ||E(t,p)^T \alpha||_2.$$

On the other hand, suppose $a_1(t,p)^T\alpha \geq ||E(t,p)^T\alpha||_2$, then for every $z = (\xi,\tau)^T \in \mathbb{R}^{n+1}$ we deduce, by using the Cauchy Schwarz formula, e. g.

$$-\tau \left(E(t,p)^T\alpha\right)^T \xi \geq -|\tau| \left|\langle E(t,p)^T\alpha, \xi\rangle_2\right| \geq -|\tau| \left\|E(t,p)^T\alpha\right\|_2 \|\xi\|_2$$

that

$$\begin{aligned} z^T \mathcal{A}(t,p;\alpha) z &= a_1(t,p)^T \alpha\ ||\xi||_2^2 - 2\tau\ \alpha^T E(t,p)\xi + a_1(t,p)^T \alpha\ \tau^2 \\ &\geq a_1(t,p)^T \alpha\ ||\xi||_2^2 - 2|\tau| \underbrace{\left\|E(t,p)^T\alpha\right\|_2}_{\leq a_1(t,p)^T\alpha} \|\xi\|_2 + a_1(t,p)^T \alpha\ \tau^2 \\ &\geq a_1(t,p)^T \alpha \left(\tau^2 - 2|\tau| ||\xi||_2 + ||\xi||_2^2\right) \ \geq\ a_1(t,p)^T \alpha \left(|\tau| - ||\xi||_2\right)^2 \ \geq\ 0 \end{aligned}$$

Thus, $a_1(t,p)^T\alpha \geq ||E(t,p)^T\alpha||_2$ implies $\mathcal{A}(t,p;\alpha) \succeq 0$. □

For solving the semi–infinite conic optimization problem (RP–SOCP) or equivalently the semi–infinite linear matrix inequality problem (RP–SDP), the parameter space of the constraints is discretized and in each step the worst case constraint violation is computed by solving optimization problem (CV–SOCP). Intuitively, for computing the worst case constraint violation of the equivalent problem (RP–SDP) we need to solve in every step the following eigenvalue problem:

$$\min_{(t,p)\in[0,T]\times P} \lambda_{\min}\left(\mathcal{A}(t,p;\alpha)\right), \qquad \text{(CV–SDP)}$$

where $\alpha \in \mathbb{R}^n$ is assumed to be given, $\mathcal{A}(t,p;\alpha) \in \mathbb{S}^{n+1}$ is defined by (4.1) and $\lambda_{\min}(\mathcal{A}(t,p;\alpha))$ denotes the minimal eigenvalue of $\mathcal{A}(t,p;\alpha)$. Using the next lemma we can show the equivalence of (CV–SOCP) and the eigenvalue problem (CV–SDP).

LEMMA 4.2. *Let $I_n \in \mathbb{R}^{n\times n}$ be the identity matrix, $d \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ and*

$$A := \begin{pmatrix} \beta I_n & d \\ d^T & \beta \end{pmatrix} \in \mathbb{S}^{n+1},$$

then the eigenvalues $\lambda_i(A)$, $i = 1,\ldots,n+1$ of A are given by:

$$\lambda_{\min}(A) := \lambda_1(A) = \beta - ||d||_2, \quad \lambda_2(A) = \ldots = \lambda_n(A) = \beta, \quad \lambda_{\max}(A) := \lambda_{n+1}(A) = \beta + ||d||_2.$$

Proof. First we assume that $\lambda \neq \beta$ ($\lambda \in \mathbb{R}$) and determine the roots of the characteristic polynomial of $A - \lambda I_{n+1}$, e. g.

$$\det\left(A - \lambda I_{n+1}\right) = \det\begin{pmatrix} (\beta-\lambda)I_n & d \\ d^T & \beta-\lambda \end{pmatrix} = 0,$$

by applying a Gaussian elimination step on the last row of $A - \lambda I_{n+1}$. Defining $0^T = (0,\ldots,0)^T \in \mathbb{R}^n$, we get

$$\begin{aligned}\det\left(A - \lambda I_{n+1}\right) = \det\begin{pmatrix} (\beta-\lambda)I_n & d \\ 0^T & (\beta-\lambda) - \sum_{i=1}^n \frac{d_i^2}{\beta-\lambda} \end{pmatrix} &= (\beta-\lambda)^n\left((\beta-\lambda) - \sum_{i=1}^n \frac{d_i^2}{\beta-\lambda}\right) \\ = 0\end{aligned}$$

which in turn is equivalent to

$$0 = (\beta-\lambda)^{n-1}\left((\beta-\lambda)^2 - ||d||^2\right) \qquad \Longleftrightarrow \qquad (\beta-\lambda)^2 = ||d||^2,$$

in case of $\lambda \neq \beta$. Thus

$$\lambda_1(A) = \beta - ||d||_2, \quad \lambda_{n+1}(A) = \beta + ||d||_2$$

are two eigenvalues of A. If $\lambda = \beta$, then by applying the Laplacian expansion theorem we deduce

$$\begin{aligned}\det\begin{pmatrix} (\beta-\lambda)I_n & d \\ d^T & \beta-\lambda \end{pmatrix} &= \det\begin{pmatrix} 0_{n\times n} & d \\ d^T & 0 \end{pmatrix} \\ &= \sum_{j=1}^{n+1}(-1)^{n+1+j} d_j \det\left(A_{j,n+1}\right) \\ &= 0,\end{aligned}$$

where $A_{j,n+1} \in \mathbb{R}^{n\times n}$ is a submatrix of A obtained by deleting the j–th row and the $(n+1)$–th column of A. Thus, β is an eigenvalue of A and it is straightforward to show that the multiplicity of this eigenvalue is equal to $n-1$. Hence we have

$$\lambda_2(A) = \ldots = \lambda_n(A) = \beta.$$

Ordering the eigenvalues $\lambda_1, ..., \lambda_{n+1}$ yields the desired result. □

The result of the following lemma proves that the objective function values of (CV–SOCP) and (CV–SDP) coincide.

LEMMA 4.3. *Let $\mathcal{A}(t,p;\alpha) \in \mathbb{S}^{n+1}$ be defined by (4.1), then*

$$\begin{array}{ll} i) & \lambda_{\min}\left(\mathcal{A}(t,p;\alpha)\right) = a_1(t,p)^T\alpha - \|E(t,p)^T\alpha\|_2 \\ ii) & \qquad \textit{(CV–SOCP)} \quad \Longleftrightarrow \quad \textit{(CV–SDP)}. \end{array}$$

Proof. Applying Lemma 4.2 to $\mathcal{A}(t,p;\alpha)$ the result immediately follows. □

Due to the previous lemma, the nonlinear minimization problem (CV–SOCP) is equivalent to the minimization of the minimal eigenvalue of $\mathcal{A}(t,p;\alpha)$, where this matrix depends in a very nonlinear fashion on the parameters $(t,p) \in [0,T] \times P$. In general such an eigenvalue minimization problem is a very hard non–smooth optimization problem (see e. g. [4], [5]).

In some cases it is well known, that eigenvalue optimization problems can be transformed to a SDP. For example the minimization of the maximal eigenvalue of a linear matrix function can be restated as a semidefinite program (see e. g. [31]). However, in our case we are interested in minimizing the *minimal* eigenvalue of a nonlinear matrix function, which contrasts the typical goal in other research areas like robust control design (see e. g. [5], [13], [18], [20], [27]).

In the next theorem we show, by using duality arguments, that the problem of minimizing the minimal eigenvalue of a matrix function can also be rewritten as a matrix optimization problem with SDP constraints. To the knowledge of the authors, no general result for this case is known so far.

THEOREM 4.4. *Let $k,l,m \in \mathbb{N}$ and $\mathcal{M}(q) : Q \to \mathbb{S}^m$, $Q \subseteq \mathbb{R}^{k\times l}$ be a real symmetric matrix function, then the following two problems are equivalent:*

$$\min_{q\in Q} \lambda_{\min}\left(\mathcal{M}(q)\right), \tag{4.3}$$

$$\min_{(q,\Pi)\in Q\times\mathbb{S}^m} \left\{Tr(\Pi\, \mathcal{M}(q)) \;\mid\; \Pi \succeq 0,\; 1 - Tr(\Pi) = 0\right\}, \tag{4.4}$$

where $\lambda_{\min}(M)$ denotes the minimal eigenvalue of $M \in \mathbb{S}^m$ and $Tr(M)$ is the trace operator of $M \in \mathbb{R}^{m\times m}$, respectively. Moreover the optimal function values coincide.

Proof. We prove the theorem in two steps. First we show that the computation of the minimal eigenvalue of a matrix function is equivalent to the maximization of a scalar variable subject to a linear matrix inequality constraint. In a second step, we derive the dual of this maximization problem which completes the proof.

First, fixing $q \in Q$ we show

$$\lambda_{\min}\left(\mathcal{M}(q)\right) = \max_{\tau\in\mathbb{R}} \left\{\, \tau \;\mid\; \mathcal{M}(q) - \tau I_m \succeq 0\right\}, \tag{4.5}$$

where $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix. The Raleigh–Ritz theorem (e. g. [15, Theorem 4.2.2]) implies $M - \lambda_{\min}\, I_m \succeq 0$, where $\lambda_{\min} := \lambda_{\min}\left(\mathcal{M}(q)\right)$ and $M := \mathcal{M}(q)$. To show that $\lambda_{\min}$ is the sharpest bound satisfying the matrix inequality $M - \tau\, I_m \succeq 0$ we prove the following equivalence

$$M \succeq \tau\, I_m \quad \Longleftrightarrow \quad \lambda_{\min}(M) \geq \tau, \tag{4.6}$$

where $M \in \mathbb{S}^m$ and $\tau \in \mathbb{R}$. We diagonalize the real symmetric matrix $M \in \mathbb{S}^m$. Let $H \in \mathbb{R}^{m\times m}$ be an orthogonal matrix such that

$$H^T M H = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) =: D,$$

where $\lambda_{\min}(M) := \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ are the real eigenvalues of M and $\operatorname{diag}(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{m\times m}$ denotes a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_m$ (see e. g. [15, Corollary 2.5.14]). Using this decomposition we get

$$M - \tau\, I_m = HDH^T - \tau\, I_m = H(D - \tau\, I_m)H^T \succeq 0 \quad\Longleftrightarrow\quad D - \tau\, I_m \succeq 0,$$

which in turn is equivalent to

$$\lambda_i \geq \tau \quad \forall\, i = 1, \ldots, m \qquad\Longleftrightarrow\qquad \lambda_{\min}(M) \geq \tau.$$

This proves (4.6) and therefore (4.5).

Secondly, using (4.5), we prove the equivalence of (4.3) and (4.4) by the strong Wolfe duality theorem (see, e. g. [9, Theorem 9.5.1], [32]). The Lagrangian of the primal problem (4.5) is defined by $\ell : \mathbb{R} \times \mathbb{S}^m \to \mathbb{R}$,

$$\ell(\tau, \Pi) = \tau + \langle \Pi, \mathcal{M}(q) - \tau I_m \rangle = (1 - Tr(\Pi))\ \tau + Tr(\Pi\, \mathcal{M}(q)).$$

The first order Fréchet derivative of ℓ with respect to τ applied to $\delta\tau$ is given by

$$\ell'_\tau(\tau, \Pi)\delta\tau = \langle \delta\tau, \nabla_\tau \ell(\tau, \Pi)\rangle = (1 - Tr(\Pi))\, \delta\tau$$

and, thus, we obtain the following (necessary and sufficient) optimality conditions of problem (4.5)

$$\nabla_\tau \ell(\tau, \Pi) = 1 - Tr(\Pi) = 0, \quad \Pi \succeq 0, \quad Tr\,(\Pi\,(\mathcal{M}(q) - \tau I_m)) = 0, \quad \mathcal{M}(q) - \tau I_m \succeq 0.$$

Applying the strong duality theorem we get the corresponding dual problem of (4.5)

$$\min_{(\tau,\Pi)\in\mathbb{R}\times\mathbb{S}^m} \ \ell(\tau, \Pi), \quad \text{s. t. } \ 1 - Tr(\Pi) = 0, \quad \Pi \succeq 0.$$

Using $1 - Tr(\Pi) = 0$, the objective function of this dual problem reduces to

$$\ell(\tau, \Pi) = (1 - Tr(\Pi))\ \tau + Tr(\Pi\, \mathcal{M}(q)) = Tr(\Pi\, \mathcal{M}(q)).$$

Therefore, the final version of the dual problem is given by the following SDP

$$\min_{\Pi\in\mathbb{S}^m} \ Tr(\Pi\, \mathcal{M}(q)) \quad \text{s. t. } \ 1 - Tr(\Pi) = 0, \quad \Pi \succeq 0. \tag{4.7}$$

Moreover, the strong duality result ensures that the objective function values of the primal (linear) SDP (4.5) and the dual (linear) SDP (4.7) are equal (for fixed (but arbitrary) $q \in Q$), e. g. we have

$$\lambda_{\min}(\mathcal{M}(q)) = \max_{\tau\in\mathbb{R}} \ \{\tau \ \mid \ \mathcal{M}(q) - \tau I_m \succeq 0\} = \min_{\Pi\in\mathbb{S}^m} \ \{Tr(\Pi\, \mathcal{M}(q)) \ \mid \ 1 - Tr(\Pi) = 0,\ \Pi \succeq 0\}.$$

But this relation implies the desired result, i. e. now we know that

$$\begin{aligned}
\min_{q\in Q} \ \{\lambda_{\min}(\mathcal{M}(q))\} &= \min_{q\in Q} \left\{ \max_{\tau\in\mathbb{R}} \ \{\tau \ \mid \ \mathcal{M}(q) - \tau I_m \succeq 0\} \right\} \\
&= \min_{q\in Q} \left\{ \min_{\Pi\in\mathbb{S}^m} \ \{Tr(\Pi\, \mathcal{M}(q)) \ \mid \ 1 - Tr(\Pi) = 0,\ \Pi \succeq 0\} \right\} \\
&= \min_{(q,\Pi)\in Q\times\mathbb{S}^m} \{Tr(\Pi\, \mathcal{M}(q)) \ \mid \ 1 - Tr(\Pi) = 0,\ \Pi \succeq 0\}.
\end{aligned}$$

Hence, problem (4.3) is equivalent to (4.4). □

Now we apply the previous general theorem to the particular case of determining the worst case constraint violation for (RP–SDP). In particular we prove that, due to the nonlinearity of the matrix function $\mathcal{A}(\cdot,\cdot,\alpha)$ (α fixed), the eigenvalue problem (CV–SDP) is equivalent to a *nonlinear* semidefinite programming problem.

COROLLARY 4.5. *The problem of computing the constraint violation (CV–SOCP) is equivalent to the nonlinear semidefinite program*

$$\begin{aligned} \min_{(t,p,\Pi)\in[0,T]\times P\times\mathbb{S}^{n+1}} & \left(a_1(t,p)^T\alpha\right)\ Tr(\Pi) - 2\alpha^T E(t,p)d_\pi \\ s.t.\quad & \Pi \succeq 0, \quad Tr(\Pi) = 1, \end{aligned} \tag{CV–NSDP}$$

where, for $\Pi_1 \in \mathbb{S}^n$, $d_\pi \in \mathbb{R}^n$ *and* $\beta_\pi \in \mathbb{R}$, *the matrix* $\Pi \in \mathbb{S}^{n+1}$ *is defined by*

$$\Pi := \begin{pmatrix} \Pi_1 & d_\pi \\ d_\pi^T & \beta_\pi \end{pmatrix}.$$

Proof. From Lemma 4.3 we know that

$$\text{(CV–SOCP)} \iff \text{(CV–SDP)}.$$

Applying Theorem 4.4 to (CV–SDP), we get the equivalence of (CV–SDP) and the following NSDP

$$\begin{aligned} \min_{(t,p,\Pi)\in[0,T]\times P\times\mathbb{S}^{n+1}} & Tr\left(\Pi\ \mathcal{A}(t,p;\alpha)\right) \\ \text{s.t.}\quad & \Pi \succeq 0, \quad 1 - Tr(\Pi) = 0, \end{aligned} \tag{4.8}$$

Due to the special structure of $\mathcal{A}(t,p;\alpha) \in \mathbb{S}^{n+1}$, set

$$\mathcal{A}(t,p;\alpha) = \begin{pmatrix} a_1(t,p)^T\alpha\ I_n & -E(t,p)^T\alpha \\ -\alpha^T E(t,p) & a_1(t,p)^T\alpha \end{pmatrix} =: \begin{pmatrix} \beta I_n & d \\ d^T & \beta \end{pmatrix}.$$

Then

$$\Pi\mathcal{A}(t,p;\alpha) = \begin{pmatrix} \Pi_1 & d_\pi \\ d_\pi^T & \beta_\pi \end{pmatrix}\begin{pmatrix} \beta I_n & d \\ d^T & \beta \end{pmatrix} = \begin{pmatrix} \beta\Pi_1 + d_\pi d^T & \Pi_1 d + \beta d_\pi \\ \beta d_\pi^T + \beta_\pi d^T & d_\pi^T d + \beta_\pi\beta \end{pmatrix},$$

$\beta_\pi = \Pi_{n+1,n+1}$ and $Tr(d_\pi d^T) = d_\pi^T d$ implies

$$\begin{aligned} Tr\left(\Pi\ \mathcal{A}(t,p;\alpha)\right) &= Tr\left(\beta\Pi_1 + d_\pi d^T\right) + d_\pi^T d + \beta_\pi\beta \\ &= \beta\ Tr\left(\Pi_1\right) + \beta\beta_\pi + Tr\left(d_\pi d^T\right) + d_\pi^T d \\ &= \beta\ Tr\left(\Pi\right) + 2d_\pi^T d \end{aligned}$$

by using properties of the trace operator. Hence, we can rewrite the objective function of (4.8) to

$$Tr\left(\Pi\ \mathcal{A}(t,p;\alpha)\right) = \left(a_1(t,p)^T\alpha\right)\ Tr(\Pi) - 2\alpha^T E(t,p)d_\pi$$

which proves the corollary. □

Combining all the previous results we can restate Algorithm 3.4 in an equivalent SDP–NSDP formulation. While Algorithm 3.4 successively solves a sequence of second order cone programs (SOCP) and nonlinear programming problems (NLP), the restated version iteratively solves a sequence of SDPs and NSDPs.

ALGORITHM 4.6. *Let* $M_1 \subset [0,T]\times P$ *and* $M_2 \subset [0,D]$, $|M_1|, |M_2| < \infty$ *be given initial grids. Further let* $\epsilon > 0$ *be a suitable convergence tolerance and* $k = 0$.

(S1) Calculate an optimal solution α^k of the (linear) semidefinite program

$$\begin{aligned} &\min_{\alpha\in\mathbb{R}^n} c^T\alpha \\ s.t.\quad &\mathcal{A}(t,p;\alpha)\succeq 0 \quad \forall\ (t,p)\in M_1 \\ &a_2(s)^T\alpha \geq b_2(s) \quad \forall\ s\in M_2 \\ &\alpha_i^{lb}\leq \alpha_i\leq \alpha_i^{ub}, \quad i=1,...,n \end{aligned} \qquad \text{(RP–SDP–DISCR)}$$

(S2) Determine the constraint violation of α^k for problem (RP–SDP) by minimizing the slack-functions at α^k:

$$\delta_1 = \min_{(t,p,\Pi)\in[0,T]\times P\times\mathcal{S}^{n+1}} \left\{(a_1(t,p)^T\alpha^k)Tr(\Pi) - 2(\alpha^k)^T E(t,p)d_\pi \ \mid\ \Pi\succeq 0,\ Tr(\Pi)=1\right\} \qquad \text{(CV–NSDP)}$$

$$\delta_2 = \min_{s\in[0,D]} a_2(s)^T\alpha^k - b_2(s)$$

If $\min(\delta_1,\delta_2)\geq -\epsilon$ *then STOP.*

(S3) Add the minimizers of the slack functions (the most violating constraints) to M_1, M_2. Set $k\to k+1$ and go to step (S1).

Comparing Algorithm 3.4 and 4.6, the nonlinear SOCP–constraint in problem (RP–SOCP–DISCR) is replaced by a linear matrix inequality in problem (RP–SDP–DISCR). Furthermore the SOCP–nonlinearity is also removed from problem (CV–SOCP) for the price of an additional matrix variable in the objective function and simple linear matrix constraints in problem (CV–NSDP). In particular the objective functions of the constraint violation problems look very similar.

Furthermore, by applying Theorem 3.5, we immediately obtain the following convergence theorem for Algorithm 4.6.

COROLLARY 4.7. *Assume that Assumption 3.2 holds. Then Algorithm 4.6 is well defined. Furthermore, if $\epsilon = 0$, every limit point of the sequence $(\alpha^k)_{k\in\mathbb{N}}$ is an optimal solution of problem (RP–SDP).*

Proof. We prove the corollary by showing that each step of Algorithm 4.6 is equivalent to the corresponding step in Algorithm 3.4. For step (S1), the equivalence of (RP–SDP–DISCR) and (RP–SOCP–DISCR) follows directly by the proof of Lemma 4.1. The equivalence of steps (S2) was proven in Corollary 4.5. Therefore, we deduce the convergence of the algorithm from Theorem 3.5. □

5. Conclusions. In the previous paragraphs we have developed a successive SDP–NSDP algorithm solving the robust semi–infinite optimization problem arising in static hedging of barrier options which additionally takes model errors into account. The convergence of the method was obtained by showing that the standard theory of linear semi–infinite optimization can be applied to the equivalent SOCP–NLP formulation.

The presented algorithm solves a sequence of discretized versions of the underlying semi–infinite SDP and nonlinear minimization problems for the computation of the worst case constraint violation of the iterates. As it turned out, the latter problem can be reformulated as the minimization of the minimal eigenvalue of a nonlinear matrix function. This eigenvalue problem was proven to be equivalent to a nonlinear semidefinite program where the nonlinearity is introduced by the parameters of a parabolic differential equation.

As a byproduct we obtained a reformulation of the minimization of the minimal eigenvalue of a general matrix function as a semidefinite matrix optimization problem with linear matrix constraints.

Although we presented the solution procedure for a particular financial market application, the proposed method can be applied in analogy to other optimization problems with an infinite number of linear constraints and ellipsoidal uncertainty sets around the problem parameters.

As a key finding of this paper, nonlinear semidefinite programs naturally arise in the computation of the constraint violation of robust optimization problems such that the development of general NSDP–solvers is highly desired.

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