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CBS CONSTANTS FOR GRAPH-LAPLACIANS AND APPLICATION TO MULTILEVEL METHODS FOR DISCONTINUOUS GALERKIN SYSTEMS

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ABSTRACT. The goal of this work is to derive and justify a multilevel preconditioner for symmetric discontinuous approximations of second order elliptic problems. Our approach is based on the following simple idea. The finite element space $\mathcal{V}$ of piece-wise polynomials of certain degree that are discontinuous on the partition $\mathcal{T}_0$ is projected onto the space of piece-wise constants on the same partition. This will constitute the finest space in the multilevel method. The projection of the discontinuous Galerkin system on this space is associated to the so-called "graph-Laplacian". In 2-D this is a very simple M-matrix with -1 as off diagonal entries and current diagonal entries equal to the number of the neighbours through the interfaces of the current finite element. Then after consecutive aggregation of the finite elements we produce a sequence of spaces of piece-wise constant functions. We develop the concept of hierarchical splitting of the unknowns and using local analysis we derive uniform estimates for the constant in the strengthen Cauchy-Bunyakowski-Schwarz (CBS) inequality. As a measure of the angle between the spaces of the splitting, this further is used to justify a multilevel preconditioner of the discontinuous Galerkin system in spirit of the work [4] of Axelsson and Vassilevski.

KEY WORDS: discontinuous Galerkin, second order elliptic equation, graph-Laplacian, multilevel preconditioning, CBS constant

1. Introduction

Consider a second order elliptic problem on a polygonal domain $\Omega \subset R^d$, $d = 2, 3$:

$$
\begin{aligned}
-\nabla \cdot (a(x)\nabla u) &= f(x) \quad \text{in } \Omega, \\
u(x) &= g_D \quad \text{on } \Gamma_D, \\
\partial_N u(x) \equiv a\nabla u \cdot \mathbf{n} &= g_N \quad \text{on } \Gamma_N.
\end{aligned}
\tag{1.1}
$$

Here $\mathbf{n}$ is the exterior unit normal vector to $\partial\Omega \equiv \Gamma$. The boundary is assumed to be decomposed into two disjoint parts Γ_D and Γ_N, $\Gamma_D \cap \Gamma_N = \emptyset$ and the boundary data g_D, g_N are smooth. For the formulation below we shall need the existence of the traces of u and $a\nabla u \cdot \mathbf{n}$ on certain interfaces in Ω. Thus, the solution u is assumed to have the required regularity. To simplify our exposition we assume that the set Γ_D is not empty and its R^{d-1}-dimensional measure in nonzero.

In Section 2 we introduce two discontinuous Galerkin FEM for the above problem that lead to symmetric and positive definite algebraic problems. Such approximations were studied by [1, 12, 14, 20, 22] and are particular cases of a more general class of discontinuous Galerkin schemes for elliptic problems of second order.

Below we comment on some of the existing works on fast solution methods for system linear equations arising in discontinuous Galerkin approximations [11, 16, 17, 18, 21].

In [16] Gopalakrishnan and Kanschat discuss and study multigrid method (MG) for a symmetric discontinuous Galerkin method. A multigrid with variable number of smoothing steps is considered under the standard assumptions: (1) there is a sequence of nested triangulations $\mathcal{T}_1 \subset$

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$\mathcal{T}_2 \subset \dots \mathcal{T}_J$ and multilevel spaces $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots \mathcal{V}_J$ of piece-wise polynomials that define a sequence of operators $A_1, A_2, \dots, A_J$ and corresponding projections $P_{k-1} : \mathcal{V}_k \mapsto \mathcal{V}_{k-1}$; (2) P_{k-1} has certain weak approximation property: for all $u \in \mathcal{V}_k$, $k = 2, \dots, J$: $|||u - P_{k-1}u|||_k \le Ch_k \|A_k u\|_{-1+\beta}$. It is shown that if the number $m(k)$ of smoothings increases as k decreases, say $m(k) = 2^{J-k}$, then MG preconditioner has optimal arithmetic complexity comparable to the complexity of W-cycle.

Hemker, Hoffmann and van Raalte [18] have presented a local mode analysis of the MG convergence for discontinuous Galerkin systems. The results are obtained by Fourier analysis of the discretized Poisson equation in one space dimension. Two different ways for block-tridiagonal partitioning of the discrete operator and the related block-relaxation MG smoothers are considered. Though limited to 1-D and periodic boundary conditions the conclusion is that the point-wise block partitioning shows a much better convergence than the usual cell-wise block-partitioning.

Brenner and Zhao [11] considered rectangular partitions and bilinear finite element spaces in the discontinuous Galerkin method for the above problem. Main result of Brenner and Zhao for V-cycle could be summarized as follows: if the solution satisfies the a priori estimate $\|u\|_{H^{1+\alpha}} \le C\|f\|_{H^{-1+\alpha}},\ \alpha \in (1/2, 1]$, then there is m_0 which is independent of the number of levels k such that the norm of the multigrid error propagation operator E_{mg} satisfies the estimate $|||E_{mg}||| \le C/m^\alpha,\ k \ge 1,\ m \ge m_0$. Similar results are obtained for the W-cycle. This result is remarkable with the fact that the MG convergence improves with the smoothness of the solution.

Recently Lazarov, Vassilevski and Zikatanov [21] have studied multilevel methods for discontinuous Galerkin systems. Their approach is based on the following idea. First, apply the classical two grid method involving the original discontinuous Galerkin space $\mathcal{V}$ and a second space $\mathcal{V}^{(0)}$ of piece-wise polynomials defined on the same mesh. Three different choices for the space $\mathcal{V}^{(0)}$ have been proposed and studied: (1) continuous linear finite elements; (2) nonconforming Crouzeix-Raviart elements, (3) space of discontinuous piece-wise constant functions. Then the convergence of the two-grid method is considered in the general framework of algebraic two-grid methods established in [15]. Under certain assumptions it has been proved that the two-grid method converges independently of the mesh size. Further a multilevel extension based on a generalization of the algebraic multilevel iteration (AMLI) of Axelsson and Vassilevski [4, 23] has been proposed and studied. The third choice of spaces led to a novel and interesting from mathematical view point problem. The discontinuous Galerkin scheme on the space $\mathcal{V}^{(0)}$ of discontinuous piece-wise constant functions produces a symmetric and positive definite matrix, called "graph-Laplacian".

In the past such algebraic problems were generated by cell-centered approximations of elliptic equations on rectangular grids. Optimal preconditioners for such problems were studied in [9]. On rectangular grids the "graph Laplacian" approximates the Laplacian and the analysis could use some of the tools of the general multigrid theory. A multigrid preconditioners of optimal complexity were analyzed and tested in [9]. On an irregular grid the corresponding linear problem does not approximate an elliptic problem and the study of optimal preconditioners does not fit into the general framework of multigrid or multilevel methods.

The AMLI method is suitable for such situations since it could be analyzed by algebraic means, see e.g. [4, 6, 7, 13, 23]. One way to construct and justify optimal preconditioners is to introduce multilevel splitting of the unknowns as proposed originally by Bank and Dupont in [5] for standard finite element approximations. When applying to systems like one generated by the graph Laplacian the hierarchical splitting should have certain properties. These include the locality of the new basis, such that the condition number of the pivot block in the two-level matrix has a uniformly bounded condition number. This block correspond to the unknowns which are complementary to the coarse grid unknowns. The related second diagonal block can be viewed as a certain aggregation of the current two-level matrix. Within the introduced parametric setting, we will require this block to be not only associated but equal to the coarse grid matrix. The key role in the derivation of optimal convergence rate estimates plays the constant γ in the strengthened CBS inequality, associated with the angle between the two subspaces of the hierarchical splitting. It turns out that only existence of uniform estimates for this constant is not enough and accurate quantitative bounds for γ have to be found as well. More precisely, the value of the upper bound for $\gamma \in (0,1)$ is a part of the construction of the multilevel extension of the related two-level method.

The paper is organized as follows. In Section 2 we describe two symmetric discontinuous Galerkin finite element approximations of second order elliptic problem. The two-grid algorithm which reduces the problem to a graph-Laplacian system is introduced in the next section. Sections 4, 5 contain the needed setting of the AMLI method and the theory of the CBS constant. The derived new estimates of the CBS constant for graph-Laplacians and the related optimal multilevel preconditioners are presented in Section 6. Some concluding remarks are given at the end.

2. Discontinuous Galerkin FE approximation

Let $\mathcal{T}$ be a partitioning of Ω into finite number of open subdomains (finite elements) K with boundaries ∂K. We assume that the partition is quasi uniform and regular. For each finite element we denote by h_K its size and further $h = \max_{K\in\mathcal{T}} h_K$. Let $e = \overline{K}_1 \cap \overline{K}_2$ be the interface of two adjacent subdomains K_1, K_2. The set of all such interfaces is denoted by $\mathcal{E}_0$, note that these interfaces are inside Ω. Further, $\mathcal{E}_D$ and $\mathcal{E}_N$ will be the faces/edges of finite elements on the boundary Γ_D and Γ_N, respectively. Finally, $\mathcal{E}$ will be the set of all faces/edges: $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_D \cup \mathcal{E}_N$. Here we allow finite elements of polygonal or polyhedral shape, with hanging nodes etc. The important assumption here is that if e is a side or a face of a finite element $K \in \mathcal{T}$ then $|e| \approx h_K$ for $d = 2$ and $|e|^{\frac{1}{2}} \approx h_K$ for $d = 3$. In other words we do not allow very small edges or faces.

On the partition $\mathcal{T}$ we define the finite element space

$$\mathcal{V} := \mathcal{V}(\mathcal{T}) := \{v \in L^2(\Omega) : v|_K \in P_r(K),\ K \in \mathcal{T}\},$$

where P_r is the set of polynomials of degree $r \ge 0$.

For each $e = \bar{K} \cap \bar{K}' \in \mathcal{E}_0$ we define the jump $[\![v]\!]$ of any function $v \in \mathcal{V}$ as the vector

$$[\![v]\!]_e := \begin{cases} v|_K \cdot \mathbf{n} + v|_{K'}\mathbf{n}', & e = \bar{K} \cap \bar{K}',\ \text{i.e. } e \in \mathcal{E}_0, \\ v|_K \cdot \mathbf{n}, & e = \bar{K} \cap \Gamma_D,\ \text{i.e. } e \in \mathcal{E} \setminus \mathcal{E}_0. \end{cases}$$

Here $\mathbf{n}$ and $\mathbf{n}'$ are the external unit vectors to K and K', respectively.

We shall also need the following notation for the average value of the traces of the normal component of a vector function $\mathbf{v} \in \mathcal{V}$ on $e = \bar{K} \cap \bar{K}'$

$$\{\mathbf{v}\}|_e := \begin{cases} \frac{1}{2}\{\mathbf{v}|_K \cdot \mathbf{n} + \mathbf{v}|_{K'} \cdot \mathbf{n}'\}, & e = \bar{K} \cap \bar{K}',\ \text{i.e. } e \in \mathcal{E}_0, \\ \mathbf{v}|_K \cdot \mathbf{n}, & e = \bar{K} \cap \Gamma_D,\ \text{i.e. } e \in \mathcal{E} \setminus \mathcal{E}_0 \end{cases}$$

and the piecewise constant function $h_{\mathcal{E}}$ defined on $\mathcal{E}$ as

$$h_{\mathcal{E}} = h_{\mathcal{E}}(x) = \begin{cases} |e|, \text{ for } x \in e \in \mathcal{E}, d = 2 \\ |e|^{\frac{1}{2}}, \text{ for } x \in e \in \mathcal{E}, d = 3. \end{cases}$$

Further denote by

$$(a\nabla v, \nabla v)_{\mathcal{T}} := \sum_{K\in\mathcal{T}} \int_K a\nabla u\ \nabla v\ dx, \quad \left\langle h_{\mathcal{E}}^{-1} [\![u]\!], [\![v]\!]\right\rangle_{\mathcal{E}\cup\mathcal{E}_D} := \sum_{e\in\mathcal{E}\cup\mathcal{E}_D} \int_e h_{\mathcal{E}}^{-1} [\![u]\!] \cdot [\![v]\!] ds.$$

Finally, we shall use the following norm on $\mathcal{V}$:

$$|\!|\!| v |\!|\!|_h^2 = (a\nabla v, \nabla v)_{\mathcal{T}} + \kappa \left\langle h_{\mathcal{E}}^{-1} [\![v]\!], [\![v]\!]\right\rangle_{\mathcal{E}\cup\mathcal{E}_D}. \tag{2.1}$$

We shall consider the following symmetric interior penalty discontinuous Galerkin (IP DG) finite element method: (see, e.g. [1]): find $u_h \in \mathcal{V}$ such that

$$\mathcal{A}(u_h, v) = \mathcal{L}(v), \quad \forall\ v \in \mathcal{V}, \tag{2.2}$$

where

$$\begin{aligned} \mathcal{A}(u_h, v) \equiv \;& (a\nabla u_h, \nabla v)_{\mathcal{T}} + \kappa \left\langle h_{\mathcal{E}}{}^{-1} [\![u_h]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D} \\ & -\langle \{a\nabla u_h\},\ [\![v]\!]\rangle_{\mathcal{E}\cup\mathcal{E}_D} - \langle [\![u_h]\!],\ \{a\nabla v\}\rangle_{\mathcal{E}\cup\mathcal{E}_D} \end{aligned} \tag{2.3}$$

and

$$\mathcal{L}(v) = (f, v) + \langle g_N,\ v\rangle_{\mathcal{E}_N} - \langle g_D,\, a\nabla v\cdot \mathbf{n}\rangle_{\mathcal{E}_D} + \kappa \left\langle h_{\mathcal{E}}{}^{-1} g_D,\ v\right\rangle_{\mathcal{E}_D}. \tag{2.4}$$

It is well known that if κ is sufficiently large then the bilinear from (2.3) is coercive and bounded in $\mathcal{V}$ equipped with the norm (2.1), (see, e.g. [1]).

Another symmetric discontinuous Galerkin scheme could be derived by using an approach developed in the work of Ewing, Wang, and Yang [14]. In this case we get a bilinear from

$$\begin{aligned} \mathcal{A}(u_h, v) \equiv \;& (a\nabla u_h, \nabla v)_{\mathcal{T}} + \kappa \left\langle h_{\mathcal{E}}{}^{-1} [\![u_h]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D} \\ - \;& \langle \{a\nabla u_h\},\ [\![v]\!]\rangle_{\mathcal{E}\cup\mathcal{E}_D} - \langle [\![u_h]\!],\ \{a\nabla v\}\rangle_{\mathcal{E}\cup\mathcal{E}_D} \\ - \;& \tfrac{1}{4}\kappa^{-1} \langle h_{\mathcal{E}}\, [\![a\nabla u_h\cdot\mathbf{n}]\!],\ [\![a\nabla v\cdot\mathbf{n}]\!]\rangle_{\mathcal{E}_0}, \end{aligned} \tag{2.5}$$

which is coercive for sufficiently large κ. Note that the corresponding DG scheme is slightly different from (2.2).

We summarize the main results regarding the discontinuous Galerkin method (2.2) in the following lemma:

Lemma 2.1. *Assume that the finite element partition $\mathcal{T}$ is regular and locally quasi uniform. Then the bilinear form $\mathcal{A}(\cdot,\cdot)$ defined by (2.3) or (2.5) is coercive and bounded in $\mathcal{V}$ equipped with the norm (2.1) for any sufficiently large $\kappa > 0$ and the discontinuous Galerkin method (2.2) has unique solution.*

3. Two-grid method

In [21] the following setting for preconditioning discontinuous Galerkin system (2.2) has been proposed and studied. First, one smooths and then projects the problem from $\mathcal{V}$ to an appropriate intermediate space $\mathcal{V}^{(0)}$. In order to use the general theory developed in [15] we shall first reformulate the problem (2) in terms of matrices and degrees of freedom. We introduce the space of degrees of freedom $\mathbb{V}$ and $\mathbb{V}^{(0)}$ as dual to the function spaces $\mathcal{V}$ and $\mathcal{V}^{(0)}$, once the basis in $\mathcal{V}$ and $\mathcal{V}^{(0)}$ is fixed, which we shall take to be the standard nodal basis.

To avoid proliferation of indexes further we shall skip the subindex h from our notation. That is u_h is replaced by u.

The degrees of freedom for a triangle and quadrilateral are shown on Figure 1. For linear finite elements over triangular mesh $\mathbb{V}$ could be identified with $\mathbb{R}^{3N_T}$, while for bilinear elements over quadrilateral mesh, with $\mathbb{R}^{4N_T}$.

Then we define the standard ℓ_2-inner product for elements on $\mathbb{V}$ and $\mathbb{V}^{(0)}$

$$(u, v)_{\ell_2} = \mathbf{v}^T\mathbf{u} \quad \text{for } \mathbf{v}, \mathbf{u} \in \mathbb{V},\ (\text{or } \ \mathbf{v}, \mathbf{u} \in \mathbb{V}^{(0)}).$$

Then we introduce the matrix $\tilde{A} = \tilde{A}_D + \tilde{A}_P$, where

$$(\tilde{A}_D\mathbf{u}, \mathbf{v})_{\ell_2} = (a\nabla u, \nabla v)_{\mathcal{T}},\ (\tilde{A}_P\mathbf{u}, \mathbf{v})_{\ell_2} = \kappa \left\langle h_{\mathcal{E}}^{-1} [\![u]\!],\ [\![v]\!]\right\rangle_{\mathcal{E}\cup\mathcal{E}_D}.$$

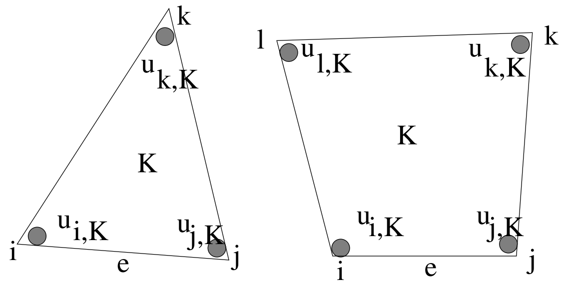

FIGURE 1. Degrees of freedom in K being a triangle or a quadrilateral

Obviously both matrices are symmetric and $\tilde{A}_D$ is semidefinite, while $\tilde{A}_P$ is positive definite. Next we define the "stiffness matrices" A and $A^{(0)}$ by the identities

$$(A\mathbf{u},\mathbf{v})_{\ell_2} = \mathcal{A}(u,v),\ \ \mathbf{v},\mathbf{u}\in\mathbb{V},\quad (A^{(0)}\mathbf{u},\mathbf{v})_{\ell_2} = \mathcal{A}(u,v),\ \ \mathbf{v},\mathbf{u}\in\mathbb{V}^{(0)}.$$

Finally, we introduce the norm

$$|||v|||^2 = |||\mathbf{v}|||^2 = (\tilde{A}\mathbf{v},\mathbf{v})_{\ell_2} = (\tilde{A}_D\mathbf{v},\mathbf{v})_{\ell_2} + (\tilde{A}_P\mathbf{v},\mathbf{v})_{\ell_2},$$

where $u,v\in\mathcal{V}$ and by duality $\mathbf{u},\mathbf{v}\in\mathbb{V}$.

From the symmetry, the coercivity and the boundedness of the bilinear form $\mathcal{A}$ it follows that A is symmetric and positive definite and spectrally equivalent to the matrix $\tilde{A}$. Then we can define the operator norm $|||A|||$ and A-orthogonal projectors $\pi_A : \mathbb{V}\mapsto\mathbb{V}^{(0)}$ of $\mathbf{u}\in\mathbb{V}$ in ℓ_2 norms:

$$(\pi_A A\mathbf{u},\mathbf{v})_{\ell_2} = \mathcal{A}(u,v)\quad \forall \mathbf{v}\in\mathbb{V}^{(0)}.$$

Let M be a smoothing matrix that satisfies the condition M^T+M-A is symmetric and positive definite. The following two-level method has been studies and justified in [15]:

Two-level algorithm:

(0) Let $\mathbf{u}_0$ be given
For $\mathbf{u}_i$ "approximating" $\mathbf{u}$, the solution of $A\mathbf{u}=\mathbf{b}$, define $\mathbf{u}_{i+1}$ as follows:
(1) Set $\mathbf{x}_1 = \mathbf{u}_i - M^{-1}(A\mathbf{u}_i-\mathbf{b})$ (pre-smooth)
(2) $\mathbf{x}_2 = \mathbf{x}_1 - (A^{(0)})^{-1}\pi_A(A\mathbf{x}_1-\mathbf{b})$ (correct)
(3) $\mathbf{u}_{i+1} = \mathbf{x}_2 - M^{-T}(A\mathbf{x}_2-\mathbf{b})$ (post-smooth).

More general two-grid methods with m pre-smoothing and m post-smoothing steps could be also justified.

In [15], the convergence of the two-level method, which is characterized by the error transfer operator E_{tg}, that is $\mathbf{u}-\mathbf{u}_{i+1} = E_{tg}(\mathbf{u}-\mathbf{u}_i)$, has been established in the following form

$$|||E_{tg}||| = 1-1/K,\ \ K\le \sup_{\mathbf{v}\in\mathbb{V}} \frac{\|I-Q\mathbf{v}\|_{\ell_2}^2}{|||\mathbf{v}|||^2},$$

where $Q:\mathbb{V}\mapsto\mathbb{V}^{(0)}$ is an ℓ_2-orthogonal projection operator. A sufficient condition for convergence, independent of the step size, is existence of an operator $Q:\ \mathbb{V}\mapsto\mathbb{V}^{(0)}$ such that

$$\|(I-Q)\mathbf{v}\|_{\ell_2}^2 \le C\,|||\mathbf{v}|||^2,\qquad \forall\mathbf{v}\in\mathbb{V}.$$

In [15], the following result has been obtained:

Theorem 3.1. *The two level method with Gauss-Seidel as a smoother and coarse space $\mathcal{V}^{(0)}$ of piece-wise constant functions is uniformly convergent with respect to the number of degrees of freedom.*

Further, this result has been extended to a multi-level method in the general framework of the AMLI of Axelsson and Vassilevski [4] and using the basic properties of the two-level projection methods of Falgout, Vassilevski, and Zikatanov [15]. Our goal is to obtain similar results by using multilevel splitting of the unknowns and establishing sharp estimate for the angle between the corresponding spaces.

Now consider the bilinear form $\mathcal{A}(\cdot,\cdot)$, defined by (2.3) or (2.5), on the space $\mathcal{V}^{(0)}$ of piece-wise constant functions. Obviously it reduces to the jump part only $\kappa \left\langle h_{\mathcal{E}}^{-1} [\![u]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D}$. Then we define $(A^{(0)}\mathbf{u},\mathbf{v})_{\ell_2} = \left\langle h_{\mathcal{E}}^{-1} [\![u]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D}$ for $u,v \in \mathcal{V}^{(0)}$ and call this matrix "graph-Laplacian". Now we associate the partition $\mathcal{T}$ with a planar graph. The finite elements are the vertexes and the interfaces of the finite elements are the edges of the graph. Then taking as degrees of freedom the values of a function in $\mathcal{V}^{(0)}$ over each finite element, we shall get a matrix that has -1 at the s graph vertices connected to a chosen graph vertex and s at the vertex itself. An example of the matrix representing "graph-Laplacian" for a particular mesh is given on Figure 2. For any partitions into quadrilaterals, regardless of the shape, we get the standard 4-point stencil with $4,-1,-1,-1,-1$, probably the reason for the name.

We note that the case of piece-wise constant space is simple and natural. It is a generalization of the technique of cell centered schemes that are still popular and frequently used in petroleum reservoir modeling using rectangular (or parallelepiped) meshes. These schemes are produced either by finite difference approximation of the elliptic problem or by mixed finite element approximations with subsequent elimination of the fluxes. Preconditioners for such systems were developed in [9]. Important ingredient of the analysis in [9] is the fact that on an orthogonal grid the cell centered scheme has approximation property on all levels.

The matrix related by the graph-Laplacian is a symmetric M-matrix. However, this matrix does not have any approximation property on an arbitrary grid. Therefore, the multigrid theory that relies on such property (see, e.g. [9]) cannot be used for designing a robust preconditioner by using "graph-Laplacian".

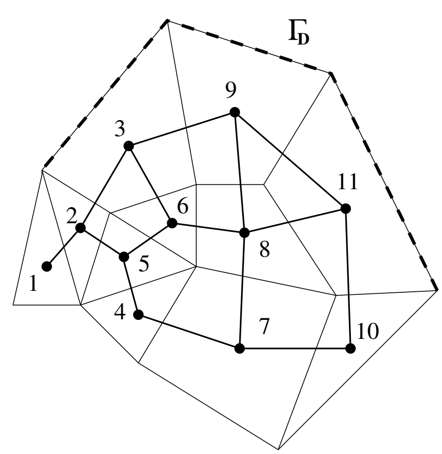

FIGURE 2. Partition $\mathcal{T}$ and related graph-Laplacian

Below is the matrix of graph-Laplacian for the mesh shown on Figure 2.

$$A^{(0)} = \begin{bmatrix} 1 & -1 & & & & & & & & & \\ -1 & 3 & -1 & & -1 & & & & & & \\ & -1 & 4 & & & -1 & & & -1 & & \\ & & & 2 & -1 & & -1 & & & & \\ & -1 & & -1 & 3 & -1 & & & & & \\ & & -1 & & -1 & 3 & & -1 & & & \\ & & & -1 & & & 3 & -1 & & -1 & \\ & & & & & -1 & -1 & 4 & -1 & & -1 \\ & & -1 & & & & & -1 & 4 & & -1 \\ & & & & & & -1 & & & 2 & -1 \\ & & & & & & & -1 & -1 & -1 & 4 \end{bmatrix}$$

4. AMLI preconditioning of graph-Laplacians

We construct hierarchical basis functions (HB) for multilevel preconditioners for the algebraic system involving graph-Laplacian. To this end we follow the framework for constructing HB two-level preconditioners for conforming FEM, as described in e.g. [5, 3], and their multilevel extensions, known as AMLI, see e.g. [4, 23]. The construction of a hierarchical decomposition for the discontinuous Galerkin FE spaces is neither obvious nor unique. To fit the classical HB construction techniques in the non-conforming case, we search for a hierarchical decomposition of the fine grid degrees of freedom, such that one part corresponds to the degrees of freedom of the coarse grid problem. Such kind of aggregation based hierarchical decompositions were recently studied in [6, 7] for the case of Crouzeix-Raviart non-conforming FEs.

Let $A^{(0)}\mathbf{u} = \mathbf{b}$ be the algebraic formulation of our problem where $A^{(0)}$ is a symmetric positive definite graph-Laplacian, corresponding to the finest discretization $\mathcal{T}_0$. Consider a sequence of nested meshes (triangulations) $\mathcal{T}_m \subset \mathcal{T}_{m-1} \subset \cdots \subset \mathcal{T}_0$ of the domain Ω, the spaces of piece-wise constant functions $\mathcal{V}^{(m)} \subset \mathcal{V}^{(m-1)} \subset \cdots \subset \mathcal{V}^{(0)}$, the spaces of degrees of freedom $\mathbb{V}^{(m)}, \mathbb{V}^{(m-1)}, \cdots, \mathbb{V}^{(0)}$, and the number of degrees of freedom $n_m < n_{m-1} < \cdots < n_0$. Further, introduce the graph-Laplacians associated with each triangulation level, $A^{(m)}, A^{(m-1)} \dots A^{(0)}$, with $(A^{(s)}u, v)_{\ell_2} = \left\langle h_{\mathcal{E}}^{-1}[u],\ [v]\right\rangle_{\mathcal{E}\cup\mathcal{E}_D}$ for $u, v \in \mathcal{V}^{(s)}$, $s = m, \dots, 0$.

We denote by $\chi^{(k)} = \{\chi_i^{(k)}\}_{i=1}^{n_k}$ the set of standard piece-wise constant basis functions on level k and by $\widehat{\chi}^{(k)} = \{\widehat{\chi}_i^{(k)}\}_{i=1}^{n_k}$ the set of properly defined hierarchical basis functions. The hierarchical basis $\widehat{\chi}^{(k)}$ is determined by a nonsingular transformation matrix $J^{(k)}$, i.e., $\widehat{\chi}^{(k)} = J^{(k)}\chi^{(k)}$. Then the hierarchical basis stiffness matrix $\widehat{A}^{(k)}$ and hierarchical basis spaces of degrees of freedom $\mathbb{V}^k$ are expressed as follows

$$\widehat{\mathbb{V}}^{(k)} = J^{(k)^{-T}}\mathbb{V}^{(k)} \text{ and } \widehat{A}^{(k)} = J^{(k)}A^{(k)}J^{(k)^T}. \tag{4.1}$$

On each level k the matrix $\widehat{A}^{(k)}$ is partitioned into a two-by-two block form

$$\widehat{A}^{(k)} = \begin{bmatrix} \widehat{A}_{11}^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & \widehat{A}_{22}^{(k)} \end{bmatrix} \begin{matrix} \}n_k - n_{k+1} \\ \}n_{k+1} \end{matrix} , \tag{4.2}$$

where n_{k+1} is the dimension of the space $\mathbb{V}^{(k+1)}$. Regarding the splitting (4.2) we make the following assumption:

Assumption 4.1. *The hierarchical basis is locally constructed so that the transformation matrix is sparse. Moreover, the following relations hold*

$$\widehat{A}_{22}^{(k)} = A^{(k+1)}, \qquad \kappa\left(\widehat{A}_{11}^{(k)}\right) = O(1). \tag{4.3}$$

Obviously the splitting (4.2) generates a splitting in the space $\mathbb{V}^{(k)}$ into two subspaces in the following manner: if $\mathbf{v} = {J^{(k)}}^T \widehat{\mathbf{v}}$ such that $\widehat{\mathbf{v}} = (\widehat{\mathbf{v}}_1^T, \widehat{\mathbf{v}}_2^T)^T \in \widehat{\mathbb{V}}^{(k)}$ where $\widehat{\mathbf{v}}_2 \in \widehat{\mathbb{V}}^{(k+1)}$, then this gives the splitting $\mathbb{V}^{(k)} = \mathbb{V}_1^{(k)} + \mathbb{V}_2^{(k)}$ where

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{V}^{(k)} \text{ with } \mathbf{v}_1 = {J^{(k)}}^T \begin{bmatrix} \widehat{\mathbf{v}}_1 \\ 0 \end{bmatrix} \in \mathbb{V}_1^{(k)},\ \mathbf{v}_2 = {J^{(k)}}^T \begin{bmatrix} 0 \\ \widehat{\mathbf{v}}_2 \end{bmatrix} \in \mathbb{V}_2^{(k)}. \tag{4.4}$$

Since the matrix $A^{(k)}$ is symmetric and positive definite it generates an inner product and geometry in $\mathbb{V}^{(k)}$. The ideal case of splitting (4.4) would be when the vectors $\mathbf{v}_1$ $\mathbf{v}_2$ are orthogonal in the $A^{(k)}$-inner product. In any case, between these spaces $\mathbb{V}_1^{(k)}$ and $\mathbb{V}_2^{(k)}$ there is an angle in the $A^{(k)}$-inner product. The cosine of the angle is defined by the constant $\gamma^{(k)}$ in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality:

$$(A^{(k)}\mathbf{v}_1, \mathbf{v}_2) \le \gamma^{(k)} \sqrt{(A^{(k)}\mathbf{v}_1, \mathbf{v}_1)} \sqrt{(A^{(k)}\mathbf{v}_2, \mathbf{v}_2)}, \quad \mathbf{v}_1 \in \mathbb{V}_1^{(k)},\ \mathbf{v}_2 \in \mathbb{V}_2^{(k)}.$$

Later in the next Section this inequality will be given a different, but equivalent form, which is more convenient for estimating $\gamma^{(k)}$.

The following assumption on the constant $\gamma^{(k)}$ plays an important role in the construction of hierarchical preconditioners:

Assumption 4.2. *There is an absolute constant γ such that the following inequality is valid for all $k \ge 0$*

$$\gamma^{(k)} \le \gamma < 1.$$

We will analyze the AMLI generalization of the multiplicative two-level method, corresponding to the introduced hierarchical setting. AMLI was originally proposed by Axelsson and Vassilevski for the case of conforming linear FEs, see [4, 23].

Algorithm 4.3. Algebraic Multi Level Iteration (AMLI) method:

$C^{(m)} = A^{(m)}$;
for $k = 0, 1, \ldots, m-1$

$$C^{(k)} = {J^{(k)}}^{-1} \begin{bmatrix} \widehat{C}_{11}^{(k)} & 0 \\ \widehat{A}_{21}^{(k)} & \tilde{A}^{(k+1)} \end{bmatrix} \begin{bmatrix} I & {\widehat{C}_{11}^{(k)}}^{-1} \widehat{A}_{12}^{(k)} \\ 0 & I \end{bmatrix} {J^{(k)}}^{-T},$$

where the blocks $\widehat{C}_{11}^{(k)}$ are symmetric positive definite approximations of $\widehat{A}_{11}^{(k)}$, and the Schur complement approximation is stabilized by

$$\tilde{A}^{(k+1)^{-1}} = \left[I - p_\beta\left({C^{(k+1)}}^{-1} A^{(k+1)}\right)\right] {A^{(k+1)}}^{-1}.$$

The acceleration polynomial is explicitly defined by

$$p_\beta(t) = \frac{1 + T_\beta\left(\dfrac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\beta\left(\dfrac{1+\alpha}{1-\alpha}\right)},$$

were $\alpha \in (0,1)$ is a properly chosen parameter, and T_β stands for the Chebyshev polynomial of degree β with L^∞-norm 1 on $(-1,1)$.

Then, the next theorem is a straightforward reformulation of the basic theorem from [4].

Theorem 4.4. *Let the assumptions 4.1 and 4.2 hold and let the integer β satisfy*

$$\frac{1}{\sqrt{1-\gamma^2}} < \beta < \rho,$$

where $\rho = \max n_k/n_{k+1}$. Then there exists $\alpha \in (0,1)$, such that the AMLI preconditioner $C^{(0)}$ defined in (4.3) has optimal condition number

$$\kappa\left({C^{(0)}}^{-1}A^{(0)}\right) = O(1),$$

and the total computational complexity is $O(n_0)$.

Remark 4.5. *Explicit formulas for the AMLI parameter α are given in* [4] *where the considered acceleration polynomials are of degree $\beta = 2$ and $\beta = 3$.*

The constant in the strengthened CBS inequality (CBS constant) $\gamma^{(k)}$ is a quantitative characterization of the HB. The remaining part of the paper is devoted to the construction of hierarchical splittings for the case of class of matrices represented by graph-Laplacian that satisfy the assumptions 4.1 and 4.2.

5. On the local estimates of the CBS constant

Let $\widehat{\mathbb{V}}^{(k)} = \widehat{\mathbb{V}}_1^{(k)} \times \widehat{\mathbb{V}}_2^{(k)}$ be the partitioning corresponding to the block two-by-two presentation (4.2) of the hierarchical basis stiffness matrix $\widehat{A}^{(k)}$. More appropriate for computation of the CBS constant is the following formula

$$\gamma^{(k)} = \sup_{\mathbf{v}_i \in \widehat{\mathbb{V}}_i^{(k)},\ i=1,2} \frac{{\mathbf{v}_1}^T \widehat{A}_{12}^{(k)} \mathbf{v}_2}{\sqrt{{\mathbf{v}_1}^T \widehat{A}_{11}^{(k)} \mathbf{v}_1}\sqrt{{\mathbf{v}_2}^T \widehat{A}_{22}^{(k)} \mathbf{v}_2}} = \sup_{\mathbf{v}_2 \in \widehat{\mathbb{V}}_2^{(k)}} \sqrt{\frac{\mathbf{v}_2^T \widehat{A}_{21}^{(k)} \left(\widehat{A}_{11}^{(k)}\right)^{-1} \widehat{A}_{12}^{(k)} \mathbf{v}_2}{\mathbf{v}_2^T \widehat{A}_{22}^{(k)} \mathbf{v}_2}}. \tag{5.1}$$

Now, let us assume that

$$\widehat{A}^{(k)} = \sum_{e\in\mathcal{F}} \widehat{A}_e^{(k)}\ ,\ \mathbf{v} = \sum_{e\in\mathcal{F}} \mathbf{v}_e,$$

where $\widehat{A}_e^{(k)}$ are symmetric positive semidefinite local matrices, $\mathcal{F}$ is some set of indices, and the summation is understood as assembling. The global hierarchical basis splitting naturally induces the block two-by-two presentation of the local matrix $\widehat{A}_e^{(k)}$, namely,

$$\widehat{A}_e^{(k)} = \begin{bmatrix} \widehat{A}_{e:11}^{(k)} & \widehat{A}_{e:12}^{(k)} \\ \widehat{A}_{e:21}^{(k)} & \widehat{A}_{e:22}^{(k)} \end{bmatrix},\ \mathbf{v}_e = \begin{bmatrix} \mathbf{v}_{e,1} \\ \mathbf{v}_{e,2} \end{bmatrix}. \tag{5.2}$$

Let $\widehat{\mathbb{V}}_e^{(k)}$ be the restriction of $\widehat{\mathbb{V}}^{(k)}$, corresponding to the local matrix $\widehat{A}_e^{(k)}$, and let $\widehat{\mathbb{V}}_e^{(k)} = \widehat{\mathbb{V}}_{e:1}^{(k)} \times \widehat{\mathbb{V}}_{e:2}^{(k)}$ be the partitioning corresponding to (5.2).

Lemma 5.1. *Assume that for all* $\mathbf{w} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in ker(\widehat{A}_e^{(k)})$, $\mathbf{v}_1 \in \widehat{\mathbb{V}}_{e:1}^{(k)}$, $\mathbf{v}_2 \in \widehat{\mathbb{V}}_{e:2}^{(k)}$, *it holds than* $\mathbf{v}_2 \in ker(\widehat{A}_{e:22}^{(k)})$. *Then the local CBS constant* $\gamma_e^{(k)}$ *is determined by*

$$\gamma_e^{(k)} = \sup_{\mathbf{v}_2 \in \widehat{\mathbb{V}}_{e:2}^{(k)} \setminus ker(\widehat{A}_{e:22}^{(k)})} \sqrt{\frac{\mathbf{v}_2^T \widehat{A}_{e:21}^{(k)} \left(\widehat{A}_{e:11}^{(k)}\right)^{-1} \widehat{A}_{e:12}^{(k)} \mathbf{v}_2}{\mathbf{v}_2^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_2}} < 1, \tag{5.3}$$

and the following estimate holds

$$\gamma^{(k)} \le \max_{e \in \mathcal{F}} \gamma_e^{(k)} \ . \tag{5.4}$$

Proof The assumption of the lemma is necessary condition for the correctness of (5.3), see e.g. [2, 13].

Now, let $\mathbf{v}_i \in \widehat{\mathbb{V}}_i^{(k)}$, and let $\mathbf{v}_{e:i} \in \widehat{\mathbb{V}}_{e:i}^{(k)}$ be the restrictions, corresponding to the local matrices $\widehat{A}_e^{(k)}$, $i = 1, 2$. Then

$$\begin{aligned}
\left|\mathbf{v}_1^T \widehat{A}_{12}^{(k)} \mathbf{v}_2\right| &= \left|\sum_{e \in \mathcal{F}} \mathbf{v}_{e:1}^T \widehat{A}_{e:12}^{(k)} \mathbf{v}_{e:2}\right| \le \sum_{e \in \mathcal{F}} \left|\mathbf{v}_{e:1}^T \widehat{A}_{e:12}^{(k)} \mathbf{v}_{e:2}\right| \\
&\le \sum_{e \in \mathcal{F}} \gamma_e^{(k)} \sqrt{\mathbf{v}_{e:1}^T \widehat{A}_{e:11}^{(k)} \mathbf{v}_{e:1}} \sqrt{\mathbf{v}_{e:2}^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_{e:2}} \\
&\le \max_{e \in \mathcal{F}} \gamma_e^{(k)} \sum_{e \in \mathcal{F}} \sqrt{\mathbf{v}_{e:1}^T \widehat{A}_{e:11}^{(k)} \mathbf{v}_{e:1}} \sqrt{\mathbf{v}_{e:2}^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_{e:2}} \\
&\le \max_{e \in \mathcal{F}} \gamma_e^{(k)} \sqrt{\sum_{e \in \mathcal{F}} \mathbf{v}_{e:1}^T \widehat{A}_{e:11}^{(k)} \mathbf{v}_{e:1}} \sqrt{\sum_{e \in \mathcal{F}} \mathbf{v}_{e:2}^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_{e:2}} \\
&\le \max_{e \in \mathcal{F}} \gamma_e^{(k)} \sqrt{\mathbf{v}_1^T \widehat{A}_{11}^{(k)} \mathbf{v}_1} \sqrt{\mathbf{v}_2^T \widehat{A}_{22}^{(k)} \mathbf{v}_2}
\end{aligned}$$

which completes the proof. ∎

Remark 5.1. *The obtained result is a straightforward generalization of the known estimate for the standard finite element method, where the local matrices are the element stiffness matrices.*

6. Estimates of the CBS constant for graph-Laplacians

Let us consider two consecutive discretizations $\mathcal{T}_k \subset \mathcal{T}_{k-1}$. In what follows we will derive uniform estimates of the CBS constant based on properly introduced construction of hierarchical basis and related decomposition of the graph-Laplacians

$$A^{(k)} = \sum_{e \in \mathcal{E}} A_e^{(k)} \ , \quad \widehat{A}^{(k)} = \sum_{e \in \mathcal{E}} \widehat{A}_e^{(k)} \ ,$$

as a sum of local matrices associated with the set of edges $\mathcal{E}$ of the coarser grid $\mathcal{T}_k$.

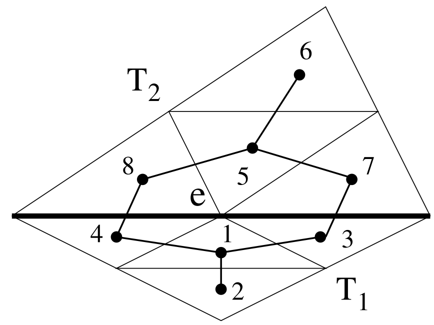

FIGURE 3. Macroelement of two adjacent triangles from $\mathcal{T}_k$

6.1. **Mesh of triangles.** Let us assume that the coarsest mesh $\mathcal{T}_m$ consists of triangles only, and each refined mesh is obtained by dividing the current triangle in four congruent triangles connecting the midpoints of its sides. Following the numbering from Figure 3, we introduce the local matrix $A_e^{(k)}$ in the form

$$A_e^{(k)} = \left[\begin{array}{cccc|cccc} 1 & -t & \frac{t-1}{2} & \frac{t-1}{2} & & & & \\ -t & t & & & & & & \\ \frac{t-1}{2} & & 1+\frac{1-t}{2} & & & -1 & & \\ \frac{t-1}{2} & & & 1+\frac{1-t}{2} & & & & -1 \\ \hline & & & & 1 & -t & \frac{t-1}{2} & \frac{t-1}{2} \\ & & & & -t & t & & \\ & & -1 & & \frac{t-1}{2} & & 1+\frac{1-t}{2} & \\ & & & -1 & \frac{t-1}{2} & & & 1+\frac{1-t}{2} \end{array}\right]. \tag{6.1}$$

This edge matrix is also associated with the macroelement $E = T_1 + T_2$ of the two adjacent triangles from $\mathcal{T}_k$ with a common side e. The role of the weight parameter $t \in (0,1)$ is correctly to distribute the contribution of the links between the interior nodes. For example, the couple (1,2) has a weight t here, but will appear also with a weight of $\frac{t-1}{2}$ in the local matrices associated with the rest two sides of the current triangle, so that the total contribution to have the right weight of one.

It is naturally to introduce the hierarchical basis locally with respect to the triangles from $\mathcal{T}_k$. Let us consider the macroelement T_1 and the set of standard piece-wise constant basis functions

$\chi_{T_1}^{(k)} = \{\chi_{T_1:i}^{(k)}\}_{i=1}^4$. We introduce the related hierarchical basis $\widehat{\chi}_{T_1}^{(k)} = \{\widehat{\chi}_{T_1:i}^{(k)}\}_{i=1}^4$ in the form

$$\begin{aligned}
\widehat{\chi}_{T_1:1}^{(k)} &= \chi_{T_1:1}^{(k)} + p\chi_{T_1:2}^{(k)} + q\chi_{T_1:3}^{(k)} + q\chi_{T_1:4}^{(k)} \\
\widehat{\chi}_{T_1:2}^{(k)} &= \chi_{T_1:1}^{(k)} + q\chi_{T_1:2}^{(k)} + p\chi_{T_1:3}^{(k)} + q\chi_{T_1:4}^{(k)} \\
\widehat{\chi}_{T_1:3}^{(k)} &= \chi_{T_1:1}^{(k)} + q\chi_{T_1:2}^{(k)} + q\chi_{T_1:3}^{(k)} + p\chi_{T_1:4}^{(k)} \\
\widehat{\chi}_{T_1:4}^{(k)} &= r\left(\chi_{T_1:1}^{(k)} + \chi_{T_1:2}^{(k)} + \chi_{T_1:3}^{(k)} + \chi_{T_1:4}^{(k)}\right)
\end{aligned} \tag{6.2}$$

where p, q are parameters to be determined later, and r is the corresponding scaling factor. Then the assembled transformation matrix $J_e^{(k)}$ is as follows

$$J_e^{(k)} = \begin{bmatrix}
1 & p & q & q & & & & \\
1 & q & p & q & & & & \\
1 & q & q & p & & & & \\
& & & & 1 & p & q & q \\
& & & & 1 & q & p & q \\
& & & & 1 & q & q & p \\
r & r & r & r & & & & \\
& & & & r & r & r & r
\end{bmatrix}, \tag{6.3}$$

and

$$\widehat{A}_e^{(k)} = J_e^{(k)} A_e^{(k)} J_e^{(k)^T} = \begin{bmatrix} \widehat{A}_{e:11}^{(k)} & \widehat{A}_{e:12}^{(k)} \\ \widehat{A}_{e:21}^{(k)} & \widehat{A}_{e:22}^{(k)} \end{bmatrix} .$$

Lemma 6.1. *Consider the hierarchical basis (6.2) for nested meshes of triangles. Then*

$$\widehat{A}_{22}^{(k)} = A^{(k+1)} \quad \text{if and only if} \quad r = \frac{\sqrt{2}}{2}.$$

Proof The definition of the last two terms in the local hierarchical basis ensures that $\widehat{A}_{e:22}^{(k)}$ has rowsums/columnsums equal to zero. Then, the equivalent statement

$$\widehat{A}_{e:22}^{(k)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

simply follows from the equalities

$$\widehat{A}_{e:22}^{(k)}(1,1) = r^2 \sum_{i,j=1}^{4} A_e^{(k)}(i,j) = 2r^2 \ , \quad \widehat{A}_e^{(k)}(2,2) = r^2 \sum_{i,j=5}^{8} A_e^{(k)}(i,j) = 2r^2 \ .$$

■

Now, it is readily seen from (5.3) that $\gamma_e^{(k)} = 1 - \lambda$ where λ is the eigenvalue (which is unique in this particular case) of the eigenproblem

$$\widehat{S}_e^{(k)} \mathbf{v} = \lambda \widehat{A}_{e:22}^{(k)} \mathbf{v} \ , \quad \mathbf{v} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $S_e^{(k)} = A_{e:22}^{(k)} - A_{e:21}^{(k)} {A_{e:11}^{(k)}}^{-1} A_{e:12}^{(k)}$.

Lemma 6.2. *Consider the hierarchical splitting (6.1), (6.3) with parameters $p = 1$, $q = -0.5$ and $t = 0.5$. Then the following estimate holds uniformly with respect to the refinement level k,*

$$\gamma^2 \leq \gamma_e^2 = \gamma_{TT}^2 = \frac{16}{25} \ . \tag{6.4}$$

Proof The construction of the hierarchical basis and all related matrices are independent of the particular edge $e \in \mathcal{E}$ and of the current refinement level. Then, the estimate of the local CBS constant follows straightforwardly by simple computations with fixed numbers. Here, γ_{TT} indicates that the interface edge is always between two triangles. ■

Remark 6.1. *Varying the parameters (p, q, t) we can get a family of hierarchical splittings. For example, the parameter set $p = -1$, $q = 0$ and $t = 1/3$ corresponds to $\gamma_e^2 = 9/13$ which leads to the condition number estimate of the related multiplicative two-level method, $\kappa < 13/4$. The later result is derived by different arguments in* [21].

6.2. **Mesh of quadrilaterals.** We assume here, that the coarsest mesh $\mathcal{T}_m$ consists of quadrilaterals only, and each next refinement is obtained by dividing the current element in four new quadrilaterals as illustrated in Figure 4 (a). Following the setting of the previous subsection and

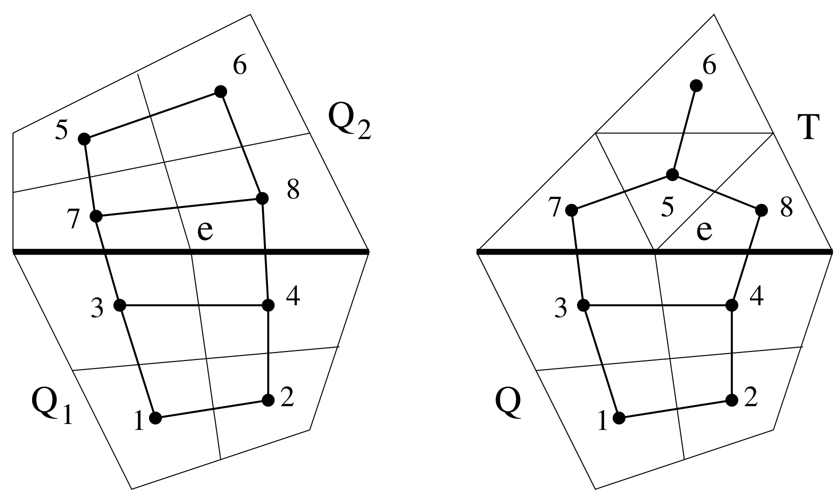

FIGURE 4. (a) Macroelement of two adjacent quadrilaterals of the mesh $\mathcal{T}_k$; (b) Macroelement of adjacent triangle and quadrilateral of $\mathcal{T}_k$

the node numbering from Figure 4, we introduce the new local matrix $A_e^{(k)}$ in the form

$$A_e^{(k)} = \left[\begin{array}{cccc|cccc} \frac{1}{2} & -s & s-\frac{1}{2} & & & & & \\ -s & \frac{1}{2} & & s-\frac{1}{2} & & & & \\ s-\frac{1}{2} & & \frac{3}{2} & -s & & & -1 & \\ & s-\frac{1}{2} & -s & \frac{3}{2} & & & & -1 \\ \hline & & & & \frac{1}{2} & -s & s-\frac{1}{2} & \\ & & & & -s & \frac{1}{2} & & s-\frac{1}{2} \\ & & -1 & & s-\frac{1}{2} & & \frac{3}{2} & -s \\ & & & -1 & & s-\frac{1}{2} & -s & \frac{3}{2} \end{array}\right]. \tag{6.5}$$

The weight parameter $s \in (0,1)$ is again responsible for the correct distribution of the contribution of the links between the interior nodes of each quadrilateral macroelements Q_i, see Figure 4 (a).

The hierarchical basis is now introduced locally with respect to the quadrilaterals from $\mathcal{T}_k$. If consider the macroelement Q_1, then the set of standard piece-wise constant basis functions is $\chi_{T_1}^{(k)} = \{\chi_{T_1:i}^{(k)}\}_{i=1}^4$, and the related hierarchical basis $\widehat{\chi}_{T_1}^{(k)} = \{\widehat{\chi}_{T_1:i}^{(k)}\}_{i=1}^4$ is introduced in the form

$$\begin{aligned} \widehat{\chi}_{T_1:1}^{(k)} &= (\chi_{T_1:1}^{(k)} + \chi_{T_1:2}^{(k)}) - (\chi_{T_1:3}^{(k)} + \chi_{T_1:4}^{(k)}) \\ \widehat{\chi}_{T_1:2}^{(k)} &= (\chi_{T_1:1}^{(k)} + \chi_{T_1:3}^{(k)}) - (\chi_{T_1:2}^{(k)} + \chi_{T_1:4}^{(k)}) \\ \widehat{\chi}_{T_1:3}^{(k)} &= (\chi_{T_1:1}^{(k)} + \chi_{T_1:4}^{(k)}) - (\chi_{T_1:2}^{(k)} + \chi_{T_1:3}^{(k)}) \\ \widehat{\chi}_{T_1:4}^{(k)} &= r\left(\chi_{T_1:1}^{(k)} + \chi_{T_1:2}^{(k)} + \chi_{T_1:3}^{(k)} + \chi_{T_1:4}^{(k)}\right) \end{aligned} \tag{6.6}$$

where r is again the corresponding scaling factor. Then the assembled transformation matrix $J_e^{(k)}$ reads as

$$J_e^{(k)} = \left[\begin{array}{cccccccc} 1 & 1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ & & & & 1 & -1 & -1 & 1 \\ & & & & 1 & -1 & 1 & -1 \\ & & & & -1 & 1 & 1 & -1 \\ r & r & r & r & & & & \\ & & & & r & r & r & r \end{array}\right], \tag{6.7}$$

We follow the local analysis scheme from the previous subsection and get the next two lemmas.

Lemma 6.3. *Consider the hierarchical basis (6.6) for nested meshes of quadrilaterals. Then* $\widehat{A}_{22}^{(k)} = A^{(k+1)}$ *if and only if* $r = \sqrt{2}/2$.

Lemma 6.4. *The estimate*

$$\gamma^2 \leq \gamma_e^2 = \gamma_{QQ}^2 \to \frac{1}{2} \tag{6.8}$$

holds uniformly with respect to the refinement level k for the hierarchical splitting (6.6) with positive weight parameter $s \to 0^+$.

Here, γ_{QQ} indicates that the interface edge is always between two quadrilaterals. The next table illustrates the behaviour of the local CBS constant varying the parameter s.

TABLE 1. The behaviour of γ_{QQ}^2

s	0.2	0.1	0.01	0.001
γ_{QQ}^2	0.6250	0.5556	0.5051	0.5005

Remark 6.2. *The result from Lemma 6.4 is asymptoticly equivalent to the condition number estimate of the related multiplicative two-level method, $\kappa < 2$. Applying a different technique, the later estimate is obtained in* [21] *for quadrilateral meshes of arbitrary space dimension.*

6.3. **Mesh of quadrilaterals and triangles.** The general case of a coarsest mesh $\mathcal{T}_m$, consisting of quadrilaterals and triangles is consider. The refinement procedure is regular, and for the particular cases, it is the same as was considered in the previous two subsections. What remains to be analyzed is the situation, where macroelements of different kind are adjacent as shown in Figure 4 (b). Combining the constructions from the previous two subsections and following the node numbering from Figure 4(b), we get the local matrix $A_e^{(k)}$ in the form

$$A_e^{(k)} = \left[\begin{array}{cccc|cccc} \frac{1}{2} & -s & s-\frac{1}{2} & & & & & \\ -s & \frac{1}{2} & & s-\frac{1}{2} & & & & \\ s-\frac{1}{2} & & \frac{3}{2} & -s & & & -1 & \\ & s-\frac{1}{2} & -s & \frac{3}{2} & & & & -1 \\ \hline & & & & 1 & -t & \frac{t-1}{2} & \frac{t-1}{2} \\ & & & & -t & t & & \\ & & -1 & & \frac{t-1}{2} & & 1+\frac{1-t}{2} & \\ & & & -1 & \frac{t-1}{2} & & & 1+\frac{1-t}{2} \end{array}\right], \tag{6.9}$$

with weight parameter $s, t \in (0,1)$. Keeping the already introduced local definitions of hierarchical bases, we write the combined transformation matrix in the form

$$J_e^{(k)} = \begin{bmatrix} 1 & 1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ & & & & 1 & p & q & q \\ & & & & 1 & q & p & q \\ & & & & 1 & q & q & p \\ r & r & r & r & & & & \\ & & & & r & r & r & r \end{bmatrix}. \tag{6.10}$$

Let us stress the attention on the fact, that all locally introduced parameters are fixed for each particular triangle/quadrilateral macroelement from $\mathcal{T}_{k+1}$, independently of what kind of neighbours it has. In this respect, it is important, that $r = \sqrt{2}/2$ in both cases of triangles and quadrilaterals, see Lemmas 6.1, 6.3.

Lemma 6.5. *Consider the local matrix, corresponding to the case of edge between quadrilateral and triangle, indicated below by "QT", and let $r = \sqrt{2}/2$ and $s \to 0^+$. Then*

$$\widehat{A}_{e:22}^{(k)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and the relation

$$\gamma_e^2 = \gamma_{QT}^2 \to 0.5814, \tag{6.11}$$

holds uniformly with respect to the refinement level k.

The table below illustrates the behaviour of γ_{QT} varying the parameter s.

TABLE 2. The behaviour of γ_{QQ}^2

s	0.2	0.1	0.01	0.001
γ_{QT}^2	0.6327	0.6022	0.5832	0.5816

The next two theorems summarize the results of Lemmas 6.4, 6.8, and 6.5.

Theorem 6.3. *Consider the hierarchical splitting of the graph-Laplacian, corresponding to the general case of nested meshes, where the coarsest one $\mathcal{T}_m$ consists of quadrilaterals and triangles.*

- (a) *Then $\widehat{A}_{22}^{(k)} = A^{(k+1)}$ if and only if $r = \sqrt{2}/2$.*
- (b) *Consider the set of parameters $p = 1$, $q = -0.5$ and $t = 0.5$, and let $0 < s < s_0$, where s_0 is small enough. Then*

$$\gamma^2 \leq \max\{\gamma_{TT}^2,\ \gamma_{QQ}^2,\ \gamma_{QT}^2\} = \frac{16}{25}, \tag{6.12}$$

uniformly with respect to the refinement level k.

As seen from Tables 1, 2, in the general case of Theorem 6.3, it is enough to take $s_0 = 0.2$.

Theorem 6.4. *Let the parameters of the hierarchical splitting of the graph-Laplacian satisfy the conditions of Theorem 6.3 (a-b). Then the related AMLI algorithm with acceleration polynomial of degree $\beta \in \{2,3\}$ has optimal condition number and the total computational complexity is $O(n_0)$.*

Proof The statement follows directly from Theorems 4.4, 6.3, taking into account that $\rho = 4$ and

$$\gamma^2 \leq \frac{16}{25} < \frac{3}{4}.$$

■

7. Concluding remarks

A general setting of multilevel preconditioning technology for symmetric discontinuous Galerkin approximations of second order elliptic problems is presented. The finite element space of piece-wise discontinuous polynomials of certain degree is first projected onto the space of piece-wise constants on the same partition. This constitutes the finest space in the multilevel method. The projection of the discontinuous Galerkin system on such a space is associated to the so-called graph-Laplacian.

A novel unified concept for hierarchical decomposition of graph-Laplacians and the related local analysis of the constant in the strengthened Cauchy-Bunyakowski-Schwarz inequality is developed. The obtained uniform estimates lead to optimal AMLI preconditioners of the discontinuous Galerkin system.

We have assumed in this study that the discontinuous Galerkin partition is obtained by a regular refinement of a given initial mesh consisting of both triangles and quadrilaterals. However, the introduced approach is more generally applicable to partitions, including pentagons, etc.

The presented scheme is further capable to construction and analysis of AMLI preconditioners for discontinuous Galerkin systems in 3-D.

8. Acknowledgements

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References

[1] D. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.*, **39** (2002), 1749–1779.

[2] O. Axelsson, *Iterative solution methods.* Cambridge University Press, 1994.

[3] O. Axelsson, I. Gustafsson, Preconditioning and two-level multigrid methods of arbitrary degree of approximations, *Mathematics of Computation*, 40 (1983), 219-242.

[4] O. Axelsson and P.S. Vassilevski, Algebraic multilevel preconditioning methods II, *SIAM J. Numer. Anal.*, 27 (1990), 1569-1590.

[5] R. Bank and T. Dupont, An optimal order process for solving finite element equations, *Math. Comp.*, 36 (1981), 427-458.

[6] R. Blaheta, S. Margenov, M. Neytcheva, Uniform estimate of the constant in the strengthened CBS inequality for anisotropic non-conforming FEM systems, *Numer. Lin. Alg. Appl.*, Vol. 11 (4) (2004), 309-326.

[7] R. Blaheta, S. Margenov, M. Neytcheva, Robust optimal multilevel preconditioners for non-conforming finite element systems, *Numer. Lin. Alg. Appl.*, Vol. 12 (2) (2005), 495-514.

[8] J.E. Bramble, *Multigrid Methods*, Longman Scientific&Technical, Pitman Research Notes in Mathematics Series, **294**, 1993.

[9] J.H. Bramble, R.E. Ewing, J.E. Pasciak, J. Shen The analysis of multigrid algorithms for cell centered finite difference methods, *Advances in Computational Mathematics*, **5(1)**, (1996), 15-29.

[10] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994.

[11] S. Brenner and J. Zhao, Convergence of multigrid algorithms for interior penalty methods, *Appl. Numer. Anal. Comput. Math.*, **2** (1) (2005), 3–18.

[12] B. Cockburn, Discontinuous Galerkin methods, *Z. Angew. Math. Mech.*, **83 (11)** (2003) 731–754.

[13] V. Eijkhout and P.S. Vassilevski, The role of the strengthened Cauchy-Bunyakowski-Schwarz inequality in multilevel methods, *SIAM Review*, 33 (1991), 405-419.

[14] R.E. Ewing, J. Wang, and Y. Yang, A stabilized discontinuous finite element method for elliptic problems, *Numer. Lin. Alg. Appl.*, **10** (2003), 83–104.

[15] R. Falgout, P.S. Vassilevski and L.T. Zikatanov, On two-grid convergence estimates, *Numer. Lin. Alg. Appl.*, **12** (2005), 471–494.

[16] J. Gopalakrishnan and G. Kanschat, A multilevel discontinuous Galerkin method, *Numerische Mathematik*, **95**, (3) (2003), 527-550.

[17] J. Gopalakrishnan and G. Kanschat, Application of unified discontinuous Galerkin analysis to preconditioning discontinuous Galerkin methods, in *Proc. 2nd MIT Conf. Computational Fluid and Solid Mechanics*, (2003) 1943-1945.

[18] P.W. Hemker, W. Hoffmann and M.H. van Raalte, Two-level Fourier analysis of multigrid approach for discontinuous Galerkin discretization, *CWI report, MAS-R0217*, 2002

[19] K. Johannsen, A symmetric smoother for the nonsymmetric interior penalty discontinuous Galerkin discretization, ICES Report 05-23, University of Texas at Austin, 2005 (submitted to *Numerical Linear Algebra with Applications*).

[20] G. Kanschat and R. Rannacher, Local error analysis of the interior penalty discontinuous Galerkin method for second order elliptic problems, *Preprint*, 2002.

[21] R.D. Lazarov, P.S. Vassilevski, and L.T. Zikatanov, Miltilevel preconditioning of second order elliptic discontinuous Galerkin problems, Preprint, 2005.

[22] B. Rivière, M.F. Wheeler, and V. Girault, A priori error estimates for finite element method based on discontinuous approximation spaces for elliptic problems. Part II, *SIAM J. Numer. Anal.*, **39** (3) (2001), 902–931.

[23] P.S. Vassilevski, Hibrid V-cycle algebraic multilevel preconditioners, *Mathematics of Computations*, 58 (198) (1992), 489-512.

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