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ON THE BALANCING PRINCIPLE FOR SOME PROBLEMS OF NUMERICAL ANALYSIS

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ABSTRACT. We discuss a choice of weight in penalization methods. The motivation for the use of penalization in Computational Mathematics is to improve the conditioning of the numerical solution. One example of such improvement is a regularization, where a penalization substitutes ill-posed problem for well-posed one. In modern numerical methods for PDE a penalization is used, for example, to enforce a continuity of approximate solution on non-matching grids. A choice of penalty weight should provide a balance between error components related with convergence and stability, which are usually unknown. In this paper we propose and analyze a simple adaptive strategy for the choice of penalty weight which does not rely on *a priori* estimates of above mentioned components. It is shown that under natural assumptions the accuracy provided by our adaptive strategy is worse only by a constant factor than one could achieve in the case of known stability and convergence rates. Finally, we successfully apply our strategy for self-regularization of Volterra-type severely ill-posed problems, such as sideway heat equation, and for the choice of a weight in interior penalty discontinuous approximation on non-matching grids. Numerical experiments on a series of model problems support theoretical results.

1. INTRODUCTION

In modern Numerical Analysis the approximation and solution methods and algorithms are becoming more sophisticated and dependent of various parameters that have to be finely tuned. These parameters are often responsible for crucial properties of the method, among the most important are stability and accuracy in various norms. However, often while enhancing stability the parameters could negatively influence the accuracy. Two prominent examples where a balance between stability and accuracy is of interest are Regularization of ill-posed problems and discontinuous Galerkin methods for elliptic PDE.

While the choice of the regularization parameter has to balance the stability with the accuracy of the regularization method, the parameters in DG schemes usually improve the stability without destroying the accuracy in the energy norm. However, some computations show loss of expected accuracy in L^∞-norm, (e.g. [20]) introduction of spurious modes in the spectrum (e.g. [26]), and other "side effects". Therefore, an analysis of the influence of the parameter(s) on the global quality of the approximate solution and algorithms for parameter selection represents an important task.

In the cases of Regularization of ill-posed problems and Interior Penalty methods for elliptic PDE an element of interest (solution of the problem) u^+ can be in principle approximated by an ideal element u_σ depending on a positive parameter σ in such a way that a norm $\|u^+ - u_\sigma\|$ goes to zero as $\sigma \to 0$. It means that there exists a non-decreasing continuous function $\varphi(\sigma) = \varphi(\sigma; u^+)$ such that $\varphi(0) = 0$, and for any $\sigma \in (0, 1)$

$$\|u^+ - u_\sigma\| \le \varphi(\sigma). \tag{1.1}$$

In practice, however, an approximation u_σ is not available. One reason for this is that the data required for constructing u_σ are given with errors. Such situation is typical for ill-posed problems. Another reason is that u_σ itself is defined as a solution of some infinite dimensional problem and become numerically feasible only after its discretization, as is the case in interior penalty methods, where u_σ arises from the weak reformulation of an appropriate boundary value problem. Somehow or other, we have at our disposal some element u_σ^Δ instead of u_σ, where Δ is a level of unavoidable error. In the case of regularization of ill-posed problems Δ is a bound for the error in given data, while in penalty methods for PDE Δ is a discretization error. If the discretization is done within a finite element space using polynomials of degree r and a quasi-uniform triangulation $\mathcal{T}$ then Δ usually has a form $\Delta = h^{s-\lambda}$, $0 \le \lambda < s \le r + 1$, where h is an appropriate mesh-size parameter. In both above mentioned cases the stability of approximation u_σ with respect to Δ-perturbation can be described in the form of inequality

$$\|u_\sigma - u_\sigma^\Delta\| \le k\frac{\Delta}{\sigma^v}, \quad \sigma \in (0, 1), \tag{1.2}$$

where k and v are some positive constants.

Combining (1.1) with (1.2) we obtain the following estimation

$$\|u^+ - u_\sigma^\Delta\| \le \varphi(\sigma) + k\frac{\Delta}{\sigma^v}, \quad \sigma \in (0, 1), \tag{1.3}$$

which tells us that a coordination between a parameter σ, governing the approximation, and the amount of unavoidable error Δ in the problems is required to obtain good accuracy. In an ideal situation such a coordination should provide a balance between approximation error (1.1)and stability (1.2), that is achieved by the choice of $\sigma = \sigma_{opt}$ solving an equation $\varphi(\sigma)\sigma^v = k\Delta$ and leading to the error estimate

$$\|u^+ - u_{\sigma_{opt}}^\Delta\| \le 2\varphi(\sigma_{opt}). \tag{1.4}$$

Note that the choice of $\sigma = \sigma_{opt}$ is possible only if φ and v are known. A knowledge of k is less important, because in practice $\varphi(\sigma)$ does not increase faster than with a polynomial rate, and then, as we will see in the next section

$$c_1\varphi(\theta_v^{-1}(\Delta)) \le \varphi(\sigma_{opt}) \le c_2\varphi(\theta_v^{-1}(\Delta)), \tag{1.5}$$

where $\theta_v(\sigma) = \varphi(\sigma)\sigma^v$, and the constants c_1, c_2 do not depend on Δ. Thus, the constant k from (1.2) has no influence on the function $\varphi(\theta_v^{-1}(\Delta))$ which indicates the optimal order of the accuracy that cannot be surpassed within the framework of assumptions (1.1),(1.2).

For classical regularization methods of the Theory of ill-posed problems the power v in a stability estimation (1.2) is usually known. If, for example, u_σ^Δ is obtained within the framework of Tikhonov-Phillips regularization then $v = 1/2$, while for Lavrentiev regularization one has $v = 1$. Nevertheless, *a priori* parameter choice $\sigma = \sigma_{opt}$ can seldom be used in practice because the smoothness properties of the unknown solution u^+ reflected in a function φ from (1.1) are generally unknown.

For classical regularization methods there is a considerable literature concerned with strategies for the choice of parameter σ without knowledge of approximation rate $\varphi(\sigma)$ in (1.1) that may not be accessible. Appropriate references can be found in the monograph [10] and in the recent paper [24].

At the same time, there are several practically important ill-posed problems where the application of classical regularization methods is not desirable. An example of such a problem is the inverse heat conduction problem, or sideways heat equation. It has been pointed out in [18] that classical regularization methods like Tikhonov-Phillips regularization tend to destroy the causal nature of this Volterra type problem, while its specialized structure suggests other particularly effective regularization methods, some taking advantage of equivalent problem formulation as an partial differential equation problem with overspecified data. One of such effective regularization methods known as a method of lines has been proposed in [8]. However, as it has been mentioned in [4], there is no stability theory for a method of lines that can be used for selecting an appropriate value of regularization parameter, i.e. the parameter σ. In other words, for this regularization method the power v in the stability estimation (1.2) is unknown, and it would be of interest to have such recipe for determining σ that does not use neither knowledge of φ nor the exact value of v.

In interior penalty methods a quantity $\mu = \sigma^{-1}$ is usually treated as a penalty weight. These methods arose from the observation that continuity of approximate solution and Dirichlet boundary conditions could be imposed weakly instead of being built into the finite element space. This makes it possible to use spaces of discontinuous piecewise polynomials for solving second order problems on non-matching grids, when different discretization techniques are utilized in different parts of the domain. For more information on this topic the reader is encouraged to consult survey [1]. The use of penalty methods can be traced back to the late 1960s, but as it is methoded in [1], since the early 1980s less attention has been paid to such methods. This might be due to the difficulty in finding optimal values for the penalty weights. In previous papers the choice of these weights was usually based on *a priori* estimates of approximation error (1.1) and stability (1.2) in the energy norm or in the Sobolev norm $\|\cdot\|_{H^1}$, which might be too pessimistic for approximation in other norms such as L_∞-norm. Note that for particular case of boundary penalty method *a posteriori* strategy for choosing the penalty parameter σ has been proposed recently in [11]. However, this strategy has been designed for approximation in energy semi-norm, where adaptive *a posteriori* choice agrees with earlier *a priori* one (See Remark 3.2 in [11]). In general it is not the case. For example, numerical tests presented in [19] show that one can take a penalty weight $\mu = \sigma^{-1}$ much larger than it is suggested by *a priori* estimates, and it will still lead to decreasing the error in all norms (See Table 3 in [19]). Moreover, from numerical results reported in [5] it follows that even for convergence in Sobolev/energy norms it is sometimes necessary to increase a penalty weight derived from *a priori* estimates by a factor 10^4, that poses a question about optimal value of such a factor (See [5], p. 371 and Remark 4.1). Thus, like in the case of non-standard regularization methods mentioned above, it is reasonable to use here adaptive strategy for the choice of parameter σ which does not rely on *a priori* estimated φ and v in (1.1),(1.2).

In this paper we propose and analyze a simple strategy of this kind. As far as we know this is the first norm-independent strategy which can be used for adaptive choice of parameter σ in any normed space where regularization or interior penalty method converges and its stability in principle admits upper and low estimations. In the next section we formulate our strategy and provide its theoretical justification. Section 3 and Section 4 contain the results of its application to above mentioned method of lines and to interior penalty discontinuous approximation of elliptic problems on non-matching grids. Numerical tests presented in both sections confirm theoretical results.

2. Balancing principle

In this section we present a general form of our adaptive strategy and analyze it under the following assumptions.

Assumption 2.1. *Assume that the estimation (1.1) is valid for some unknown function φ such that*

(1) *φ is continuous and non-decreasing function defined on the sufficiently large interval $[0, D] \supset [0, 1]$;*
(2) *$\varphi(0) = 0$;*
(3) *$\varphi(\sigma)$ satisfies Δ_2-condition, which means that for any $\sigma \in [0, D/2]$, $\varphi(2\sigma) \leq c_\varphi \varphi(\sigma)$, where the constant c_φ does not depend on σ.*

Assumption 2.2. *Assume that the following two-sided estimation of the stability holds true for any $\sigma \in (0, 1)$*

$$k_0 \frac{\Delta}{\sigma^{v_{k-d}}} \leq \|u_\sigma - u_\sigma^\Delta\| \leq k \frac{\Delta}{\sigma^{v_k}},$$

where $\Delta \in (0, 1)$, k, k_0, and an integer number $d \geq 0$ are assumed to be known. Concerning v_{k-d}, v_k it is known only that they belong to a fixed finite sequence of increasing real numbers $\{v_i\}_{i=0}^M$, $0 < v_0 < v_1 < \cdots < v_M$.

Both assumptions seem to be realistic. Assumption 1 means that an ideal approximation u_σ converges to u^+ as $\sigma \to 0$, but not faster than with a polynomial rate. In practice it is not a serious restriction.

To illustrate Assumption 2.2 in the spirit of penalty methods we consider the case when u_σ^Δ is a finite element approximation for u_σ. An illustration for regularization methods will be given in Section 3.

Illustrative example 2.1. *As we already mentioned, in a finite element case $\Delta = h^{s-\lambda}$, and a stability estimation (1.2) has usually a form*

$$\|u_\sigma - u_\sigma^\Delta\|_{H^\lambda} \leq c_{s,\lambda} h^{s-\lambda} \|u_\sigma\|_{H^s}, \tag{2.1}$$

where $\|\cdot\|_{H^t}$ is a norm in some Sobelev space H^t and the constant $c_{s,\lambda}$ does not depend on σ,h.

Let us highlight that, though the norm $\|u_\sigma\|_{H^t}$ does depend on σ, it can happen that $\|u_\sigma\|_{H^t}$ is uniformly bounded in σ with t up to a certain value $\overline{s}$, and it increases, for example as σ^{-p}, $\sigma \to 0$, for some $t = \underline{s} > s \geq \overline{s}$, i.e. $\|u_\sigma\|_{H^{\underline{s}}} \leq c\sigma^{-p}$. Then interpolation in Sobolev scale yields

$$\|u_\sigma\|_{H^s} \leq K \|u_\sigma\|_{H^{\underline{s}}}^{\frac{s-\overline{s}}{\underline{s}-\overline{s}}} \|u_\sigma\|_{H^{\overline{s}}}^{\frac{\underline{s}-s}{\underline{s}-\overline{s}}} \leq \frac{L}{\sigma^{\tau(s-\overline{s})}}, \quad \overline{s} \leq s < \underline{s}, \tag{2.2}$$

where $\tau = p/(\underline{s} - \overline{s})$, and K,L are σ-independent, but they can depend on $\underline{s}$,$\overline{s}$,r.

In view of (2.1), (2.2) the estimation (1.3) has a form

$$\|u^+ - u_\sigma^\Delta\|_{H^\lambda} \leq \varphi(\sigma) + \frac{kh^{s-\lambda}}{\sigma^{\tau(s-\overline{s})}}, \quad k = c_{s,\lambda} L.$$

Without loss of generality one can assume that in this estimation $\tau = 1$, because introducing a new parameter $\widetilde{\sigma} = \sigma^\tau$ it is always possible to rewrite this estimation as

$$\|u^+ - u^\Delta_{\widetilde{\sigma}}\|_{H^\lambda} \le \widetilde{\varphi}(\widetilde{\sigma}) + \frac{kh^{s-\lambda}}{\widetilde{\sigma}^{s-\overline{s}}},$$

where $\widetilde{\varphi}(\widetilde{\sigma}) = \varphi(\widetilde{\sigma}^{1/\tau})$.

At first glance it is reasonable to use such estimation with $s = \overline{s}$, but it can lead to a loss of accuracy. To see it assume that $\varphi(\sigma) = \sigma^r$, $r > \overline{s} - \lambda$. Then

$$\|u^+ - u^\Delta_\sigma\|_{H^\lambda} \le \sigma^r + \frac{kh^{s-\lambda}}{\sigma^{s-\overline{s}}}.$$

For $s = \overline{s}$ an optimal choice of parameter is $\sigma = \sigma_{opt}(\overline{s}) = (kh^{\overline{s}-\lambda})^{1/r}$, which gives an accuracy of order $h^{\overline{s}-\lambda}$.

On the other hand, for $s > \overline{s}$ the value of parameter balancing σ^r with $kh^{s-\lambda}/\sigma^{s-\overline{s}}$ is $\sigma = \sigma_{opt}(s) = (kh^{s-\lambda})^{1/(r+s-\overline{s})}$ guarantees an accuracy of order $h^{r(s-\lambda)/(r+s-\overline{s})}$ which is much better than $O(h^{\overline{s}-\lambda})$ obtained under the choice $\sigma = \sigma_{opt}(\overline{s})$.

This simple analysis shows that it is reasonable to rely on (2.1) with $s > \overline{s}$ in spite of the fact that $\|u_\sigma\|_{H^s}$ does not appear to be uniformly bounded in σ.

Note that the crucial value $\overline{s}$ is defined by the strongest possible singularity in u_σ and usually is not known exactly. At the same time, in some particular cases of penalty methods one can obtain more information about $\overline{s}$.

For example, the situation arising in penalty/fictitious domain method has been discussed in [16]. *Without going into detail we note that the fictitious domain/embedding method is a kind of penalty method, where the domain of the original boundary value problem is embedded into a larger domain with a more simple geometry, and a new artificial boundary value problem is constructed there. In the new, fictitious part of the domain the coefficients of a differential equation are chosen to be close to zero, if the original boundary condition is of Neumann type, or very large, in the Dirichlet case. If μ is the maximal value of these coefficients then we have $\sigma = \mu$ in the first case, and $\sigma = \mu^{-1}$ in the second one.*

It has been observed in [16] *that for penalty/fictitious domain method in polygons the crucial value $\overline{s} \in [3/2, 5/2]$ under the assumption that a fictitious part is strictly inside of the new domain. If $3/2 \le s_0 < s_1 < \cdots < s_M \le 5/2$ are our guesses concerning the value $\overline{s}$, then for $s > s_M$, $\overline{s} \in [s_{M-k}, s_{M-k+1}]$, $k = 0, 1, \ldots, M$, the norm $\|u_\sigma\|_{H^s}$ can increase as $\sigma^{-(s-s_{M-k})}$. In view of (2.1) it is natural to assume that in this case*

$$\frac{kh^{s-\lambda}}{\sigma^{s-s_{M-k+1}}} \le \|u_\sigma - u^\Delta_\sigma\|_{H^\lambda} \le \frac{kh^{s-\lambda}}{\sigma^{s-s_{M-k}}}.$$

It is nothing but the Assumption 2.2 with $k = k_0 = Lc_{s,\lambda}$, $\Delta = h^{s-\lambda}, v_k = (s - s_{M-k})$, $k = 0, 1, \ldots, M$, $d = 1$. Note that in this context the Assumption 2.2 with $d = 0$ will mean that $\overline{s}$ is assumed to be one of $\{s_i\}_{i=0}^M$, and the estimations for $\|u_\sigma - u^\Delta_\sigma\|_{H^\lambda}$ and $\|u_\sigma\|_{H^s}$ are order optimal ones.

Remark 2.1. Two-sided inequality (1.5) is a consequence of the Assumption 2.1. Indeed, for any $L \ge 1$ there is an integer i such that $2^{i-1} \le L \le 2^i$, $i \le 1 + \log_2 L$, and

$$\varphi(L\sigma) \le \varphi(2^i\sigma) \le c_\varphi \varphi(2^{i-1}\sigma) \cdots \le c_\varphi^i \varphi(\sigma) \le c_\varphi^{1+\log_2 L}\varphi(\sigma). \tag{2.3}$$

Assume that in (1.2) $k \geq 1$ (for $k < 1$ the argumentation will be similar). Then $\sigma_{opt} = \theta_v^{-1}(k\Delta) \geq \theta_v^{-1}(\Delta) = a$, and

$$k = \frac{\theta_v(\sigma_{opt})}{\theta_v(a)} = \frac{\varphi(\sigma_{opt})\sigma_{opt}^v}{\varphi(a)a^v} \geq \left(\frac{\sigma_{opt}}{a}\right)^v \Longrightarrow \sigma_{opt} \leq k^{1/v}\theta_v^{-1}(\Delta).$$

Now using (2.3) with $L = k^{1/v}$ we obtain

$$\varphi(\theta_v^{-1}(\Delta)) \leq \varphi(\sigma_{opt}) \leq \varphi(k^{1/v}\theta_v^{-1}(\Delta)) \leq c_\varphi^{(v+\log_2 k)/v}\varphi(\theta_v^{-1}(\Delta)),$$

which is just an inequality (1.5) with $c_1 = 1$, $c_2 = c_\varphi^{(v+\log_2 k)/v}$.

The following observation gives a reason for one more assumption.

If for some increasing sequence $\{v_i\}_{i=0}^M$ the Assumption 2.2 is satisfied with $d = d_0$ then it is also satisfied with $d = d_0+1, d_0+2, \ldots$, because for any $\sigma \in (0,1)$

$$\cdots k_0\frac{\Delta}{\sigma^{v_{k-d-1}}} \leq k_0\frac{\Delta}{\sigma^{v_{k-d}}} \leq \|u_\sigma - u_\sigma^\Delta\| \leq k\frac{\Delta}{\sigma^{v_k}} \leq k\frac{\Delta}{\sigma^{v_{k+1}}} \leq \cdots .$$

Therefore, the following assumption seems to be natural.

Assumption 2.3. *It is assumed that for fixed sequence $\{v_i\}_{i=0}^M$ a known quantity d is a minimal integer number, such that the Assumption 2.2 is satisfied.*

Under the Assumption 2.2 and 2.3 $v = v_k$ is the best possible choice of the power in the stability estimation (1.2) among the set $\{v_i\}_{i=1}^M$. Then the last assumption looks like a technical one. It relates unknown approximation rate expressed in terms φ with a power of stability $v = v_k$.

Assumption 2.4. *Let $\sigma_{opt} = \sigma_{opt}(v_k)$ be a solution of equation $\varphi(\sigma)\sigma^{v_k} = k\Delta$. In the sequal we will assume that $\varphi(\sigma_{opt}) < 1$ and $\varphi(1) > k\Delta$.*

Assumption 2.4 is not at all restrictive. Indeed, in view of (1.4) a condition $\varphi(\sigma_{opt}) < 1$ means that an optimal accuracy is smaller than 1, while an inequality $\varphi(1) > k\Delta$ means that a trivial choice $\sigma = 1$ is not optimal.

If the power $v = v_k$ in the stability estimation is known then even without knowledge of φ one can choose a parameter $\sigma = \sigma(v_k)$ realizing the best possible order of accuracy $\varphi(\theta_v^{-1}(\Delta))$, $v = v_k$. *A posteriori* strategy for choosing such $\sigma(v_k)$ has been presented in [24]. It can be also applied to more general form of stability estimation such as

$$\|u_\sigma - u_\sigma^\Delta\| \leq \frac{k\Delta}{\lambda(\sigma)}.$$

For $\lambda(\sigma) = \sigma^{v_k}$ the strategy [24] selects $\sigma = \sigma(v_k)$ from a geometric sequence

$$\Sigma_N = \{\sigma = \sigma_n = \sigma_0 q^n, n = 0, 1, \ldots, N\},$$

in accordance with the rule

$$\sigma(v_k) = \max\{\sigma_n \in \Sigma_N : \|u_{\sigma_n}^\Delta - u_{\sigma_l}^\Delta\| \leq \frac{4k\Delta}{\sigma_l^{v_k}}, l = 0, 1, \ldots, n-1\}, \tag{2.4}$$

where $q > 1$, and N is such that $\sigma_{N-1} = \sigma_0 q^{N-1} \leq 1 \leq \sigma_0 q^N = \sigma_N$.

Let

$$\Sigma_N^* = \{\sigma_n \in \Sigma_N : \varphi(\sigma_n) \leq k\frac{\Delta}{\sigma_n^{v_k}}\}$$

and

$$\sigma_* = \max\{\sigma_n \in \Sigma_N^*\} \in \Sigma_N.$$

Theorem 2.1. [24] *Let Σ_N be such that $\Sigma_N^* \neq \emptyset$ and $\Sigma_N \backslash \Sigma_N^* \neq \emptyset$. Then under the Assumption 2.1-2.3*

$$\sigma(v_k) \geq \sigma_*$$

and

$$\|u^+ - u^\Delta_{\sigma(v_k)}\| \leq 6q^{v_k}\varphi(\sigma_{opt}).$$

To meet the conditions of this theorem σ_0 should be small enough. Without loss of generality we assume that $v_k > v_0$ and choose

$$\sigma_0 \leq \min\{(k_0/(16k))^{1/\theta}, q^{-1/r}(\Delta k_0/6)^{\frac{1}{rv_0}}\}, \tag{2.5}$$

where

$$\theta = \min_{i=1,2,\dots,M}(v_i - v_{i-1}), \quad r = \min_{i=1,2,\dots,M}\frac{v_{i-d}}{v_i}.$$

Such choice of σ_0 and an Assumption 2.4 guarantee that the conditions of Theorem [24] are satisfied. Indeed,

$$\sigma_{opt}^{v_0} \geq \sigma_{opt}^{v_k} > \varphi(\sigma_{opt})\sigma_{opt}^{v_k} = \Delta k \Longrightarrow \sigma_{opt} > (\Delta k)^{1/v_0} > \sigma_0. \tag{2.6}$$

Then

$$\varphi(\sigma_0) \leq \varphi(\sigma_{opt}) = \frac{k\Delta}{\sigma_{opt}^{v_k}} \leq \frac{k\Delta}{\sigma_0^{v_k}},$$

that means that $\sigma_0 \in \Sigma_N^* \neq \emptyset$. On the other hand, due to the choice of σ_N

$$\varphi(\sigma_N) \geq \varphi(1) > k\Delta > \frac{k\Delta}{\sigma_N^{v_k}},$$

which means that $\Sigma_N \backslash \Sigma_N^* \neq \emptyset$.

The idea of the strategy (2.4) has its origin in the statistical paper [21] devoted to the estimation of Hölder continuous function with unknown smoothness index observed in the presence of random Gaussian white noise. In [12, 22, 24, 3] this idea has been generalized and adapted to the regularization of ill-posed problems. But in all papers just listed it has been essentially used that the right-hand side of corresponding stability estimation (1.2) is *a priori* known. For applications that we have in mind such an assumption is rather restrictive. Therefore we will combine the strategy (2.4) with successive testing of the hypotheses that the power v in a stability estimation (1.2) is not larger than some $v_i \in \{v_l\}_{l=0}^M$. For each of these hypotheses the strategy (2.4) selects the value

$$\sigma(v_i) = \max\{\sigma_n \in \Sigma_n : \|u^\Delta_{\sigma_n} - u^\Delta_{\sigma_l}\| \leq \frac{4k\Delta}{\sigma_l^{v_i}}, l = 0, 1, \dots, n-1\}.$$

The basic property of the sequence $\sigma(v_i)$, $i = 0, 1, \dots, M$, is that it is monotonic and non-decreasing :

$$\sigma(v_0) \leq \sigma(v_1) \leq \cdots \leq \sigma(v_M). \tag{2.7}$$

Indeed, for $v_i < v_{i+1}$

$$\begin{aligned}\Sigma_N^{v_i} &:= \{\sigma_n \in \Sigma_N : \|u^\Delta_{\sigma_n} - u^\Delta_{\sigma_l}\| \leq \frac{4k\Delta}{\sigma_l^{v_i}}, l = 0, 1, \dots, n-1\} \\ &\subset \Sigma_N^{v_{i+1}} := \{\sigma_n \in \Sigma_N : \|u^\Delta_{\sigma_n} - u^\Delta_{\sigma_l}\| \leq \frac{4k\Delta}{\sigma_l^{v_{i+1}}}, l = 0, 1, \dots, n-1\},\end{aligned}$$

i.e. the set $\Sigma_N^{v_i}$ becomes larger if i grows. Therefore

$$\sigma(v_i) = \max\{\sigma_n \in \Sigma_N^{v_i}\} \leq \max\{\sigma_n \in \Sigma_N^{v_{i+1}}\} = \sigma(v_{i+1}).$$

The basic idea of our balancing principle is the following. If a parameter $v_i \in \{v_l\}_{l=0}^M$ is smaller than the actual power $v_{k-d} \in \{v_l\}_{l=0}^M$ in the two-sided stability estimation of Assumption 2.2, then a corresponding $\sigma(v_i)$ is "too small". It turns out that this can be detected from approximations $u_{\sigma_n}^\Delta$, $\sigma_n \in \Sigma_N$, using a special choice of σ_0 given by (2.5).

Lemma 2.2. *Let the Assumption 2.1-2.4 hold. If σ_0 is chosen in accordance with (2.5) then for $v_i \leq v_{k-d-1}$*

$$\sigma(v_i) < \sigma_0^r \left(\frac{3k}{k_0 - 4k\sigma_0^\theta}\right)^{1/v_0}.$$

Proof. We prove the lemma considering the cases $\sigma(v_i) < \sigma_*$ and $\sigma_* \leq \sigma(v_i)$ separately, where σ_* is defined in the Theorem 2.1 presented above.

From the definition of $\sigma(v_i)$ and Assumption 2.1, 2.2 we obtain

$$\begin{aligned}
\frac{4k\Delta}{\sigma_0^{v_i}} &\geq \|u_{\sigma_0}^\Delta - u_{\sigma(v_i)}^\Delta\| \geq \|u^+ - u_{\sigma_0}^\Delta\| - \|u^+ - u_{\sigma(v_i)}^\Delta\| \\
&\geq \|u_{\sigma_0} - u_{\sigma_0}^\Delta\| - \|u^+ - u_{\sigma_0}\| - \varphi(\sigma(v_i)) - \frac{k\Delta}{[\sigma(v_i)]^{v_k}} \\
&\geq \frac{k_0\Delta}{\sigma_0^{v_{k-d}}} - 2\varphi(\sigma(v_i)) - \frac{k\Delta}{[\sigma(v_i)]^{v_k}}. \qquad (2.8)
\end{aligned}$$

Assume that $\sigma(v_i) < \sigma_*$. Then by the definition of σ_*

$$\varphi(\sigma(v_i)) \leq \varphi(\sigma_*) \leq \frac{k\Delta}{\sigma_*^{v_k}} < \frac{k\Delta}{[\sigma(v_i)]^{v_k}},$$

and we can continue (2.8) as

$$\frac{4k\Delta}{\sigma_0^{v_i}} > \frac{k_0\Delta}{\sigma_0^{v_{k-d}}} - \frac{3k\Delta}{[\sigma(v_i)]^{v_k}}.$$

Keeping in mind that $v_i \leq v_{k-d-1} < v_{k-d}$ one can rewrite it as claimed:

$$\sigma(v_i) < \sigma_0^{\frac{v_{k-d}}{v_k}} \left(\frac{3k}{k_0 - 4k\sigma_0^{v_{k-d}-v_i}}\right)^{1/v_k} \leq \sigma_0^r \left(\frac{3k}{k_0 - 4k\sigma_0^\theta}\right)^{1/v_0}.$$

Now let us consider the remaining case $\sigma(v_i) \geq \sigma_*$ and prove that it is impossible.

In the same way as in (2.8) we obtain

$$\|u_{\sigma_0}^\Delta - u_{\sigma_*}^\Delta\| \geq \frac{k_0\Delta}{\sigma_0^{v_{k-d}}} - \frac{3k\Delta}{\sigma_*^{v_k}}.$$

On the other hand,

$$\begin{aligned}
\|u_{\sigma_0}^\Delta - u_{\sigma_*}^\Delta\| &\leq \|u_{\sigma(v_i)}^\Delta - u_{\sigma_0}^\Delta\| + \|u_{\sigma(v_i)}^\Delta - u_{\sigma_*}^\Delta\| \\
&\leq \frac{4k\Delta}{\sigma_0^{v_i}} + \frac{4k\Delta}{\sigma_*^{v_i}} \leq \frac{8k\Delta}{\sigma_0^{v_i}}.
\end{aligned}$$

Thus,

$$\frac{8k\Delta}{\sigma_0^{v_i}} > \frac{k_0\Delta}{\sigma_0^{v_{k-d}}} - \frac{3k\Delta}{\sigma_*^{v_k}},$$

and repeating the previous argument we conclude that

$$\sigma_* < \sigma_0^r \left(\frac{3k}{k_0 - 8k\sigma_0^\theta}\right)^{1/v_0}. \tag{2.9}$$

At the same time, from the definition of σ_* it follows that for $q\sigma_* \in \Sigma_N$

$$\varphi(q\sigma_*)(q\sigma_*)^{v_k} > k\Delta = \varphi(\sigma_{opt})\sigma_{opt}^{v_k} \Longrightarrow q\sigma_* > \sigma_{opt}.$$

Then in view of (2.6) and (2.5) we get an estimation

$$\sigma_* > q^{-1}\sigma_{opt} > q^{-1}(k\Delta)^{\frac{1}{v_0}} > \left(\frac{6k}{k_0}\right)^{\frac{1}{v_0}} \sigma_0^r > \left(\frac{3k}{k_0 - 8k\sigma_0^\theta}\right)^{\frac{1}{v_0}} \sigma_0^r, \tag{2.10}$$

which is in contradiction with (2.9). Thus, for $v_i \leq v_{k-d-1}$ the case $\sigma(v_i) \geq \sigma_*$ is impossible. This complete the proof of the lemma. □

Now we are in a position to describe our balancing principle. Having a family of elements $\{u_{\sigma_i}^\Delta\}_{i=0}^N$ approximating u^+ we define a monotone non-decreasing sequence (2.7) and choose a parameter

$$\sigma_+ = \min\{\sigma(v_i) : \sigma(v_i) \geq \sigma_0^r (\frac{3k}{k_0 - 4k\sigma_0^\theta})^{1/v_0}\}. \tag{2.11}$$

It is clear that σ_+ corresponds to some $v_j \in \{v_i\}_{i=0}^M$ which will be denoted as v_+, i.e. $\sigma_+ = \sigma(v_j)$, $v_+ = v_j$.

We stress that unknown function φ and power v_k describing the rates of approximation and stability are not involved in the construction of σ_+, v_+. They depend only on the level of unavoidable error Δ, on the constants k_0, k which are assumed to be known, and on the two design sets of parameter Σ_N and $\{v_i\}_{i=0}^M$.

We turn to the main result of this section. The theorem below allows us to choose adaptively from the design set $\{v_i\}_{i=0}^M$ such power v that a stability estimation (1.2) is satisfied with it (See Corollary 2.1)

Theorem 2.3. *Assume that the condition of Lemma 2.2 hold. Then* $\sigma(v_{k-d}) \leq \sigma_+ \leq \sigma(v_k)$.

Proof. From (2.10) and the Theorem 2.1 presented above it follows that

$$\sigma(v_k) \geq \sigma_* \geq \left(\frac{3k}{k_0 - 8k\sigma_0^\theta}\right)^{1/v_0} \sigma_0^r > \left(\frac{3k}{k_0 - 4k\sigma_0^\theta}\right)^{1/v_0} \sigma_0^r.$$

Then by the definition $\sigma_+ \leq \sigma(v_k)$.

On the other hand, from Lemma 2.2 it follows that

$$\sigma(v_{k-d-1}) < \sigma_0^r \left(\frac{3k}{k_0 - 4k\sigma_0^\theta}\right)^{1/v_0} \leq \sigma_+.$$

In view of (2.7) it means that $\sigma(v_{k-d}) \leq \sigma_+$, and this complete the proof. □

Corollary 2.1. *If* $v_+ = v_j \in \{v_i\}_{i=0}^M$ *then* $v_{j+d} \geq v_k$ *and* $\sigma(v_{j+d}) \geq \sigma(v_k)$.

If the Assumption 2.2 is satisfied with $d = 0$ *then* $v_+ = v_k$, $\sigma_+ = \sigma(v_k)$, *and*

$$\|u^+ - u_{\sigma_+}^\Delta\| \leq 6q^{v_k}\varphi(\sigma_{opt}) \leq 6q^{v_k} c_\varphi^{(v_k + \log_2 k)/v_k} \varphi(\theta_{v_k}^{-1}(\Delta)),$$

where c_φ, k *are the constants from the Assumption 2.1, 2.2, and* q *is a denominater of the geometric sequence* Σ_N.

Proof. The first statement of the corollary is a direct consequence of Theorem 2.3 and (2.7). The second statement follows from Theorem 2.3, Remark 2.1 and Theorem 2.1 presented above. □

We would like to note that our balancing principle (2.11) is a generalization of adaptive strategy for the choice of regularization parameter presented in [13]. This statistical paper has dealt with a problem of linear functional estimation from indirectly observed data blurred by random noise, and an analog of the Assumption 2.2 with $d = 0$ has been discussed there. Corollary 2.1 tells us that in this particular case balancing principle (2.11) allows to obtain the best possible order of accuracy without knowledge of exact rates of approximation and stability.

In the general case $d > 0$ Corollary 2.1 allows to find adaptively the power v in the stability estimation (1.2), but it may be larger than an optimal one. Nevertheless in the next section we will see that for severely ill-posed problems our balancing principle still leads to the accuracy of optimal order.

3. Regularization by discretization for solving severely ill-posed problems

In an abstract formulation a discretization of the problem, given as a linear operator equation

$$Au = v \tag{3.1}$$

with an operator $A : U \to V$ acting between Banach spaces U and V, is a rule for assigning an approximate solution u_m to the problem instances A and v.

Distinguishing feature of the discretization is that an approximate solution $u_m = u_m(A, v)$ is always an element of some fixed m-dimensional subspace $U_m \subset U$, and it converges to the solution $u^+ = u^+(A, v)$ of the problem (3.1) as $m \to \infty$. Then there exists an increasing continuous function $\psi(\lambda) = \psi(A, v; \lambda)$ such that $\psi(0) = 0$ and

$$\|u^+ - u_m(A, v)\|_U \le \psi(m^{-1}). \tag{3.2}$$

Assume that the problem (3.1) is ill-posed, which means, in particular, that its solution $u^+(A, v)$ does not depend continuously on v, and noisy data v_δ with arbitrarily small noise level δ, $\|v - v_\delta\|_V \le \delta$, can lead to disproportionately large deviations in the solution.

Moreover, a perturbed problem

$$Au = v_\delta \tag{3.3}$$

may have no solution in the space U.

In practice data will almost never be available exactly, and discretization can provide a way to regularize a perturbed problem (3.3).

Indeed, linearity of the original problem (3.1) is usually inherited by a discretization in such a way, that $v \to u_m(A, v)$ is a linear mapping into m-dimensional subspace U_m, i.e.

$$u_m(A, v) = \sum_{i=1}^{m} l_i(v)\chi_i,$$

where $\{\chi_i\}$ is a basis in U_m, and $l_i(v)$, $i = 1, 2, \ldots, m$ are some linear bounded functionals defined on the Range(A). In view of Hahn-Banach theorem all $l_i(v)$ can be considered as linear bounded functionals defined on the whole space V, which allows to extend a mapping $v \to u_m(A, v)$ up to some linear bounded operator acting from V into U_m. Therefore, without a loss of generality one can assume that a discretization procedure is well-defined for a perturbed problem (3.3) and comes up with an approximate solution $u_n(A, v_\delta)$ such that

$$\|u_m(A, v) - u_m(A, v_\delta)\|_U \le c_m \|v - v_\delta\|_V \le c_m \delta, \tag{3.4}$$

where c_m is a norm of the above mentioned extended linear operator.

Obviously, one could consider $u_m(A, v_\delta)$ as an approximate solution of the original problem (3.1), but the fact is that usually the norms c_m in (3.4) tend to infinity as $m \to \infty$, which reflects an ill-posedness of the original problem.

Nevertheless, a coordination between discretization parameter m and the amount of noise δ in the problem (3.3) allows us to obtain a regularization effect; no additional regularization of the problem is needed. This is sometimes called self-regularization property of discretizations, or regularization by discretization.

Indeed, in view of (3.2),(3.4) the discretization error can be estimated by

$$\|u^+ - u_m(A, v_\delta)\|_U \le \psi(m^{-1}) + c_m \delta,$$

and the essence of self-regularization consists in the balance between discretization error bounded by $\psi(m^{-1})$ and a rate of the noise propagation estimated by $c_m\delta$. The main part of the literature on self-regularization is devoted to *a priori* estimations of the rates of approximation $\psi(m^{-1})$ and noise propagation $c_m\delta$ for some special cases of ill-posed problems and discretization schemes. Obviously, when such *a priori* estimates are obtained, balance between them can be easily found.

At the same time, *a priori* estimation of an approximation rate $\psi(m^{-1})$ presupposes that the smoothness of unknown solution u^+ is somehow known. In practice it is not always the case.

Adaptive strategies for choosing discretization parameter n, which yield an optimal order of accuracy for self-regularization without using knowledge of $\psi(m^{-1})$, were proposed in [15],[14]. But these strategies still use *a priori* estimates for c_m.

It is also worth noting that the previous study of self-regularization was mainly restricted to the case of the so-called moderately ill-posed problems that corresponds to a polynomial rate of $\psi(m^{-1})$ and c_m, i.e. $c_m \sim m^a$, and

$$\lambda^{\mu_1} \le \psi(\lambda) \le \lambda^{\mu_2}, \quad \lambda \in [0, 1], \tag{3.5}$$

where a, μ_1, μ_2 are some positive constants.

At the same time, in practice one often is faced with the so-called severely ill-posed problems, when (3.5) is satisfied, but the rate of the noise propagation $c_m\delta$ is exponential. One typical example, when it is assumed to be the case, is an inverse heat conduction problem (IHCP), which will be considered in our numerical experiments below.

As it has been pointed out in a survey [18], it is an open question whether discretization methods alone are sufficient to handle the severely ill-posed problems without the use of additional regularization techniques. The main problem here is that for severely ill-posed problems the rate of the noise propagation $c_m\delta$ is usually overestimated. In such a situation an application of our balancing principle is a way out.

In this section we present the balancing principle in the form suitable for a self-regularization and show that it provides a positive answer for the question discussed in [18].

To the best of our knowledge only in [6] an order-optimal adaptive strategy for the self-regularization of severely ill-posed problems can be found. This strategy does not use any *a priori* estimates of $\psi(m^{-1})$ and c_m, but it has been developed only for a discretization by the so-called dual least squares projection method. At the same time, in [18] it has been noted that discretization methods of this type destroy a non-anticipatory structure of the Volterra type problems such as IHCP. Therefore, it is not desirable to use them.

It will be seen that the balancing principle covers a strategy [6], and can be also applied to discretization methods preserving the Volterra structure of the original problem.

In our analysis we use the same assumption as in [6]. Namely, we assume that the noise in discrete approximation really propagate with an exponential rate such that for some $a > 0$, $p > 1$

$$\delta e^{am} \leq \|u_m(A, v) - u_m(A, v_\delta)\|_U \leq \delta e^{apm}, \qquad m = 1, 2, \dots. \tag{3.6}$$

The following example shows that in the context of self-regularization for severely ill-posed problems this assumption is rather natural.

Illustrative example 3.1. *In a Hilbert space setting severely ill-posed problems often arise in the form (3.1) with infinitely smoothing operator A, which means that its singular values $\{s_k\}_{k=1}^{\infty}$, arranged in a non-increasing order, tend to zero with an exponential rate. More precisely, A admits a singular value expansion*

$$A = \sum_{k=1}^{\infty} s_k y_k \langle x_k, \cdot \rangle,$$

where

$$\ln \frac{1}{s_k} \asymp k, \tag{3.7}$$

and $\{x_k\}$, $\{y_k\}$ are orthonormal systems in Hilbert spaces U and V respectively. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and corresponding norm on each of the Hilbert spaces U and V. It will be always clear from the context which space is concerned.

A so-called spectral cut-off scheme is usually used as a model example for regularization by discretization. Within the framework of this ideal discretization scheme

$$u_m(A, v) = \sum_{k=1}^{m} s_k^{-1} x_k \langle y_k, v \rangle.$$

A data error $v - v_\delta$ is usually simulated as a random white noise, which means that Fourier coefficients $\langle y_k, v - v_\delta \rangle$ of the noise are uncorrelated random variables such that $\mathbf{E}\langle y_k, v - v_\delta \rangle^2 = \delta^2$, $k = 1, 2, \dots,$ where $\mathbf{E}$ means the expected value. Then

$$\mathbf{E}\|u_m(A, v) - u_m(A, v_\delta)\|^2 = \mathbf{E}\|\sum_{k=1}^{m} s_k^{-1} \langle y_k, v - v_\delta \rangle x_k\|^2 = \delta^2 \sum_{k=1}^{m} s_k^{-2}.$$

Now from (3.7) it follows that there are some constants $c_1, c_2 > 0$ such that $e^{c_1 k} \le s_k^{-1} \le e^{c_2 k}$, and

$$\delta^2 e^{2c_1 m} \le \mathbf{E}\|u_m(A,v) - u_m(A,v_\delta)\|^2 \le \delta^2 \sum_{k=1}^{m} e^{2c_2 k} \le \delta^2 e^{(2c_2+1)m}.$$

It means that in considered case on estimate (3.6) holds true with a large probability at least for $a = c_1$ and $p = \frac{2c_2+1}{2c_1}$.

Thus, our illustrative example shows that an assumption (3.6) reflects a typical noise propagation in the discrete approximation of the solution of severely ill-posed problem. If the operator A is not well studied then the exponent a in (3.6) is rarely known exactly. In this case p reflects the magnitude of a gap in our knowledge of A and is assumed to be known.

To apply our balancing principle we rewrite (3.6) in the form of two-sided stability estimation presented in Assumption 2.2.

Let $\overline{m}$ be a maximal value of the parameter m used in the discretization procedure. Then we choose a geometric sequence Σ_N with $N = \overline{m} - 1$, $\sigma_0 = e^{-\overline{m}}$, $q = e$, $\sigma_n = e^{-(\overline{m}-n)}$, $n = 0, 1, 2, \ldots, N$, and define

$$u_{\sigma_n} = u_{\overline{m}-n}(A,v), \quad u_{\sigma_n}^{\Delta} = u_{\overline{m}-n}(A,v_\delta), \quad \Delta = \delta.$$

It follows from (3.2) that

$$\begin{aligned} \|u^+ - u_{\sigma_n}\| = \|u^+ - u_{\overline{m}-n}(A,v)\| &\le \psi((\overline{m}-n)^{-1}) \\ &= \psi(\ln^{-1}\frac{1}{\sigma_n}). \end{aligned} \tag{3.8}$$

If we assume that $\psi(\lambda)$ satisfies Δ_2-condition then (3.8) is just an Assumption 2.1 with $\varphi(\sigma) = \psi(\ln^{-1}\frac{1}{\sigma})$.

Consider one more geometric sequence $v_i = v_0 p^i$, $i = 0, 1, \ldots, M$, where $p > 1$ is the same as in (3.6), and it is assumed to be known. Without loss of generality we can also assume that v_0 is so small and M is so large that

$$0 < v_0 < a < ap < v_0 p^{M-2}.$$

Then there is $k \in \{2, 3, \ldots, M-2\}$ such that

$$v_{k-2} \le a \le v_{k-1} \le ap \le v_k. \tag{3.9}$$

Using (3.9) one can rewrite (3.6) as

$$\delta e^{v_{k-2}(\overline{m}-n)} \le \|u_{\overline{m}-n}(A,v) - u_{\overline{m}-n}(A,v_\delta)\| \le \delta e^{v_k(\overline{m}-n)},$$

or

$$\frac{\Delta}{\sigma_n^{v_{k-2}}} \le \|u_{\sigma_n} - u_{\sigma_n}^{\Delta}\| \le \frac{\Delta}{\sigma_n^{v_k}}. \tag{3.10}$$

It is just an Assumption 2.2 with $k_0 = k_1 = 1$, $d = 2$. Keeping in mind that for chosen sequence $\{v_i\}$, $\theta = \min v_i - v_{i-1} = \min v_{i-1}(p-1) = v_0(p-1)$, $r = \min \frac{v_{i-2}}{v_i} = p^2$, one can also rewrite (2.5) as

$$\sigma_0 = e^{-\overline{m}} \le \min\{2^{-4/v_0(p-1)}, e^{-p^2}(\delta/6)^{p^2/v_0}\},$$

or

$$\overline{m} \ge \max\{\frac{4\ln 2}{v_0(p-1)}, p^2(\frac{1}{v_0}\ln(\frac{6}{\delta}+1))\}. \tag{3.11}$$

Let now $\{u_m(A, v_\delta)\}_{m=1}^{\overline{m}}$ be a sequence of approximate solution constructed within the framework of a discretization procedure for perturbed severely ill-posed problem (3.3). By analogy with (2.7) one can define a sequence

$$1 \leq m(v_M) \leq m(v_{M-1}) \leq \ldots \leq m(v_1) \leq m(v_0) \leq \overline{m},$$

where

$$m(v_i) = \min\{m : \|u_m(A, v_\delta) - u_n(A, v_\delta)\| \leq 4\delta e^{nv_i}, n = \overline{m}, \overline{m}-1, \ldots, m+1\},$$

which is equivalent to

$$\begin{aligned}\sigma(v_i) = \max\{\sigma_n = e^{-(\overline{m}-n)} : \|u^\Delta_{\sigma_n} - u^\Delta_{\sigma_l}\| &= \|u_{\overline{m}-n}(A, v_\delta) - u_{\overline{m}-l}(A, v_\delta)\| \\ &\leq \frac{4\delta}{\sigma_l^{v_i}}, l = 0, 1, , \ldots, n-1\}.\end{aligned}$$

Then in accordance with our balancing principle (2.11) we select

$$\sigma_+ = \sigma(v_j) = \min\{\sigma(v_i) = e^{-m(v_i)} \geq e^{-\overline{m}p^{-2}} \left(\frac{3}{1 - 4e^{-\overline{m}(p-1)v_0}}\right)^{1/v_0}\},$$

and it gives us

$$m_+ = m(v_j) = \max\{m(v_i) : m(v_i) \leq \frac{\overline{m}}{p^2} - \frac{1}{v_0} \ln\left(\frac{3e^{\overline{m}(p-1)v_0}}{e^{\overline{m}(p-1)v_0} - 4}\right)\}.$$

Note that in view of (3.8), (3.10)

$$\|u^+ - u^\Delta_{\sigma_n}\| \leq \|u^+ - u_{\sigma_n}\| + \|u_{\sigma_n} - u^\Delta_{\sigma_n}\| \leq \psi(\ln^{-1} \frac{1}{\sigma_n}) + \frac{\delta}{\sigma_n^{v_k}}, \tag{3.12}$$

and an optimal choice for σ_n would be $\sigma_n = \sigma_{opt}$ satisfying an equation

$$\psi(\ln^{-1} \frac{1}{\sigma_{opt}})\sigma_{opt}^{v_k} = \delta, \tag{3.13}$$

since such σ_{opt} balances both terms in the estimate (3.12). This optimal choice would provide an accuracy estimated as

$$\|u^+ - u^\Delta_{opt}\| \leq 2\psi(\ln^{-1} \frac{1}{\sigma_{opt}}). \tag{3.14}$$

Of course, σ_{opt} cannot be found without knowledge of ψ and v_k.

At the same time $\sigma_+ = \sigma(v_j)$ and $m_+ = m(v_j)$ can be found using only the value of p. Then we can easily find

$$\widehat{\sigma}_+ = \sigma(v_{j+2}), \quad \widehat{m}_+ = m(v_{j+2}).$$

For example, $\sigma(v_{j+2})$ is the second term after $\sigma_+ = \sigma(v_j)$ in the sequence (2.7).

If $\overline{m}$ is chosen in accordance with (3.11) then from a Corollary 2.1 it follows that $v_{j+2} \geq v_k$. Moreover, in view of (3.12)

$$\|u^+ - u^\Delta_{\sigma_n}\| \leq \psi(\ln^{-1} \frac{1}{\sigma_n}) + \frac{\delta}{\sigma_n^{v_{j+2}}}.$$

In contrast to (3.12) v_{j+2} is known here. It allows us to apply a Theorem 2.1 presented above. We put there $q = e$, $\varphi(\sigma) = \psi(\ln^{-1} \frac{1}{\sigma})$ and replace v_k by v_{j+2}. Then

$$\|u^+ - u^\Delta_{\widehat{\sigma}_+}\| = \|u^+ - u_{\widehat{m}_+}(A, v_\delta)\| \leq 6e^{v_{j+2}}\psi(\ln^{-1} \frac{1}{\overline{\sigma}}), \tag{3.15}$$

where $\overline{\sigma}$ satisfies an equation

$$\psi(\ln^{-1}\frac{1}{\overline{\sigma}})\overline{\sigma}^{v_{j+2}} = \delta. \tag{3.16}$$

From (3.16) and (3.5) we conclude that

$$\ln^{-1}\frac{1}{\overline{\sigma}} = v_{j+2}\ln^{-1}\frac{1}{\delta} + \overline{\overline{o}}(\ln^{-1}\frac{1}{\delta}).$$

Moreover, from (3.13) and (3.5) we have

$$\ln^{-1}\frac{1}{\sigma_{opt}} = v_k\ln^{-1}\frac{1}{\delta} + \overline{\overline{o}}(\ln^{-1}\frac{1}{\delta}).$$

Then

$$\ln^{-1}\frac{1}{\overline{\sigma}} = p^{j+2-k}v_k\ln^{-1}\frac{1}{\delta} + \overline{\overline{o}}(\ln^{-1}\frac{1}{\delta}) \le p^{M-k}\ln^{-1}\frac{1}{\sigma_{opt}},$$

and from (3.15) and (2.3) it follows that for $\widehat{\sigma}_+ = \sigma(v_{j+2})$

$$\|u^+ - u^{\Delta}_{\widehat{\sigma}}\| \le 6e^{v_{j+2}}\psi(p^{M-k}\ln^{-1}\frac{1}{\sigma_{opt}}) \le 6e^{v_{j+2}}c_\psi^{1+(M-k)\log_2 p}\psi(\ln^{-1}\frac{1}{\sigma_{opt}}),$$

where c_ψ is a constant from Δ_2-condition $\psi(2\lambda) \le c_\psi\psi(\lambda)$. This estimation together with (3.14) leads to the statement formulated as follows

Theorem 3.1. *Let (3.2) be satisfied with ψ meeting (3.5) and Δ_2-condition. Assume that a stability estimate (3.6) holds true for $m = 1, 2, \dots, \overline{m}$, where $\overline{m}$ is chosen as in (3.11). Then a choice of a discretization parameter $m = \widehat{m}_+$ based on a balancing principle automatically provides an optimal order of accuracy without knowledge of the rates of approximation and noise propagation, which means that*

$$\|u^+(A, v) - u_{\widehat{m}_+}(A, v_\delta)\| \le c\psi(m_{opt}^{-1}) \le c\psi(\ln^{-1}\frac{1}{\delta}),$$

where $m_{opt} = \ln\frac{1}{\sigma_{opt}}$, and a constant c does not depend on δ.

In order to demonstrate the practicability of a balancing principle for ill-posed problems, from the richness of possible applications we choose an inverse heat conduction problem, the sideways heat equation. It is a model of a problem where one wishes to determine the temperature on the surface of a body while the surface itself is inaccessible for measurements. In this case one is restricted to interior measurements, and from these one wants to compute the surface temperature.

In a one-dimensional setting, assuming that the body is large, this situation can be modelled as the following problem for the heat equation in the quarter plane: Determine the temperature $u(x,t)$ for $x \in [0,1)$ from the temperature measurements $v(t) = u(1,t)$, heat-flux measurements $v_x(t) = \frac{\partial}{\partial x}u(1,t)$, and initial condition $w(x) = u(x,0)$, when $u(x,t)$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(k(x,t)\frac{\partial u}{\partial x}), & t \ge 0, \\ u(x,0) = w(x), & x \ge 0, \\ u(1,t) = v(t), & t \ge 0, \\ \frac{\partial u}{\partial x}(1,t) = v_x(t), & t \ge 0, \end{cases} \tag{3.17}$$

$u|_{x\to\infty}$ bounded, where $k(x,t)$ is a termal diffusivity coefficient.

Without loss of generality we will discuss (3.17) with $w(x) \equiv 0$. A problem with non-zero initial data can easily be reduced to this case by introducing a new unknown $u_{new}(x,t) = u(x,t) - u^w(x,t)$, where $u^w(x,t)$ is a solution of (3.17) with

$v(t) \equiv v_x(t) \equiv 0$. However, some of our numerical experiments will be done for the general case.

Note that, although we seek to recover $u(x,t)$ only for $x \in [0,1)$, the problem specification includes the heat equation for $x > 1$ together with the boundedness as $x \to \infty$. Since we can obtain $u(x,t)$ for $x > 1$ by solving the well-posed quarter plane problem with data $u(1,t) = v(t)$ for $x \geq 1$, also $\frac{\partial u}{\partial x}(1,t) = v_x(t)$ is determined (in principle and numerically) by data $u(1,t) = v(t)$. (See [4] for details.)

For the same reason $u(x,t)$ for $x \in [0,1]$ is well-determined by the data $u(0,t) = u^+(t)$. Thus, (3.17) with $w(x) \equiv 0$ defines the linear operators $v(t) \to u(x,t)$, $v(t) \to u(0,t)$, and for reconstruction of $u(x,t)$ it is sufficient to recover from $v(t)$ only the boundary data $u^+(t) = u(0,t)$.

But from [9] it is known that such recovery is a severely ill-posed problem if instead of $v(t)$ noisy data $v_\delta(t)$ are available.

In the constant coefficient case $k(x,t) = \kappa^2$ boundary data $u^+(t)$ and $v(t)$ are related by a Volterra operator

$$Au^+(t) := \int_0^t a(t,\tau)u^+(\tau)d\tau = v(t), \tag{3.18}$$

where

$$a(t,\tau) = a(t-\tau) = \frac{1}{2\kappa\sqrt{\pi}} \exp\left[-\frac{1}{4\kappa^2(t-\tau)}\right](t-\tau)^{-3/2}.$$

Note that a Volterra equation of the first kind (3.18) cannot be reduced into an equation of the second kind by differentiation, since $\frac{\partial^l a}{\partial t^l}(t,t) = 0$ for all $l = 1, 2, \ldots$.

In [7] a Tikhonov regularization was used for solving (3.18). As it has been pointed out in [18], a disadvantage of such an approach is that Tikhonov regularization does not retain the Volterra structure of (3.18). Typical numerical realization of the Volterra operator A leads to a lower-triangular matrix, so that the solution of discretizated problem may be handled by efficient, sequential methods. In contrast, Tikhonov regularization leads to a full matrix.

Several regularization methods which specifically preserve the Volterra structure can be found in [17, 18]. Stolz method is among them. Within the framework of this method we seek for an approximate solution in the form

$$u_m(A, v_\delta) = u_m(t) = \sum_{i=1}^m l_i(v_\delta)\chi_i(t),$$

where $\chi_i(t)$ is the characteristic function defined by $\chi_i(t) = 1$, for $t \in (t_{i-1}, t_i]$, and $\chi_i(t) = 0$ otherwise; $t_i = i/m$, $i = 1, 2, \ldots, m$; $t_0 = 0$, $\chi_1(t_0) = 1$. We then define the values $l_i = l_i(v_\delta)$ solving the collocation equation

$$Au_m(t_j) = v_\delta(t_j), \quad j = 1, 2, \ldots, m.$$

Keeping in mind that

$$Au_m(t_j) = \sum_{i=1}^j l_i a_{j-i+1}, \quad a_k = \int_0^{t_1} a(t_k - \tau)d\tau,$$

TABLE 1. Numerical results for $\kappa = 2$, $\delta = 10^{-7}$, $\overline{m} = 32$, $b_0 = 0.7$.

q	$\widehat{m}_+$
1.1	14
1.3	8
1.5	4

TABLE 2. Numerical results for $\kappa = 20$, $\delta = 10^{-7}$, $\overline{m} = 32$, $b_0 = 0.7$.

q	$\widehat{m}_+$
1.1	15
1.3	8
1.5	5

we may rewrite the collocation equation in matrix form as $A_m l = v_\delta^m$, where

$$A_m = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix}, \quad l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_m \end{pmatrix}, \quad v_\delta^m = \begin{pmatrix} v_\delta(t_1) \\ v_\delta(t_2) \\ v_\delta(t_3) \\ \vdots \\ v_\delta(t_m) \end{pmatrix}.$$

It is easy to see that in the case under discussion the diagonal entries a_1 of A_m are non-zero and thus the Stolz equations may be solved via forward substitution for l_i, $i = 1, 2, \ldots, m$. However, as it has been reported in [17], for even moderate values of m the Stolz approach is an unstable method for solving an inverse heat conduction problem. We choose it to demonstrate an ability of the balancing principle to find a value of a discretization parameter stabilizing even such unstable dsicretization.

For our numerical experiments we use an example which is similar to one from [7], and consider (3.18) with the exact solution

$$u^+(t) = \begin{cases} 0.3 + 1.7\sin(\pi \frac{t}{900}), & 0 \le t \le \frac{5}{90}, \\ 0.0022\left(\frac{t}{90}\right)^2 - 0.125\frac{t}{90} + 2.57, & \frac{5}{90} \le t \le \frac{28.4}{90}, \\ u^+(28.4/90) - 0.00032\frac{t}{90}, & \frac{28.4}{90} \le t \le 1, \end{cases}$$

Unperturbed data $v(t)$ are given by (3.18), while a perturbation v_δ of v is computed for each t_i to be $v_\delta(t_i) = v(t_i) + \xi_i$, where $\xi_i \in [-\delta, \delta]$ is produced by a uniform random number generator. These simulations have been performed for $\delta = 10^{-7}$ and two values of a thermal diffusivity coefficient $\kappa = 2$ and $\kappa = 20$.

We apply the balancing principle with $\overline{m} = 32$, $v_0 = 0.7$, and try three values of the parameter $p : p = 1.1, p = 1.3$ and $p = 1.5$. The values of the discretization parameter $\widehat{m}_+$ chosen for these values of p in accordance with Theorem 3.1 are presented in Tables 1 and 2.

As it has been expected, the discretization parameter $\widehat{m}_+$ chosen for the smallest value $p = 1.1$ provides better accuracy. Corresponding approximate solutions are displayed in Fig 1 and 2. An evidence of instability of the Stolz method with inappropriate value a discretization paramter is given in Fig 3, where an approximate solution corresponding to $m = 32$ is displayed. Comparing it with Fig 1 and 2,

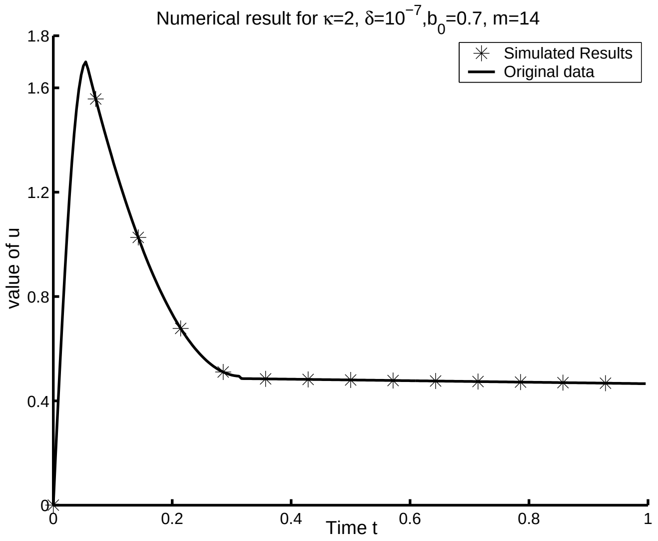

FIGURE 1

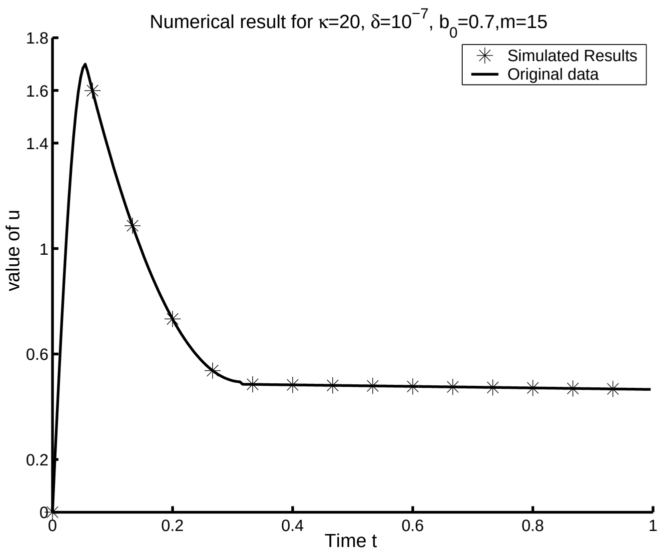

FIGURE 2

we can conclude that using the balancing principle one can really find a value of a discretization paramter providing a self-regularization effect.

For the problem (3.17) with a non-constant coefficient $k(x,t)$ one cannot use methods based on its reformulating as an integral equation of the first kind (3.18), since the kernel function $a(t,\tau)$ is not explicitly known in this case. At the same time such problem can be treated by a method of lines proposed by Eldén [8].

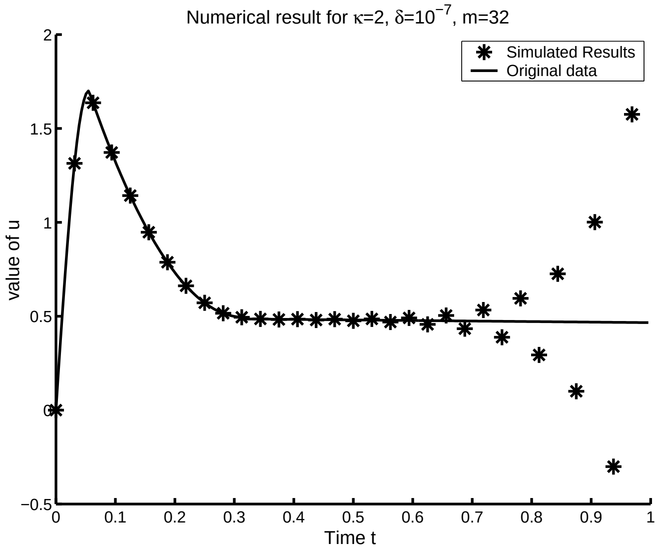

FIGURE 3

Within the framework of this method one discretizes (3.17) in time, using an equidistant discretization $\{t_i = i/m\}_{i=0}^{m}$ of the time interval normalized to $[0,1]$. The time-derivative is approximated by finite differences

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t_i) &\approx \frac{m}{2}(u(x,t_{i+1}) - u(x,t_{i-1})), \quad i = 1,2,\ldots,m-1, \\ \frac{\partial u}{\partial t}(x,t_i) &\approx m(u(x,t_m) - u(x,t_{m-1})). \end{aligned}$$

In this way we obtain a well-posed initial-boundary value problem for a system of ordinary differential equations (ODE)

$$(3.19)\begin{cases} \frac{d}{dx}(k_i(x)\frac{dz_i(x)}{dx}) = \frac{m}{2}(z_{i+1}(x) - z_{i-1}(x)), & i = 1,2,\ldots,m-1, \\ \frac{d}{dx}(k_m(x)\frac{dz_m(x)}{dx}) = m(z_m(x) - z_{m-1}(x)), & \\ z_0(x) = w(x), & \\ z_i(1) = v_\delta(t_i), & i = 1,2,\ldots,m, \\ \frac{dz_i(1)}{dx} = v_{x,\delta}(t_i), & i = 1,2,\ldots,m, \end{cases}$$

where $z_i(x) = u(x,t_i)$, $k_i(x) = k(x,t_i)$, $i = 1,2,\ldots,m$ and v_δ, $v_{x,\delta}$ are interior temperature and heat-flux measurements contaminated by a measurement noise. The discretized system (3.19) can be reduced to the ODE system of the first order and solved numerically by any standard ODE solver. Then the exact heat source $u^+(t) = u(0,t)$ can be approximated by

$$u_m(t) = \sum_{i=0}^{m} z_i(0)\chi_i(t),$$

where $\chi_i(t)$ are piecewise linear continuous functions such that $\chi_i(t_j) = 0$, for $i \neq j$, and $\chi_i(t_i) = 1$.

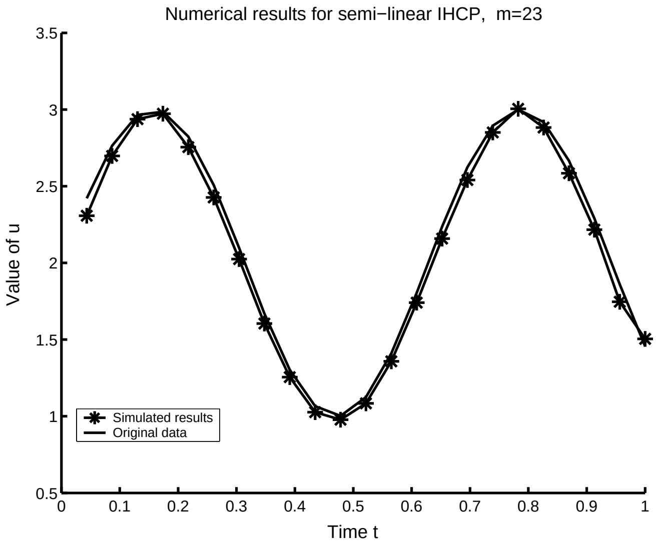

FIGURE 4

As it has been mentioned in [4], for the moment there is no stability theory, specific for the method of lines, that can be used for selecting a discretization parameter m. However, since (3.17) is known to be a severely ill-posed problem, an assumption (3.6) is appropriate, and the balancing principle described in Theorem 3.1 can be applied.

To test its viability we consider (3.17) with

$$k(x,t) = \frac{5(x^2+1)\cos 10t}{2+\sin 10t}, \quad w(x) = 2(x^2+1),$$
$$v(t) = 4(1+\sin 10t), \quad v_x(t) = 4(1+\sin 10t).$$

We are interested in the exact heat source $u^+(t) = u(0,t)$. In considered case $u^+(t) = 2 + \sin 10t$.

We apply the method of lines to noisy data $v_\delta(t_i)$, $v_{x,\delta}(t_i)$ simulated in the same way as above, but this time $\delta = 10^{-3}$. The discretization parameter takes the values $m = 1, 2, \ldots, 65$, i.e. $\overline{m} = 65$.

Using our experience with (3.17) in the constant coefficient case we take in (3.11) $p = 1.1$, but decrease v_0 up to the value 0.2, because a noise level is larger now. Using these values of parameters within the framework of a choice procedure based on the balancing principle we obtain the value $\widehat{m}_+ = 23$. Corresponding approximate solution $u_{\widehat{m}_+}(t)$ is displayed in Fig 4.

In Figs. 5 and 6 one can see the approximate solutions given by the method of lines for $m = 27$ and $m = 32$. These graphs show that for $m > 27$ the method becomes unstable. At the same time, Figs. 4-6 bear witness to reliability of the balancing principle. Using this principle one can choose a value of discretization parameter providing accurate reconstruction beside the instability regime of the reconstruction method.

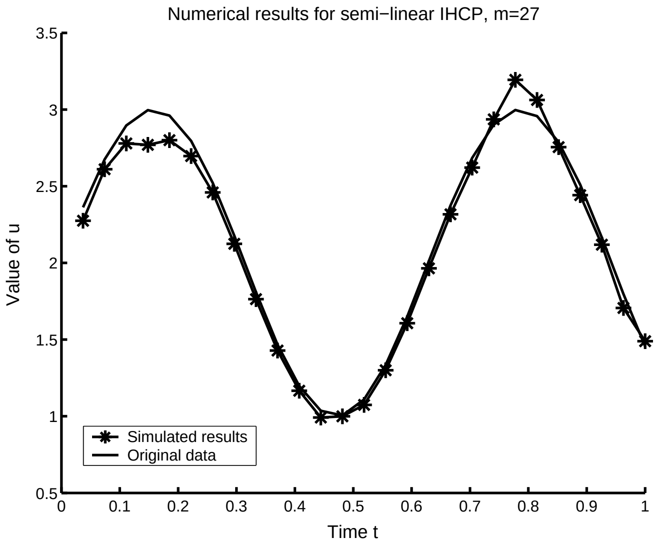

FIGURE 5

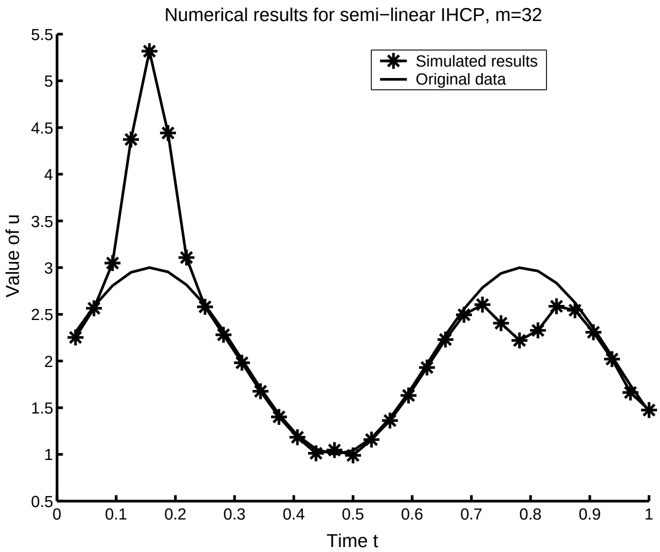

FIGURE 6

4. Application to interior penalty approximation

In this section we discuss the application of our balancing principle to interior penalty discontinuous approximation of elliptic problem

$$\begin{cases} -\nabla(a(x)\nabla u(x)) + b(x)u(x) = f(x), & x \in \Omega \\ u(x) = g(x), & x \in \partial\Omega \end{cases} \tag{4.1}$$

on non-matching grids. Here Ω is a bounded polygon in $\mathbf{R}^2$, $a(x)$ is a symmetric, uniformly positive definite and bounded coefficient. Moreover, a, b, f, g are assumed to be such that (4.1) has a unique solution $u = u^+(x)$.

The need for discretization of (4.1) on non-matching grids is motivated partially from the desire for parallel adaptive discretization methods, which is a much easier task if non-matching grids are allowed across the subdomain boundaries. A simple method for constructing such composite discretization has been proposed recently in [20]. The advantage of this method over the competing schemes is that for the original symmetric positive definite problem (4.1) it always leads to a symmetric algebraic problem. Its detailed analysis can be found in [19]. Similar scheme has been studied in [5].

4.1. **Symmetric inconsistent interior penalty Galerkin method.** To introduce a method [20](See also [5],[19]) we assume that the polygon Ω is split into Q non-overlapping shape regular polyhedrons Ω_i, $i = 1, 2, \ldots, Q$, and denote by r_{ij} the interface between Ω_i and Ω_j. We specify a "master" side of each interface r_{ij} so on r_{ij} the jump of some function u is always defined as $[u]_{r_{ij}} = u|_{\Omega_i} - u|_{\Omega_j}$, where u_{Ω_i} is a trace of $u(x)$ on r_{ij} from the Ω_i-side.

To simplify the overall exposition, we reduce our considerations to the case of homogeneous Dirichlet data, i.e. $g(x) \equiv 0$, $x \in \partial\Omega$. However, most of our numerical experiments will be done for the general case.

We define a Hilbert space $V = \bigcup_i W_2^1(\Omega_i) \bigcap \overset{\circ}{W}{}_2^1(\Omega)$ with an inner product

$$(u, w)_V := \sum_i \int_{\Omega_i} (\nabla u \cdot \nabla w + u \cdot w) dx$$

and corresponding norm $\|\cdot\|_V$. Here we use the standard notation $W_2^s(\Omega_i)$, $0 \le s < \infty$, for Sobolev spaces of functions defined in a bounded domain Ω_i. $\overset{\circ}{W}{}_2^s(\Omega)$ is a space of functions in $W_2^s(\Omega)$, which vanish on $\partial\Omega$. We approximate the original problem (4.1) with $g \equiv 0$ by the following problem in a weak interior penalty formulation: find $u \in V$ such that for all $w \in V$

$$\begin{aligned} A^\sigma_{sym}(u, w) \quad &:= \quad \sum_i \int_{\Omega_i} \left(a(x)\nabla u(x) \cdot \nabla w(x) + b(x)u(x)w(x)\right) dx \\ &+ \quad \sigma^{-1} \sum_{i,j} \int_{r_{ij}} [u]_{r_{ij}} [w]_{r_{ij}} dx = \int_\Omega f(x)w(x)dx. \end{aligned} \tag{4.2}$$

The formulation (4.2) allows discontinuous solution $u = u_\sigma$ along the interfaces r_{ij}. We have introduced a penalty term with a large parameter σ^{-1} in order to prevent a large solution jumps $[u_\sigma]_{r_{ij}}$, and to force in the limit as σ tends to zero, the continuity of the solution. It follows from [19](See also [5]) that for $u^+ \in W_2^s(\Omega)$, $3/2 < s \le 2$

$$\|u^+ - u_\sigma\|_V \le c\|u^+\|_{W_2^s(\Omega)} \sigma^{s-1} (\log \frac{1}{\sigma})^{2s-3}, \tag{4.3}$$

where the constant c is independent of σ.

Of course, u_σ becomes numerically feasible only after appropriate discretization. A discretization of (4.2) by the finite element method using meshes that generally do not align along the interfaces r_{ij} has been studied in [19]. This situation arises when each subdomain Ω_i is meshed independently of the other by a quasi-uniform and shape-regular triangulation $\mathcal{T}_i$, and consequently, the whole domain Ω has a finite element splitting $\mathcal{T} = \bigcup_i \mathcal{T}_i$. Quasi-uniformity of the mesh means that for any $\tau \in \mathcal{T}$

$$\mathrm{diam}\tau \approx \max_{\tau \in \mathcal{T}} \mathrm{diam}\tau = h.$$

Remark 4.1. If the mesh is not a quasi-uniform then it is more appropriate to replace $[u(x)]_{r_{ij}}[w(x)]_{r_{ij}}$ by $\varepsilon(x)[u(x)]_{r_{ij}}[w(x)]_{r_{ij}}$ in (4.2), where $\varepsilon(x) = [\mathrm{diam}\tau]^{-r}$, $x \in \tau \cap r_{ij}$, $r > 0$.

Let V_i be a finite element space of piecewise linear functions associated with a triangulation $\mathcal{T}_i$, and let V_Δ be a finite element space on $\mathcal{T}$ such that $V_\Delta|_{\Omega_i} = V_i$. The functions in V_Δ are , in general, discontinuous across r_{ij}, because there is no assumption that along r_{ij} the triangulations $\mathcal{T}_i$ and $\mathcal{T}_j$ produce the same mesh. However for $u, w \in V_\Delta$ all integrals in (4.2) are well-defined. Then the interior penalty finite element methods reads as: find $u \in V_\Delta$ such that (4.2) is satisfied for all $w \in V_\Delta$.

By construction the bilinear form in the right-hand side of (4.2) is symmetric and positive definite on $V_\Delta \times V_\Delta$. Therefore, the corresponding finite element system has unique solution $u = u_\sigma^\Delta \in V_\Delta$. Under the assumption that the solution u^+ of (4.1) belongs to $W_2^s(\Omega)$, $3/2 < s \leq 2$, it has been proved in [19] that

$$\|u_\sigma - u_\sigma^\Delta\|_V \leq c\|u^+\|_{W_2^s(\Omega)} \frac{h^{s-1/2}}{\sigma^{1/2}}. \tag{4.4}$$

Combining (4.3) and (4.4) one ends up with the estimate

$$\|u^+ - u_\sigma^\Delta\|_V \leq c(\sigma^{s-1}(\log\frac{1}{\sigma})^{2s-3} + \frac{h^{s-1/2}}{\sigma^{1/2}})\|u^+\|_{W_2^s(\Omega)}.$$

In the right-hand side of this estimate one can see that letting $\sigma \sim h$ would give a balance for all terms up to log-factor. On the basis of this observation in [19] it has been suggested to choose the penalty parameter σ in proportion to a global mesh-size parameter h. However, numerical experiments reported in [19] and [20] show that for $\sigma = h^{3/2}$ V-norm error $\|u^+ - u_\sigma^\Delta\|_V$ is smaller than for $\sigma = h$ and it is fairly constant for σ ranging from $h^{3/2}$ to $h^{5/2}$. These results are in a good agreement with numerical experiments presented in [11] for the boundary penalty method, where it has also been observed that the error in energy norm, which is similar to V-error, is not sensitive to decreasing a penalty parameter $\sigma < h$ (See Figure 1 in [11]). In [11] it has been explained by the fact that a stability form $\|u_\sigma - u_\sigma^\Delta\|$ is usually overestimated due to the inequalities used to derive it. At the same time, numerical experiments reported in [19] for L_∞-norm clearly indicate an ever increasing value of the error as σ tends to zero. It hints at L_∞-stability in the form of Assumption 2.2.

There is no theory available for the interior penalty discontinuous approximation in L_∞, with the exception of the fact that from (4.3) and the embedding $V \hookrightarrow L_\infty(\Omega)$ one can derive a convergence $\|u^+ - u_\sigma\|_{L_\infty} \to 0$ as σ tends to zero. In this situation it is reasonable to apply our balancing principle for the choice of penalty parameter, because it mainly relays on a qualitative information such as convergence and stability, but does not require the knowledge of the convergence rate or the exact power v in the stability estimations.

We present 4 numerical examples to verify the theoretical results. As in [19] we consider (4.1) with $a(x) \equiv 1$, $b(x) \equiv 0$ and $\Omega = [0,1] \times [0,1]$. The domain Ω

is split into four equal subdomains $\Omega_i = [\frac{l-1}{2}, \frac{l}{2}] \times [\frac{m-1}{2}, \frac{m}{2}]$, $i = 2(l-1)+m$, $l, m = 1, 2$, that are triangulated independently so that the meshes do not match along the interface $\{x = (x_1, x_2) : x_1 = 1/2 \vee x_2 = 1/2\}$. More precisely, let $M_R(l_1, l_2) = \{x_{t,\tau} = (x_{1,t}, x_{2,\tau}) : x_{1,t} = \frac{l_1-1}{2} + \frac{t}{2R}, x_{2.\tau} = \frac{l_2-1}{2} + \frac{\tau}{2R}\}_{t,\tau=0}^{R}$. Then the subdomain $\Omega_1 \cup \Omega_4$ is meshed by the grid $M_{15}(1,1) \cup M_{15}(2,2)$, while in the subdomain $\Omega_2 \cup \Omega_3$ we use the grid $M_{10}(1,2) \cup M_{10}(2,1)$.

To apply a general balancing principle (2.11) in the considered situation we should specify the level of unavoidable error Δ, and the constants k_0, k, d in the Assumption 2.2. Moreover, we should choose two design set Σ_N and $\{v_i\}$.

As to Δ, it is determined by the best possible order of piecewise linear approximation on a given grid. We choose $\Delta = (1/30)^2$. The reason to take here the minimal mesh-size $h_{\min} = 1/30$ instead of the maximal $h_{\max} = 1/20$ has the following explanation. Non-uniform grids arise usually after adaptive discretization that presumes the same order of the error of piecewise linear approximation in subdomains triangulated by fine and by coarse grids. Then the estimation of the global error order is given by Approximation Theory in terms of a mesh-size $h_{\min}$ of the finest grids as $h_{\min}^2$.

A design set $\{v_i\}$ should be sufficiently representative. For our tests we choose it as $\{v_i = \frac{i+1}{11}\}_{i=0}^{6}$. Concerning the set $\Sigma_N = \Sigma_N(q) = \{\sigma_n = \sigma_0 q^n, n = 0, 1, \ldots, N\}$ we note that only σ_0 has an influence on the threshold value in (2.11), and it should be chosen small enough. In our calculations we use $\sigma_0 = 10^{-9}$ and three sets $\sum_{50}(1.5)$, $\sum_{40}(1.7)$, $\sum_{33}(1.9)$. The constants k_0, k, d in Assumption 2.2 are usually fixed on the base of computational experience. One possible way to specify these constants is the application of the balancing principle to the problem (4.1) with a known solution u^+. In this case one can try several values of k_0, k, d and calculate the error $\|u^+ - u_{\sigma_+}^{\Delta}\|_{L_\infty}$, $\sigma_+ = \sigma_+(k_0, k, d)$, exactly. Then the value k_0, k, d corresponding to the smallest error can be used for the application of balancing principle to the problem (4.1) with unknown solution, but with the same or similar coefficients and domain Ω.

Taking (4.1) with the solution $u^+(x) = u^+(x_1, x_2) = x_1 + x_2$, and $a(x) \equiv 1$, $b(x) \equiv 0$ as such calibrating problem we fix the constants in the Assumption 2 as follows: $k_0 = 3/4$, $k = 1$, $d = 0$.

The application of the balancing principle (2.11) with parameters specified above gives the results presented in Table 3.

For considered problems the following values of the penalty parameter σ_+ have been selected in accordance with a balancing principle (2.11): $\sigma_+ = 0.0111 \in \Sigma_{50}(1.5)$, $\sigma_+ = 0.0139 \in \Sigma_{40}(1.7)$, $\sigma_+ = 0.0093 \in \Sigma_{33}(1.9)$.

The error surface for one of the approximate solutions is displayed in Fig. 7. Note that the problem 3 with $u^+(x_1, x_2) = x_1^2 - x_2^2$ has been also used as a test example in [19], where the same subdomains Ω_i, $i = \overline{1,4}$, have been meshed by more fine non-matching girds with 833 nodes. It has been reported in [19] that a penalty parameter $\sigma_h = h = 0.04$ suggested as an optimal one for approximation in V-norm leads to L_∞-error 0.13809. Moreover, from [19] it follows that the interior penalty approximations with penalty parameters $\sigma_1 = 0.2\sigma_h = 0.008$, $\sigma_2 = 0.1\sigma_h = 0.004$, $\sigma_3 = 10^{-3}\sigma_h = 4 \cdot 10^{-5}$ produce in L_∞ the errors with norms 0.03429, 0.01859 and 0.02523 respectively. At the same time, the Table 3 shows that in considered case our balancing principle automatically selects a value of penalty parameter leading to a smaller error even for a coarser grid.

TABLE 3. Numerical results for four problems (4.1).

Problems Number	Solution $u^+(x_1, x_2)$	L_∞-error for $\sigma_+ \in \Sigma_{50}(1.5)$	L_∞-error for $\sigma_+ \in \Sigma_{40}(1.7)$	L_∞-error for $\sigma_+ \in \Sigma_{33}(1.9)$	v
1	$x_1 + x_2$	0.0096	0.0115	0.0082	2/11
2	$x_1^2 + x_2^2$	0.0098	0.0118	0.0085	2/11
3	$x_1^2 - x_2^2$	0.0091	0.0112	0.0077	2/11
4	$50 \sin^2 \frac{\pi x_1}{6} \sin^2 \frac{\pi x_2}{5}$	0.0286	0.0336	0.0252	4/11

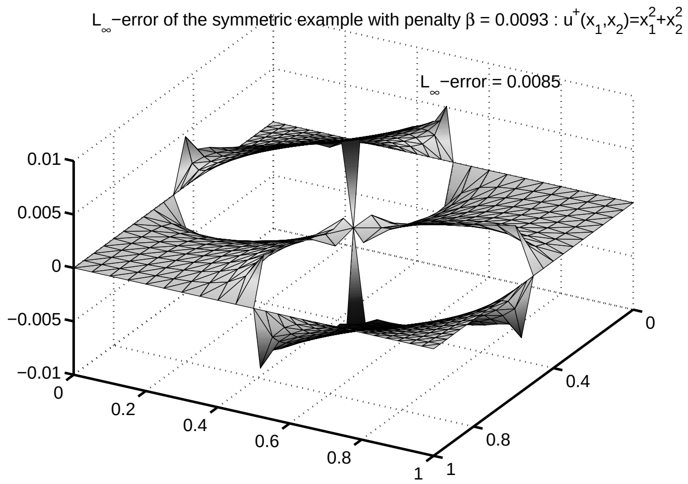

FIGURE 7

4.2. **Consistent and non-symmetric interior penalty scheme.** Observe that the numerical method discussed above is non-consistent, because the exact solution of (4.1) does not satisfy the formulation (4.2). This inconsistency can be seen as a payment for the symmetry of the bilinear form (4.2). To get a consistent, but non-symmetric interior penalty scheme one can consider a bilinear form

$$\begin{aligned} A^\sigma_{nonsym}(u, w) \quad := \quad & A^\sigma_{sym}(u, w) - \sum_{i,j} \int_{r_{ij}} \{a(x)\nabla u(x) \cdot n_{ij}(x)\}_{r_{ij}} [w]_{r_{ij}} dx \\ + \quad & \sum_{i,j} \int_{r_{ij}} \{a(x)\nabla w(x) \cdot n_{ij}(x)\}_{r_{ij}} [u]_{r_{ij}} dx \\ - \quad & \int_{\partial\Omega} (a(x)\nabla u(x) \cdot n_{\partial\Omega}(x)) w(x) dx \\ + \quad & \int_{\partial\Omega} (a(x)\nabla w(x) \cdot n_{\partial\Omega}(x)) u(x) dx, \end{aligned}$$

and a linear form

$$L(w) = \int_\Omega f(x)w(x)dx + \int_{\partial\Omega} (a(x)\nabla w(x) \cdot n_{\partial\Omega}(x)) g(x) dx,$$

where n_{ij} is a unit normal vector associated with the edge r_{ij} which is exterior to Ω_i, $n_{\partial\Omega}$ is taken to be the unit outward vector normal to $\partial\Omega$, and

$$\{v\}_{r_{ij}} = \frac{1}{2}(v|_{\Omega_i})|_{r_{ij}} + \frac{1}{2}(v|_{\Omega_j})|_{r_{ij}}.$$

The fact that the following weak interior penalty formulation: find $u \in V$ such that for all $w \in V$

$$A^{\sigma}_{nonsym}(u, w) = L(w), \tag{4.5}$$

is consistent with the problem (4.1) has been shown in [25]. Corresponding discrete problem for the approximation u^{Δ}_{σ} to the solution u_σ of (4.5) thus reads: find $u \in V_\Delta$ such that (4.5) is satisfied for all $w \in V_\Delta$.

Note that a consistency of the formulation (4.5) makes a difference in the application of the balancing principle to the non-symmetric interior penalty Galerkin method. Indeed, from Lemma 2.1 [25] it follows that the solution $u = u^+$ of (4.1) satisfies (4.5), and the converse is also true if u^+ is smooth enough. It means that in (1.1) $u_\sigma = u^+$, and $\varphi(\sigma) \equiv 0$. Then a straightforward application of the balancing principle, as it is described in the Subsection 4.1, will look like an attempt to balance $\|u_\sigma - u^{\Delta}_{\sigma}\| = \|u^+ - u^{\Delta}_{\sigma}\|$ with zero, that, of course, does not make sense.

The way out of this situation is related with a decomposition

$$\|u^+ - u^{\Delta}_{\sigma}\| \le \|u^+ - \widetilde{u}_\Delta\| + \|\widetilde{u}_\Delta - u^{\Delta}_{\sigma}\|. \tag{4.6}$$

Following [25] we can use here the interpolant $\widetilde{u}_\Delta \in V_\Delta \cap C(\Omega)$ approximating $u^+ \in W^s_2(\Omega)$ in L_2-norm with an accuracy $O(h^s)$, $0 < s \le 2$. Moreover, using the arguments from the proof of Theorem 3.1 and Theorem 3.2 [25] one can observe that a constant $c = c(\frac{1}{\sigma})$ in the estimation for the error $\widetilde{u}_\Delta - u^{\Delta}_{\sigma}$ increases with a penalty weight σ^{-1} as $O(\sigma^{-v})$. It hints at the estimate of the form

$$\|\widetilde{u}_\Delta - u^{\Delta}_{\sigma}\| \le k\frac{\Delta}{\sigma^v}, \tag{4.7}$$

but v here is unknown.

Note that dealing with a consistent interior penalty Galerkin method one expects to obtain an accuracy of the same order as $\|u^+ - \widetilde{u}_\Delta\|$. On the other hand, the inequalities (4.6), (4.7) can provide the estimate of a right order only when $\Delta = o(\|u^+ - \widetilde{u}_\Delta\|)$, because a penalty parameter σ also goes to zero as $h \to 0$. Keeping in mind that for a piecewise linear interpolant $\widetilde{u}_\Delta$, $\|u^+ - \widetilde{u}_\Delta\|$ cannot be better than $O(h^2_{\min})$ one can take $\Delta = h^s_{\min}$ with $s > 2$. A design parameter s regulates the closeness of u^{Δ}_{σ} to the best possible approximation $\widetilde{u}_\Delta$. Our numerical experiments with a calibrating problem suggest the value $s = 3.5$ (for $s > 3.5$ the value σ balancing $\|u^+ - \widetilde{u}_\Delta\|$ with $\|\widetilde{u}_\Delta - u^{\Delta}_{\sigma}\|$ becomes too small).

Observe that the estimate (4.6), (4.7) has a form (1.3) that is necessary for applying the balancing principle, but this time $\varphi(\sigma) \equiv$ const. Such a function meets the Assumption 2.4. As to the Assumption 2.1, it is also satisfied, except for (2). Examining the proof of Theorem 2.3 one can see that it is still valid for such functions. Thus, in considered case the balancing principle can be also used.

To demonstrate it we consider (4.1) with $a(x) = b(x) \equiv 1$. The domain Ω and its splitting into the subdomains are the same as in Subsection 4.1. Moreover, in the beginning the subdomains are meshed by the same grids, i.e. now $\Delta = (30)^{-3.5} \simeq 0.67 \cdot 10^{-5}$.

TABLE 4. Numerical results for non-symmetric interior penalty scheme

u^+	$\sigma_+ \in \sum_{50}(1.5)$	L_2 error	$\sigma_+ \in \sum_{40}(1.7)$	L_2 error	$\sigma_+ \in \sum_{33}(1.9)$	L_2 error	v
$x_1^2 - x_2^2$	0.0111	$2.052 \cdot 10^{-4}$	0.0237	$2.044 \cdot 10^{-4}$	0.0093	$2.051 \cdot 10^{-4}$	$\frac{49}{132}; \frac{50}{132}$
$x_1^2 + x_2^2$	0.0166	$5.874 \cdot 10^{-4}$	0.0139	$5.886 \cdot 10^{-4}$	0.0093	$5.915 \cdot 10^{-4}$	$\frac{46}{132}$

Using old design set $\{v_i = \frac{i+1}{11}\}_{i=0}^{6}$ we have observed comparatively large differences between corresponding values $\sigma(v_i)$ in (2.7). It suggests to switch to more fine set $\{v_i\}$. Therefore, in the sequel we use a design set $\{v_i = \frac{i+40}{132}\}_{i=0}^{30}$. All other parameters are the same as in Subsection 4.1.

In Table 4 we present the results of numerical experiments with the non-symmetric interior penalty Galerkin method based on the formulation (4.5), where a penalty parameter σ is chosen in accordance with the balancing principle (2.11). These results correspond to the case when L_2-norm is used within the framework of the procedure generating the sequence (2.7). The errors there are also measured in L_2-norm. The results for L_∞-norm look similar, the order of L_∞-error is also 10^{-4}. Only the values of the power v in the stability estimation (4.7) detected by the balancing principle for L_∞-norm are larger than for L_2-norm. For example, in the case of L_∞-norm and $u^+(x_1,x_2) = x_1^2 - x_2^2$, $\sigma_+ = 0.0111$ is also selected from $\sum_{50}(1.5)$, but this time it corresponds to $v_+ = 67/132$, while in the case of L_2-norm it was $v_+ = 49/132$. This difference can be easily explained by the fact that L_∞-norm is stronger than L_2-norm, i.e.

$$\|u^+ - u^\Delta_{\sigma_+}\|_{L_2} \le \frac{\Delta}{\sigma_+^{49/132}} < \|u^+ - u^\Delta_{\sigma_+}\|_{L_\infty} \le \frac{\Delta}{\sigma_+^{67/132}}.$$

Using Table 4 one can check that the value $\sigma = \sigma_+$ chosen in accordance with (2.11) really balance the stability bound $k\frac{\Delta}{\sigma^v}$ from (4.7) with the error $\|u^+ - u^\Delta_\sigma\|$. For example, in the case $u^+(x_1,x_2) = x_1^2 - x_2^2$, $\sigma = \sigma_+ = 0.0237 \in \sum_{40}(1.7)$, $v_+ = 50/132$, $\Delta/\sigma^v \simeq 2.8 \cdot 10^{-4}$, that is in a good agreement with $\|u^+ - u^\Delta_\sigma\| \simeq 2.0 \cdot 10^{-4}$.

Another interesting observation is related with a common belief that for consistent interior penalty methods a penalty parameter should have a form $\sigma = \beta^{-1} h_{\min}$, where β does not depend on a mesh-size. Our numerical experiments with the balancing principle support it. As an example, we consider a problem with the exact solution $u^+(x_1,x_2) = x_1^2 - x_2^2$ and choose two grids $G_1 = M_{15}(1,1) \cup M_{15}(2,2) \cup M_{10}(1,2) \cup M_{10}(2,1)$, with $h_{\min} = 1/30$, and $G_2 = M_{10}(1,1) \cup M_{10}(2,2) \cup M_5(1,2) \cup M_5(2,1)$, with $h_{\min} = 1/20$. The value of a penalty parameter chosen in accordance with the balancing principle for consistent interior penalty Galerkin approximation based on a coarse grid G_2 is $\sigma_+ = 0.0177 \in \sum_{33}(1.9)$. If one represents it as $\sigma = \beta^{-1} h_{\min}$, $h_{\min} = 1/20$, then $\beta^{-1} = 0.354$. Now the value of a penalty parameter for a fine grid G_1 with $h_{\min} = 1/30$ can be calculated as $\sigma = \beta^{-1} h_{\min} = 0.354/30 = 0.0118$. It is in a good agreement with $\sigma_+ = 0.0093 \in \sum_{33}(1.9)$, or $\sigma_+ = 0.0111 \in \sum_{50}(1.5)$, chosen in accordance with the balancing principle for consistent interior penalty method based on the grid G_1 (see Table 4).

In conclusion we compare the non-symmetric interior penalty Galerkin method discussed above with the discontinuous Galerkin approximation without penalty

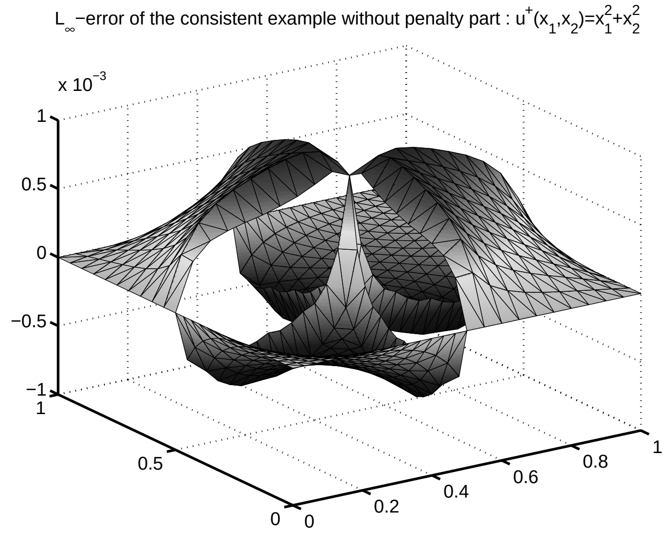

FIGURE 8

introduced in [2], [23]. In the present context the latter approximation can be seen as $u = u_\infty^\Delta \in V_\Delta$ such that (4.5) with $\sigma = \infty$ is satisfied for all $w \in V_\Delta$. The fact that this scheme is also consistent with the problem (4.1) has been shown in [2]. The error surface $|u^+(x_1,x_2) - u_\infty^\Delta(x_1,x_2)|$ for u_∞^Δ approximating the solution $u^+(x_1,x_2) = x_1^2 + x_2^2$ is displayed in Fig. 8. At the same time, in Fig. 9 one can see the error surface $|u^+(x_1,x_2) - u_{\sigma_+}^\Delta(x_1,x_2)|$, where $u = u_{\sigma_+}^\Delta(x_1,x_2)$ satisfies the formulation (4.5) for all $w \in V_\Delta$ and $\sigma = \sigma_+ = 0.0093 \in \sum_{33}(1.9)$ selected in accordance with the balancing principle (see Table 4). These pictures show that $u_{\sigma_+}^\Delta$ is not only more accurate, but it is also much more smooth.

Thus, presented numerical experiments confirm our theoretical results and show that a combination of balancing principle with interior penalty discontinuous approximation on non-matching grids allows to improve the performance of this method.

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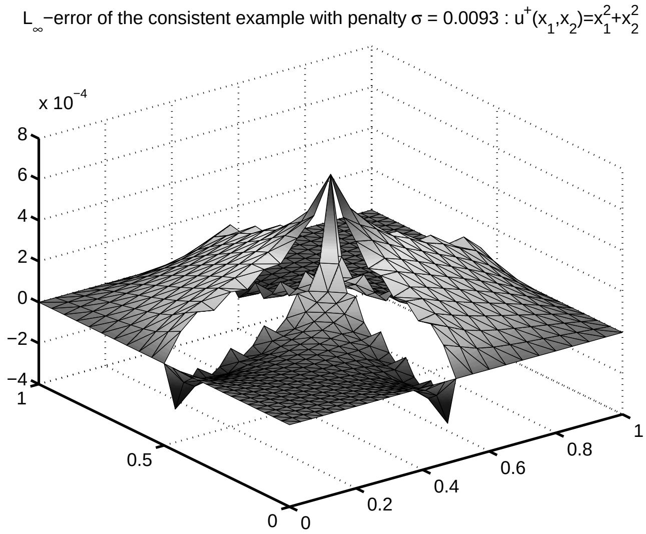

FIGURE 9

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