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RICAM-Report 2005-22

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# Domain Decomposition Preconditioning for Elliptic Problems with Jumps in Coefficients

Sungmin Cho * S. V. Nepomnyaschikh† Eun-Jae Park ‡

**Abstract**

In this paper, we propose an effective iterative preconditioning method to solve elliptic problems with jumps in coefficients. The algorithm is based on the additive Schwarz method (ASM). First, we consider a domain decomposition method without 'cross points' on interfaces between subdomains and the second is the 'cross points' case. In both cases the main computational cost is an implementation of preconditioners for the Laplace operator in whole domain and in subdomains. Iterative convergence is independent of jumps in coefficients and mesh size.

## 1 Introduction

In this paper we suggest a technique of constructing effective preconditioning operators for elliptic problems with jumps in the coefficients. We design a preconditioning operator for the following elliptic equation [4]

$$\begin{cases} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x)u = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Gamma, \end{cases} \tag{1}$$

where $\Omega$ is a bounded and polygonal domain with the boundary $\Gamma$.
The following case is considered n section 2. Let $\Omega$ be a union of $n+1$ nonoverlapping subdomains $\Omega_i$, such that

$$\overline{\Omega} = \bigcup_{i=0}^{n} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j.$$

Here we have the polygonal subdomains $\Omega_i$ in the interior of $\Omega$. Their boundaries are given by $\Gamma_i$, $i = 1, ..., n$ such as $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$. The domain $\Omega_0$ is defined to be multiply connected having the boundary $\Gamma \bigcup (\bigcup_{i=1}^{n} \Gamma_i)$. We assume that

$$\operatorname{diam}(\Omega_i) \leq \alpha_0 H_i, \quad \text{where} \quad 0 < H_i \leq 1, \quad i = 1, \cdots, n,$$

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and $\alpha_0$ is a constant independent of the parameter $H_i$. Furthermore, for any subdomain $\Omega_i$ if there exists a subdomain $\Omega_j$ such that

$$\mathrm{dist}(\Omega_i, \Omega_j) \le \alpha_1 H_i$$

holds, then the conditions

$$H_j = O(H_i), \quad \text{and} \quad \alpha_2 H_i \le \mathrm{dist}(\Omega_i, \Omega_j)$$

must be fulfilled, where $\alpha_1$ and $\alpha_2$ are constants which are independent of $H_i, i = 1, ..., n$. For the problem (1), introduce the bilinear form,

$$a(u, v) = \int_\Omega \left( \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x) uv \right) dx.$$

We assume that the coefficients of the problem (1) are such that $a(u, v)$ is a symmetric bilinear form in the Sobolev space $H_0^1(\Omega)$. Let the inequalities

$$\alpha_3 a(u, u) \le \int_\Omega \epsilon(x) |\nabla u|^2 \le \alpha_4 a(u, u), \quad \forall u \in H_0^1(\Omega)$$

be fulfilled with positive constants $\alpha_3$, $\alpha_4$, which are independent of $\epsilon$. Here we fix

$$\epsilon(x) = \mathrm{const} = \epsilon_i, \quad \forall x \in \Omega_i,$$

where we have

$$\epsilon_0 = 1, \quad 0 < \epsilon_i \le 1, \quad i = 1, ..., n. \tag{2}$$

In section 3, we consider elliptic problems with cross points on $\cup_{i=1}^n \partial\Omega_i$ when $\Omega$ consists of nonoverlapping subdomains $\Omega_i$ which is independent of a small parameter $H_i$ and $\bar{\Omega} = \cup_{i=1}^n \bar{\Omega}_i$. We assume that there exist constants $\alpha_5$, $\alpha_6$ which are independent of coefficient $p$ such that

$$\alpha_5 a(u, u) \le \int_\Omega p(x)(\, |\nabla u|^2 + |u|^2 \,)\, dx \le \alpha_6 a(u, u), \quad \forall u \in H_0^1(\Omega), \tag{3}$$

where $p(x) = p_i = \mathrm{const} > 0, \quad x \in \Omega_i$.

**Remark 1.1** *Results of this paper are easy generalized for the case of coefficients such that*

$$\alpha_5' a(u, u) \le \int_\Omega (p(x)|\nabla u|^2 + q|u|^2) dx \le \alpha_6' a(u, u), \quad \forall u \in H_0^1(\Omega) \tag{4}$$

*where $p(x) \le q \equiv$ constant for any $x \in \Omega$. The assumption (4) is typical, for instance, for parabolic problems.*

The weak formulation of (1) is given as follows : Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = l(v), \quad \forall v \in H_0^1(\Omega), \tag{5}$$

where $l(v)$ is the linear functional

$$l(v) = \int_\Omega f(x) v dx.$$

In section 2 we use some of the ideas suggested in [10] for the first case domain decompositions, i.e., without cross points and explain this technique. The algorithms, suggested in this section are very simple and do not use explicit extension operators, exact solvers in subdomains, and hierarchical structure of grids.
The cross points case were considered in [2, 12, 14], but suggested methods are not optimal for arbitrary distribution of coefficients. An optimal preconditioner based on domain decomposition technique for elliptic problems with jumps in the coefficients and cross points on interfaces between subdomains was suggested in [8], but an implementation of this algorithm is rather complicated in practice. In section 3, we suggest the algorithms, using the same idea as in section 2.
To demonstrate the main idea of construction of preconditioning operators for problems with cross points, we consider the model examples when $\Omega$ is a rectangular domain which is decomposed into four subdomains $\bar{\Omega} = \bigcup_{i=1}^{4} \bar{\Omega}_i$ according to Figure 1, and coefficient $p(x) = p_i = \text{constant} > 0$ in $\Omega_i$, including the so-called chess case, and suggest optimal algorithms for arbitrary distribution of coefficients.
In section 4, numerical results are presented for both cases without cross points and with cross points.

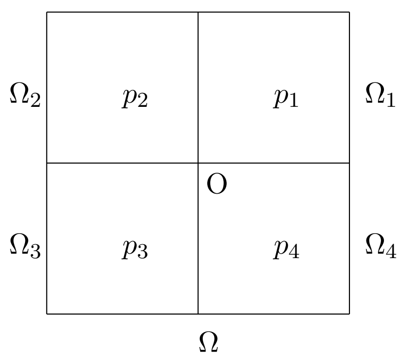



Figure 1: The domain with cross point O.

## 2 Preconditioning to the Problem without Cross Points

Let $\Omega^h = \bigcup_{i=0}^{n} \Omega_i^h$ be a quasiuniform triangulation of the domain $\Omega$ which can be characterized by a parameter $h$. Define $W$ be the space of real–valued continuous functions being linear on the triangles of the triangulation $\Omega^h$. Using the finite element method, the variational formulation (5) can be transferred to the linear algebraic equation

$$Au = f, \tag{6}$$

where the matrix $A$ is such as

$$(Au, v) = a(u^h, v^h), \quad \forall u^h, v^h \in W.$$

Here vectors $u, v$ correspond to $u^h, v^h \in W$ and in evident cases we identify vectors $u$ and functions $u^h$. The condition number of the matrix $A$ depends on $h, H_i, \epsilon_i$ and can be large. Our goal is the design of a preconditioner $B$ for the problem (6) such that the following inequalities are valid:

$$c_1(Bu,u) \leq (Au,u) \leq c_2(Bu,u), \quad \forall u \in R^N.$$

Here $N$ is the dimension of $W$, the positive constants $c_1, c_2$ are independent of $h, H_i$, $\epsilon_i$, and the action of $B^{-1}$ on a vector can be implemented at low cost. We introduce the following notation $X \preceq Y$ which means that there exists a constant $c$ such that

$$X \leq cY,$$

where $c$ is independent of "bad" parameters as $h, H_i, \epsilon_i$. The goal of this section is the design of a domain decomposition preconditioning operator for the problem (1) without using of the extension operator from $\cup_{i=1}^n \partial\Omega_i$ into $\Omega$ and exact solvers in the subdomains $\Omega_i$.

Define the restriction operator $r_{\Omega_i} : W \to W_{\Omega_i}$,

$$(r_{\Omega_i} u^h)(x) = u^h(x), \quad x \in \Omega_i, \quad u^h \in W,$$

and denote $W_{\Omega_i} = r_{\Omega_i} W, \quad i = 0, 1, \cdots, n.$
Let $C_{\Omega_i},\ i = 0, 1, ..., n$ be preconditioning operators in the finite element subspaces of $H^1(\Omega_i)$. Hence, we have : for any $u^h \in W_{\Omega_i} \cap H_0^1(\Omega_i),\ i = 1, 2, \cdots, n$ and $u^h \in W_{\Omega_0}$,

$$\|u^h\|^2_{H^1(\Omega_i)} \preceq (C_{\Omega_i} u, u) \preceq \|u^h\|^2_{H^1(\Omega_i)}. \tag{7}$$

For instance, these operators $C_{\Omega_i}$ can be constructed using the fictitious space lemma in [9], [11], [15]. We extend the operator $C_{\Omega_i}$ by zero outside of $\Omega_i$ and denote by $\check{C}^+_{\Omega_i}$ pseudo-inverse operator for this extension of $C_{\Omega_i}$. And let $C$ be a preconditioning operator in the finite element space $W$ :

$$\|u^h\|^2_{H^1(\Omega)} \preceq (Cu, u) \preceq \|u^h\|^2_{H^1(\Omega)}, \quad \forall u^h \in W.$$

Denote

$$B_{ov}^{-1} = C^{-1} + \frac{1}{\epsilon_1}\check{C}^+_{\Omega_1} + ... + \frac{1}{\epsilon_n}\check{C}^+_{\Omega_n},$$

where $\check{C}^+_{\Omega_i}$ is from (7). The following theorem fulfills.

**Theorem 2.1** *The following inequalities hold*

$$(B_{ov}u, u) \preceq (Au, u) \preceq (B_{ov}u, u), \quad \forall u \in R^N.$$

To prove this, first we introduce the nonoverlapping preconditioner $B_{nov}^{-1}$, which uses extension operators. And then, using this preconditioner, we prove Theorem 2.1 in the end of this section.
Let us decompose $W$ into a sum of subspace, $W = W_0 + W_1$ and use Additive Schwarz Method (ASM) [5, 6, 11, 13]. For convenience, we give the statements of the following general lemmas [7] in order to explain using of the ASM in this paper.

**Lemma 2.1** *Let the Hilbert space $Q$ with the scalar product $(\cdot,\cdot)$ be decomposed into a vector sum of subspaces*

$$Q = Q_1 + Q_2 + \cdots + Q_m,$$

*$A : Q \to Q$ be a linear, self-adjoint, bounded, and positive definite operator, $P_i,\ i = 1, 2, \cdots, m$, be operators of orthogonal projection of $Q$ onto $Q_i$ with respect to the scalar product $(\cdot,\cdot)_A$ generated by the operator $A$*

$$(u, v)_A = (Au, v).$$

*Assume that positive constants $\alpha$ and $\beta$ exist such that for any element $p \in Q$ there exits $p_i \in Q_i$ such that*

$$p_1 + p_2 + \cdots + p_m = p,$$

$$\alpha((p_1, p_1)_A + (p_2, p_2)_A + \cdots (p_m, p_m)_A) \leq (p, p)_A,$$

$$((P_1 + P_2 + \cdots + P_m)p, p)_A \leq \beta(p, p)_A.$$

*Also, let operators $B_i, i = 1, 2, \cdots, m$, self-adjoint in $Q$ be determined such that $ImB_i = Q_i$, where $Im\ B_i = \{q \in Q_i \mid q = B_i p,\ \forall p \in Q\}$,*

$$\check{c}(B_i p, p) \leq (Ap, p) \leq \hat{c}(B_i p, p), \quad \forall p \in Q_i.$$

*Then,*

$$\alpha\check{c}(A^{-1}p, p) \leq (B^{-1}p, p) \leq \beta\hat{c}(A^{-1}p, p), \quad \forall p \in Q,$$

*where $B^{-1} = B_1^+ + B_2^+ + \cdots + B_m^+$, and $B_i^+$ is a pseudo-inverse operator for $B_i$.*

**Lemma 2.2** *Let $\Lambda$ and $Q$ be Hilbert spaces and $\Sigma$ and $S$ be self-adjoint, positive operators in $\Lambda$ and $Q$, respectively. Denote by*

$$(\lambda, \mu)_\Sigma = (\Sigma\lambda, \mu), \quad (p, q)_S = (Sp, q)$$

*both scalar products generated by $\Sigma$ and $S$, and let $t$ be a linear operater acting from $\Lambda$ into $Q$ such that*

$$\alpha(\lambda, \lambda)_\Sigma \leq (t\lambda, t\lambda)_S \leq \beta(\lambda, \lambda)_\Sigma, \quad \forall\lambda \in \Lambda.$$

*Here $\alpha, \beta$ are positive constants. Set*

$$C^+ = t\Sigma^{-1}t^*,$$

*where $t^*$ is adjoint to the operator $t$ with respect to basic scalar products in $\Lambda$ and $Q$. Then,*

$$\alpha(Cp, p) \leq (p, p)_S \leq \beta(Cp, p), \quad \forall p \in Im\ t.$$

We divide the nodes of the triangulation $\Omega^h$ into two groups: those which lie inside of $\Omega_i^h, i = 1, ..., n$ and those which lie on $\overline{\Omega}_0^h$. The subspace $W_0$ corresponds to the first set. Let

$$\gamma = \bigcup_{i=1}^{n} \partial\Omega_i^h,$$

$$W_0 = \left\{u^h \in W | \quad u^h(x) = 0, \quad x \in \overline{\Omega}_0^h\right\},$$

$$W_{0,i} = \{u^h \in W_0 | \quad u^h(x) = 0, \quad x \notin \Omega_i^h\}, \quad i = 1, 2, \ldots, n.$$

It is clear that $W_0$ is the direct sum of the orthogonal subspaces $W_{0,i}$ with respect to the scalar product in $H_0^1(\Omega)$ :

$$W_0 = W_{0,1} \oplus \cdots \oplus W_{0,n}.$$

The subspace $W_1$ corresponds to the second group of nodes $\Omega^h$ and can be defined in the following way. First, define $V$ which is the space of traces of functions from $W$ on $\gamma$ :

$$V = \{\varphi^h | \quad \varphi^h(x) = u^h(x), \quad x \in \gamma, \quad u^h \in W\},$$

and the trace operator $r_\gamma : W \to V$,

$$(r_\gamma u^h)(x) = \varphi^h(x), \quad x \in \gamma.$$

To define $W_1$, let us use a norm-preserving extension into the subdomains $\Omega_i, \ i = 1, 2, \cdots, n$ of function $\varphi^h$ from the space $V$. Denote this operator by $t_\gamma$ (for instance $t_\gamma$ can be defined as the harmonic extension operator), then we have

$$\sum_{i=1}^{n} \|t_\gamma \varphi^h\|^2_{H^1(\Omega_i)} \preceq \|\varphi^h\|^2_{H^{\frac{1}{2}}(\gamma)}, \quad \forall \varphi^h \in V, \tag{8}$$

where

$$\|\varphi^h\|^2_{H^{\frac{1}{2}}(\gamma)} = \sum_{i=1}^{n} \|\varphi^h\|^2_{H^{\frac{1}{2}}(\Gamma_i)},$$

$$\begin{aligned}
\|\varphi^h\|^2_{H^{1/2}(\Gamma_i)} &= H_i \|\varphi^h\|^2_{L_2(\Gamma_i)} + |\varphi^h|^2_{H^{1/2}(\Gamma_i)}, \\
\|\varphi^h\|^2_{L_2(\Gamma_i)} &= \int_{\Gamma_i} (\varphi^h(x))^2 dx, \\
|\varphi^h|^2_{H^{1/2}(\Gamma_i)} &= \int_{\Gamma_i} \int_{\Gamma_i} \frac{(\varphi^h(x) - \varphi^h(y))^2}{|x-y|^2} dx dy.
\end{aligned}$$

Now we can define subspace $W_1$

$$\begin{aligned}
W_1 = \{u^h | \ u^h(x) &= (t_\gamma \varphi^h)(x), \ x \in \Omega_i, \ i = 1, ..., n, \ \varphi^h(x) = v^h(x), \ x \in \gamma, \\
u^h(x) &= v^h(x), \quad x \in \Omega_0^h, \quad \forall \, v^h \in W\}.
\end{aligned}$$

It is obvious that $W = W_0 + W_1$, and this decomposition of the space $W$ is stable in the following sense.

**Lemma 2.3** *For any function $u^h \in W$ there exist $u_i^h \in W_i, \ i = 0, 1,$ such that*

$$u_0^h + u_1^h = u^h,$$

$$a(u_0^h, u_0^h) + a(u_1^h, u_1^h) \preceq a(u^h, u^h).$$

*Proof* :
In our problem,

$$a(u_0^h, u_0^h) = \int_{\Omega_0} (|u_0^h|^2 + |\nabla u_0^h|^2) dx + \sum_{i=1}^{n} \epsilon_i \int_{\Omega_i} (|u_0^h|^2 + |\nabla u_0^h|^2) dx.$$

Because of $0 < \epsilon \leq 1$, we have

$$a(u_0^h, u_0^h) \leq \|u_0^h\|^2_{H^1(\Omega)}.$$

And by the trace lemma for $H^1(\Omega)$ and the inequality (8)

$$\begin{aligned} \|u_0^h\|^2_{H^1(\Omega)} &= \|u^h\|^2_{H^1(\Omega_0)} + \sum_{i=1}^{n} \|t_\gamma \varphi^h\|^2_{H^1(\Omega_i)} \\ &\preceq \|u^h\|^2_{H^1(\Omega_0)} + \|\varphi^h\|^2_{H^{\frac{1}{2}}(\gamma)} \\ &\preceq \|u^h\|^2_{H^1(\Omega_0)}. \end{aligned}$$

Thus

$$a(u_0^h, u_0^h) \preceq a(u^h, u^h).$$

We define $u_1^h = u^h - u_0^h$. Since the bilinear form $a(u, u)$ is an inner product we can use a standard triangle inequality and we have the following estimate

$$\begin{aligned} a(u_1^h, u_1^h) &= \|u_1^h\|_a^2 \\ &\leq (\|u^h\|_a + \|u_0^h\|_a)^2 \\ &\leq 2(\|u^h\|_a^2 + \|u_0^h\|_a^2) \\ &\preceq \|u^h\|_a^2 \quad = \quad a(u^h, u^h). \end{aligned}$$

Hence

$$a(u_0^h, u_0^h) + a(u_1^h, u_1^h) \preceq a(u^h, u^h).$$

Now we define the extension operator $t : W_{\Omega_0} \to W_1$ such that

$$(tu^h)(x) = \begin{cases} u^h(x), & x \in \bar{\Omega}_0, \\ (t_\gamma \varphi^h)(x), & x \in \Omega_i, \quad i = 1, \cdots, n, \end{cases}$$

where $\varphi^h = r_\gamma u^h \in V$ is the trace of $u^h$ on $\gamma$. Denote

$$B_{nov}^{-1} = tC_{\Omega_0}^{-1} t^* + \frac{1}{\epsilon_1} \check{C}^+_{\Omega_1} + ... + \frac{1}{\epsilon_n} \check{C}^+_{\Omega_n},$$

where $t^*$ is an operator adjoint to $t$. By ASM and [10], we have

**Theorem 2.2** *The following inequalities hold*

$$(B_{nov} u, u) \preceq (Au, u) \preceq (B_{nov} u, u), \quad \forall u \in R^N.$$

Now we can prove the Theorem 2.1 by use Theorem 2.2.
In the case of $\epsilon_i = 1,\ i = 1, \cdots, n$, using Theorem 2.2,

$$(C^{-1}u,u) \preceq ((tC_{\Omega_0}^{-1}t^* + \check{C}_{\Omega\Omega_1}^{+} + \cdots + \check{C}_{\Omega_n}^{+})u,u) \preceq (C^{-1}u,u)$$

holds for all $u \in R^N$. Hence, we have

$$\begin{aligned}
(B_{nov}^{-1}u,u) &= ((tC_{\Omega_0}^{-1}t^* + \frac{1}{\epsilon_1}\check{C}_{\Omega_1}^{+} + \cdots + \frac{1}{\epsilon_n}\check{C}_{\Omega_n}^{+})u,u) \\
&\leq ((tC_{\Omega_0}^{-1}t^* + \check{C}_{\Omega_1}^{+} + \cdots + \check{C}_{\Omega_n}^{+} + \frac{1}{\epsilon_1}\check{C}_{\Omega_1}^{+} + \cdots + \frac{1}{\epsilon_n}\check{C}_{\Omega_n}^{+})u,u) \\
&\preceq ((C^{-1} + \frac{1}{\epsilon_1}\check{C}_{\Omega_1}^{+} + \cdots + \frac{1}{\epsilon_n}\check{C}_{\Omega_n}^{+})u,u) \quad = \quad (B_{ov}^{-1}u,u) \\
&\preceq ((tC_{\Omega_0}^{-1}t^* + \check{C}_{\Omega_1}^{+} + \cdots + \check{C}_{\Omega_n}^{+} + \frac{1}{\epsilon_1}\check{C}_{\Omega_1}^{+} + \cdots + \frac{1}{\epsilon_n}\check{C}_{\Omega_n}^{+})u,u) \\
&\preceq (B_{nov}^{-1}u,u).
\end{aligned}$$

## 3 Problems with Cross Points

In this section, we consider an elliptic problem with jumps in coefficients in a domain $\Omega$,

$$\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i, \quad \Omega_i \bigcap \Omega_j = \emptyset, \quad i \neq j$$

with, in general, cross points on $\cup_{i=1}^{n} \partial\Omega_i$. Let $\Omega^h = \cup_{i=1}^{n} \Omega_i^h$ be a quasiuniform triangulation of the domain $\Omega$ which can be characterized by a parameter $h$. Define $W$ be the space of real–valued continuous functions being linear on the triangles of the triangulation $\Omega^h$. Using the finite element method, we can derive to the linear algebraic equation

$$Au = f.$$

Decompose this domain $\Omega$ into overlapping subdomains $\tilde{\Omega}_j$

$$\Omega = \bigcup_{j=1}^{m} \tilde{\Omega}_j, \quad \text{such that} \quad \Omega^h = \bigcup_{j=1}^{m} \tilde{\Omega}_j^h. \tag{9}$$

Here, the subdomains $\tilde{\Omega}_j$ are chosen in such a way that the following conditions hold:

- There exists $c_1$ which is independent of h, such that for all $x \in \Omega$ there exists $\tilde{\Omega}_j$

$$\text{dist}(x, \partial\tilde{\Omega}_j) \geq c_1.$$

- For each $\tilde{\Omega}_j$ there exists at most one cross point $x_0^{(j)} \in \tilde{\Omega}_j$.

Define

$$W_j = \{u^h \in W \quad | \quad u^h(x) = 0, \quad x \notin \tilde{\Omega}_j\}.$$

Since (9) is a decomposition with "good" overlapping and the inequalities (3) are fulfilled, then the decomposition $W = W_1 + W_2 + \cdots + W_m$ is stable :

**Lemma 3.1** *For all $u^h \in W$, there exist $u_j^h \in W_j$ such that $u_1^h + u_2^h + \cdots + u_m^h = u^h$,*

$$\int_{\tilde{\Omega}_1} p(x)(|\nabla u_1^h|^2 + \|u_1^h\|)^2)dx + \cdots + \int_{\tilde{\Omega}_m} p(x)(|\nabla u_m^h|^2 + |u_m^h|)^2 dx \preceq \int_\Omega p(x)(|\nabla u^h|^2 + |u^h|^2)dx.$$

Then according to ASM, if

$$(B_i u, u) \preceq (Bu, u) \preceq (B_i u, u), \quad \forall u^h \in W_i,$$

where

$$(Bu, u) = \int_\Omega p(x)(|\nabla u^h|^2 + |u^h|^2)dx, \quad \forall u^h \in W,$$

then

$$(B^{-1}u, u) \preceq ((B_1^+ + B_2^+ + \cdots + B_m^+)u, u) \preceq (B^{-1}u, u) \quad \forall u.$$

More interesting is the construction of preconditioning operators in subdomains $W_j$ which correspond subdomains $\tilde{\Omega}_j$ with a cross point.
Let $G$ be a part of the triangulation $\Omega^h$. Introduces matrices $C_G$ and $\check{C}_G$ such that

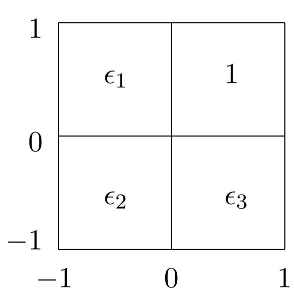
$$\begin{aligned} \|u^h\|^2_{H^1(G)} &\preceq (C_G u, u) \preceq \quad \|u^h\|^2_{H^1(G)}, \quad \forall u^h \in W_G, \\ \|u^h\|^2_{H^1(G)} &\preceq (\check{C}_G u, u) \preceq \quad \|u^h\|^2_{H^1(G)}, \quad \forall u^h \in W_G \cap H_0^1(G). \end{aligned}$$

Here vectors $u$ corresponds to $u^h$ from $W \cap H^1(G)$ and $W \cap H_0^1(G)$ respectively. We extend $\check{C}_G$ by zero outside of $G$ and denote by $\check{C}_G^+$ pseudo-inverse operator for this extension of $\check{C}_G$.

**Example 1** :
For the simple example of cross points case, we consider the following problem

$$\begin{cases} -\text{div}(p(x)\nabla u) &= f(x), \quad x \in \Omega \\ \qquad u(x) &= 0, \qquad x \in \Gamma \end{cases} \tag{10}$$

and the following distribution of the coefficients in $\Omega = (-1, 1) \times (-1, 1)$.




$p_1 = 1, \qquad p_2 = \epsilon_1,$

$p_3 = \epsilon_2, \qquad p_4 = \epsilon_3 \qquad$ and

$1 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3$

In this case constructions of the decomposition of the space $W$ and the corresponding preconditioners are more sequential and simple. In section 2 we define the preconditioner

into two steps. But in this Example 1, a preconditioner is defined into four steps. To design the preconditioning operator, we decompose $W$ into a sum of subspace $W = W_1 + W_2$, and then again decompose $W_2$ into a sum of subspaces.
First, define $\Gamma_1 = \partial\Omega_1$, and trace space $V_1$ such as

$$V_1 = r_{\Gamma_1} W = \{\varphi^h \mid \varphi^h(x) = u^h(x),\ x \in \Gamma_1,\ u^h \in W\}.$$

We use a norm-preserving extension operator $t_{\Gamma_1} : V_1 \to W_{\Omega\setminus\Omega_1}$ for regular elliptic second order problems such that

$$\|t_{\Gamma_1}\varphi_1^h\|^2_{H^1(\Omega\setminus\Omega_1)} \preceq \|\varphi_1^h\|^2_{H^{\frac{1}{2}}(\Gamma_1)}, \quad \forall \varphi_1^h \in V_1,$$

and define $W_1$

$$\begin{aligned} W_1 = \{u^h \in W \mid u^h(x) = (t_{\Gamma_1}\varphi^h)(x),\ x \in \Omega_i,\ i = 2,3,4,\ \varphi^h(x) = v^h(x),\ x \in \Gamma_1, \\ u^h(x) = v^h(x),\ x \in \Omega_1,\ \forall v^h \in W\}. \end{aligned} \tag{11}$$

Also define the extension operator $t_1 : W_{\Omega_1} \to W$,

$$(t_1 u_1^h)(x) = \begin{cases} u_1^h(x), & x \in \Omega_1 \\ (t_{\Gamma_1}\varphi_1^h)(x), & x \in \Omega \setminus \Omega_1 \end{cases} \qquad \forall u_1^h \in W_{\Omega_1},\ \forall \varphi_1^h = r_{\Gamma_1} u_1^h.$$

Introduce the subspace $W_2$

$$W_2 = \{u^h \in W \mid u^h(x) = 0,\ x \in \bar{\Omega}_1\}.$$

To decompose $W_2$, define $\Gamma_2 = \partial\Omega_2$, and the trace space $V_2$ on $\Gamma_2$ of functions from $W_2$. Let $t_{\Gamma_2} : V_2 \to W_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)}$ is a norm-preserving extension operator and $\tilde{W}_2$ :

$$\begin{aligned} \tilde{W}_2 = \{u^h \in W_2 \mid u^h(x) &= (t_{\Gamma_2}\varphi^h)(x),\ x \in \Omega_i,\ i = 3,4,\ \varphi^h(x) = v^h(x),\ x \in \Gamma_2, \\ u^h(x) &= v^h(x),\ x \in \Omega_2,\ \forall v^h \in W_2\}. \end{aligned}$$

Now define

$$W_3 = \{u^h \in W_3 \mid u^h(x) = 0,\ x \in \bar{\Omega}_1 \cup \bar{\Omega}_2\}$$

and

$$\begin{aligned} \tilde{W}_3 = \{u^h \in W_3 \mid u^h(x) &= (t_{\Gamma_3}\varphi^h)(x),\ x \in \Omega_4,\ \varphi^h(x) = v^h(x),\ x \in \Gamma_3, \\ u^h(x) &= v^h(x),\ x \in \Omega_3,\ \forall v^h \in W_3\}, \end{aligned}$$

where $\Gamma_3 = \partial\Omega_3, t_{\Gamma_3} : V_3 \to W_{\Omega_4}$. Here $V_3$ is the trace space on $\Gamma_3$ of functions from $W_3$, and $t_{\Gamma_3}$ is a norm-preserving operator as above. Finally, introduce

$$W_4 = \{u^h \in W \mid u^h(x) = 0,\ x \in \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \bar{\Omega}_3\}.$$

**Lemma 3.2** *For any function $u^h \in W$ there exist $u_1 \in W_1,\ \tilde{u}_i^h \in \tilde{W}_i,\ i = 2,3,\ u_4^h \in W_4$ such that*

$$u_1^h + \tilde{u}_2^h + \tilde{u}_3^h + u_4^h = u^h$$
$$a(u_1^h, u_1^h) + a(\tilde{u}_2^h, \tilde{u}_2^h) + a(\tilde{u}_3^h, \tilde{u}_3^h) + a(u_4^h, u_4^h) \preceq a(u^h, u^h).$$

*proof* :
From the assumption $\epsilon_i$, the stability of the following decompositions are evident :

$$\begin{aligned} W &= W_1 + W_2, \\ W_2 &= \tilde{W}_2 + W_3, \\ W_3 &= \tilde{W}_3 + W_4. \end{aligned}$$

The proof of Lemma 3.2 follows from stability of these decompositions.

**Theorem 3.1** *The following inequalities hold*

$$(A^{-1}u, u) \preceq (B_{ov}^{-1}u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N,$$

*where*

$$B_{ov}^{-1} = C_\Omega^{-1} + \frac{1}{\epsilon_1}\check{C}^+_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon_2}\check{C}^+_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)} + \frac{1}{\epsilon_3}\check{C}^+_{\Omega_4}.$$

*proof* :
It follows from the proof of Lemma 2.3 that the decomposition $W = W_1 + W_2$ is stable. Denote $A_2$ such as

$$(A_2 u, u) = a(u^h, u^h), \quad \forall u^h \in W_2,$$

and pseudo-inverse operator $A_2^+$. From ASM

$$(A^{-1}u, u) \preceq ((t_1 C_{\Omega_1}^{-1} t_1^* + A_2^+)u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N,$$

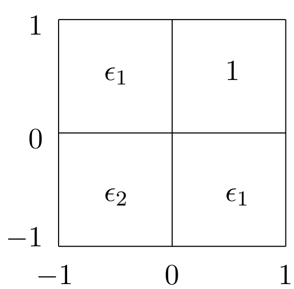
and by the same technique from section 2

$$(A^{-1}u, u) \preceq ((C_\Omega^{-1} + A_2^+)u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N. \tag{12}$$

Now we construct equivalent operators for $A_2^+$. Using stability of the decomposition $W_2$, $W_2 = \tilde{W}_2 + \tilde{W}_3 + W_4$ and the technique from section 2, we have

$$(A_2^+ u_2, u_2) \preceq ((\frac{1}{\epsilon_1}\check{C}^+_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon_2}\check{C}^+_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)} + \frac{1}{\epsilon_3}\check{C}^+_{\Omega_4})u_2, u_2) \preceq (A_2^+ u_2, u_2), \quad \forall u_2^h \in W_2.$$

**Example 2** :
Now consider the following distribution of the coefficients such as on the figure.




$p_1 = 1, \qquad p_2 = \epsilon_1,$

$p_3 = \epsilon_2, \qquad p_4 = \epsilon_1$ and

$1 \geq \epsilon_1 \geq \epsilon_2$

As in the previous example 1, we start with the subdomain $\Omega_1$ which corresponds to the biggest coefficient of the problem and define the trace space $V_1$ on $\Gamma_1 = \partial\Omega_1$ and a norm-preserving extension operator $t_{\Gamma_1} : V_1 \to W$, subspace $W_1 \subset W$. Now define subspace

$$W_{2,4} = \{u^h \in W \mid u^h(x) = 0, \ x \in \bar{\Omega}_1\}.$$

Denote $\Gamma_i = \partial\Omega_i$ and the trace space $V_i$ on $\Gamma_i,\ i = 2, 4$ of functions from $W_{2,4}$ Define norm preserving extension operator $t_{\Gamma_{2,4}} : V_2 \cup V_4 \to W_{2,4}$ such that

$$||u^h_{2,4}||^2_{H^1(\Omega_3)} = ||t_{\Gamma_{2,4}}\varphi^h_{2,4}||^2_{H^1(\Omega_3)} \preceq (||\varphi^h_{2,4}||^2_{H^{\frac{1}{2}}(\Gamma_2)} + ||\varphi^h_{2,4}||^2_{H^{\frac{1}{2}}(\Gamma_4)}),$$

and subspace $\tilde{W}_{2,4}$ :

$$\begin{aligned}\tilde{W}_{2,4} = \{u^h \mid u^h(x) &= (t_{\Gamma_{2,4}}\varphi^h_{2,4})(x),\ x \in \Omega_3,\ \varphi^h_{2,4}(x) = v^h(x),\ x \in \Gamma_2 \cup \Gamma_4,\\ u^h(x) &= v^h(x),\ x \in \Omega_2 \cup \Omega_4,\ \forall v^h \in W_{2,4}\}.\end{aligned}$$

At last, determine $W_3 = \{u^h \in W \mid u^h(x) = 0,\ x \in \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \bar{\Omega}_4\}$.

**Lemma 3.3** *For any function $u^h \in W$ there exist $u^h_i \in W_i,\ i = 1, 3,\ u^h_{2,4} \in \tilde{W}_{2,4}$ such that*

$$u^h_1 + \tilde{u}^h_{2,4} + u^h_3 = u^h,$$

$$a(u^h_1, u^h_1) + a(\tilde{u}^h_{2,4}, \tilde{u}^h_{2,4}) + a(u^h_4, u^h_4) \preceq a(u^h, u^h).$$

*proof* :
To prove this Lemma, we use the stability of the decomposition

$$W = W_1 + W_2, \quad W_2 = \tilde{W}_{2,4} + W_3,$$

and these stable decompositions give the proof of Lemma 3.3.

**Theorem 3.2** *The following inequalities hold*

$$(A^{-1}u, u) \preceq (B^{-1}_{ov}u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N,$$

*where*

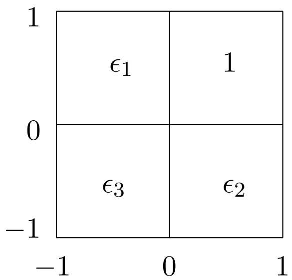
$$B^{-1}_{ov} = C^{-1}_{\Omega} + \frac{1}{\epsilon_1}(\check{C}^+_{\Omega\setminus\Omega_1} + \check{C}^+_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)}) + \frac{1}{\epsilon_2}\check{C}^+_{\Omega_3}.$$

*proof* :
Again, the proof of this theorem is based on Lemma 3.3 and the same technique as above. Here $\frac{1}{\epsilon_1}(\check{C}^+_{\Omega\setminus\Omega_1} + \check{C}^+_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)})$ corresponds to the subspace $\tilde{W}_{2,4}$.

**Remark 3.1** *In order to simplify the preconditioner $B^{-1}_{ov}$, we can define an spectrally equivalent preconditioner (see Lemma 3.6)*

$$B^{-1}_{ov,s} = C^{-1}_{\Omega} + \frac{1}{\epsilon_1}\check{C}^+_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon_2}\check{C}^+_{\Omega_3}.$$

**Remark 3.2** *If the distribution of the coefficients is as on the picture*




$$p_1 = 1, \qquad p_2 = \epsilon_1,$$

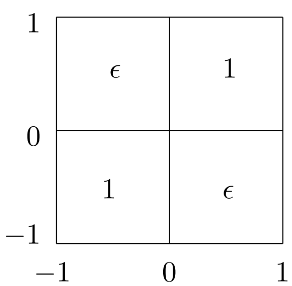
$$p_3 = \epsilon_3, \qquad p_4 = \epsilon_2 \qquad \text{and}$$

$$1 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3$$

*then, slightly modifying the above analysis, the optimality of the following preconditioning can be proved*

$$B_{ov}^{-1} = C_{\Omega}^{-1} + \frac{1}{\epsilon_1}\check{C}^{+}_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon_2}\check{C}^{+}_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)} + \frac{1}{\epsilon_3}\check{C}^{+}_{\Omega_4}.$$

**Example 3** :
We consider the following distribution of the coefficients.




$$p_1 = 1, \qquad p_2 = \epsilon,$$

$$p_3 = 1, \qquad p_4 = \epsilon \qquad \text{and}$$

$$1 \geq p_1$$

This case is the most difficult. Again we start with the subdomain $\Omega_1$ and as above difine the trace space $V_1$ on $\Gamma_1$ and $t_{\Gamma_1} : V_1 \to W$ and the subspace $W_1$ :

$$W_1 = \{u^h \in W \mid u^h(x) = (t_{\Gamma_1}\varphi^h)(x),\ x \in \Omega_i,\ i = 2,3,4,\ \varphi^h(x) = v^h(x),\ x \in \Gamma_1,$$
$$u^h(x) = v^h(x),\ x \in \Omega_1,\ \forall v^h \in W\}$$

Now we introduce the subspace $W_3$ which is the subspace of functions corresponds to $L$-shape domain $\Omega \setminus \Omega_1$, it means that $u^h(x) = 0$, for any $x \in \bar{\Omega}_1, u^h \in W_3$. To decompose $W_3$, define $\Gamma_3 = \partial\Omega_3$, and trace space $V_3$ on $\Gamma_3$ of functions from $W_3$. In the space $V_3$, let us define the harmonic extension operator $t_{\Gamma_3} : V_3 \to W_3$ such that

$$\|t_{\Gamma_3}\varphi_3^h\|^2_{H^1(\Omega_2)} + \|t_{\Gamma_3}\varphi_3^h\|^2_{H^1(\Omega_4)} = inf_{w^h\in W_3, r_{\Gamma_3}w^h=\varphi_3^h}(|w^h|^2_{H^1(\Omega_2)} + |w^h|^2_{H^1(\Omega_4)}).$$

**Remark 3.3** *In examples 1,2, we can use as optimal with respect to condition numbers preconditioners $B_{nov}$ and $B_{ov}$. To use $B_{nov}^{-1}$, we need to use effective extensions operators $t_{\Gamma_i}, t^*_{\Gamma_i}$. For this, implicit norm-preserving operators, suggested in [3], [7], can be used with the arithmetical costs of implementations is proportional to the number of degrees of freedom. Now we use $t_{\Gamma_3}$ only for a theoretical analysis of $B_{ov}$ and use for this goal the harmonic extension operator $t_{\Gamma_3}$.*

And define the subspace $\hat{W}_{\Omega_3} \subset W_{\Omega_3}$ and the extension operator $t_3 : \hat{W}_{\Omega_3} \to W_3$

$$\hat{W}_{\Omega_3} = \{\hat{u}_3^h \in W_{\Omega_3} = r_{\Omega_3}W \mid \hat{u}_3^h(0,0) = 0\},$$

$$(t_3\hat{u}_3^h)(x) = \begin{cases} \hat{u}^h(x), & x \in \bar{\Omega}_3 \\ (t_{\Gamma_3}\varphi_3^h(x)), & x \in \Omega_i, \quad i = 2, 4, \end{cases}$$

where $\varphi_3^h = r_{\Gamma_3}\hat{u}_3^h$ for any $\hat{u}_3^h \in \hat{W}_{\Omega_3}$.
Then we define $\tilde{W}_3$ such that

$$\tilde{W}_3 = \{u^h \in W_3 \mid u^h(x) = (t_{\Gamma_3}\varphi_3^h)(x),\ x \in \Omega_i,\ i = 2, 4,\ \varphi_3^h(x) = v^h(x),\ x \in \Gamma_3,$$
$$u^h(x) = v^h(x),\ x \in \Omega_3,\ \forall v^h \in W_3\},$$

and define subspaces

$$W_i = \{u^h \in W_3 \mid u^h(x) = 0, \quad \forall x \notin \Omega_i\}, \quad i = 2, 4.$$

Because $W = W_1 + W_3$, $W_3 = \tilde{W}_3 + W_2 + W_4$, it is obvious that $W = W_1 + \tilde{W}_3 + W_2 + W_4$.

**Lemma 3.4** *For any function $u^h \in W$ there exist $u_i^h \in W_i,\ i = 1, 2, 4,$ and $\tilde{u}_3^h \in \tilde{W}_3$ such that*

$$u_1^h + \tilde{u}_3^h + u_2^h + u_4^h = u^h,$$
$$a(u_1^h, u_1^h) + a(\tilde{u}_3^h, \tilde{u}_3^h) + a(u_2^h, u_2^h) + a(u_4^h, u_4^h) \preceq a(u^h, u^h).$$

*proof* :
For any $u^h \in W$, define $u_1^h$

$$u_1^h = t_1 r_{\Omega_1} u^h.$$

In our problem,

$$a(u_1^h, u_1^h) = \int_{\Omega_1\cup\Omega_3} (|u_1^h|^2 + |\nabla u_1^h|^2)dx + \epsilon \int_{\Omega_2\cup\Omega_4} (|u_1^h|^2 + |\nabla u_1^h|^2)dx.$$

Because of $0 < \epsilon \leq 1$, we have

$$a(u_1^h, u_1^h) \leq \|u_1^h\|^2_{H^1(\Omega)}.$$

And by the trace lemma for $H^1(\Omega)$,

$$\begin{aligned} \|u_1^h\|^2_{H^1(\Omega)} &= \|u^h\|^2_{H^1(\Omega_1)} + \|t_1 r_{\Omega_1} u^h\|^2_{H^1(\Omega\setminus\Omega_1)} \\ &\preceq \|u^h\|^2_{H^1(\Omega_1)}. \end{aligned}$$

Thus

$$a(u_1^h, u_1^h) \preceq a(u^h, u^h).$$

Since the bilinear form $a(u, u)$ is an inner product, we can use a standard triangle inequality

$$a(u_3^h, u_3^h) \preceq a(u^h, u^h) \quad \text{for } u_3^h = u^h - u_1^h,$$

Now define

$$\tilde{u}_3^h = t_3 r_{\Omega_3} u_3^h,$$

and

$$a(\tilde{u}_3^h, \tilde{u}_3^h) \leq a(u_3^h, u_3^h) \preceq a(u^h, u^h).$$

Then define $u_i^h \in W_i$

$$u_i^h(x) = \begin{cases} u_3^h(x) - \tilde{u}_3^h(x), & x \in \Omega_i, \\ 0, & x \notin \Omega_i, \end{cases} \quad i = 2, 4$$

and again using a triangle inequality, we complete the proof of Lemma 3.4.
For subdomains $\Omega_1, \Omega_2, \Omega_4$, we introduce matrices $C_{\Omega_1}, \check{C}^+_{\Omega_2}, \check{C}^+_{\Omega_4}$, respectively, according to (7) and $B_3$ such that

$$(B_3 u, v) = a(t_3 \hat{u}_3^h, t_3 \hat{v}_3^h), \quad \forall \hat{u}_3^h,\ \hat{v}_3^h \in \hat{W}_{\Omega_3}.$$

Put

$$B_{nov}^{-1} = t_1 C_{\Omega_1}^{-1} t_1^* + t_3 B_3^{-1} t_3^* + \frac{1}{\epsilon} \check{C}^+_{\Omega_2} + \frac{1}{\epsilon} \check{C}^+_{\Omega_4}.$$

Using the same technique as in the previous section 2 (ASM), we can show

$$(A^{-1} u, u) \preceq (B_{nov}^{-1} u, u) \preceq (A^{-1} u, u), \quad \forall u \in R^N.$$

The problem of using the preconditioner $B_{nov}^{-1}$ is the multiplication of $B_3^{-1}$ by vectors or a construction of specially equivalent operators. Instead of $B_{nov}^{-1}$ we suggest more simple preconditioner $B_{ov}^{-1}$, as in the examples 1,2, but in this case $B_{ov}^{-1}$ is not optimal. An advantage is that the operator $B_{ov}^{-1}$ is very simple to implement, for instance, we do not need hierarchical grids.
Denote

$$\begin{aligned} \gamma_2 &= \{(x_1, x_2) \mid -1 \le x_1 \le 0,\ x_2 = 0\}, \\ \gamma_4 &= \{(x_1, x_2) \mid x_1 = 0,\ -1 \le x_2 \le 0\}. \end{aligned}$$

Define a norm in $V_3$

$$\|\varphi_3^h\|^2_{H_a^{1/2}(\Gamma_3)} = \|\varphi_3^h\|^2_{H^{1/2}(\Gamma_3)} + \epsilon \|\varphi_3^h\|^2_{H_{00}^{1/2}(\gamma_2)} + \epsilon \|\varphi^h\|^2_{H_{00}^{1/2}(\gamma_4)}$$

with the standard norm in $H_{00}^{1/2}(\gamma_2)$

$$\|\varphi_3^h\|^2_{H_{00}^{1/2}(\gamma_2)} = \|\varphi_3^h\|^2_{H^{1/2}(\gamma_2)} + \int_{-1}^{0} \frac{(\varphi_3^h(x_1))^2}{x_1(x_1+1)} dx_1$$

and in the same way in $H_{00}^{1/2}(\gamma_4)$.
Then for any function $u_3^h \in W_3$,

$$\|\varphi_3^h\|^2_{H_a^{1/2}(\Gamma_3)} \preceq a(u_3^h, u_3^h),$$

where $\varphi_3^h \in V_3$ is the trace of $u_3^h$ on $\Gamma_3$. Conversely, for any function $\varphi_3^h \in V_3$ there exists $u_3^h \in W_3$ such that

$$\begin{aligned} u_3^h(x) &= \varphi_3^h(x), \qquad \forall x \in \Gamma_3, \\ a(u_3^h, u_3^h) &\preceq \|\varphi_3^h\|^2_{H_a^{1/2}(\Gamma_3)}. \end{aligned}$$

Also we need the following well-known result (see, for instance, [1], [16]).

**Lemma 3.5** *The following inequalities hold.*

$$\|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_2)} + \|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_4)} \preceq log^2 h^{-1} \|\varphi_3^h\|^2_{H^{1/2}(\Gamma_3)}, \quad \forall \varphi_3^h \in V_3.$$

Note that,

$$\begin{aligned}
\|\tilde{u}_3^h\|^2_{H^1(\Omega)} &\preceq \|\tilde{u}_3^h\|^2_{H^1(\Omega_3)} + \|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_2)} + \|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_4)} &\preceq \|\tilde{u}_3^h\|^2_{H^1(\Omega)}, \quad \forall \tilde{u}_3^h \in \tilde{W}_3,\\
a(\tilde{u}_3^h, \tilde{u}_3^h) &\preceq \|\tilde{u}_3^h\|^2_{H^1(\Omega_3)} + \epsilon\|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_2)} + \epsilon\|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_4)} &\preceq a(\tilde{u}_3^h, \tilde{u}_3^h), \quad \forall \tilde{u}_3^h \in \tilde{W}_3,
\end{aligned}$$

where $\varphi_3^h = r_{\Gamma_3}\tilde{u}_3^h$. Define a symmetric matrix $\hat{B}_3$

$$(\hat{B}_3 u_3, u_3) = \|\hat{u}_3^h\|^2_{H^1(\Omega_3)} + \|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_2)} + \|\varphi_3^h\|^2_{H^{1/2}_{00}(\gamma_4)}, \quad \forall u_3^h \in \hat{W}_{\Omega_3},$$

where $\varphi_3^h = r_{\Gamma_3}\hat{u}_3^h$.
Then

$$(\hat{B}_3 u_3, u_3) \preceq \|t_3 \hat{u}_3^h\|^2_{H^1(\Omega)} \preceq (\hat{B}_3 u_3, u_3), \quad \forall \hat{u}_3^h \in \hat{W}_{\Omega_3},$$

$$(\check{C}^+_{\Omega\setminus\Omega_1} u, u) \preceq (t_3 \hat{B}_3^{-1} t_3^* u, u) + (\check{C}^+_{\Omega_2} u, u) + (\check{C}^+_{\Omega_4} u, u) \preceq (\check{C}^+_{\Omega\setminus\Omega_1} u, u), \quad \forall u \in R^N.$$

Put

$$B_{ov}^{-1} = C_\Omega^{-1} + \check{C}^+_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4}.$$

**Theorem 3.3** *The following inequalities hold.*

$$\frac{1}{log^2 h^{-1}}(A^{-1}u, u) \preceq (B_{ov}^{-1} u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N.$$

*proof* :
Using Lemma 3.5

$$\begin{aligned}
(B_{nov}^{-1} u, u) &= ((t_1 C_{\Omega_1}^{-1} t_1^* + t_3 B_3^{-1} t_3^* + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u)\\
&\leq ((t_1 C_{\Omega_1}^{-1} t_1^* + t_3 B_3^{-1} t_3^* + \check{C}^+_{\Omega_2} + \check{C}^+_{\Omega_4} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u)\\
&\preceq log^2 h^{-1}((t_1 C_{\Omega_1}^{-1} t_1^* + t_3 \hat{B}_3^{-1} t_3^* + \check{C}^+_{\Omega_2} + \check{C}^+_{\Omega_4} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u)\\
&\preceq log^2 h^{-1}((t_1 C_{\Omega_1}^{-1} t_1^* + \check{C}^+_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u)\\
&\preceq log^2 h^{-1}((C_\Omega^{-1} + \check{C}^+_{\Omega\setminus\Omega_1} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u) \quad = \quad log^2 h^{-1}(B_{ov}^{-1} u, u)\\
&\preceq log^2 h^{-1}((C_\Omega^{-1} + t_3 \hat{B}_3^{-1} t_3^* + \check{C}^+_{\Omega_2} + \check{C}^+_{\Omega_4} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u)\\
&\preceq log^2 h^{-1}((C_\Omega^{-1} + t_3 B_3^{-1} t_3^* + \check{C}^+_{\Omega_2} + \check{C}^+_{\Omega_4} + \frac{1}{\epsilon}\check{C}^+_{\Omega_2} + \frac{1}{\epsilon}\check{C}^+_{\Omega_4})u, u)\\
&\preceq 2 log^2 h^{-1}(B_{nov}^{-1} u, u).
\end{aligned}$$

Here the evident (since $0 < \epsilon \leq 1$) inequality,

$$(B_3 u_3, u_3) \preceq (\hat{B}_3 u_3, u_3), \quad \forall u_3,$$

is used. To simplify $B_{ov}^{-1}$, define

$$B_{ov,s}^{-1} = C_\Omega^{-1} + \frac{1}{\epsilon}\check{C}_{\Omega_2}^{+} + \frac{1}{\epsilon}\check{C}_{\Omega_4}^{+}.$$

**Lemma 3.6** *The following inequalities hold.*

$$(C_\Omega^{-1}u, u) \preceq ((C_\Omega^{-1} + \check{C}_{\Omega\setminus\Omega_1}^{+})u, u) \preceq (C_\Omega^{-1}u, u).$$

*proof* :
Note that

$$(C_\Omega^{-1}u, u) \preceq ((tC_{\Omega_1}^{-1}t^* + \check{C}_{\Omega\setminus\Omega_1}^{+})u, u) \preceq (C_\Omega^{-1}u, u)$$

Then we have

$$\begin{aligned}(C_\Omega^{-1}u, u) &\leq ((C_\Omega^{-1} + \check{C}_{\Omega\setminus\Omega_1}^{+})u, u)\\ &\preceq (((tC_{\Omega_1}^{-1}t^* + \check{C}_{\Omega\setminus\Omega_1}^{+}) + \check{C}_{\Omega\setminus\Omega_1}^{+})u, u)\\ &\preceq ((tC_{\Omega_1}^{-1}t^* + \check{C}_{\Omega\setminus\Omega_1}^{+})u, u)\\ &\preceq (C_\Omega^{-1}u, u).\end{aligned}$$

Then the following theorem fulfills.

**Theorem 3.4** *The following inequalities hold :*

$$\frac{1}{log^2 h^{-1}}(A^{-1}u, u) \preceq (B_{ov,s}^{-1}u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N.$$

Now we suggest more complicated preconditioner $B_{opt}$ for example 3, but optimal with respect to the condition number of $B_{opt}^{-1}A$.
As above, we start with the subdomains $\Omega_1$ and can prove that

$$(A^{-1}u, u) \preceq ((C_\Omega^{-1} + A_3^{+})u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N.$$

Here $A_3$ is the matrix such that

$$(A_3u_3, v_3) = a(u_3^h, v_3^h), \quad \forall u_3^h, v_3^h \in W_3,$$

and $A_3^+$ is the pseudo-inverse operator for $A_3$,

$$W_3 = \{u^h \in W \mid u^h(x) = 0, \quad x \in \Omega_1\}.$$

Now our goal is a construction of the preconditioner for $A_3$. To do it, we use ASM and explicit norm-preserving extension operators $\bar{t}_{\Gamma_2}, \bar{t}_{\Gamma_3}, \bar{t}_{\Gamma_4}$, for instance, from [3], [7], such that

$$\bar{t}_i : V_{\Gamma_i} \to W_{\Omega_i},$$

$$\|\bar{t}_{\Gamma_i}\varphi_i^h\|_{H^1(\Omega_i)}^2 \preceq \|\varphi_i^h\|_{H^{1/2}(\Gamma_i)}^2, \quad \forall \varphi_i^h \in V_{\Gamma_i} = r_{\Gamma_i}W, \quad i = 2, 3, 4.$$

Introduce

$$V_{\gamma_i} = r_{\gamma_i}W_3, \quad i = 2, 4,$$

and define the extension operators $t_s : V_{\gamma_2} \to V_{\Gamma_3}$

$$\begin{aligned}(t_s\varphi^h_{\gamma_2})(x) = \{\varphi^h_3 \in V_{\Gamma_3} \mid \varphi^h_3(x_1,0) &= \varphi^h_{\gamma_2}(x_1,0), \quad -1 \le x_1 \le 0,\\ \varphi^h_3(0,x_1) &= \varphi^h_{\gamma_2}(0,x_1), \quad -1 \le x_1 \le 0, \quad \forall \varphi^h_{\gamma_2} \in V_{\gamma_2}\}\end{aligned}$$

and $t_{\gamma_2} : V_{\gamma_2} \to W_3$,

$$(t_{\gamma_2}\varphi^h_2)(x) = \begin{cases} u^h(x) = (\bar t_{\Gamma_2}\tilde\varphi^h_{\gamma_2})(x), & x \in \Omega_2,\\ u^h(x) = (\bar t_{\Gamma_3} t_s\varphi^h_{\gamma_2})(x), & x \in \Omega_3,\\ u^h(x) = (\bar t_{\Gamma_4}\tilde r_{\gamma_4}(t_s\varphi^h_{\gamma_2}))(x), & x \in \Omega_4, \quad \forall \varphi^h_{\gamma_2 \in V_{\gamma_2}}, \end{cases}$$

where $\tilde\varphi^h_{\gamma_2}$ is the extension by zero on $\Gamma_2 \setminus \gamma_2$ of $\varphi^h_{\gamma_2}$ and $\tilde r_{\gamma_4}$ is the extension of the operator $r_{\gamma_4}$ by zero on $\Gamma_4 \setminus \gamma_4$. Then define the subspaces

$$\begin{aligned}\tilde W_{\gamma_2} &= t_{\gamma_2} V_{\gamma_2},\\ \check W_2 &= \{u^h \in W_3 \mid u^h(x) = 0, \quad x \notin \Omega_2\},\\ \check W_{3,4} &= \{u^h \in W_3 \mid u^h(x) = 0, \quad x \in \Omega_2\}.\end{aligned}$$

The following decomposition

$$W_3 = \tilde W_{\gamma_2} + \check W_2 + \check W_{3,4}$$

is stable :

**Lemma 3.7** *For any function $u^h \in W_3$ there exist $\tilde u^h_{\gamma_2} \in \tilde W_{\gamma_2}, \check u^h_2 \in \check W_2, \check u^h_{3,4} \in \check W_{3,4}$ such that*

$$\tilde u^h_{\gamma_2} + \check u^h_2 + \check u^h_{3,4} = u^h,$$

$$a(\tilde u^h_{\gamma_2}, \tilde u^h_{\gamma_2}) + a(\check u^h_2, \check u^h_2) + a(\check u^h_{3,4}, \check u^h_{3,4}) \preceq a(u^h, u^h).$$

For $u^h \in W_3$, define $\varphi^h_2 = r_{\gamma_2} u^h \in V_{\gamma_2}$ and put $\tilde u^h_{\gamma_2} = t_{\gamma_2}\tilde\varphi^h_{\gamma_2}$. Denote

$$\|\varphi^h_{\gamma_2}\|^2_{\hat H^{1/2}(\gamma_2)} = \|\varphi^h\|^2_{\hat H^{1/2}(\gamma_2)} + \int_{-1}^0 \frac{\varphi^2_{\gamma_2}(x_1)}{(x_1+1)} dx_1.$$

By trace theorem [16], and by the definition of $t_{\gamma_2}$, we have

$$\begin{aligned}\epsilon\|t_{\gamma_2}\tilde\varphi^h_{\gamma_2}\|^2_{H^1(\Omega_2)} &\preceq \epsilon\|\varphi^h_{\gamma_2}\|^2_{H^{1/2}_{00}(\gamma_2)},\\ \|t_{\gamma_2}\tilde\varphi^h_{\gamma_2}\|^2_{H^1(\Omega_3)} &\preceq (\|\varphi^h_{\gamma_2}\|^2_{\hat H^{1/2}(\gamma_2)} + \|t_s\varphi^h_{\gamma_2}\|^2_{\hat H^{1/2}(\gamma_4)} + \int_{-1}^0 \frac{(\varphi^h_{\gamma_2}(x_1) - t_s\varphi^h_{\gamma_2}(x_1))^2}{x_1} dx_1)\\ &= (\|\varphi^h_{\gamma_2}\|^2_{\hat H^{1/2}(\gamma_2)} + \|\varphi^h_{\gamma_2}\|^2_{\hat H^{1/2}(\gamma_2)} + 0),\\ \epsilon\|t_{\gamma_2}\tilde\varphi^h_{\gamma_2}\|^2_{H^1(\Omega_4)} &\preceq \epsilon\|t_s\varphi^h_{\gamma_2}\|^2_{H^{1/2}_{00}(\gamma_4)}\\ &= \epsilon\|\varphi^h_{\gamma_2}\|^2_{H^{1/2}_{00}(\gamma_2)}.\end{aligned}$$

Here $\|\cdot\|_{\hat H^{1/2}(\gamma_4)}$ is defined in the same way as $\|\cdot\|_{\hat H^{1/2}(\gamma_2)}$. Define

$$\begin{aligned}\check u^h_2(x) &= u^h(x) - (t_{\gamma_2}\varphi^h_{\gamma_2})(x), \quad x \in \bar\Omega_2,\\ \check u^h_{3,4}(x) &= u^h(x) - (t_{\gamma_2}\varphi^h_{\gamma_2})(x), \quad x \in \bar\Omega_3 \cup \bar\Omega_4,\end{aligned}$$

and using a standard triangle inequality, we complete the proof of Lemma 3.7.
Introduce a matrix $\Sigma$ such that

$$(\Sigma\varphi_{\gamma_2}, \varphi_{\gamma_2}) \preceq \|\varphi^h_{\gamma_2}\|^2_{\hat{H}^{1/2}(\gamma_2)} + \epsilon\|\varphi^h_{\gamma_2}\|^2_{H^{1/2}_{00}(\gamma_2)} \preceq (\Sigma\varphi_{\gamma_2}, \varphi_{\gamma_2}), \quad \forall\varphi^h_{\gamma_2} \in V_{\gamma_2}. \tag{13}$$

**Theorem 3.5** *The following inequalities hold :*

$$(A^{-1}u, u) \preceq (B^{-1}_{opt}u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N,$$

where

$$B^{-1}_{opt} = C^{-1}_{\Omega} + t_{\gamma_2}\Sigma^{-1}t^*_{\gamma_2} + \frac{1}{\epsilon}\check{C}_{\Omega_2} + \check{C}^{+}_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)} + \frac{1}{\epsilon}\check{C}^{+}_{\Omega_4}.$$

*proof* :
Using ASM and the special structure of the subspace $W_{\gamma_2}$ and (13), we have

$$(A^{-1}u, u) \preceq ((C^{-1}_{\Omega} + t_{\gamma_2}\Sigma^{-1}t^*_{\gamma_2} + \frac{1}{\epsilon}\check{C}^{+}_{\Omega_2} + \check{A}^{+}_{3,4})u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N,$$

where

$$(A_{3,4}u, v) = \int_{\Omega_3} \nabla u^h \cdot \nabla v^h dx + \epsilon \int_{\Omega_4} \nabla u^h \cdot \nabla v^h dx, \quad \forall u^h, v^h \in W_{\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)} \cup H^1_0(\Omega\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)).$$

Using above approach to construct a preconditioner for $A_{3,4}$,

$$(\check{A}^{+}_{3,4}u, u) \preceq ((\check{C}^{+}_{\Omega\setminus(\Omega_1\cup\Omega_2)} + \frac{1}{\epsilon}\check{C}_{\Omega_4})u, u) \preceq (\check{A}^{+}_{3,4}u, u), \quad \forall u \in R^N$$

is hold. This completes the proof the theorem.

To define a matrix $\Sigma$ satisfying to (13), let us consider the following model case on the unit interval $I$

$$I = \{x \mid 0 < x < 1\}$$

and on $I$ consider Sobolev space $\hat{H}^{1/2}(I)$ with the norm, generated by the following inner product

$$(\varphi, \psi)_{\hat{H}^{1/2}(I)} = (\varphi, \psi)_{L_2(I)} + \int_I\int_I \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^2} dxdy + \int_0^1 \frac{\varphi(x)\psi(x)}{(1 - x)} dx$$

and $H^{1/2}_{00}(I)$ with standard inner product and norm. Denote for $0 < \epsilon \leq 1$,

$$a_\epsilon(\varphi, \psi) = (\varphi, \psi)_{\hat{H}(I)} + \epsilon(\varphi, \psi)_{H^{1/2}_{00}(I).}$$

Introduce an uniform grid on $I$ with the grid size $h = 2^{-L}$ and denote by $V_h \subset H^1(I)$ piecewise-linear finite element space such that

$$V_h = \{\varphi^h \mid \varphi^h(0) = 0,\ \varphi^h(1) = 0\}.$$

Let $x_l = 2^{-l}$ for $0 \leq l < L$ and $x_L = 0$ nodes of the coarse grid, $T_l = (x_l, x_{l-1})$ for $1 \leq l \leq L$ (see Figure 2) and denote $V_{ML} \subset V_h$ finite element subspace which consists of functions $\varphi_{ML}$,

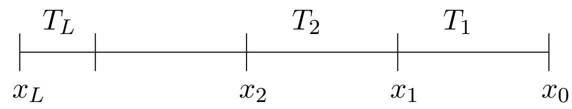



Figure 2: A coarse grid

linear on intervals $T_l,\ 1 \le l \le L$. Note that $\dim(V_{ML}) = O(logh^{-1})$. Define $A_\epsilon : V_h \to V_h$

$$(A_\epsilon \varphi^h_, \psi^h)_{L_2(I)} = a_\epsilon(\varphi^h, \psi^h), \quad \forall \varphi^h, \psi^h \in V_h,$$

$Q_{ML} : V_h \to V_{ML}$ is $L_2$-orthoprojector from $V_h$ onto $V_{ML}$ and $a_\epsilon(\varphi, \psi)$-orthoprojector from $V_h$ onto $V_{ML}$ by $P_{ML}$ :

$$a_\epsilon(P_{ML}\varphi^h, \psi^h_{ML}) = a_\epsilon(\varphi^h, \psi^h_{ML}), \quad \forall \varphi^h \in V_h,\ \psi^h_{ML} \in V_{ML}.$$

Then for operator $P_{ML}$ we have the following representation

$$P_{ML} = Q_{ML} A^{-1}_{ML} Q_{ML} A_\epsilon,$$

where $A_{ML} : V_{ML} \to V_{ML}$ is a restriction of $A_\epsilon$ on $V_{ML}$ such that

$$\begin{aligned} a(\varphi^h_{ML}, \psi^h_{ML}) &= (A_{ML}\varphi^h_{ML}, \psi^h_{ML})_{L_2(I)} \\ &= (A_\epsilon \varphi^h_{ML}, \psi^h_{ML})_{L_2(I)} \\ &= a(\varphi^h_{ML}, \psi^h_{ML}), \quad \forall \varphi^h_{ML}, \psi^h_{ML} \in V_{ML}. \end{aligned}$$

Let $A_{00} : V_h \to V_h$ such that

$$\|\varphi^h\|^2_{H^{1/2}_{00}(I)} \preceq (A_{00}\varphi^h, \varphi^h)_{L_2(I)} \preceq \|\varphi^h\|^2_{H^{1/2}_{00}(I)}, \quad \forall \varphi^h \in V_h,$$

**Lemma 3.8** *If we let*

$$\Sigma^{-1} = Q_{ML} A^{-1}_{ML} Q_{ML} + A^{-1}_{00},$$

*then (Joachim Sehöberl, private communication)*

$$(A^{-1}_\epsilon \varphi^h, \varphi^h)_{L_2(I)} \preceq (\Sigma^{-1}\varphi^h, \varphi^h)_{L_2(I)} \preceq (A^{-1}_\epsilon \varphi^h, \varphi^h)_{L_2(I)}, \quad \forall \varphi^h \in V_h.$$

Now we consider the problem (10) in more general case with domain $\Omega$ consists of n non-overlapping subdomains $\Omega_i$ and with an arbitrary distribution of piecewise-constant coefficients $p(x) = \epsilon_i = \text{const} > 0,\ x \in \Omega_i$, and there is a unique cross point $O$ in $\Omega,\ O = \cap^n_{i-1}\partial\Omega_i$ (see Figure 3).

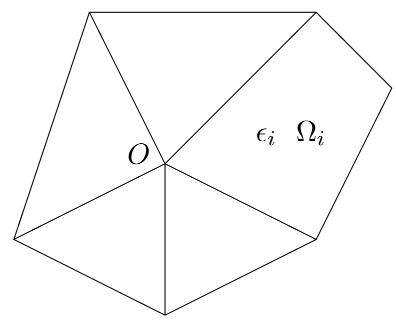



Figure 3: A sample of several subdomains $\Omega_i$.

We assume that the coefficients $\epsilon_i$ such that

$$\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_n \tag{14}$$

To construct a preconditioning operator for this case, the same technique as in examples 1-3 can be used.
Denote $G_i = \Omega \setminus \cup_{j=1}^{i-1} \bar{\Omega}_j,\ i = 2, 3, \cdots, n$. We start with domain $\Omega_1$ where the coefficient $\epsilon_1$ is the biggest, $\Gamma_1 = \partial\Omega_1$, the trace space $V_1$ and the subspace $W_1$ (11). Then define for $i = 2, 3, \cdots, n$,

$$\begin{aligned} W_i &= \{u^h \in W \mid u^h(x) = 0, \quad x \in \Omega \setminus G_i\}, \\ \Gamma_i &= \partial\Omega_i, \\ V_i &= \{\varphi^h \mid \varphi^h(x) = u^h(x), \quad x \in \Gamma_i, \quad u^h \in W_i\}. \end{aligned}$$

In the space $V_i$, let us define the explicit norm-preserving extension operator $t_{\Gamma_i} : V_i \to W_i$, [3], [7]. Introduce the matrix $A_{G_i}$

$$(A_{G_i} u_i, v_i) = a(u_i^h, v_i^h), \quad \forall u_i^h, v_i^h \in W_{G_i} \cap H_0^1(G_i) \quad i = 2, 3, \cdots, n,$$

and extend $A_{G_i}$ by zero outside of $G_i$ and denote by $\breve{A}_{G_i}^+$ pseudo-inverse operator for this extension of $A_{G_i}$, for $i = 2, 3, \cdots, n$.
For the subdomain $\Omega_1$ with the coefficient $\epsilon_1$, as above, we can prove that

$$(A^{-1}u, u) \preceq ((\frac{1}{\epsilon_1} C_\Omega^{-1} + \breve{A}_{G_2}^+)u, u) \preceq (A^{-1}u, u), \quad \forall u \in R^N. \tag{15}$$

Let us consider $G_i$ for some subindex $i$, $2 \leq i \leq n-1$. We assume that $G_i$ is a connected set. Otherwise we consider each connected component of $G_i$.

**Lemma 3.9** *Let $\bar{\Omega}_i \cap (\Omega \setminus G_i) \neq \emptyset$. Then*

$$(\breve{A}_{G_i}^+ u, u) \preceq ((\frac{1}{\epsilon_i} \breve{C}_{G_i}^+ + \breve{A}_{G_{i+1}}^+)u, u) \preceq (\breve{A}_{G_i}^+ u, u),$$

*for any vector $u$ related to $u^h \in W_i$.*

*prrof :*
The assumption $\bar{\Omega}_i \cap (\Omega \setminus G_i) \neq \emptyset$ means that the subdomain $\Omega_i$ with the biggest coefficient $\epsilon_i$ is a neighbor subdomain with $\Omega \setminus G_i$ and $r_{(\partial\Omega_i \cap \partial(\Omega\setminus G_i))} u^h = 0$ for any $u^h \in W_i$. Then

$$a(t_{\Gamma_i}\varphi_i^h,\ t_{\Gamma_i}\varphi_i^h) \preceq \epsilon_i \|u_i^h\|^2_{H^1(\Omega_i)}$$

for any $u_i^h \in W_i$, $\varphi_i^h = r_{\Gamma_i} u_i^h$. Using the same technique as in examples 1,2 for construction of a preconditioner for $A_{G_i}$, we get the statement of Lemma 3.9.

Now let $\Omega_i$ is an interior subdomain of $G_i$ and denote $\gamma_{i_l} = \partial\Omega_{i_l} \cap \partial\Omega_i$, $\gamma_{i_r} = \partial\Omega_i \cap \partial\Omega_{i_r}$ (see Figure 4), and let $G_{i_l}$, $G_{i_r}$ are subdomains of $G_i$ such that

$$G_i = (G_{i_l} \cup \gamma_{i_l} \cup G_{i_r}) \setminus \{O\},\ \Omega_i \subset G_{i_r}.$$

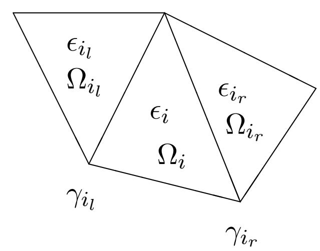



Figure 4: A sample of $\Omega_i$ and neighbor subdomains.

Let $G_i$ consists of $q$ subdomains $\Omega_{i,j}$ and let us introduce new numeration of subdomains in $G_i$ (see Figure 5). Here $\Omega_i = \Omega_{i,m}$ for some $m$ such that $1 < m < q$.

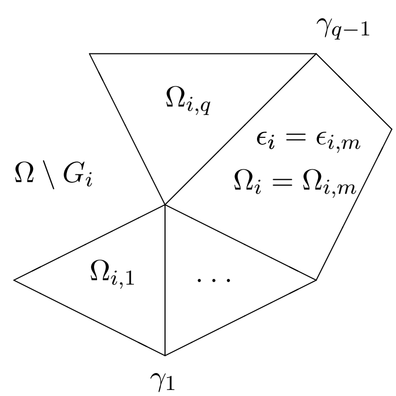



Figure 5: A sample of $G_i$.

Denote

$$\begin{aligned} \gamma_j &= (\partial\Omega_{i,j}) \cap (\partial\Omega_{i,j+1}), \quad j = 1, 2, \cdots q-1, \\ \gamma_{i_l} &= \gamma_{m-1}, \quad \gamma_{i_r} = \gamma_m. \end{aligned}$$

Without loss of generality, we can define decomposition (9) such that each $\gamma_j$ has the same number of nodes of the triangulation $\Omega^h$ and denote these nodes by $x_{j,k}$, $j = 1, 2, \cdots, q-1$,

$k = 0, 1, \cdots, m$, using the natural ordering of nodes on $\gamma_j$ such that $x_{j,0} = O$ and $x_{j,m} \in \partial\Omega_{i,j}$. Introduce trace space $V_{\gamma_j} = r_{\gamma_j} W_i,\ j = 1, 2, \cdots, q-1$, denote $J_i = \cup_{j=1}^{q} \partial\Omega_{i,j}$ and $V_{J_i} = r_{J_i} W_i$. Define the extension operator $t_{s,i} : V_{\gamma_{i_l}} \to V_{J_i}$

$$(t_{s,i}\varphi^h_{\gamma_{i_l}})(x_{j,k}) = \varphi^h(x_{m-1,k}),$$

$j = 1, 2, \cdots q-1,\ k = 0, 1, \cdots, m$, and $t_{\gamma_{i_l}} : V_{\gamma_{i_l}} \to W_i$,

$$(t_{\gamma_{i_l}}\varphi^h_{\gamma_{i_l}})(x) = u^h(x) = (\bar{t}_{\Gamma_{i,j}} r_{\Gamma_{i,j}}(t_{s,i}\varphi^h_{\gamma_{i_l}}))(x), \quad x \in \Omega_{i,j}, \quad \forall \varphi^h_{\gamma_{i_j}} \in V_{\gamma_{i_l}},$$

where $\bar{t}_{\Gamma_{i,j}} : V_{\Gamma_{i,j}} \to W_{\Omega_{i,j}}$ are explicit norm preserving extension operators

$$\|\bar{t}_{\Gamma_{i,j}}\varphi^h_i\|^2_{H^1(\Omega_{i,j})} \preceq \|\varphi^h_i\|^2_{H^{1/2}(\Gamma_{i,j})}, \quad \forall \varphi^h_{i,j} \in V_{\Gamma_{i,j}} = r_{\Gamma_{i,j}} W.$$

It is easy to see that [16]

$$\begin{aligned}\sum_{j=1}^{q} \epsilon_{i,j}\|t_{s,i}\varphi^h_{\gamma_{i_l}}\|^2_{H^{1/2}(\Gamma_{i,j})} &\preceq \epsilon_{i,1}\|t_{s,i}\varphi^h_{\gamma_{i_l}}\|^2_{H^{1/2}_{00}(\gamma_1)} + \epsilon_i\|t_{s,i}\varphi^h_{\gamma_{i_l}}\|^2_{H^{1/2}(\Gamma_i)} + \epsilon_{i,q}\|t_{s,i}\varphi^h_{\gamma_{i_l}}\|^2_{H^{1/2}_{00}(\gamma_{q-1})} \\ &\preceq \sum_{j=1}^{q} \epsilon_{i,j}\|t_{s,i}\varphi^h_{\gamma_{i_l}}\|^2_{H^{1/2}(\Gamma_{i,j})}.\end{aligned} \tag{16}$$

Define subspaces

$$\begin{aligned}\tilde{W}_{\gamma_{i_l}} &= t_{\gamma_{i_l}} V_{\gamma_{i_l}}, \\ \check{W}_{G_{i_l}} &= \{u^h \in W_i \mid u^h(x) = 0,\ x \notin G_{i_l}\}, \\ \check{W}_{G_{i_r}} &= \{u^h \in W_i \mid u^h(x) = 0,\ x \notin G_{i_r}\}.\end{aligned}$$

As in example 3, the following decomposition $W_i = \tilde{W}_{\gamma_{i_l}} + \check{W}_{G_{i_l}} + \check{W}_{G_{i_r}}$ is stable :

**Lemma 3.10** *For any $u^h \in W_i$ there exist $\tilde{u}^h_{\gamma_{i_l}} \in \tilde{W}_{\gamma_{i_l}},\ \check{u}^h_{G_{i_l}} \in \check{W}_{G_{i_l}},\ \check{u}^h_{G_{i_r}} \in W_{G_{i_r}}$ such that*

$$\tilde{u}^h_{\gamma_{i_l}} + \check{u}^h_{G_{i_l}} + \check{u}^h_{G_{i_r}} = u^h,$$

$$a(\tilde{u}^h_{\gamma_{i_l}}, \tilde{u}^h_{\gamma_{i_l}}) + a(\check{u}^h_{G_{i_l}}, \check{u}^h_{G_{i_l}}) + a(\check{u}^h_{G_{i_r}}, \check{u}^h_{G_{i_r}}) \preceq a(u^h, u^h).$$

*proof :*
For $u^h \in W_i$, define $\varphi^h_{\gamma_{i_l}} = r_{\gamma_{i-l}} u^h \in V_{\gamma_{i_l}}$ and put

$$\begin{aligned}\tilde{u}^h_{\gamma_l} &= t_{\gamma_{i_l}}\varphi^h_{\gamma_{i_l}}, \\ \check{u}^h_{G_{i_l}}(x) &= u^h(x) - \tilde{u}^h_{\gamma_{i_l}}(x), \quad x \in G_{i_l}, \\ \check{u}^h_{G_{i_r}}(x) &= u^h(x) - \tilde{u}^h_{\gamma_{i_l}}(x), \quad x \in G_{i_r}.\end{aligned}$$

To complete the proof, we can use the same technique as in Lemma 3.8 and (16).
Denote $\tilde{\epsilon}_i = \max\{\epsilon_{i,1}, \epsilon_{i,q}\}$ and let a matrix $\Sigma_i$ such that

$$(\Sigma_i\varphi_{\gamma_{i_l}}, \varphi_{\gamma_{i-l}}) \preceq \epsilon_i\|\varphi^h_{\gamma_{i_l}}\|^2_{\check{H}^{1/2}(\gamma_{i_l})} + \tilde{\epsilon}_i\|\varphi^h_{\gamma_{i_l}}\|^2_{H^{1/2}_{00}(\gamma_{i_l})} \preceq (\Sigma_i\varphi_{\gamma_{i_l}}, \varphi_{\gamma_{i_l}}), \quad \forall \varphi^h_{\gamma_{i_l}} \in V_{\gamma_{i_l}}. \tag{17}$$

**Lemma 3.11** *If $\Omega_i$ is an interior subdomain of $G_i$, then the following inequalities hold :*

$$(\check{A}^+_{G_i}u,u) \preceq ((t_{\gamma_{i_l}}\Sigma_i^{-1}t^*_{\gamma_{i_l}} + \check{A}^+_{G_{i_l}} + \frac{1}{\epsilon_i}\check{C}^+_{G_{i_r}} + \check{A}^+_{(G_{i_r}\setminus\bar{\Omega}_i)})u,u) \preceq (\check{A}^+_{G_i}u,u)$$

*for any vector $u$ related to $u^h \in W_i$.*

*proof :*
Since $\epsilon_i$ is a biggest coefficient in $G_{i_r}$, we have

$$(\check{A}^+_{G_{i_r}}u,u) \preceq ((\frac{1}{\epsilon_i}\check{C}^+_{G_{i_r}} + \check{A}^+_{(G_{i_r}\setminus\bar{\Omega}_i)})u,u) \preceq (\check{A}^+_{G_{i_r}}u,u).$$

Then using ASM, (17), and the same approach as in Theorem 3.5, we get Lemma 3.11.
For the case $\bar{\Omega}_i \cap (\Omega \setminus G_i) \neq \emptyset$, denote $\gamma_{i_l} = \partial\Omega_i \cap \partial G_{i+1}$ and consider a preconditioner for this case, which is equivalent to the preconditioner from Lemma 3.9 and has the same structure as in Lemma 3.11. The following lemma holds.

**Lemma 3.12** *Let $\bar{\Omega}_i \cap (\Omega \setminus G_i) = \emptyset$. Then*

$$(\check{A}^+_{G_i}u,u) \preceq ((\frac{1}{\epsilon_i}\check{C}^+_{G_i} + t_{\gamma_{i_l}}\Sigma_i^{-1}t^*_{\gamma_{i_l}} + \check{A}^+_{G_{i+1}})u,u) \preceq (\check{A}^+_{G_i}u,u),$$

*for any vector $u$ related to $u^h \in W_i$.*

*proof :*
By the Lemma 3.9

$$(\check{A}^+_{G_i}u,u) \preceq ((\frac{1}{\epsilon_i}\check{C}^+_{G_i} + t_{\gamma_{l_i}}\Sigma_i^{-1}t^*_{\gamma_{l_i}} + \check{A}^+_{G_{i+1}})u,u)$$

is hold. Since $\gamma_{i_l}$ is from inside of $G_i$, and $\tilde{\epsilon}_i = \epsilon_i$, then from the previous analysis we have

$$((t_{\gamma_{l_i}}\Sigma_i^{-1}t^*_{\gamma_{l_i}})u,u) \preceq (A^+_{G_i}u,u)$$

for any $u^h \in W_i$ and evidently, Lemma 3.12 holds.
According to Section 2 and Examples 1-3, to simplify a preconditioner in (15), it is enough to construct an effective preconditioner for $A_{G_2}$ in subspace $W_2$. Then, to simplify a preconditioner for $A_{G_2}$, we can use Lemma 3.9 or Lemma 3.11 and step by step a number of subdomains in corresponding $G_{i+1}$ or $G_{i_l}$, $G_{i_r} \setminus \bar{\Omega}_i$ is less then in $G_i$. Finally, note that last $\check{A}^+_n$ can be defined as $\check{A}^+_n = \frac{1}{\epsilon_n}\check{C}^+_n$ and an optimal preconditioner $B^{-1}_{opt}$ for the case of the coefficients (14) can be constructed.
To define more uniform structure of $B^{-1}_{opt}$, put

$$B^{-1}_{opt} = \frac{1}{\epsilon_1}C^{-1}_\Omega + \Sigma^n_{i=2}\frac{1}{\epsilon_i}\check{C}^+_{G_i} + \Sigma^{n-1}_{i=1}t_{\gamma_{i_l}}\Sigma_i^{-1}t^*_{\gamma_{i_l}}, \tag{18}$$

Then from Lemma 3.11and 3.12 we have

**Theorem 3.6** *The following inequalities hold :*

$$(A^{-1}u,u) \preceq (B^{-1}_{opt}u,u) \preceq (A^{-1}u,u), \quad \forall u \in R^N,$$

*where $B^{-1}_{opt}$ is from (18).*

Note that the arithmetical cost of implementation of $B_{opt}^{-1}$ from (18) is more then construction of a preconditioner step by step using Lemma 3.9, Lemma 3.11, but the arithmetical cost of implementation of $B_{opt}^{-1}$ is still proportional to the number of degrees of freedom in the original problem.

# 4 Numerical Experiments

In this section we consider the two different cases as the domain has a cross point or not.

## 4.1 Numerical Experiments for the Problems without a Cross Point

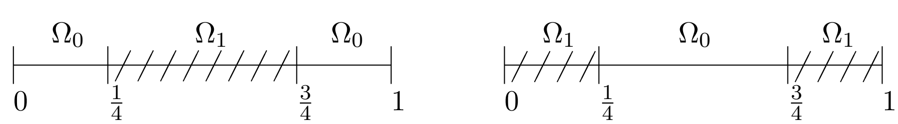
Here we present the results of two different test cases. One test case having small coefficients $\epsilon$ in the inner subdomain and $\epsilon = 1$ in the outer subdomain supports the efficiency of the overlapping domain decomposition method introduced above (Case 1). The other test case having opposite coefficients in the subdomains is added for merely the purpose of a comparison with the former case (Case 2).
For these examples we consider the following boundary value problem :

$$\begin{cases} -\mathrm{div}(p(x)\nabla u) + q(x)u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $p(x) = \begin{cases} 1 & \text{in } \Omega_0, \\ \epsilon & \text{in } \Omega_1, \end{cases}$ $\quad \overline{\Omega} = \overline{\Omega}_0 \bigcup \overline{\Omega}_1$ , $\Omega_0 \bigcap \Omega_1 = \emptyset$,
and $q(x) = 0$ or 1.

- **1 – Dimensional examples**

For 1-dimensional example, we consider the shape of domain $\Omega = (0, 1)$ decomposed as in Figure 6 and discretize the domain uniformly using finite element method with mesh size $h = 1/n$ and nodes $x_i = ih$, $i = 1, 2, \cdots, n-1$.




Case 1 Case 2

Figure 6: Two test cases of 1-D examples

To solve the above problem we use preconditioned conjugate gradient method with initial vector $\exp(10x_i)\sin(\pi x_i)u_1(x_i)$ with the following preconditioning operator

$$B_{ov}^{-1} = C^{-1} + \frac{1}{\epsilon}C_1^{+},$$

$C$ and $C_1$ are the finite element analogs of the operator $-\Delta + q \cdot I$ in $\Omega$ and $\Omega_1$ respectively, where $\Delta$ is the Laplace operator and I is the identity operator. Here $u_1$ is 1 and -1 where the node number of $x_i$ is odd and even, respectively, such as

$$u_1(x_i) = (-1)^i,\ i = 1, 2, \cdots, n-1.$$

And for the stoping criterion, when TOL=$10^{-5}$ is used for the tolerance,

$$(Au^k, u^k)^{\frac{1}{2}} \leq TOL(Au^0, u^0)^{\frac{1}{2}}$$

where $u_k$ is a corresponding iterate after $k$ steps of this preconditioned conjugate gradient method.

The left table of Table 1 shows the change of iteration number when $\epsilon = 10^{-2}$ and $\epsilon = 10^{-4}$. We can see that the iteration number is not sensitive to mesh size h and coefficient $\epsilon$. And the right table represents the iteration number of the problem $-(p(x)\tau u^{'})^{'} + u = 0$ which is a model for solving parabolic problems. Here, we can see the result that the iteration number decreases as time step parameter $\tau$ decreases.

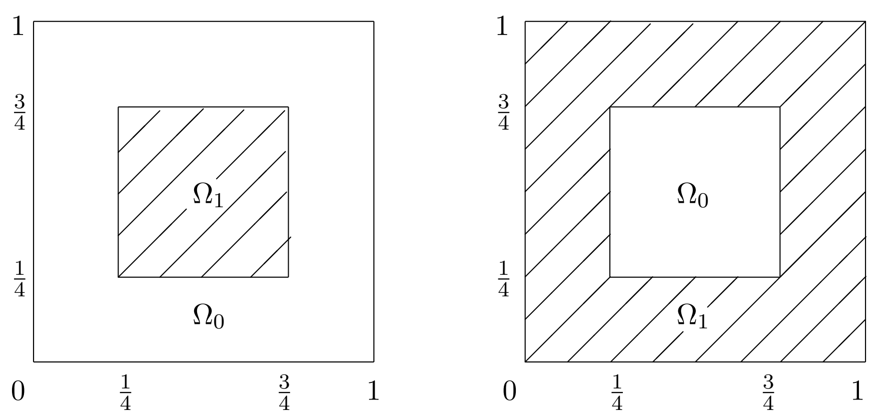
| $\epsilon$ | h | Case 1 | | Case 2 | |
|---|---|---|---|---|---|
| | | $q=0$ | $q=1$ | $q=0$ | $q=1$ |
| $10^{-2}$ | 1/8 | 3 | 6 | 4 | 5 |
| | 1/16 | 3 | 6 | 4 | 6 |
| | 1/32 | 3 | 6 | 4 | 6 |
| $10^{-4}$ | 1/8 | 3 | 5 | 4 | 5 |
| | 1/16 | 3 | 6 | 4 | 6 |
| | 1/32 | 3 | 6 | 4 | 6 |

| $\epsilon$ | $\tau$ | h | Case 1 | Case 2 |
|---|---|---|---|---|
| | | | $q=1$ | $q=1$ |
| $10^{-2}$ | $10^{-2}$ | 1/8 | 6 | 5 |
| | | 1/16 | 9 | 7 |
| | | 1/32 | 9 | 9 |
| | $10^{-5}$ | 1/8 | 5 | 3 |
| | | 1/16 | 5 | 5 |
| | | 1/32 | 6 | 6 |

Table 1: Numerical results of 1-D examples

- **2 – Dimensional examples**

For 2-dimensional case we consider the shape of domain $\Omega = (0,1) \times (0,1)$ such as the following Figure 7.




Case 1 Case 2

Figure 7: Two test cases of 2-D examples

We discretize the domain with uniform triangulation and solve $Au = 0$ as in the 1-dimensional examples, and for the conjugate gradient method,we take initial vector

$\exp(10x_i)\sin(\pi x_i)u_1(x_i)\sin(\pi y_j)u_1(y_j)$, $i, j = 1, 2, \cdots, n-1$. Results of numerical experiments are a shown Table 2:

| $\epsilon$ | h | Case 1 | | Case 2 | |
|---|---|---|---|---|---|
| | | $q=0$ | $q=1$ | $q=0$ | $q=1$ |
| $10^{-2}$ | 1/8 | 8 | 11 | 10 | 15 |
| | 1/16 | 10 | 12 | 12 | 15 |
| | 1/32 | 11 | 13 | 13 | 16 |
| $10^{-4}$ | 1/8 | 8 | 11 | 10 | 16 |
| | 1/16 | 10 | 12 | 13 | 16 |
| | 1/32 | 11 | 13 | 14 | 17 |

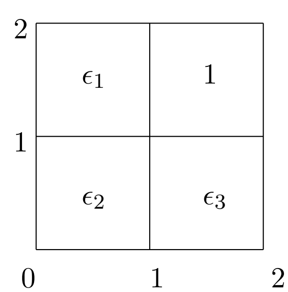
| $\epsilon$ | $\tau$ | h | Case 1 | Case 2 |
|---|---|---|---|---|
| | | | $q=1$ | $q=1$ |
| $10^{-2}$ | $10^{-2}$ | 1/8 | 12 | 14 |
| | | 1/16 | 13 | 16 |
| | | 1/32 | 14 | 17 |
| | $10^{-5}$ | 1/8 | 6 | 6 |
| | | 1/16 | 7 | 7 |
| | | 1/32 | 9 | 9 |

Table 2: Numerical results of 2-D examples

Table 2 show that the iteration number is not very sensitive to mesh size $h$ and $\epsilon$ likewise 1-dimensional case.

## 4.2 Numerical Experiments for the Problems with a Cross Point

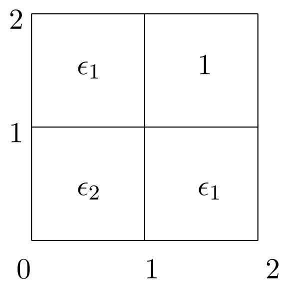
For these examples we consider the problem (10), where $\Omega = (0, 2) \times (0, 2)$ and coefficient $p(x)$ are represented in each figure. We discretize the domain with uniform triangulation and solve $Au = 0$. The following tables which correspond to the left figure, represent a iteration number with the different initial data $u_1, u_2, u_3$. Here $u_1$ is $u_1(x_i, y_j) = (-1)^i(-1)^j$, $i, j = 1, 2 \cdot\cdot, 2n-1$, $u_2 = \sin(\pi y_i)u_1(y_i)$ and $u_3 = \exp(10x_i)\sin(\pi x_i)u_1(x_i)u_2(y_i)$. For the stoping criterion, TOL $= 10^{-6}$ is used and the results are as follows. By numerical experiments, we can see that the suggested algorithms are effective.




$\epsilon_2 = \epsilon_1^2$, $\epsilon_3 = \epsilon_1^3$

| $\epsilon_1$ | h | initial data | | |
|---|---|---|---|---|
| | | $u_1$ | $u_2$ | $u_3$ |
| $10^{-2}$ | 1/4 | 9 | 6 | 3 |
| | 1/8 | 10 | 7 | 3 |
| | 1/16 | 10 | 7 | 3 |
| $10^{-4}$ | 1/4 | 7 | 6 | 2 |
| | 1/8 | 8 | 7 | 2 |
| | 1/16 | 8 | 7 | 2 |

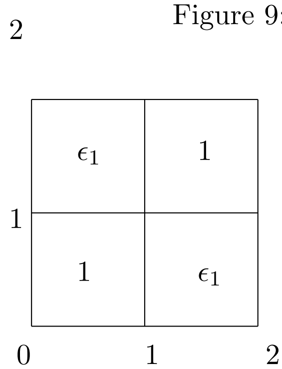
Figure 8: Numerical experiment of Example 1




$\epsilon_2 = \epsilon_1^2$

| $\epsilon_1$ | h | initial data | | |
|---|---|---|---|---|
| | | $u_1$ | $u_2$ | $u_3$ |
| $10^{-2}$ | 1/4 | 8 | 6 | 3 |
| | 1/8 | 8 | 7 | 3 |
| | 1/16 | 9 | 7 | 3 |
| $10^{-4}$ | 1/4 | 6 | 6 | 2 |
| | 1/8 | 7 | 7 | 2 |
| | 1/16 | 7 | 7 | 2 |

Figure 9: Numerical experiment of Example 2




| $\epsilon_1$ | h | initial data | | |
|---|---|---|---|---|
| | | $u_1$ | $u_2$ | $u_3$ |
| $10^{-2}$ | 1/4 | 3 | 3 | 4 |
| | 1/8 | 3 | 3 | 4 |
| | 1/16 | 3 | 3 | 4 |
| $10^{-4}$ | 1/4 | 3 | 3 | 4 |
| | 1/8 | 3 | 3 | 4 |
| | 1/16 | 4 | 4 | 4 |

Figure 10: Numerical experiment of Example 3

**Acknowledgement:** The paper was completed during the Special Semester on Computational Mechanics in Linz 2005. The second author thanks the RICAM for the hospitality during his stay in Linz.

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