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Topological sensitivity analysis in the context of ultrasonic nondestructive testing

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TOPOLOGICAL SENSITIVITY ANALYSIS IN THE CONTEXT OF ULTRASONIC NONDESTRUCTIVE TESTING

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ABSTRACT. The aim of the topological sensitivity analysis is to determine an asymptotic expansion of a shape functional with respect to the variation of the topology of the domain. In this paper, we consider a state equation of the form div $(A\nabla u) + k^2 u = 0$ in dimensions 2 and 3. For that problem, the topological asymptotic expansion is obtained for a large class of cost functions and two kinds of topology perturbation: the creation of arbitrary shaped holes and cracks on which a Neumann boundary condition is prescribed. These results are illustrated by some numerical experiments in the context of the detection of defects in metallic plates by means of ultrasonic probing.

1. INTRODUCTION

Inspection problems can generally be seen as shape inversion problems. If techniques borrowed from shape optimization are now commonly accepted as good theoretical candidates to address shape inversion problems, their applications to inspection problems such as nondestructive testing or medical imaging are today relatively restricted. Let us give a brief overview of the existing shape optimization methods. The most widespread, the so-called classical shape optimization method [25], consists in deforming continuously the boundary of the domain to be optimized so as to decrease the criterion of interest. The main drawback of this approach is that it does not allow any topology changes: the final shape and the initial one, the "initial guess", contain the same number of holes. The consequence is that this method is not suitable either for defect(s) detection problems or for most optimal structural design problems. To get round this limitation, some techniques have been especially constructed to allow for topology variations. In the context of structural mechanics, several authors [1, 2, 3, 19] have introduced some intermediate material by using the homogenization theory. Then, to retrieve an admissible shape, the removing of matter is done by applying penalization techniques. The range of application of this approach being restricted to very particular cost functions, global optimization techniques like genetic algorithms and simulated annealing are used to handle more general problems (see *e.g.* [33]). Unfortunately, these methods have a high computational cost and can hardly be applied to industrial problems. An other approach relies on the topological sensitivity analysis that directly deals with the variable "topology". It has been mainly introduced by Friedman and Vogelius [9] in the case of shape inversion and by Schumacher [32], Sokolowski and Zochowski [34] in structural optimization. The principle is the following. Let us consider a cost function $\mathcal{J}(\Omega) = J_\Omega(u_\Omega)$ where u_Ω is the solution of a partial differential equation defined in the domain $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, a point $x_0 \in \Omega$ and a fixed open and bounded subset ω of $\mathbb{R}^N$ containing the origin. The "topological asymptotic expansion" is an expression of the form

$$\mathcal{J}(\Omega \setminus (\overline{x_0 + \rho\omega})) - \mathcal{J}(\Omega) = f(\rho)g(x_0) + o(f(\rho)), \tag{1.1}$$

where $f(\rho)$ is a positive function tending to zero with ρ. Therefore, to minimize the criterion, we have interest to remove matter where the "topological gradient" (also called "topological derivative") g is negative. This remark leads to new topology optimization algorithms.

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A general framework enabling to calculate the topological asymptotic for a large class of shape functionals has been worked out by Masmoudi [23]. It is based on an adaptation of the adjoint method and a domain truncation technique providing an equivalent formulation of the PDE in a fixed functional space. Using this framework, Garreau, Guillaume, Masmoudi and Sididris [12, 16, 17] have obtained the topological asymptotic expansions for several problems associated to linear and homogeneous differential operators. For such operators, but with a different approach, more general shape functionals are considered in [26]. We also refer the reader to [18, 24, 11] for a complete study of the asymptotic behavior of the solution $u_{\Omega\setminus(\overline{x_0+\rho\omega})}$ in various situations. The link between the shape and the topological derivatives has been stated by Feijóo *et al* [28, 8]. This gives rise to a generic method for deriving the latter. However, it seems rather restricted to circular or spherical holes. For the first time a topological sensitivity analysis for a non-homogeneous operator was performed in [31]. The case of a circular hole with a Dirichlet condition imposed on its boundary was considered.

In this paper, the physical problem of interest is related to nondestructive testing with ultrasound in the context of elastodynamics. Therefore, the governing equations at a fixed frequency involve a non-homogeneous differential operator of the form

$$u \mapsto \text{div }(A\nabla u) + k^2 u, \tag{1.2}$$

where A is a symmetric positive definite tensor. For such a problem, the topological asymptotic expansion is determined in dimensions 2 and 3 with respect to the creation of an arbitrary shaped hole and an arbitrary shaped crack on which a Neumann condition is prescribed. For the sake of simplicity, the mathematical study is presented for the Helmholtz operator ($A = I$), but it applies in the same way to any operator of the form (1.2) by taking the fundamental solution of the principal part of the operator as the kernel of the integral equations involved. We introduce an adjoint method that takes into account the variation of the functional space, so that a domain truncation is not needed. This formalism brings several technical simplifications, notably for the study of criteria depending explicitely on Ω, for which the truncation necessitates to transport the cost function in the fixed domain (see [16]). Compared to [31], not only the setting is more general and the approach is original, but the boundary condition on the inclusion, which plays a crucial role in the analysis, is different.

The paper is organized as follows. The adjoint method is presented in Section 2. The framework of the mathematical study is described in Section 3. The topological asymptotic analysis for a hole and a crack are carried out in Sections 4 and 5, respectively, the most technical proofs being reported in Section 8. The case of some particular cost functions is examined in Section 6. Section 7 is devoted to numerical experiments that highlight the relevance of the topological sensitivity approach for nondestructive testing applications.

2. An appropriate adjoint method

In this section, the adjoint method is generalized to a class of problems for which the solution belongs to a functional space that varies with the variable of control. Let $(\mathcal{V}_\rho)_{\rho\geq 0}$ be a family of Hilbert spaces on the complex field such that

$$\mathcal{V}_0 \subset \mathcal{V}_\rho \qquad \forall \rho \geq 0.$$

As we will see in the next section, the PDEs involved in topological sensitivity analysis can be formulated in such functional spaces provided that a Neumann condition is prescribed on the boundary of the inclusion. For all $\rho \geq 0$, let a_ρ be a sesquilinear and continuous form on $\mathcal{V}_\rho$ and let l_ρ be a semilinear and continuous form on $\mathcal{V}_\rho$. We assume that, for all $\rho \geq 0$, the variational problem

$$\begin{cases} u_\rho \in \mathcal{V}_\rho, \\ a_\rho(u_\rho, v) = l_\rho(v) \qquad \forall v \in \mathcal{V}_\rho \end{cases} \tag{2.1}$$

admits a unique solution. We consider the following assumption.

Hypothesis 2.1. *For all $\rho \geq 0$, there exist on $\mathcal{V}_\rho$ a sesquilinear and continuous form $\tilde{a}_\rho$ and a semilinear and continuous form $\tilde{l}_\rho$ such that*

$$\tilde{a}_\rho(u_0, v) = \tilde{l}_\rho(v) \qquad \forall v \in \mathcal{V}_\rho. \tag{2.2}$$

Consider now a criterion $j(\rho) = J_\rho(u_\rho) \in \mathbb{R}$. For all $\rho \geq 0$, the function J_ρ is supposed to be "$\mathbb{R}$-differentiable" at the point u_0, that is: there exists a linear and continuous form on $\mathcal{V}_\rho$ denoted by L_ρ such that

$$J_\rho(u_0 + h) - J_\rho(u_0) = \Re L_\rho(h) + o(\|h\|_{\mathcal{V}_\rho}), \tag{2.3}$$

were $\Re z$ denotes the real part of the complex number z.
Furthermore, we assume that, for all $\rho \geq 0$, the problem

$$\left\{ \begin{array}{l} v_\rho \in \mathcal{V}_\rho, \\ a_\rho(u, v_\rho) = -L_\rho(u) \qquad \forall u \in \mathcal{V}_\rho \end{array} \right. \tag{2.4}$$

admits a unique solution. We call v_0 the adjoint state. The two hypotheses that will furnish the first variation of the criterion are the following.

Hypothesis 2.2. *There exist two complex numbers δ_a and δ_l and a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, when ρ tends to zero,*

$$\begin{array}{rcl} f(\rho) & \longrightarrow & 0, \\ (a_\rho - \tilde{a}_\rho)(u_0, v_\rho) & = & f(\rho)\delta_a + o(f(\rho)), \\ (l_\rho - \tilde{l}_\rho)(v_\rho) & = & f(\rho)\delta_l + o(f(\rho)). \end{array}$$

Hypothesis 2.3. *There exists a real number δ_J such that*

$$J_\rho(u_\rho) - J_0(u_0) = \Re L_\rho(u_\rho - u_0) + f(\rho)\delta_J + o(f(\rho)).$$

Then, the asymptotic expansion of $j(\rho)$ is provided by the theorem below.

Theorem 2.1. *If hypotheses 2.1, 2.2 and 2.3 hold, then*

$$j(\rho) - j(0) = f(\rho)\Re(\delta_a - \delta_l + \delta_J) + o(f(\rho)).$$

Proof. Using Equation (2.1) and Hypothesis 2.1, we obtain

$$\begin{array}{rcl} j(\rho) - j(0) & = & J_\rho(u_\rho) - J_0(u_0) + \Re(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) + \Re a_\rho(u_\rho - u_0, v_\rho) \\ & & -\Re(l_\rho - \tilde{l}_\rho)(v_\rho). \end{array}$$

Hypotheses 2.2 and 2.3 and Equation (2.4) yield

$$\begin{array}{rcl} j(\rho) - j(0) & = & \Re L_\rho(u_\rho - u_0) + f(\rho)\delta_J + f(\rho)\Re\delta_a - \Re L_\rho(u_\rho - u_0) \\ & & -f(\rho)\Re\delta_l + o(f(\rho)), \end{array}$$

from which we deduce the announced result. □

3. The topological sensitivity problem

3.1. **Problem formulation.** Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, $N = 2$ or 3, with smooth boundary Γ. We assume for simplicity that Γ is piecewise of class $\mathcal{C}^\infty$, but this hypothesis could be considerably weakened. We consider a function $u_0 \in H^1(\Omega)$ satisfying the state equations

$$\left\{ \begin{array}{rcll} \Delta u_0 + k^2 u_0 & = & 0 & \text{in} \quad \Omega, \\ \partial_n u_0 & = & S u_0 + \sigma & \text{on} \quad \Gamma. \end{array} \right. \tag{3.1}$$

Here, $\mathbf{n}$ denotes the outward unit normal of Γ, $k \in \mathbb{C}$, $S \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ and $\sigma \in H^{-1/2}(\Gamma)$. For a given $x_0 \in \Omega$ and a small parameter $\rho > 0$, we denote by Ω_ρ the perturbed domain. We consider two situations.

- In the case of a perforation, let ω be a fixed open and bounded subset of $\mathbb{R}^N$ containing the origin and whose boundary Σ is the union of two graphs of functions of class $\mathcal{C}^1$ from $\mathbb{R}^{N-1}$ into $\mathbb{R}$ (this technical hypothesis could also be weakened). We define $\omega_\rho = x_0 + \rho\omega$, $\Sigma_\rho = \partial\omega_\rho$ and $\Omega_\rho = \Omega \setminus \overline{\omega_\rho}$ (see Figure 1 (a)).
- In the case of the creation of a crack, let Σ be a bounded manifold of dimension $N-1$ which can be represented by the graph of a function of class $\mathcal{C}^1$ from $\mathbb{R}^{N-1}$ into $\mathbb{R}$. We define $\Sigma_\rho = x_0 + \rho\Sigma$ and $\Omega_\rho = \Omega \setminus \overline{\Sigma_\rho}$ (see Figure 1 (b)).

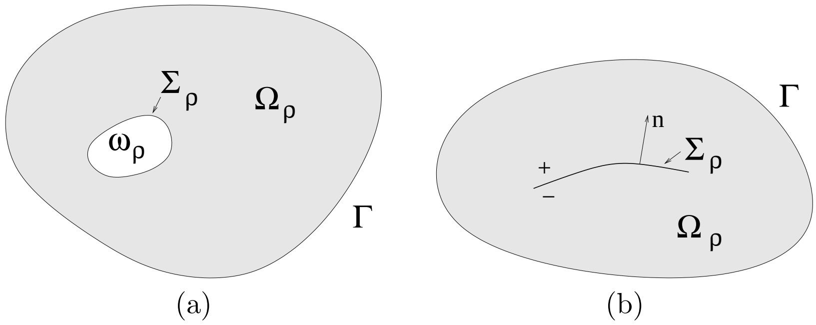


Figure 1. The perturbed domain: (a) perforated domain, (b) cracked domain.

Possibly changing the coordinate system, we suppose henceforth that $x_0 = 0$. In both cases, the new function $u_\rho \in H^1(\Omega_\rho)$ is assumed to solve the PDE

$$\left\{\begin{array}{rcll} \Delta u_\rho + k^2 u_\rho & = & 0 & \text{in} \quad \Omega_\rho, \\ \partial_n u_\rho & = & S u_\rho + \sigma & \text{on} \quad \Gamma, \\ \partial_n u_\rho & = & 0 & \text{on} \quad \Sigma_\rho. \end{array}\right. \tag{3.2}$$

3.2. **Well-posedness.** The variational formulation of System (3.2) writes in the standard form (2.1) with

$$\mathcal{V}_\rho = H^1(\Omega_\rho),$$

$$\left\{\begin{array}{ll} a_\rho(u,v) = \displaystyle\int_{\Omega_\rho} (\nabla u.\overline{\nabla v} - k^2 u\bar{v})dx - \int_\Gamma S u\bar{v}ds & \forall u, v \in \mathcal{V}_\rho, \\ l_\rho(v) = \displaystyle\int_\Gamma \sigma\bar{v}ds & \forall v \in \mathcal{V}_\rho. \end{array}\right. \tag{3.3}$$

As usual in analysis, the duality product between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ is denoted by an integral. This formulation applies also to Problem (3.1) when $\rho = 0$ by setting $\Omega_0 = \Omega$.

To insure well-posedness, we suppose that S verifies the following hypothesis.

Hypothesis 3.1. *The operator S is split into $S = S_0 + S_1$ where*

- $S_0 \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ *and satisfies*
  (1) $\displaystyle\int_\Gamma S_0\varphi\bar{\psi}ds = \overline{\int_\Gamma S_0\psi\bar{\varphi}ds} \qquad \forall \varphi, \psi \in H^{\frac{1}{2}}(\Gamma),$
  (2) $\displaystyle\int_\Gamma S_0\varphi\bar{\varphi}ds \leq 0 \qquad \forall \varphi \in H^{\frac{1}{2}}(\Gamma),$
  (3) $\left\{\displaystyle\int_\Gamma S_0\varphi\bar{\varphi}ds = 0\right\} \Rightarrow \{\varphi = 0$ *on a piece of nonzero measure of* $\Gamma\}$,
- $S_1 \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))$.

Let us give two examples of such an operator.

- If the physical problem under consideration is concerned by electromagnetic or acoustic waves propagating in the free space, then we come down to a PDE posed in a bounded domain by taking S as the Dirichlet-to-Neumann operator associated to the truncation on Γ of the Sommerfeld condition at infinity. When $k \in \{k \in \mathbb{C}, \Im k < 0\} \cup \mathbb{R}_+^*$ and Ω is a disc (2D case), it is proved in [22] that Hypothesis 3.1 holds.
- If the boundary condition on Γ is of the form
$$\partial_n u - ik'u = \Phi,$$
where $k' \in \mathbb{C}$ (transmission condition), then $Su = ik'u$ and Hypothesis 3.1 is automatically checked by setting $S_0 = 0$.

Then, we split a_ρ into $a_\rho = a_\rho^0 + a_\rho^1$ with

$$\begin{cases} a_\rho^0(u,v) = \displaystyle\int_{\Omega_\rho} \nabla u.\overline{\nabla v} dx - \int_\Gamma S_0 u\bar{v} ds, \\ a_\rho^1(u,v) = -k^2 \displaystyle\int_{\Omega_\rho} u\bar{v} dx - \int_\Gamma S_1 u\bar{v} ds. \end{cases} \tag{3.4}$$

We assume the following uniqueness property.

Hypothesis 3.2. *There exists $\rho_0 > 0$ such that for all $\rho \leq \rho_0$,*

$$\{a_\rho(u,v) = 0 \ \forall v \in \mathcal{V}_\rho\} \Rightarrow \{u = 0\},$$
$$\{a_\rho(u,v) = 0 \ \forall u \in \mathcal{V}_\rho\} \Rightarrow \{v = 0\}.$$

For the examples of operator S given above, Hypothesis 3.2 is satisfied (see *e.g.* [22] and [31]).

We consider a cost function J_ρ "$\mathbb{R}$-differentiable" in the sense of Equation (2.3). The following proposition is proved in Appendix 9.1.

Proposition 3.1. *If Hypotheses 3.1 and 3.2 are satisfied, then for all $\rho \leq \rho_0$*

- *the sesquilinear form a_ρ^0 is coercive on $\mathcal{V}_\rho$,*
- *the sesquilinear form a_ρ satisfies the inf-sup conditions*
$$\inf_{u\neq 0} \sup_{v \neq 0} \frac{|a_\rho(u,v)|}{\|u\|_{\mathcal{V}_\rho}\|v\|_{\mathcal{V}_\rho}} > 0, \qquad \inf_{v\neq 0} \sup_{u \neq 0} \frac{|a_\rho(u,v)|}{\|u\|_{\mathcal{V}_\rho}\|v\|_{\mathcal{V}_\rho}} > 0,$$
- *Problem (2.1) and Problem (2.4) are uniquely solvable.*

Remark 3.1. *The boundary condition on Γ could be replaced without any influence on the topological asymptotic analysis by any condition insuring that Problems (2.1) and (2.4) are well-posed. This can be observed in the proof of Lemma 8.3, which only requires the elliptic regularity property for the solutions of variational problems defined with the help of the sesquilinear form a_0.*

We wish now to apply Theorem 2.1 in this context. The imbedding $\mathcal{V}_0 \subset \mathcal{V}_\rho$ is defined by the restriction $u \in \mathcal{V}_0 \mapsto u_{|\Omega_\rho} \in \mathcal{V}_\rho$. To simplify the writing, the function $u_{|\Omega_\rho}$ will be still denoted by u. The analysis will be carried out in three steps:

(1) to define $\tilde{a}_\rho$ and $\tilde{l}_\rho$ such that Hypothesis 2.1 holds,
(2) to determine the function $f(\rho)$ and the complex numbers δ_a and δ_l such that Hypothesis 2.2 holds,
(3) for some examples of cost function, to determine δ_J such that Hypothesis 2.3 holds.

For the first two points, the cases of a perforation and of a crack will be studied separately (Sections 4 and 5). According to Theorem 2.1 the topological gradient at the origin will be

$$g(0) = \Re(\delta_a - \delta_l + \delta_J).$$

Then, a shift of the coordinate system will provide immediately $g(x)$ for any $x \in \Omega$.

4. Creation of a hole

In this section, we focus on the case of a perforation: $\Omega_\rho = \Omega \setminus \overline{\omega_\rho}$, $\Sigma_\rho = \partial\omega_\rho$.

4.1. **Formulation of the initial problem in the perforated domain.** For all $\rho \geq 0$, we define the sesquilinear form

$$b_\rho(u,v) = \int_{\omega_\rho} (\nabla u.\overline{\nabla v} - k^2 u\bar{v})dx \qquad \forall u,v \in H^1(\omega_\rho).$$

Using the Poincaré inequality, it is easy to check that, when ρ is sufficiently small (namely k diam $(\omega_\rho) < 1$), b_ρ is coercive on $H^1_0(\omega_\rho)$. For such a ρ and $\varphi \in H^{1/2}(\Sigma_\rho)$, let $h^\varphi_\rho \in H^1(\omega_\rho)$ be the solution of

$$\begin{cases} \Delta h^\varphi_\rho + k^2 h^\varphi_\rho &= 0 \quad \text{in} \quad \omega_\rho, \\ h^\varphi_\rho &= \varphi \quad \text{on} \quad \Sigma_\rho. \end{cases} \tag{4.1}$$

We set for all $u,v \in H^1(\Omega_\rho)$

$$\begin{cases} \tilde{a}_\rho(u,v) = a_\rho(u,v) + b_\rho(h^u_\rho, h^v_\rho), \\ \tilde{l}_\rho(v) = l_\rho(v). \end{cases}$$

It is then standard in PDE theory that Equation (2.2) holds.

Since $\tilde{l}_\rho = l_\rho$, we have by construction

$$\delta_l = 0.$$

For obtaining the general expression of the topological asymptotic, it remains to determine $f(\rho)$ and δ_a, that is, to calculate the first order expansion with respect to ρ of the quantity

$$(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) = -b_\rho(u_0, h^{v_\rho}_\rho).$$

In this equality, we have taken into account the fact that, by uniqueness, $h^{u_0}_\rho = u_0$. The first step is to estimate $h^{v_\rho}_\rho$.

4.2. **Preliminary calculus.** We set

$$w_\rho = v_\rho - v_0.$$

In order to estimate w_ρ, we need the following assumption on the right hand side of the adjoint equation.

Hypothesis 4.1. *There exists a function L of regularity $\mathcal{C}^0 \cap H^1$ in the vicinity of the origin such that for all $u \in H^1(\Omega)$ and for all ρ small enough,*

$$L_0(u) = L_\rho(u_{|\Omega_\rho}) + \int_{\omega_\rho} L u dx. \tag{4.2}$$

The following lemma provides in particular the variational problem solved by w_ρ.

Lemma 4.1. *For all ρ sufficiently small, we have*

(1) *in the sense of distributions,*

$$\Delta v_0 + \bar{k}^2 v_0 = \overline{L} \qquad \text{in } \omega_\rho, \tag{4.3}$$

(2) *for all $u \in H^1(\omega_\rho)$,*

$$b_\rho(u, v_0) = \int_{\Sigma_\rho} \overline{\partial_n v_0} u ds - \int_{\omega_\rho} L u dx, \tag{4.4}$$

(3) *for all $u \in H^1(\Omega_\rho)$,*

$$a_\rho(u, w_\rho) = \int_{\Sigma_\rho} \overline{\partial_n v_0} u ds. \tag{4.5}$$

Proof. (1) To obtain (4.3), it suffices to consider Equation (2.4) with $\rho = 0$ and a test function v of class $\mathcal{C}^\infty$ and whose support is included in ω_ρ.

(2) Equation (4.4) results from the Green formula and Equation (4.3).
(3) Let $\tilde{u} \in H^1(\Omega)$ be any extension of u in ω_ρ. We have

$$\begin{aligned} a_\rho(u, v_0) &= a_0(\tilde{u}, v_0) - b_\rho(\tilde{u}, v_0) \\ &= -L_0(\tilde{u}) - \int_{\Sigma_\rho} \overline{\partial_n v_0} \tilde{u} ds + \int_{\omega_\rho} L\tilde{u} dx \\ &= -L_\rho(u) - \int_{\Sigma_\rho} \overline{\partial_n v_0} u ds. \end{aligned}$$

The fact that $a_\rho(u, w_\rho) = a_\rho(u, v_\rho) - a_\rho(u, v_0)$ leads to (4.5).

□

Let $S^* \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ be the adjoint operator of S, defined by

$$\int_\Gamma S\varphi\bar{\psi} ds = \overline{\int_\Gamma S^*\psi\bar{\varphi} ds} \qquad \forall \varphi, \psi \in H^{\frac{1}{2}}(\Gamma). \tag{4.6}$$

Then, the classical formulation associated to Problem (4.5) reads:

$$\left\{ \begin{array}{rcll} \Delta w_\rho + \bar{k}^2 w_\rho &=& 0 & \text{in} \quad \Omega_\rho, \\ \partial_n w_\rho &=& -\partial_n v_0 & \text{on} \quad \Sigma_\rho, \\ \partial_n w_\rho &=& S^* w_\rho & \text{on} \quad \Gamma. \end{array} \right. \tag{4.7}$$

For all $\rho \geq 0$ and $\varphi \in H^{1/2}(\Sigma_\rho)$, we define the function l_ρ^φ as the solution in $H^1(\omega_\rho)$ to

$$\left\{ \begin{array}{rcll} \Delta l_\rho^\varphi &=& 0 & \text{in} \quad \omega_\rho, \\ l_\rho^\varphi &=& \varphi & \text{on} \quad \Sigma_\rho. \end{array} \right.$$

The following lemma provides another expression of $(a_\rho - \tilde{a}_\rho)(u_0, v_\rho)$ which is more convenient for the asymptotic analysis.

Lemma 4.2.

$$\begin{aligned} (a_\rho - \tilde{a}_\rho)(u_0, v_\rho) \quad = \quad & -\int_{\Sigma_\rho} \overline{\partial_n v_0}(u_0 - u_0(0)) ds + \int_{\omega_\rho} L(u_0 - u_0(0)) dx \\ & + k^2 u_0(0) \int_{\omega_\rho} \overline{v_0} dx - \int_{\Sigma_\rho} \overline{\partial_n l_\rho^{w_\rho}} (u_0 - u_0(0)) ds \\ & + k^2 \int_{\omega_\rho} u_0 \overline{l_\rho^{w_\rho}} dx. \end{aligned} \tag{4.8}$$

Proof. The calculus is based on the Green formula and Lemma 4.1:

$$
\begin{aligned}
(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) &= -b_\rho(u_0, h_\rho^{v_\rho}) \\
&= -\int_{\Sigma_\rho} \partial_n u_0 \overline{v_\rho} ds \\
&= -\int_{\Sigma_\rho} \partial_n u_0 \overline{v_0} ds - \int_{\Sigma_\rho} \partial_n u_0 \overline{w_\rho} ds \\
&= k^2 \int_{\omega_\rho} (u_0 - u_0(0)) \overline{v_0} dx - \int_{\omega_\rho} \nabla u_0 . \overline{\nabla v_0} dx \\
&\quad + k^2 u_0(0) \int_{\omega_\rho} \overline{v_0} dx - \int_{\Sigma_\rho} \partial_n u_0 \overline{l_\rho^{w_\rho}} ds \\
&= -\int_{\omega_\rho} (u_0 - u_0(0))(\overline{\Delta v_0} - L) dx - \int_{\omega_\rho} \nabla u_0 . \overline{\nabla v_0} dx \\
&\quad + k^2 u_0(0) \int_{\omega_\rho} \overline{v_0} dx + k^2 \int_{\omega_\rho} u_0 \overline{l_\rho^{w_\rho}} dx \\
&\quad - \int_{\omega_\rho} \nabla (u_0 - u_0(0)) . \overline{\nabla l_\rho^{w_\rho}} .
\end{aligned}
$$

We derive the announced equality by applying twice again the Green formula. □

4.3. **Asymptotic calculus.** Our purpose now is to determine the first order expansion of the expression given by Lemma 4.2. To improve the readability, all error estimates are reported in Section 8.

4.3.1. *Approximation of w_ρ.* A satisfactory approximation of w_ρ is expected to be provided by the function

$$
p_\rho(x) = \rho P\left(\frac{x}{\rho}\right),
$$

where the function $P \in W^1(\mathbb{R}^N \setminus \bar{\omega})$, independent of ρ, is the solution of

$$
\left\{
\begin{array}{rcllr}
\Delta P &=& 0 & \text{in} & \mathbb{R}^N \setminus \bar{\omega}, \\
P &=& O(1/r^{N-1}) & \text{at} & \infty, \\
\partial_n P(x) &=& -\nabla v_0(0).\mathbf{n} & \text{on} & \Sigma.
\end{array}
\right. \tag{4.9}
$$

The definition of the Sobolev space W^1 is recalled in Appendix 9.2, and some useful results about exterior Laplace problems are gathered in Appendix 9.3. The existence and uniqueness of the solution of Problem (4.9) comes basically from the fact that $\int_{\partial\omega} \nabla v_0(0).\mathbf{n} ds = 0$. The function P can be written with the help of the single layer potential:

$$
P(x) = \int_\Sigma \lambda(y) E(x-y) ds(y) \qquad \forall x \in \mathbb{R}^N \setminus \bar{\omega}, \tag{4.10}
$$

where the density $\lambda \in H_0^{-1/2}(\Sigma)$ is the unique solution of the boundary integral equation

$$
\frac{\lambda(x)}{2} + \int_\Sigma \lambda(y) \partial_{n_x} E(x-y) ds(y) = -\nabla v_0(0).\mathbf{n} \qquad \forall x \in \Sigma. \tag{4.11}
$$

We refer again the reader to Appendix 9.3 for the definitions of the fundamental solution E and of the space $H_0^{-1/2}(\Sigma)$.

4.3.2. *Topological asymptotic expansion.* First, we write Equation (4.8) in the form

$$\begin{aligned}(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) &= -\int_{\Sigma_\rho} \overline{\partial_n v_0}(u_0 - u_0(0))ds + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} \\ &\quad -\int_{\Sigma_\rho} \overline{\partial_n l_\rho^{p_\rho}}(u_0 - u_0(0))ds + \sum_{i=1}^{4} \mathcal{E}_i(\rho),\end{aligned}$$

where

$$\mathcal{E}_1(\rho) = -\int_{\Sigma_\rho} (\overline{\partial_n l_\rho^{w_\rho}} - \overline{\partial_n l_\rho^{p_\rho}})(u_0 - u_0(0))ds, \quad \mathcal{E}_2(\rho) = \int_{\omega_\rho} L(u_0 - u_0(0))dx,$$

$$\mathcal{E}_3(\rho) = k^2 u_0(0)\left[\int_{\omega_\rho} \overline{v_0}dx - \rho^N|\omega|\overline{v_0(0)}\right], \quad \mathcal{E}_4(\rho) = k^2\int_{\omega_\rho} u_0\overline{l_\rho^{w_\rho}}dx.$$

For all $\varphi \in H^{1/2}(\Sigma)$, let l^φ be the solution of

$$\begin{cases} \Delta l^\varphi = 0 & \text{in} \quad \omega, \\ l^\varphi = \varphi & \text{on} \quad \Sigma. \end{cases}$$

For all $x \in \Sigma_\rho$, we have

$$l_\rho^{p_\rho}(x) = \rho l^P\left(\frac{x}{\rho}\right) \qquad \text{and} \qquad \partial_n l_\rho^{p_\rho}(x) = \partial_n l^P\left(\frac{x}{\rho}\right).$$

The second jump relation of Theorem 9.1 (in appendix) yields

$$\lambda(y) = -\nabla v_0(0).\mathbf{n} - \partial_n l^P(y) \qquad \forall y \in \Sigma.$$

Thus, we can write

$$\begin{aligned}(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) &= \int_{\Sigma_\rho} \overline{\lambda\left(\frac{x}{\rho}\right)}(u_0 - u_0(0))ds + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} \\ &\quad -\int_{\Sigma_\rho} (\overline{\partial_n v_0} - \overline{\nabla v_0(0).\mathbf{n}})(u_0 - u_0(0))ds + \sum_{i=1}^{4} \mathcal{E}_i(\rho) \\ &= \rho^{N-1}\int_{\Sigma} \overline{\lambda(x)}(u_0(\rho x) - u_0(0))ds + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} \\ &\quad -\rho^{N-1}\int_{\Sigma} (\overline{\partial_n v_0(\rho x)} - \overline{\nabla v_0(0).\mathbf{n}})(u_0(\rho x) - u_0(0))ds \\ &\quad +\sum_{i=1}^{4} \mathcal{E}_i(\rho).\end{aligned}$$

Finally, denoting

$$\mathcal{E}_5(\rho) = -\rho^{N-1}\int_{\Sigma} (\overline{\partial_n v_0(\rho x)} - \overline{\nabla v_0(0).\mathbf{n}})(u_0(\rho x) - u_0(0))ds,$$

$$\mathcal{E}_6(\rho) = \rho^{N-1}\int_{\Sigma} \overline{\lambda(y)}(u_0(\rho y) - u_0(0) - \nabla u_0(0).\rho y)ds(y),$$

we obtain

$$(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) = \rho^N \nabla u_0(0).\int_{\Sigma} \overline{\lambda(x)}x ds(x) + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} + \sum_{i=1}^{6} \mathcal{E}_i(\rho).$$

In Subsection 8.2, we prove that $|\mathcal{E}_i(\rho)| = o(\rho^N)$ for all $i = 1, ..., 6$. Therefore, we set

$$f(\rho) = \rho^N,$$

$$\delta_a = \nabla u_0(0).\int_\Sigma \overline{\lambda(x)}x ds(x) + k^2|\omega|u_0(0)\overline{v_0(0)}.$$

We are going to rewrite this expression in order to make appear explicitely the dependence of δ_a with respect to the adjoint state. Thanks to the linearity of Equation (4.11), it comes

$$\int_\Sigma \lambda(x)x ds(x) = -\mathcal{A}\nabla v_0(0)$$

where the matrix $\mathcal{A}$ is defined by

$$\mathcal{A}V = \int_\Sigma \eta(x)x ds(x) \qquad \forall V \in \mathbb{C}^N, \tag{4.12}$$

the density $\eta \in H_0^{-1/2}(\Sigma)$ being the unique solution of

$$\frac{\eta(x)}{2} + \int_\Sigma \eta(y)\partial_{n_x}E(x-y)ds(y) = V.\mathbf{n} \qquad \forall x \in \Sigma. \tag{4.13}$$

Since $\mathcal{A}$ maps a vector of $\mathbb{R}^N$ to a vector of $\mathbb{R}^N$, its coefficients are real numbers. It is called a polarization tensor [29]. It is proved *e.g.* in [9] that it is symmetric positive definite. Then, we obtain the following result as a consequence of Theorem 2.1.

Theorem 4.1. *We assume that*

- *the cost function satisfies Hypothesis 2.3 with* $f(\rho) = \rho^N$,
- *Hypotheses 3.1, 3.2 and 4.1 are satisfied,*
- *the adjoint state* v_0 *solves*

$$\begin{cases} v_0 \in H^1(\Omega), \\ a_0(u, v_0) = -L_0(u) \qquad \forall u \in H^1(\Omega), \end{cases} \tag{4.14}$$

- *the polarization tensor* $\mathcal{A}$ *is defined by (4.12).*

Then the cost function admits the asymptotic expansion:

$$j(\rho) - j(0) = \rho^N \Re\left(-\nabla u_0(0).\mathcal{A}\overline{\nabla v_0(0)} + k^2|\omega|u_0(0)\overline{v_0(0)} + \delta_J\right) + o(\rho^2). \tag{4.15}$$

4.4. **Spherical hole.** The case where $\omega = B(0,1)$, the unit ball of $\mathbb{R}^N$, is of particular interest for the applications. By using spherical (polar in 2D) coordinates, or by solving explicitly the associated exterior and interior problems and by calculating the density as the jump between the normal derivatives, one can check that the solution of Equation (4.13) is

$$\eta(x) = \frac{N}{N-1}V.x \qquad \forall x \in \Sigma$$

and consequently that

$$\mathcal{A} = \frac{N}{N-1}|\omega| I.$$

Here, $|\omega|$ denotes the Lebesgue measure of ω, that is, $|\omega| = 4\pi/3$ in 3D, $|\omega| = \pi$ in 2D. Hence, the topological asymptotic becomes

$$j(\rho) - j(0) = |\omega|\rho^N \Re\left(-\frac{N}{N-1}\nabla u_0(0).\overline{\nabla v_0(0)} + k^2 u_0(0)\overline{v_0(0)} + \frac{\delta_J}{|\omega|}\right) + o(\rho^N). \tag{4.16}$$

5. Creation of a Crack

We address now the case of the creation of a crack: $\Omega_\rho = \Omega \setminus \overline{\Sigma_\rho}$.

5.1. **Formulation of the initial problem in the cracked domain.** We set for all $\rho \geq 0$ and all $u, v \in H^1(\Omega_\rho)$

$$\left\{ \begin{array}{l} \tilde{a}_\rho(u,v) = a_\rho(u,v), \\ \tilde{l}_\rho(v) = a_\rho(u_0,v). \end{array} \right.$$

It is then obvious that Hypothesis 2.1 holds. We have in this case by construction

$$\delta_a = 0,$$

and we shall determine $f(\rho)$ and δ_l.

5.2. **Preliminary calculus.** We obtain thanks to the Green formula essentially

$$\begin{array}{rcl} (l_\rho - \tilde{l}_\rho)(v_\rho) & = & l_\rho(v_\rho) - a_\rho(u_0, v_\rho) \\ & = & \displaystyle\int_\Gamma \sigma \overline{v_\rho} ds - \int_{\Omega_\rho} \left(\nabla u_0 . \overline{\nabla v_\rho} - k^2 u_0 \overline{v_\rho} \right) dx + \int_\Gamma S u_0 \overline{v_\rho} ds \\ & = & \displaystyle\int_{\Sigma_\rho} \partial_n u_0 [\overline{v_\rho}] ds \\ & = & \displaystyle\int_{\Sigma_\rho} \partial_n u_0 [\overline{w_\rho}] ds \\ & = & \rho^{N-1} \displaystyle\int_\Sigma \partial_n u_0(\rho x) [\overline{w_\rho(\rho x)}] ds, \end{array}$$

where $[v_\rho] = v^+_{\rho|\Sigma_\rho} - v^-_{\rho|\Sigma_\rho} \in H^{1/2}_{00}(\Sigma_\rho)$ (see Figure 1 and Definition (9.6)) and $w_\rho = v_\rho - v_0$. We make the following assumption on the cost function.

Hypothesis 5.1. *For all ρ sufficiently small and all $u \in H^1(\Omega)$,*

$$L_0(u) = L_\rho(u_{|\Omega_\rho}). \tag{5.1}$$

Moreover, L_0, as a distribution, is of regularity H^1 in a neighborhood of the origin.

Then, the function w_ρ satisfies:

$$\left\{ \begin{array}{rcll} \Delta w_\rho + \bar{k}^2 w_\rho & = & 0 & \text{in} \quad \Omega_\rho, \\ \partial_n w_\rho & = & -\partial_n v_0 & \text{on} \quad \Sigma_\rho, \\ \partial_n w_\rho & = & S^* w_\rho & \text{on} \quad \Gamma. \end{array} \right. \tag{5.2}$$

5.3. **Asymptotic calculus.**

5.3.1. *Approximation of w_ρ.* We will show later that a suitable approximation of w_ρ is provided by the function

$$p_\rho(x) = \rho P_\rho \left(\frac{x}{\rho} \right),$$

where $P_\rho \in W^1(\mathbb{R}^N \setminus \Sigma)$ is the solution of

$$\left\{ \begin{array}{rcll} \Delta P_\rho & = & 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\Sigma}, \\ P_\rho & = & O(1/r^{N-1}) & \text{at} \quad \infty, \\ \partial_n P_\rho(x) & = & -\partial_n v_0(\rho x) & \text{on} \quad \Sigma. \end{array} \right.$$

This function P_ρ can be written with the help of the double layer potential:

$$P_\rho(x) = \int_\Sigma \mu_\rho(y) \partial_{n_y} E(x-y) ds(y) \qquad \forall x \in \mathbb{R}^N \setminus \Sigma, \tag{5.3}$$

where the density $\mu_\rho \in H^{1/2}_{00}(\Sigma)$ is defined by

$$\mu_\rho = T(-\partial_n v_0(\rho x)), \tag{5.4}$$

the map T being an isomorphism from $H_{00}^{1/2}(\Sigma)'$ into $H_{00}^{1/2}(\Sigma)$ (see Theorem 9.2 in appendix). Then, we approximate μ_ρ by

$$\mu = T(-\nabla v_0(0).\mathbf{n}). \tag{5.5}$$

5.3.2. *Topological asymptotic expansion.* Denoting by

$$\mathcal{E}_1(\rho) = \rho^{N-1}\int_\Sigma \partial_n u_0(\rho x)[\overline{(w_\rho - p_\rho)(\rho x)}]ds,$$

we have

$$(l_\rho - \tilde{l}_\rho)(v_\rho) = \rho^N \int_\Sigma \partial_n u_0(\rho x)[\overline{P_\rho}]ds + \mathcal{E}_1(\rho).$$

According to the jump relation of Theorem 9.2, $[P_\rho] = -\mu_\rho$. Hence,

$$\begin{aligned}(l_\rho - \tilde{l}_\rho)(v_\rho) &= -\rho^N \int_\Sigma \partial_n u_0(\rho x)\overline{\mu_\rho}ds + \mathcal{E}_1(\rho)\\ &= -\rho^N \int_\Sigma \partial_n u_0(\rho x)\overline{\mu}ds + \mathcal{E}_1(\rho) + \mathcal{E}_2(\rho)\end{aligned}$$

with

$$\mathcal{E}_2(\rho) = -\rho^N \int_\Sigma \partial_n u_0(\rho x)(\overline{\mu_\rho - \mu})ds.$$

Finally, setting

$$\mathcal{E}_3(\rho) = -\rho^N \int_\Sigma (\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n})\overline{\mu}ds,$$

we get

$$(l_\rho - \tilde{l}_\rho)(v_\rho) = -\rho^N \int_\Sigma \nabla u_0(0).\mathbf{n}\overline{\mu}ds + \sum_{i=1}^{3}\mathcal{E}_i(\rho).$$

In Subsection 8.3, we prove that $|\mathcal{E}_i(\rho)| = o(\rho^N)$ for all $i = 1, ..., 3$. Therefore, we set

$$f(\rho) = \rho^N,$$

$$\delta_l = \int_\Sigma \nabla u_0(0).\mathbf{n}\overline{\mu}ds.$$

Again, it is convenient to introduce the polarization matrix $\mathcal{B}$ defined by

$$\mathcal{B}V = \int_\Sigma \eta\mathbf{n}ds \qquad \forall V \in \mathbb{C}^N, \tag{5.6}$$

where

$$\eta = T(V.\mathbf{n}). \tag{5.7}$$

We obtain the following result as a consequence of Theorem 2.1.

Theorem 5.1. *We assume that*

- *the cost function satisfies Hypothesis 2.3 with* $f(\rho) = \rho^N$,
- *Hypotheses 3.1, 3.2 and 5.1 are satisfied,*
- *the adjoint state* v_0 *is solution of (4.14),*
- *the polarization tensor* $\mathcal{B}$ *is defined by (5.6).*

Then the cost function admits the asymptotic expansion :

$$j(\rho) - j(0) = \rho^N \Re\left(-\nabla u_0(0).\mathcal{B}\overline{\nabla v_0(0)} + \delta_J\right) + o(\rho^N). \tag{5.8}$$

5.4. Linear and planar cracks.

- *Linear crack (2D).* Let Σ be the line segment $\{(s,0), -1 < s < 1\}$. Using Theorem 9.2 in appendix, it can be checked quite easily that the solution of Equation (5.7) is
$$\eta(x) = 2\sqrt{1-s^2}(V.\mathbf{n}) \quad \forall x = (s,0) \in \Sigma.$$
- *Planar circular crack (3D).* Consider now the planar unit disc $\Sigma = \{(r\cos\theta, r\sin\theta, 0), 0 \leq r < 1, 0 \leq \theta < 2\pi\}$. By means of polar coordinates whose origin is located at the singularity of the integrand, one can check by a technical calculus that the corresponding density is
$$\eta(x) = \frac{4}{\pi}\sqrt{1-r^2}(V.\mathbf{n}) \quad \forall x \in \Sigma,\ |x| = r.$$

The integration over Σ of the above densities leads to the polarization matrix
$$\mathcal{B} = \alpha \mathbf{n} \otimes \mathbf{n},$$
where $\mathbf{n} \otimes \mathbf{n} := \mathbf{n}\mathbf{n}^T$ and
$$\alpha = \begin{cases} \pi & \text{in 2D,} \\ \dfrac{8}{3} & \text{in 3D.} \end{cases}$$
Therefore, the topological asymptotic expansion reads
$$j(\rho) - j(0) = \rho^N \Re\left(-\alpha(\nabla u_0(0).\mathbf{n})(\overline{\nabla v_0(0).\mathbf{n}}) + \delta_J\right) + o(\rho^N). \tag{5.9}$$

Consider now the special case where $\delta_J = 0$. For a linear (or planar) crack of normal $\mathbf{n}$, the topological gradient at the origin is
$$g(0,\mathbf{n}) = -\alpha\Re(\nabla u_0(0).\mathbf{n})(\overline{\nabla v_0(0).\mathbf{n}}) = -\alpha M(0)\mathbf{n}.\mathbf{n},$$
where $M(0)$ is the hermitian matrix
$$M(0) = \frac{\nabla u_0(0) \otimes \overline{\nabla v_0(0)} + \nabla v_0(0) \otimes \overline{\nabla u_0(0)}}{2}.$$
The topological gradient is minimal when the normal $\mathbf{n}$ is an eigenvector associated to the greatest eigenvalue λ_1 of the symmetric matrix $\Re M(0)$. For this orientation, $g(0,\mathbf{n}) = -\alpha\lambda_1$.

We have synthesized in Table 1 the obtained results corresponding to the insertion of a circular (resp. spherical) hole of radius ρ, and a linear crack of length 2ρ (resp. planar circular crack of radius ρ) and of unit normal $\mathbf{n}$. We recall that for an arbitrary shaped hole or crack, the topological gradient expresses by means of a polarization tensor that can be computed numerically. When the principal part of the differential operator is different from the laplacian, the adequate fundamental solution must be used for solving the integral equations (4.13) and (5.7). The term δ_J, that depends on the chosen criterion, is explicited for some particular choices in the following section.

	$f(\rho)$	$g(x_0)$
hole 2D	ρ^2	$-\pi\Re\left(2\nabla u_0(x_0).\overline{\nabla v_0(x_0)} - k^2 u_0(x_0)\overline{v_0(x_0)}\right) + \delta_J$
crack 2D	ρ^2	$-\pi\Re\left((\nabla u_0(x_0).\mathbf{n})(\overline{\nabla v_0(x_0).\mathbf{n}})\right) + \delta_J$
hole 3D	ρ^3	$-\frac{4\pi}{3}\Re\left(\frac{3}{2}\nabla u_0(x_0).\overline{\nabla v_0(x_0)} - k^2 u_0(x_0)\overline{v_0(x_0)}\right) + \delta_J$
crack 3D	ρ^3	$-\frac{8}{3}\Re\left((\nabla u_0(x_0).\mathbf{n})(\overline{\nabla v_0(x_0).\mathbf{n}})\right) + \delta_J$

TABLE 1. Expressions of the topological asymptotic for a circular hole, a linear crack, a spherical hole and a planar crack, respectively.

6. Particular cost functions

The following theorem is proved in Subsection 8.4.

Theorem 6.1. *For the following cost functions, Hypotheses 2.3, 4.1 and 5.1 hold for the values of δ_J indicated below.*

(1) *First example. The easiest case consists in a cost function of the form*

$$J_\rho(u_\rho) = J(u_{\rho|D_R}).$$

where $D_R = \Omega \setminus \overline{B(0,R)}$, R being a fixed radius such that $\overline{B(0,R)} \subset \Omega$. We assume that there exists $L_0 \in (H^1(D_R))'$ such that, when $h \in H^1(D_R)$,

$$J(u_{0|D_R} + h) - J(u_{0|D_R}) = \Re L_0(h) + O(\|h\|^2_{1,D_R}).$$

For such a criterion, we have

$$\delta_J = 0.$$

(2) *Second example. It consists in the quadratic cost function*

$$J_\rho(u) = \int_{\Omega_\rho} |u - u_d|^2 dx,$$

where u_d belongs to $H^1(\Omega)$ and it is continuous in the vicinity of the origin. In this case,

$$\delta_J = \begin{cases} -|\omega||u_0(0) - u_d(0)|^2 & \textit{for a hole,} \\ 0 & \textit{for a crack.} \end{cases}$$

7. Numerical experiments

7.1. **Description of the problem and of the recovery method.** It is of particular interest to apply the topological asymptotic approach to the equations of elastodynamics. Indeed many target detection methods involved in fields such as non destructive testing, submarine detection or medical imaging, use the so-called pulse-echo method with acoustic or elastic waves at ultrasonic frequencies. The basic principle is the one of echography. A short pulse source is sent through the medium with an emitter/receiver apparatus and the variation of elastic properties of the medium (characterizing the kind of target) generates scattered waves that are recorded by the receiving apparatus. In the case of air bubbles, cracks or delaminations in solids, a Neumann boundary condition is involved at the edge of the defect. The major issue is to be able to read the results so as to detect, localize and characterize the target(s). The topological gradient is a great prospect for the automatic interpretation of these kind of results. It is clear that the pulse-echo method is intrinsically a transient phenomenon, then in order to mimic it we need to derive asymptotic formulas for the elastodynamics equations in the time domain.

To do so we extend the formulas obtained in the time-harmonic case to the dynamic problem by using the duality of the frequency and time domains through the Fourier transform. The time domain problem associated to the linear elasticity problem reads

$$\rho_d \frac{\partial^2 u}{\partial t^2} - \text{div } \sigma(u) = 0. \tag{7.1}$$

The corresponding time-harmonic problem is

$$-\rho_d \, \nu^2 \hat{u} - \text{div } \sigma(\hat{u}) = 0,$$

where $\hat{u}(x,\nu) = \int_{\mathbb{R}} u(t,x) e^{-i\nu t} dt$ is the Fourier transform of the displacement field $u(x,t)$. The notations ρ_d and ν standing for the density of the material and the pulsation, respectively, are adopted to avoid any confusion with the previously introduced notations ρ (the radius of the infinitesimal perforation) and ω (the hole of unitary size). We recall that in the context of linear elasticity, which is adequate for our applications, the stress tensor $\sigma(u)$ is a linear function of the first spatial derivatives of u, characterized by Hooke's tensor which is well-known to be symmetric positive definite. Hence we are in the scope of the analysis developed beforehand.

Starting with the cost function of the time domain problem and using successively Fubbini's theorem and Parseval's equality, it comes

$$J(u) = \frac{1}{2}\int_{\mathbb{R}}(\int_{\Gamma_m} |u - u_m|^2 dx)\, dt = \int_{\mathbb{R}}(\frac{1}{2}\int_{\Gamma_m} |\hat{u} - \hat{u}_m|^2 dx)\, d\nu = \int_{\mathbb{R}} J^\nu(\hat{u}(.,\nu))\, d\nu. \tag{7.2}$$

Here, Γ_m denotes the sensors locations, e.g. a part of the border of Ω where the measurements are performed and u_m is the measured displacement field. At a given frequency, the topological asymptotic expansion for $J^\nu(\hat{u}(\cdot,\nu))$ is known. Starting from

$$J(u_\rho) - J(u_0) \;=\; \int_{\mathbb{R}}(J^\nu(\hat{u}_\rho(.,\nu)) - J^\nu(\hat{u}_0(.,\nu)))\, d\nu, \tag{7.3}$$

then using (7.2) and Parseval's equality, and assuming that $\int_{\mathbb{R}} o(\rho^2)\, d\nu \sim o(\rho^2)$, one obtains the expressions for the time domain problem. Denoting $\hat{u}_0 = \hat{u}_0(x_0,\nu)$ to simplify the writing, one has for instance for a circular hole created around the point x_0 (see Theorem 4.1 with the polarization tensor replaced by the one obtained by Garreau et al [15]):

$$\begin{aligned} & J(u_\rho) - J(u_0) \\ = \;& \pi\rho^2 \int_{\mathbb{R}} \left(-\frac{(\mu+\eta)}{2\mu\eta}\,(4\mu\sigma(\hat{u}_0) : \varepsilon(\hat{v}_0) + (\eta - 2\mu)\mathrm{tr}\sigma(\hat{u}_0)\mathrm{tr}\varepsilon(\hat{v}_0)) + \rho_d\, \nu^2\, \hat{u}_0.\hat{v}_0 \right) d\nu + o(\rho^2) \\ = \;& \pi\rho^2 \int_{\mathbb{R}} \left(-\frac{(\mu+\eta)}{2\mu\eta}\,(4\mu\sigma(u_0) : \varepsilon(v_0) + (\eta - 2\mu)\mathrm{tr}\sigma(u_0)\mathrm{tr}\varepsilon(v_0)) - \rho_d\, \partial_t u_0 . \, \partial_t v_0 \right) dt + o(\rho^2). \end{aligned} \tag{7.4}$$

where ε is the strain tensor of the material, and η is a combination of the Lamé coefficients λ and μ. In the plane stress case $\eta = \dfrac{\mu(3\lambda+2\mu)}{\lambda+2\mu}$.
The topological gradient at any point $x_0 \in \Omega$ is then

$$g(x_0) = \int_{\mathbb{R}} \left(-\frac{(\mu+\eta)}{2\mu\eta}\,(4\mu\sigma(u_0) : \varepsilon(v_0) + (\eta - 2\mu)\mathrm{tr}\sigma(u_0)\mathrm{tr}\varepsilon(v_0)) - \rho_d\, \partial_t u_0 . \, \partial_t v_0 \right) dt,$$

where all the quantities in the integrand are evaluated at the point x_0. Practically we will not have access to the solutions for $t \in \mathbb{R}$, but only over an interval $[0,T]$. Then T must be taken large enough so that the amplitude of the fields in the computation domain after the time T is weak enough to be neglected when computing the topological gradient.

7.2. **The forward solver.** It can be shown that the adjoint problem can be solved with the forward solver provided attention is paid to the fact that the adjoint problem solves backward in time, from $t = T$ to $t = 0$.

We use a finite difference C++ code following Virieux's numerical scheme [35] which is accurate at the order 2 in space and time and intrinsically centered. It allows one to take into account abrupt ruptures of elastic properties or density such as fluid/solid interfaces. This code is integrated to the software ACEL developed by M. Tanter [36] and which is dedicated to the simulation of acoustic and elastic wave propagation. The boundary conditions at the edges of the computation domain are either of the classical Dirichlet and Neumann type, or of absorbing type to simulate unbounded propagation. The implemented absorbing conditions are Perfectly Matched Layers following Collino and Tsogka [6].

7.3. **Numerical results.** In this section we present numerical results relative to non destructive testing. The measurement step is up to now replaced by a numerical solving of the forward problem in the presence of the obstacles. The presented results are 2D since the 3D code is still being developed.

Unique defect in an isotropic solid. The considered medium is an isotropic aluminium slab of density $\rho_d = 2572\,kg.m^{-3}$, the compressional (index p) and shear (index s) speeds of propagation are $v_p = 6408\,m.s^{-1}$ and $v_s = 3228\,m.s^{-1}$. The ultrasonic linear array is placed at the bottom of the slab. We use a 55 sensors array, all of them being used in emission and receive. Absorbing conditions are positioned at the boundaries of the computation domain, except at the bottom where a Dirichlet condition models the presence of the sensors.
The emitted signal is a pulse of 1 μs at the central frequency of 2 MHz (fig. 2). The defect

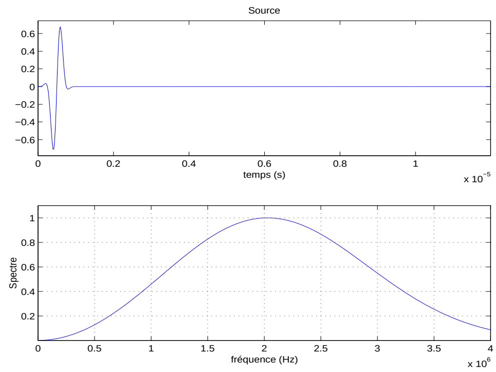
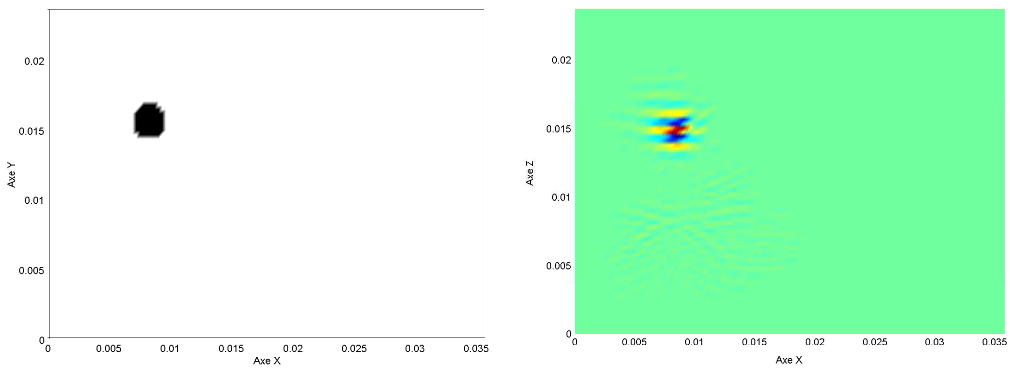


FIGURE 2. Source : temporal signal (top), frequency spectrum (bottom)

is as shown on figure 3(a), it corresponds to a cylindrical hole whose size is of the order of the compressional wavelength λ_p. Then the boundary condition at the edges of the defect is 2D Neumann.



(a) (b)

FIGURE 3. Detection of a unique defect. (a) Position of the defect, (b) Levels of the topological gradient

The position of the defect is clearly pointed out by the high level values of the topological gradient. The negative values (in red) indicate the bottom of the defect. Indeed, since we insonify from the bottom of the slab, it is clear that we have information about the shape of the bottom of the defect, and poor information in the acoustical shadow zones.

Multiple shaped defects. Let us now test the ability of the method to detect multiple defects of different sizes and shapes. We put five defects of various shapes in the aluminium slab (fig. 4(a)). Their horizontal sizes vary from $\frac{\lambda_p}{5}$ to $\frac{3\lambda_p}{2}$. These defects are well resolved since they are separated from more than a wavelength. We use the same linear array and source as in the previous example. In order to draw nearer to experimental non destructive testing conditions, we have added white noise to the simulated measurements. Figures 4(b)(c)(d) show the levels of the topological gradient when the noise level is respectively of 0%, 5% and 10% of the maximum value of the emitted signal, corresponding respectively to signal to noise ratios of ∞, 5 and 2.5 on the signal scattered by the defects. In each presented result, the five defects are detected and localized. The approximate sizes and shapes of the obstacles are obtained, except in the shadow zones. It is very interesting to see that the method has a robust behavior upon addition of noise to the simulated measurements. It allows one to be optimistic as for the application of the method to experimental measurements that are intrinsically noisy.

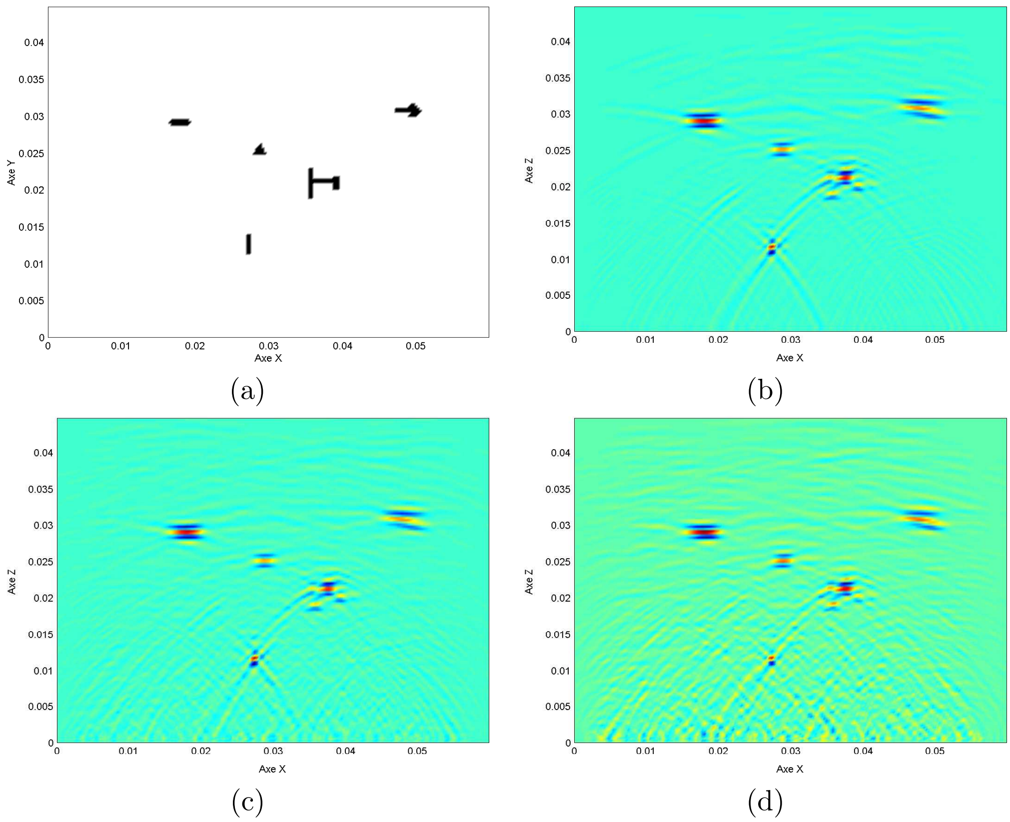


FIGURE 4. Detection of multiple shaped defects. (a) Positions of the defects, (b)-(d) Levels of the topological gradient (b) with no added noise, (c) with 5% of noise, (d) with 10% of noise

8. Proofs

The aim of this section is to prove Theorems 4.1, 5.1 and 6.1. We recall that, for any fixed radius $R > 0$, $D_R = \Omega \setminus \overline{B(0,R)}$. The letter c denotes any positive constant that may change from place to place but that never depends on ρ.

8.1. **Preliminary lemmas.** The following lemmas are valid for both types of domain perturbation. We will use the notations:

- for a perforation,

$$\mathcal{T}(\Sigma) = H_0^{-1/2}(\Sigma), \qquad \mathcal{T}(\Sigma_\rho) = H_0^{-1/2}(\Sigma_\rho),$$

$$\mathcal{U} = \mathbb{R}^N \setminus \overline{\omega}, \qquad \mathcal{U}_\rho = \mathbb{R}^N \setminus \overline{\omega_\rho},$$

- for a crack,

$$\mathcal{T}(\Sigma) = H^{1/2}_{00}(\Sigma)', \qquad \mathcal{T}(\Sigma_\rho) = H^{1/2}_{00}(\Sigma_\rho)',$$
$$\mathcal{U} = \mathbb{R}^N \setminus \overline{\Sigma}, \qquad \mathcal{U}_\rho = \mathbb{R}^N \setminus \overline{\Sigma_\rho}.$$

We refer to Appendix 9.3 for the definitions of those functional spaces.

Lemma 8.1. *Consider $g \in \mathcal{T}(\Sigma)$ and let $z \in W^1(\mathcal{U})$ be the solution of the problem*

$$\begin{cases} \Delta z &= 0 & \text{in} & \mathcal{U}, \\ z &= O(1/r^{N-1}) & \text{at} & \infty, \\ \partial_n z &= g & \text{on} & \Sigma. \end{cases}$$

There exists $c > 0$ such that

$$|z|_{1,\frac{1}{\rho}D_R} \le c\rho^{\frac{N}{2}} \|g\|_{\mathcal{T}(\Sigma)}.$$

Proof. Let us first consider the case of a hole. By Theorem 9.1 (in appendix), there exists $\lambda \in H_0^{-1/2}(\Sigma)$ such that

$$z(x) = \int_\Sigma \lambda(y) E(x-y) ds(y), \qquad \forall x \in \mathbb{R}^N \setminus \bar{\omega},$$

where λ depends continuously on g. Using a Taylor expansion of E computed at the point x, we obtain that

$$|\nabla z(x)| \le \frac{c}{|x|^N} \|g\|_{-1/2,\Sigma},$$

from which we deduce easily the wanted estimate. For a crack, we obtain from Theorem 9.2 the existence of $\mu \in H^{1/2}_{00}(\Sigma)$ such that

$$z(x) = \int_\Sigma \mu(y) \partial_{n_y} E(x-y) ds(y), \qquad \forall x \in \mathbb{R}^N \setminus \overline{\Sigma}.$$

The reasoning is then similar to the previous case. □

Lemma 8.2. *For all ρ and all $g \in \mathcal{T}(\Sigma_\rho)$, the solution $z_\rho \in W^1(\mathcal{U}_\rho)$ to the problem*

$$\begin{cases} \Delta z_\rho &= 0 & \text{in} & \mathcal{U}_\rho, \\ z_\rho &= O(1/r^{N-1}) & \text{at} & \infty, \\ \partial_n z_\rho &= g & \text{on} & \Sigma_\rho \end{cases}$$

satisfies the estimates

$$\|z_\rho\|_{0,\Omega_\rho} \le c\rho^{\frac{N}{2}+1} \|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$
$$|z_\rho|_{1,\Omega_\rho} \le c\rho^{\frac{N}{2}} \|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$
$$|z_\rho|_{1,D_R} \le c\rho^{N} \|g(\rho x)\|_{\mathcal{T}(\Sigma)}.$$

Proof. Let us set $Z_\rho(x) = z_\rho(\rho x)$. We have

$$\begin{cases} \Delta Z_\rho &= 0 & \text{in} & \mathcal{U}, \\ Z_\rho &= O(1/r^{N-1}) & \text{at} & \infty, \\ \partial_n Z_\rho &= \rho g(\rho x) & \text{on} & \Sigma. \end{cases}$$

By elliptic regularity, we have

$$\|Z_\rho\|_{W^1(\mathcal{U})} \le c \|\rho g(\rho x)\|_{\mathcal{T}(\Sigma)}.$$

A change of variable yields

$$\|z_\rho\|_{0,\Omega_\rho} \le c\rho^{\frac{N}{2}} \|\rho g(\rho x)\|_{\mathcal{T}(\Sigma)},$$
$$|z_\rho|_{1,\Omega_\rho} \le c\rho^{\frac{N}{2}-1} \|\rho g(\rho x)\|_{\mathcal{T}(\Sigma)}.$$

The last inequality to be proved results from Lemma 8.1 and a change of variable. □

Lemma 8.3. *Consider $\sigma \in H^{-1/2}(\Gamma)$, $\rho \geq 0$, $f \in L^2(\Omega)$ and let $z_\rho \in H^1(\Omega_\rho)$ be the solution of the problem*

$$\left\{\begin{array}{rcll} \Delta z_\rho + k^2 z_\rho &=& f & \text{in} \quad \Omega_\rho, \\ \partial_n z_\rho &=& S z_\rho + \sigma & \text{on} \quad \Gamma, \\ \partial_n z_\rho &=& 0 & \text{on} \quad \Sigma_\rho. \end{array}\right.$$

There exist $\rho_2 > 0$ and a constant $c > 0$ independent of ρ, f and σ such that for all $\rho < \rho_2$

$$\|z_\rho\|_{1,\Omega_\rho} \leq c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}.$$

This result remains true if we replace S by S^* and k by $\bar{k}$.

Proof. We present the proof in the most technical case of a perforation. We choose a parameter ρ_1 such that $k \operatorname{diam}(\omega_{\rho_1}) < 1$. Let ρ_2 be such that $\omega_\rho \subset \omega_{\rho_1}$ for all $\rho < \rho_2$. For all $u \in H^1(\Omega_{\rho_1})$, we define the function $\hat{u} \in H^1(\Omega)$ by

$$\hat{u} = \left\{\begin{array}{l} u \text{ in } \Omega_{\rho_1}, \\ h^u_{\rho_1} \text{ in } \omega_{\rho_1}, \end{array}\right.$$

where h^φ_ρ is defined by (4.1). For all $v \in H^1(\Omega)$, we have

$$\begin{array}{rcl} a_0(\widehat{z_{\rho|\Omega_{\rho_1}}}, v) &=& a_\rho(z_\rho, v) + \left[a_0(\widehat{z_{\rho|\Omega_{\rho_1}}}, v) - a_\rho(z_\rho, v)\right] \\ &=& -\displaystyle\int_{\Omega_\rho} f\bar{v}dx + \int_\Gamma \sigma\bar{v}ds + A_\rho + B_\rho \end{array}$$

with

$$A_\rho = \int_{\omega_{\rho_1}\setminus\omega_\rho} \nabla(h^{z_\rho}_{\rho_1} - z_\rho).\overline{\nabla v}dx - k^2 \int_{\omega_{\rho_1}\setminus\omega_\rho} (h^{z_\rho}_{\rho_1} - z_\rho)\bar{v}dx,$$

$$B_\rho = \int_{\omega_\rho} \nabla h^{z_\rho}_{\rho_1}.\overline{\nabla v}dx - k^2 \int_{\omega_\rho} h^{z_\rho}_{\rho_1}\bar{v}dx.$$

Let us first estimate B_ρ. We have

$$\|h^{z_\rho}_{\rho_1}\|_{1,\omega_\rho} \leq c\rho^{\frac{N}{2}}\|h^{z_\rho}_{\rho_1}\|_{\mathcal{C}^1(\omega_\rho)} \leq c\rho^{\frac{N}{2}}\|z_\rho\|_{\frac{1}{2},\Sigma_{\rho_1}} \leq c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}}. \tag{8.1}$$

Thus,

$$|B_\rho| \leq c\|h^{z_\rho}_{\rho_1}\|_{1,\omega_\rho}\|v\|_{1,\omega_\rho} \leq c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}}\|v\|_{1,\Omega}.$$

Let us now estimate A_ρ. We have

$$|A_\rho| \leq c\|h^{z_\rho}_{\rho_1} - z_\rho\|_{1,\omega_{\rho_1}\setminus\omega_\rho}\|v\|_{1,\omega_{\rho_1}\setminus\omega_\rho}.$$

The function $y_\rho = h^{z_\rho}_{\rho_1} - z_\rho$ solves

$$\left\{\begin{array}{rcll} \Delta y_\rho + k^2 y_\rho &=& -f & \text{in} \quad \omega_{\rho_1} \setminus \overline{\omega_\rho}, \\ y_\rho &=& 0 & \text{on} \quad \Sigma_{\rho_1}, \\ \partial_n y_\rho &=& \partial_n h^{z_\rho}_{\rho_1} & \text{on} \quad \Sigma_\rho. \end{array}\right.$$

It can be split into $y_\rho = y^1_\rho + y^2_\rho$ with

$$\left\{\begin{array}{rcll} \Delta y^1_\rho &=& 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\omega}_\rho, \\ y^1_\rho &=& O(1/r^{N-1}) & \text{at} \quad \infty, \\ \partial_n y^1_\rho &=& \partial_n h^{z_\rho}_{\rho_1} & \text{on} \quad \Sigma_\rho, \end{array}\right.$$

$$\left\{\begin{array}{rcll} \Delta y^2_\rho + k^2 y^2_\rho &=& -k^2 y^1_\rho - f & \text{in} \quad \omega_{\rho_1} \setminus \overline{\omega}_\rho, \\ y^2_\rho &=& -y^1_\rho & \text{on} \quad \Sigma_{\rho_1}, \\ \partial_n y^2_\rho &=& 0 & \text{on} \quad \Sigma_\rho. \end{array}\right.$$

According to Lemma 8.2,

$$\|y^1_\rho\|_{1,\omega_{\rho_1}\setminus\overline{\omega}_\rho} \leq c\rho^{\frac{N}{2}}\|\partial_n h^{z_\rho}_{\rho_1}(\rho x)\|_{-\frac{1}{2},\Sigma} \leq c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}}. \tag{8.2}$$

Using Proposition 9.1 and the Lax Milgram Theorem, it is easy to prove the following elliptic regularity property:

$$\|y_\rho^2\|_{1,\omega_{\rho_1}\setminus\overline{\omega}_\rho} \leq c\|k^2 y_\rho^1 + f\|_{0,\omega_{\rho_1}\setminus\overline{\omega}_\rho} + c\|y_\rho^1\|_{\frac{1}{2},\Sigma_{\rho_1}}. \tag{8.3}$$

Gathering Equations (8.2) and (8.3) yields

$$\|y_\rho\|_{1,\omega_{\rho_1}\setminus\overline{\omega}_\rho} \leq c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}} + c\|f\|_{0,\omega_{\rho_1}\setminus\overline{\omega}_\rho}$$

and

$$|A_\rho| \leq (c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}} + c\|f\|_{0,\omega_{\rho_1}\setminus\overline{\omega}_\rho})\|v\|_{1,\omega_{\rho_1}\setminus\omega_\rho}.$$

It follows

$$|a_0(\widehat{z_{\rho|\Omega_{\rho_1}}}, v)| \leq (c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma})\|v\|_{1,\Omega}. \tag{8.4}$$

As Equation (8.4) holds for any $v \in H^1(\Omega)$, we obtain by using the first inf-sup condition of Proposition 3.1 that

$$\|\widehat{z_{\rho|\Omega_{\rho_1}}}\|_{1,\Omega} \leq c\rho^{\frac{N}{2}}\|z_\rho\|_{1,\Omega_{\rho_1}} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}.$$

Hence,

$$(1 - c\rho^{\frac{N}{2}})\|z_\rho\|_{1,\Omega_{\rho_1}} \leq c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}.$$

So, for ρ sufficiently small (possibly decreasing ρ_2),

$$\|z_\rho\|_{1,\Omega_{\rho_1}} \leq c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}.$$

Using again Proposition 9.1 and the Lax Milgram Theorem, we obtain easily that

$$\|z_\rho\|_{1,\omega_{\rho_1}\setminus\omega_\rho} \leq c\|f\|_{0,\Omega_\rho} + c\|z_\rho\|_{1,\Omega_{\rho_1}},$$

which completes the proof. For a crack, the reasoning is similar, ω_{ρ_1} remaining a fixed domain containing the crack. In this case $B_\rho = 0$. □

Lemma 8.4. *Consider $\sigma \in H^{-1/2}(\Gamma)$, $\rho \geq 0$, $g \in \mathcal{T}(\Sigma_\rho)$, $f \in L^2(\Omega_\rho)$ and let $z_\rho \in H^1(\Omega_\rho)$ be the solution of the problem*

$$\left\{\begin{array}{rclll} \Delta z_\rho + k^2 z_\rho &=& f & \textit{in} & \Omega_\rho, \\ \partial_n z_\rho &=& S z_\rho + \sigma & \textit{on} & \Gamma, \\ \partial_n z_\rho &=& g & \textit{on} & \Sigma_\rho. \end{array}\right. \tag{8.5}$$

There exist some constants independent of ρ, σ, f and g such that for all ρ sufficiently small

$$\begin{array}{rcl} \|z_\rho\|_{0,\Omega_\rho} &\leq& c\rho^{\frac{N}{2}+1}\|g(\rho x)\|_{\mathcal{T}(\Sigma)} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}, \\ |z_\rho|_{1,\Omega_\rho} &\leq& c\rho^{\frac{N}{2}}\|g(\rho x)\|_{\mathcal{T}(\Sigma)} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}, \\ \|z_\rho\|_{1,D_R} &\leq& c\rho^{\frac{N}{2}+1}\|g(\rho x)\|_{\mathcal{T}(\Sigma)} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}. \end{array}$$

This result remains true if we replace S by S^* and k by $\bar{k}$.

Proof. We split z_ρ into $z_\rho = z_\rho^1 + z_\rho^2$ with

$$\left\{\begin{array}{rcllr} \Delta z_\rho^1 &=& 0 & \text{in} & \mathcal{U}_\rho, \\ z_\rho^1 &=& O(1/r^{N-1}) & \text{at} & \infty, \\ \partial_n z_\rho^1 &=& g & \text{on} & \Sigma_\rho, \end{array}\right.$$

$$\left\{\begin{array}{rcllr} \Delta z_\rho^2 + k^2 z_\rho^2 &=& -k^2 z_\rho^1 + f & \text{in} & \Omega_\rho, \\ \partial_n z_\rho^2 &=& S z_\rho^2 + S z_\rho^1 - \partial_n z^1 + \sigma & \text{on} & \Gamma, \\ \partial_n z_\rho^2 &=& 0 & \text{on} & \Sigma_\rho. \end{array}\right.$$

By Lemma 8.2,

$$\|z_\rho^1\|_{0,\Omega_\rho} \leq c\rho^{\frac{N}{2}+1}\|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$

$$|z_\rho^1|_{1,\Omega_\rho} \leq c\rho^{\frac{N}{2}} \|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$
$$\|z_\rho^1\|_{1,D_R} \leq c\rho^N \|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$

and by lemma 8.3,

$$\begin{aligned}
\|z_\rho^2\|_{1,\Omega_\rho} &\leq c\| - k^2 z_\rho^1 + f\|_{0,\Omega_\rho} + c\|S z_\rho^1 - \partial_n z_\rho^1 + \sigma\|_{-\frac{1}{2},\Gamma} \\
&\leq c\|z_\rho^1\|_{0,\Omega_\rho} + c\|f\|_{0,\Omega_\rho} + c\|z_\rho^1\|_{1,D_R} + c\|\sigma\|_{-\frac{1}{2},\Gamma} \\
&\leq c\rho^{\frac{N}{2}+1}\|g(\rho x)\|_{-\frac{1}{2},\Sigma} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}.
\end{aligned}$$

The desired inequalities follow straightforwardly. □

8.2. **Proof of Theorem 4.1 (Topological asymptotic for a hole).** We shall successively prove that $\mathcal{E}_i(\rho) = o(\rho^N)$ for $i = 1, ..., 6$.

(1) We set $e_\rho = p_\rho - w_\rho$. We have, using basically the fact that u_0 is of class $\mathcal{C}^\infty$ in the vicinity of the origin,

$$\begin{aligned}
|\mathcal{E}_1(\rho)| &= \left| \int_{\omega_\rho} \overline{\nabla l_\rho^{e_\rho}}.\nabla u_0 dx \right| \\
&\leq |l_\rho^{e_\rho}|_{1,\omega_\rho} |u_0|_{1,\omega_\rho} \\
&\leq c\rho^{N-1} |l_\rho^{e_\rho}(\rho x)|_{1,\omega} \|u_0\|_{\mathcal{C}^1(\omega_\rho)} \\
&\leq c\rho^{N-1} |l^{e_\rho(\rho x)}|_{1,\omega} \\
&\leq c\rho^{N-1} \|e_\rho(\rho x)\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}}.
\end{aligned}$$

Yet, the function e_ρ solves

$$\left\{ \begin{array}{rcll}
\Delta e_\rho + \bar{k}^2 e_\rho &=& \bar{k}^2 p_\rho & \text{in} \quad \Omega_\rho, \\
\partial_n e_\rho &=& -\nabla v_0(0).\mathbf{n} + \partial_n v_0 & \text{on} \quad \Sigma_\rho, \\
\partial_n e_\rho &=& S^* e_\rho + \partial_n p_\rho - S^* p_\rho & \text{on} \quad \Gamma.
\end{array} \right.$$

Hence, by Lemma 8.4,

$$\begin{aligned}
|e_\rho|_{1,\Omega_\rho} &\leq c\rho^{\frac{N}{2}} \| - \nabla v_0(0).\mathbf{n} + \partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} + c\|\bar{k}^2 p_\rho\|_{0,\Omega_\rho} \\
&\quad + c\|\partial_n p_\rho - S^* p_\rho\|_{-\frac{1}{2},\Gamma} \\
&\leq c\rho^{\frac{N}{2}} \| - \nabla v_0(0).\mathbf{n} + \partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} + c\|p_\rho\|_{0,\Omega_\rho} + c\|p_\rho\|_{1,D_R}.
\end{aligned}$$

Using Equation (4.3) and the interior regularity theorem, we obtain that there exists $\rho_3 > 0$ such that $v_0 \in H^3(\omega_{\rho_3}) \subset \mathcal{C}^1(\omega_{\rho_3})$. Thus, we have

$$\lim_{\rho \to 0} \| - \nabla v_0(0).\mathbf{n} + \partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} = 0. \tag{8.6}$$

Thanks to Lemma 8.2,

$$\|p_\rho\|_{0,\Omega_\rho} + \|p_\rho\|_{1,D_R} \leq c\rho^{\frac{N}{2}+1} \|\nabla v_0(0).\mathbf{n}\|_{-\frac{1}{2},\Sigma} \leq c\rho^{\frac{N}{2}+1}.$$

Hence

$$|e_\rho|_{1,\Omega_\rho} = o(\rho^{\frac{N}{2}}).$$

Using successively the trace theorem, Lemma 9.2 (in appendix) and a change of variable and denoting by B some ball containing ω, we obtain that

$$\|e_\rho(\rho x)\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} \leq c\|e_\rho(\rho x)\|_{H^1(B\backslash\bar{\omega})/\mathbb{C}} \leq c|e_\rho(\rho x)|_{1,B\backslash\bar{\omega}} \leq c\rho^{1-\frac{N}{2}} |e_\rho|_{1,\Omega_\rho},$$

from which we deduce that $\mathcal{E}_1(\rho) = o(\rho^N)$.

(2) The fact that, in the vicinity of the origin, L is continuous and u_0 is of class $\mathcal{C}^\infty$ yields directly
$$|\mathcal{E}_2(\rho)| \leq c\rho^{N+1},$$

(3) We have
$$\mathcal{E}_3(\rho) = k^2 u_0(0) \int_{\omega_\rho} (\overline{v_0} - \overline{v_0(0)})dx.$$
Since v_0 is of class $\mathcal{C}^1$ in the vicinity of the origin, we obtain immediately with the help of a Taylor expansion that
$$|\mathcal{E}_3(\rho)| \leq c\rho^{N+1}.$$

(4) We get by a change of variable
$$|\mathcal{E}_4(\rho)| = k^2\rho^N \left| \int_\omega u_0(\rho x)\overline{l^{w_\rho(\rho x)}}dx \right| \leq c\rho^N \|l^{w_\rho(\rho x)}\|_{0,\omega}.$$
The elliptic regularity, the trace theorem and a change of variable bring successively
$$\begin{aligned} |\mathcal{E}_4(\rho)| &\leq c\rho^N \|w_\rho(\rho x)\|_{\frac{1}{2},\Sigma} \\ &\leq c\rho^N \|w_\rho(\rho x)\|_{1,\frac{1}{\rho}\Omega_\rho} \\ &\leq c\rho^N \left(\rho^{-\frac{N}{2}}\|w_\rho\|_{0,\Omega_\rho} + \rho^{1-\frac{N}{2}}|w_\rho|_{1,\Omega_\rho}\right). \end{aligned}$$
Then, Lemma 8.4 and the fact that v_0 is of class $\mathcal{C}^1$ in the vicinity of the origin furnish
$$|\mathcal{E}_4(\rho)| \leq c\rho^{N+1}\|\partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} \leq c\rho^{N+1}.$$

(5) We have
$$|\mathcal{E}_5(\rho)| \leq \rho^{N-1}\|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{-\frac{1}{2},\Sigma}\|u_0(\rho x) - u_0(0)\|_{\frac{1}{2},\Sigma}.$$
Equation (8.6) and the regularity of u_0 near the origin yield $\mathcal{E}_5(\rho) = o(\rho^N)$.

(6) We have
$$|\mathcal{E}_6(\rho)| \leq \rho^{N-1}\|\lambda\|_{-\frac{1}{2},\Sigma}\|u_0(\rho y) - u_0(0) - \nabla u_0(0).\rho y\|_{\frac{1}{2},\Sigma}.$$
With the help of a Taylor expansion, we derive
$$|\mathcal{E}_6(\rho)| \leq c\rho^{N+1},$$
which achieves the proof of the theorem. □

8.3. **Proof of Theorem 5.1 (Topological asymptotic for a crack).** We have here to prove that $\mathcal{E}_i(\rho) = o(\rho^N)$ for all $i = 1, ..., 3$.

(1) Setting $e_\rho = p_\rho - w_\rho$, we have
$$\begin{aligned} |\mathcal{E}_1(\rho)| &\leq \rho^{N-1}\|\partial_n u_0(\rho x)\|_{(H_{00}^{\frac{1}{2}}(\Sigma))'}\|[e_\rho(\rho x)]\|_{H_{00}^{\frac{1}{2}}(\Sigma)} \\ &\leq c\rho^{N-1}\|[e_\rho(\rho x)]\|_{H_{00}^{\frac{1}{2}}(\Sigma)}. \end{aligned}$$
The function e_ρ solves
$$\left\{ \begin{array}{rcll} \Delta e_\rho + \bar{k}^2 e_\rho &=& \bar{k}^2 p_\rho & \text{in} \quad \Omega_\rho, \\ \partial_n e_\rho &=& 0 & \text{on} \quad \Sigma_\rho, \\ \partial_n e_\rho &=& S^* e_\rho + \partial_n p_\rho - S^* p_\rho & \text{on} \quad \Gamma. \end{array} \right.$$
So, Lemma 8.3 yields
$$\begin{aligned} \|e_\rho\|_{1,\Omega_\rho} &\leq c\|\bar{k}^2 p_\rho\|_{0,\Omega_\rho} + c\|\partial_n p_\rho - S^* p_\rho\|_{-\frac{1}{2},\Gamma} \\ &\leq c\|p_\rho\|_{0,\Omega_\rho} + c\|p_\rho\|_{1,D_R}. \end{aligned}$$

Yet, according to Lemma 8.2,

$$\|p_\rho\|_{0,\Omega_\rho} + \|p_\rho\|_{1,D_R} \le c\rho^{\frac{N}{2}+1}\|\partial_n u_0(\rho x)\|_{(H^{\frac{1}{2}}_{00}(\Sigma))'} \le c\rho^{\frac{N}{2}+1}.$$

Thus,

$$\|e_\rho\|_{1,\Omega_\rho} \le c\rho^{\frac{N}{2}+1}.$$

Using the trace theorem and Lemma 9.2 (in appendix) and denoting by B some ball containing Σ, we get

$$\|[e_\rho(\rho x)]\|_{H^{\frac{1}{2}}_{00}(\Sigma)} \le c\|e_\rho(\rho x)\|_{H^1(B\backslash\Sigma)/\mathbb{C}} \le c|e_\rho(\rho x)|_{1,B\backslash\Sigma} \le c\rho^{1-\frac{N}{2}}|e_\rho|_{1,\Omega_\rho}.$$

It follows

$$|\mathcal{E}_1(\rho)| \le c\rho^{N+1}.$$

(2) We have

$$\begin{aligned} |\mathcal{E}_2(\rho)| &\le \rho^N\|\partial_n u_0(\rho x)\|_{(H^{\frac{1}{2}}_{00}(\Sigma))'}\|\mu_\rho - \mu\|_{H^{\frac{1}{2}}_{00}(\Sigma)} \\ &\le c\rho^N\|\mu_\rho - \mu\|_{H^{\frac{1}{2}}_{00}(\Sigma)}. \end{aligned}$$

By continuity of the operator T, it comes

$$\|\mu_\rho - \mu\|_{H^{\frac{1}{2}}_{00}(\Sigma)} \le c\|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{(H^{\frac{1}{2}}_{00}(\Sigma))'}.$$

Hypothesis 5.1 guarantees that v_0 is still of class $\mathcal{C}^1$ in the vicinity of the origin. Thus,

$$\lim_{\rho\to 0}\|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{(H^{\frac{1}{2}}_{00}(\Sigma))'} = 0 \tag{8.7}$$

and $\mathcal{E}_2(\rho) = o(\rho^N)$.

(3) We have

$$|\mathcal{E}_3(\rho)| \le \rho^N\|\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n}\|_{(H^{\frac{1}{2}}_{00}(\Sigma))'}\|\mu\|_{H^{\frac{1}{2}}_{00}(\Sigma)}.$$

Next, as u_0 is of class $\mathcal{C}^\infty$ in the vicinity of the origin, we derive

$$|\mathcal{E}_3(\rho)| \le c\rho^{N+1},$$

which ends the proof of the theorem. □

8.4. **Proof of Theorem 6.1 (Examples of cost functions).** The following lemma will be useful.

Lemma 8.5. *We have the estimates*

$$\|u_\rho - u_0\|_{0,\Omega_\rho} = O(\rho^{\frac{N}{2}+1}) \quad \text{and} \quad \|u_\rho - u_0\|_{1,D_R} = O(\rho^{\frac{N}{2}+1}).$$

Proof. It suffices to apply Lemma 8.4 to the function $u_\rho - u_0$ restricted to Ω_ρ. □

Let us now turn to the proof of Theorem 6.1.

(1) For the first category of cost functions, the result is an immediate application of Lemma 8.5.

(2) For the second example, we present only the case of a perforation. The case of a crack can be treated in a similar way. We have

$$\begin{aligned} J_\rho(u_\rho) - J_0(u_0) &= \int_{\Omega_\rho}(|u_\rho - u_d|^2 - |u_0 - u_d|^2)dx - \int_{\omega_\rho}|u_0 - u_d|^2 dx \\ &= \int_{\Omega_\rho}\left[|u_\rho - u_0|^2 + 2\Re((\overline{u_0 - u_d})(u_\rho - u_0))\right]dx \\ &\quad - \int_{\omega_\rho}|u_0 - u_d|^2 dx. \end{aligned}$$

Lemma 8.4, the regularity of u_0 and u_d near the origin and the fact that

$$L_\rho(u) = \int_{\Omega_\rho} 2(\overline{u_0 - u_d})u dx \qquad \forall u \in H^1(\Omega_\rho)$$

yield

$$J_\rho(u_\rho) - J_0(u_0) = \Re L_\rho(u_\rho - u_0) - \rho^N \text{meas}(\omega)|u_0(0) - u_d(0)|^2 dx + o(\rho^N),$$

which achieves the proof.

9. Appendix

9.1. **Proof of Proposition 3.1 (Well-posedness).** We need the following lemma which is a kind of generalization of the Poincaré inequality. We refer the reader to [30] for a very similar proof.

Lemma 9.1. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, whose boundary Γ is piecewise of class $\mathcal{C}^1$. Let V be a closed subspace of $H^1(\Omega)$ and $\|.\|$ be some norm on V satisfying for all $u \in V$:*

$$|u|_{1,\Omega} \le \|u\| \le c\|u\|_{1,\Omega}.$$

Then, on V, the norm $\|.\|$ is equivalent to the norm $\|.\|_{1,\Omega}$.

We will now prove Proposition 3.1.

(1) First, let us prove that a_ρ^0 is coercive. For all $u \in H^1(\Omega_\rho)$, we have

$$a_\rho^0(u,u) = \|\nabla u\|_{0,\Omega_\rho}^2 - \int_\Gamma S_0 u\bar{u} ds. \tag{9.1}$$

Inequality (9.1), Hypothesis 3.1 and the continuity of a_ρ^0 yield

$$|u|_{1,\Omega_\rho} \le \sqrt{a_\rho^0(u,u)} \le c_\rho \|u\|_{1,\Omega_\rho}.$$

Therefore, the coercivity of a_ρ^0 results from Lemma 9.1 if we prove that $u \mapsto \sqrt{a_\rho^0(u,u)}$ is a norm on $H^1(\Omega_\rho)$. It is obvious that the sesquilinear form a_ρ^0 is hermitian and positive. To prove that it is definite, let us assume that $a_\rho^0(u,u) = 0$. Inequality (9.1) implies $\|\nabla u\|_{0,\Omega_\rho} = 0$ and $\int_\Gamma S_0 u\bar{u} ds = 0$. Then, from Hypothesis 3.1 and the fact that Ω_ρ is connected, we obtain that $u = 0$ in Ω_ρ.

(2) Let us now prove the first inf-sup condition and the existence of solutions to Problem (2.1). For Problem (2.4) and the second inf-sup condition, the proofs should be analogous. From the Lax Milgram theorem and the fact that the imbeddings $H^1(\Omega_\rho) \hookrightarrow L^2(\Omega_\rho)$ and $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ are compact, we obtain (see *e.g.* [31] for more details) that for all $u, v \in H^1(\Omega_\rho)$,

$$a_\rho(u,v) = a_\rho^0((I - C_\rho)u, v),$$

where C_ρ is a compact endomorphism of $H^1(\Omega_\rho)$. The Fredholm alternative and Hypothesis 3.2 imply that $I - C_\rho$ is bijective. Then, we deduce the existence of solutions to Problem (2.1). The inf-sup condition comes from the fact that for all $u \in H^1(\Omega_\rho)$

$$\begin{aligned} a_\rho(u, (I - C_\rho)u) &= a_\rho^0((I - C_\rho)u, (I - C_\rho)u) \\ &\ge c_\rho \|(I - C_\rho)u\|_{1,\Omega_\rho}^2 \\ &\ge c_\rho \|u\|_{1,\Omega_\rho} \|(I - C_\rho)u\|_{1,\Omega_\rho}. \end{aligned}$$

9.2. **Quotient and weighted Sobolev spaces.**

9.2.1. *Quotient Sobolev spaces.* Let $\mathcal{S}$ be a Sobolev space on the complex field. The space $\mathcal{S}/\mathbb{C}$ denotes the quotient space of $\mathcal{S}$ by the constants. It is endowed with the following norm and semi-norm:

$$\|u\|_{\mathcal{S}/\mathbb{C}} = \inf_{v-u=C^{st}} \|v\|_{\mathcal{V}}, \qquad |u|_{\mathcal{S}/\mathbb{C}} = |u|_{\mathcal{S}}. \tag{9.2}$$

The following standard result can be considered as a consequence of Lemma 9.1.

Lemma 9.2. *If Ω is an open, bounded and connected subset of $\mathbb{R}^N$ then, in the space $H^1(\Omega)/\mathbb{C}$, the semi-norm is equivalent to the norm.*

9.2.2. *A weighted Sobolev space.* Let B' be an exterior domain : $B' = \mathbb{R}^N \setminus \overline{B}$, where B is an open and bounded subset of $\mathbb{R}^N$. The space $W^1(B')$ is defined by (see *e.g.* [20, 14, 7]):

$$W^1(B') = \left\{ u \in \mathcal{D}'(B'), \frac{u}{(1+r)\ln r} \in L^2(B') \text{ and } \nabla u \in L^2(B') \right\} \qquad \text{in 2D,}$$

$$W^1(B') = \left\{ u \in \mathcal{D}'(B'), \frac{u}{1+r} \in L^2(B') \text{ and } \nabla u \in L^2(B') \right\} \qquad \text{in 3D.}$$

It is endowed with the norm and the semi-norm defined in a natural way.

9.3. **Solution of exterior Laplace problems.** In this section, we gather some useful results about the solution of exterior Laplace problems.

9.3.1. *The Neumann problem for a hole.* Let ω be an open, bounded and connected subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \bar{\omega}$ is connected. Its boundary $\partial\omega$ is supposed to be piecewise of class $\mathcal{C}^1$. We are interested in the following problem:

$$\left\{ \begin{array}{rcllll} \Delta u & = & 0 & \text{in} & \mathbb{R}^N \setminus \bar{\omega}, \\ \partial_n u & = & g & \text{on} & \partial\omega, \\ u & = & O(1/r^{N-1}) & \text{at} & \infty. \end{array} \right. \tag{9.3}$$

We will need the fundamental solution of the Laplace operator

$$E(x) = \left\{ \begin{array}{ll} \dfrac{1}{2\pi} \ln|x| & \text{in 2D,} \\ -\dfrac{1}{4\pi|x|} & \text{in 3D.} \end{array} \right.$$

and the space

$$H_0^{-\frac{1}{2}}(\partial\omega) = \left\{ u \in H^{-\frac{1}{2}}(\partial\omega), \int_{\partial\omega} u ds = 0 \right\}.$$

We recall the following classical theorem. We refer *e.g.* to [7] for the proof.

Theorem 9.1. *Consider $g \in H_0^{-1/2}(\partial\omega)$.*

(1) *Problem (9.3) admits a unique solution $u \in W^1(\mathbb{R}^N \setminus \bar{\omega})$ and the map $g \mapsto u$ is linear and continuous from $H_0^{-\frac{1}{2}}(\partial\omega)$ into $W^1(\mathbb{R}^N \setminus \bar{\omega})$.*

(2) *The boundary integral equation: find $\lambda \in H_0^{-1/2}(\partial\omega)$ such that*

$$\frac{\lambda(x)}{2} + \int_{\partial\omega} \lambda(y) \partial_{n_x} E(x-y) ds(y) = g(x) \qquad \forall x \in \Gamma$$

has a unique solution. Moreover, the map $g \mapsto \lambda$ is a continuous automorphism of $H_0^{-1/2}(\partial\omega)$.

(3) *The single layer potential of density λ satisfies:*

$$\int_{\partial\omega} \lambda(y) E(x-y) ds(y) = \left\{ \begin{array}{l} u_i(x) \text{ if } x \in \omega, \\ u(x) \text{ if } x \in \mathbb{R}^N \setminus \bar{\omega}, \end{array} \right. \tag{9.4}$$

where u is the solution of Problem (9.3) and u_i is a function of $H^1(\omega)$ verifying $\Delta u_i = 0$ in ω.

(4) *We have the jump relations across the boundary* $\partial\omega$ *for the functions* u *and* u_i *defined by (9.4):*

$$u - u_i = 0 \qquad \text{and} \qquad \partial_n u - \partial_n u_i = \lambda.$$

The considered normal is everywhere outward with respect to ω.

9.3.2. *The Neumann problem for a crack.* We consider a bounded manifold Σ of dimension $N-1$. We are interested in the problem:

$$\left\{\begin{array}{rcll} \Delta u & = & 0 & \text{in} \quad \mathbb{R}^N \setminus \overline{\Sigma}, \\ \partial_n u & = & g & \text{on} \quad \Sigma, \\ u & = & O(1/r^{N-1}) & \text{at} \quad \infty. \end{array}\right. \tag{9.5}$$

We assume that $\overline{\Sigma} \subset \tilde{\Sigma}$, $\tilde{\Sigma}$ being a manifold of dimension $N-1$ and of class $\mathcal{C}^1$. We define the space (see [21])

$$H^{\frac{1}{2}}_{00}(\Sigma) = \left\{ u_{|\Sigma}, u \in H^{\frac{1}{2}}(\tilde{\Sigma}),\ \text{Supp } (u) \subset \overline{\Sigma} \right\}, \tag{9.6}$$

where Supp (u) denotes the support of u. It is endowed with the norm

$$\|u_{|\Sigma}\|_{H^{\frac{1}{2}}_{00}(\Sigma)} = \|u\|_{\frac{1}{2},\tilde{\Sigma}}.$$

We have the following theorem.

Theorem 9.2. *Consider* $g \in (H^{1/2}_{00}(\Sigma))'$.

(1) *Problem (9.5) admits a unique solution* $u \in W^1(\mathbb{R}^N \setminus \overline{\Sigma})$ *and the map* $g \mapsto u$ *is linear and continuous from* $(H^{1/2}_{00}(\Sigma))'$ *into* $W^1(\mathbb{R}^N \setminus \overline{\Sigma})$.

(2) *The solution of Problem (9.5) is the double layer potential*

$$u(x) = \int_\Sigma \mu(y) \partial_{n_y} E(x-y) ds(y) \qquad \forall x \in \mathbb{R}^N \setminus \overline{\Sigma},$$

where $\mu = Tg$, T *being a certain isomorphism from* $(H^{1/2}_{00}(\Sigma))'$ *into* $H^{1/2}_{00}(\Sigma)$.

(3) *We have the jump relation across* Σ *(see Figure 1(b) for the orientation convention):*

$$[u] = u^+ - u^- = -\mu.$$

(4) *If* Σ *is a linear (planar in 3D) crack, then the map* T^{-1} *is defined by the following relations, valid for all* $\mu \in (H^{1/2}_{00} \cap \mathcal{C}^1)(\Sigma)$ *and* $\varphi \in \mathcal{D}(\Sigma)$

$$< T^{-1}\mu, \varphi >= - \int_\Sigma \int_\Sigma \frac{d\mu}{ds}(x) \frac{d\varphi}{ds}(y) E(x-y) ds(x) ds(y) \qquad \text{in 2D},$$

$$< T^{-1}\mu, \varphi >= - \int_\Sigma \int_\Sigma curl_\Sigma\ \mu(x).curl_\Sigma\ \varphi(y) E(x-y) ds(x) ds(y) \qquad \text{in 3D}.$$

In this latter expression, we use the notation

$$curl_\Sigma\ u = \mathbf{n} \times \nabla \tilde{u},$$

where $\tilde{u}$ *is an arbitrary lifting of* u *in* $\mathbb{R}^N \setminus \overline{\Sigma}$.

Proof. For what concerns the well-posedness, the integral representation of the solution and the jump relation, we refer the reader to [14]. The explicit formulation of the integral equation for a linear (or planar) crack is obtained *e.g.* in [27] in 2D, in [7] in 3D. All those works address the case of an unbounded manifold but the changes due to the presence of a boundary are minor (because the density vanishes on this boundary) and can be retrieved easily. □

9.4. **A Poincaré inequality for perforated domains.** Let Ω be an open and bounded subset of $\mathbb{R}^N$. We denote by Γ its boundary and by d its diameter. We consider an open set ω included in Ω and we denote $D = \Omega \setminus \bar{\omega}$. We suppose that there exist two real functions f_1 and f_2 defined on an open and bounded subset I of $\mathbb{R}^{N-1}$ such that

$$\omega = \{(x, y) \in I \times \mathbb{R}, f_1(x) < y < f_2(x)\}.$$

Then, the following inequality holds. As it will appear in the proof, this result would remain true if ω was replaced by a crack that can be represented as the graph of a continuous function.

Proposition 9.1. *For all $u \in H^1(D)$ such that $u_{|\Gamma} = 0$, we have*

$$\|u\|_{0,D} \leq d|u|_{1,D}.$$

Proof. For simplicity, it is presented in 2D. Consider $(x, y) \in D, y > 0$. We call $\Delta(x) = \{x\} \times \mathbb{R}$ and $h(x, y)$ the real number defined by

$$(x, h(x, y)) \in \Gamma \text{ and } \{x\} \times [y, h(x, y)] \subset \bar{D} \text{ (see Figure 5).}$$

We have

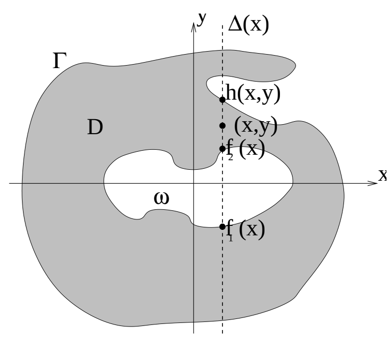


FIGURE 5. The perforated domain.

$$u(x, y) = \int_y^{h(x,y)} \partial_y u(x, y')dy'.$$

Obviously, we have an analogous result if $y < 0$. Therefore, for all $(x, y) \in D$,

$$|u(x, y)| \leq \left| \int_{\Delta(x)\cap D} \partial_y u(x, y')dy' \right|.$$

The Cauchy-Schwartz inequality yields

$$|u(x, y)|^2 \leq d \int_{\Delta(x)\cap D} |\partial_y u(x, y')|^2 dy'.$$

Hence,

$$\int_{\Delta(x)\cap D} |u(x, y)|^2 dy \leq d^2 \int_{\Delta(x)\cap D} |\partial_y u(x, y)|^2 dy.$$

This leads to the desired result by integrating with respect to x. $\square$

References

[1] G. Allaire, E. Bonnetier, G. Francfort and F. Jouve, *Shape optimization by the homogenization method*, Numer. Math. 76, pp. 27-68, 1997.

[2] G. Allaire and R. Kohn, *Optimal design for minimum weight and compliance in plane stress using extremal microstructures*, European Journal of Mechanics, A/Solids, 12(6), pp. 839-878, 1993.

[3] M. P. Bendsoe, N. Kikuchi, *Generating optimal topologies in structural design using an homogeneization method*, Comput. Methods Appl. Engrg. 71, pp. 197-224, 1988.

[4] H. Brézis, *Analyse fonctionnelle*, Dunod, 1999.

[5] J. Céa, *Conception optimale ou identification de forme, calcul rapide de la dérivée directionnelle de la fonction coût*, M.A.A.N., 20(3), pp. 371-402, 1986.

[6] F. Collino, C. Tsogka, *Application of the Perfectly Matched Layer Model to the Linear Elastodynamic Problem in Anisotropic Heterogeneous Media*, Geophysics 66(1), pp. 294-307, 2001.

[7] R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Masson, collection CEA, 1987.

[8] R.A. Feijóo, A.A Novotny, C. Padra and E. Taroco, *The topological derivative for the Poisson's problem*, M3AS 13(12), pp. 1825-1844, 2003.

[9] A. Friedman and M.S. Vogelius, *Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem of continuous dependence*, Arch. Rational Mech. Anal. 105(4), pp. 299-326, 1989.

[10] A. Friedman and M.S. Vogelius, *Determining cracks by boundary measurements*, Indiana Univ. Math. J. 38(3), pp. 527-556, 1989.

[11] R.R. Gadyl'shin, *Existence and asymptotics of poles with a small imaginary part for the Helmholtz resonator*, Russian Math. Surveys, 52(1), pp. 1-72, 1997

[12] S. Garreau, Ph. Guillaume and M. Masmoudi, *The topological asymptotic for PDE systems: the elasticity case*, SIAM J. Control Optim., 39(6), pp. 1756-1778, 2001.

[13] J. Giroire, *Formulation variationnelle par équations intégrales de problèmes aux limites extérieurs*, Rapport interne du Centre de Mathématiques Appliquées de l'Ecole Polytechnique, no. 6, 1976.

[14] J. Giroire and J.-C. Nédélec, *Numerical solution of an exterior Neumann problem using a double layer potential*, Mathematics of computation, 32(144), pp. 973-990, 1978.

[15] S. Garreau, Ph. Guillaume, M. Masmoudi, *The topological Asymptotic for PDE systems: the elasticity case*, SIAM J. Control Optim. 39(6), pp. 1756-1778, 2001.

[16] Ph. Guillaume and K. Sididris, *The topological asymptotic expansion for the Dirichlet problem*, SIAM J. Control and Optim., 41(4), pp. 1042-1072, 2002.

[17] Ph. Guillaume and K. Sididris, *Topological sensitivity and shape optimization for the Stokes equations*, SIAM J. Control Optim. 43(1), pp. 1-31, 2004.

[18] A.M. Il'in, *Matching of asymptotic expansions of solutions of boundary value problems*, Translations of mathematical monographs, Vol. 102, 1992.

[19] J. Jacobsen, N. Olhoff and E. Ronholt, *Generalized shape optimization of three-dimensional structures using materials with optimum microstructures*, Technical report, Institute of Mechanical Engineering, Aalborg University, 1996.

[20] M. Jaoua, *Equations intégrales pour un problème singulier dans le plan*, Thèse, Université Pierre et Marie Curie, 1977.

[21] J.L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, 1968.

[22] M. Masmoudi, *Numerical Solution for Exterior Problems*, Numerische Mathematik 51, pp. 87-101, Springer-Verlag, 1987.

[23] M. Masmoudi, *The Toplogical Asymptotic*, Computational Methods for Control Applications, R. Glowinski, H. Kawarada and J. Periaux eds., GAKUTO Internat. Ser. Math. Sci. Appl. Vol. 16, pp. 53-72, 2001.

[24] V. Maz'ya, S. Nazarov and B. Plamenevskij, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*, Operator theory, advances and applications, Vol. 101, Birkhäuser Verlag, 2000.

[25] F. Murat and J. Simon, *Sur le contrôle par un domaine géométrique*, PhD thesis, Paris, 1976.

[26] S.A. Nazarov and J. Sokolowski, *Asymptotic analysis of shape functionals*, J. Math. Pures Appl. 82, pp. 125-196, 2003.

[27] N. Nishimura and S. Kobayashi, *A boundary integral equation method for an inverse problem related to crack detection*, Int. J. Num. Methods Engrg. 32, pp. 1371-1387, 1991.

[28] A.A. Novotny, R.A. Feijóo, C. Padra and E. Taroco, *Topological sensitivity analysis*, Comput. Methods Appl. Mech. Engrg. 192, pp. 803-829, 2003.

[29] G. Pólya and G. Szegö, *Isoperimetric inequalities in Mathematical Physics*, Annals of Mathematical Studies no. 27, Princeton University Press, 1951.

[30] P.-A. Raviart and J.-M. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson 1988.

[31] B. Samet, S. Amstutz and M. Masmoudi, *The topological asymptotic for the Helmholtz equation*, SIAM J. Control Optim. Vol. 42(5), pp. 1523-1544, 2003.

[32] A. Schumacher, V.V. Kobolev and H.A. Eschenauer, *Bubble method for topology and shape optimization of structures*, Journal of structural optimization no. 8, pp. 42-51, 1994.

[33] M. Shoenauer, L. Kallel and F. Jouve, *Mechanics inclusions identification by evolutionary computation*, Revue européenne des éléments finis 5(5-6), pp. 619-648, 1996.

[34] J. Sokolowski and A. Zochowski, *On the topological derivative in shape optimization*, SIAM J. Control Optim. 37, pp. 1241-1272, 1999.

[35] J. Virieux, *P-SV wave propagation in heterogeneous media: Velocity-stress finite-difference method*, Geophysics 51 (4), pp. 889-901, 1986.

[36] M. Tanter, Software URL : http://www.loa.espci.fr/∼michael/fr/acel/aceltest.htm

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