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OPTIMAL CONTROL IN NONCONVEX DOMAINS: A PRIORI DISCRETIZATION ERROR ESTIMATES

T. APEL*, A. RÖSCH†, AND G. WINKLER‡

Abstract. An optimal control problem for a 2-d elliptic equation is investigated with pointwise control constraints. The domain is assumed to be polygonal but non-convex. The corner singularities are treated by a priori mesh grading. Approximations of the optimal solution of the continuous optimal control problem will be constructed by a projection of the discrete adjoint state. It is proved that these approximations have convergence order h^2.

Key words. Linear-quadratic optimal control problems, error estimates, elliptic equations, non-convex domains, corner singularities, control constraints, superconvergence.

AMS subject classifications. 49K20, 49M25, 65N30, 65N50

1. Introduction. The paper is concerned with the discretization of the 2-d elliptic optimal control problem

$$J(\bar{u}) = \min_{u \in U_{\text{ad}}} J(u), \tag{1.1}$$

$$J(u) := F(Su, u), \tag{1.2}$$

$$F(y, u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}, \tag{1.3}$$

where the associated state $y = Su$ to the control u is the weak solution of the state equation

$$Ly = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{1.4}$$

and the control variable is constrained by

$$a \le u(x) \le b \qquad \text{for a.a. } x \in \Omega. \tag{1.5}$$

Here, y_d is the desired state, a and b are real numbers, and the regularization parameter $\nu > 0$ is a fixed positive number. Moreover, $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with boundary Γ, the second order elliptic operator L is defined by

$$Ly(x) := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}y(x)\right) + \sum_{i=1}^{2} a_i(x)\frac{\partial}{\partial x_i}y(x) + a_0(x)y(x), \tag{1.6}$$

and the set of admissible controls is

$$U_{\text{ad}} := \{u \in L^2(\Omega) : \ a \le u \le b \ \text{ a.e. in } \Omega\}.$$

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We discuss here the full discretization of the control and the state equations by a finite element method, where we focus on the case that the domain Ω has concave corners. The asymptotic behavior of the discretized problem is studied, and superconvergence results are established.

The approximation theory of discretized optimal control problems is well developed, we refer to Falk [9], Geveci [11], Malanowski [14], Arada, Casas, and Tröltzsch [3], and Casas, Mateos, and Tröltzsch [6] for piecewise constant controls. Piecewise linear controls are investigated by Casas [5], Casas and Tröltzsch [7], Meyer and Rösch [16] and Rösch [20],[21]. In all those papers, a family of quasi-uniform meshes is discussed, and the convergence is of order $\alpha = 1$ or $\alpha = \frac{3}{2}$ in the discretization parameter h,

$$\|\bar{u} - u_h\|_{L_2(\Omega)} \le Ch^\alpha,$$

provided that the solution is sufficiently smooth.

Moreover, it is possible to achieve the approximation order $\alpha = 2$ for elliptic optimal control problems in *convex* domains by at least two different approaches. Hinze [12] discretizes only the state equation and its adjoint. The control is obtained by a projection of the adjoint state to the set of admissible controls. In the superconvergence approach by Meyer and Rösch [15], the control variable is discretized as well, but a postprocessing step generates the final approximation of $\bar{u}$.

If the domain is not convex the solution of elliptic boundary value problems contains corner singularities leading to a reduced convergence order of numerical algorithms. Adapted methods have been developed in order to retain the optimal convergence orders. Local mesh grading was introduced by Oganesyan and Rukhovets [18], Raugel [19], Babuška, Kellogg and Pitkäranta [4], and Schatz and Wahlbin [23]. All these papers treat only the case of piecewise linear finite elements, higher order finite elements were discussed by Apel in [1]. The results were extended to systems of differential equations and to the three-dimensional case by Apel, Sändig and Whiteman [2]. Other methods include the singular function method (augmenting technique) described in Strang and Fix [24].

In this paper, we derive error estimates for elliptic optimal control problem (1.1)–(1.5) in non-convex domains. We integrate the local mesh grading technique into the above mentioned approach by Meyer and Rösch [15] where a control $\tilde{u}_h$ is calculated by the projection of the adjoint state $\bar{p}_h$ in a post-processing step. We will show that this post-processing step improves the convergence order from $\alpha = 1$ to $\alpha = 2$.

The paper is organized as follows: In Section 2 we recall regularity results for the elliptic boundary value problem with differential operator L from (1.6). In Section 3 the discretization is introduced and the main results are stated. Section 4 contains results from the finite element theory. The proofs of the superconvergence results are placed in Section 5. The paper ends with results from a numerical experiment.

In this paper, we denote by c a generic constant which may have a different value at each occurrence but is independent of the discretization parameter h.

2. Regularity. An appropriate discretization is adapted to the solution that has to be computed. Therefore we formulate our assumptions in more detail and provide some regularity results for the solution.

Throughout this paper, $\Omega \subset \mathbb{R}^2$ denotes a bounded polygonal domain. For simplicity, let us assume that there is exactly one reentrant corner with interior angle $\omega > \pi$ located in the origin. We denote by

$$r := \sqrt{x_1^2 + x_2^2}$$

the distance to this corner.

The coefficients a_{ij}, a_i and a_0 of the operator L are smooth in $\bar{\Omega}$ and satisfy the ellipticity condition

$$m_0|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \qquad \forall(\xi, x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad m_0 > 0,$$

and the usual condition

$$a_0 - \frac{1}{2}\sum_{i=1}^{2} \frac{\partial a_i}{\partial x_i} \ge 0 \qquad \forall x \in \Omega$$

ensuring coercivity. Moreover, we require $a_{12}(x) = a_{21}(x)$ and $y_d \in L^\infty(\Omega)$. This assumption on the regularity of the desired state y_d simplifies the further presentation but could be weakened.

The regularity of the solution of the elliptic boundary value problem can be described favorably by using the weighted Sobolev spaces

$$V_\beta^{k,p}(\Omega) := \{v \in \mathcal{D}'(\Omega) : \ \|v\|_{V_\beta^{k,p}(\Omega)} < \infty\},$$

with $k \in \mathbb{N}$, $p \in [1, \infty]$, $\beta \in \mathbb{R}$. By using the standard multi-index notation, the norm is defined by

$$\|v\|_{V_\beta^{k,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \le k} r^{p(\beta - k + |\alpha|)} |D^\alpha u|^p \, dx \right)^{1/p}$$

with the standard modification for $p = \infty$. We will make use of the fact that $c_1|r^\beta v|_{W^{k,p}(\Omega)} \le \|v\|_{V_\beta^{k,p}(\Omega)} \le c_2|r^\beta v|_{W^{k,p}(\Omega)}$. For proving the desired regularity result, we follow here the outline by Sändig in [22].

Remark 2.1. *The regularity of the solution y of the elliptic boundary value problem*

$$Ly = g \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma, \tag{2.1}$$

is characterized by one particular eigenvalue of an operator pencil, which is obtained by an integral transformation of the Dirichlet boundary value problem for the equation $L_0 y = g$ in Ω, where the operator L_0 is obtained from the principal part of the operator L by freezing the coefficients in the corner point (here the origin of the coordinate system). That means, the regularity is not influenced by the lower order terms with the coefficients a_i, $i = 0, 1, 2$.

Moreover, the coefficient functions $a_{ij}(x)$ are of interest only in the origin, see, for example, [22]. In that paper the eigenvalue of interest is denoted by $\lambda_- \in \mathbb{C}$. For our purposes we introduce the real quantity $\lambda = -\operatorname{Im} \lambda_-$.

In the case of the Dirichlet problem for the Laplace operator and a two-dimensional domain with a reentrant corner with interior angle $\omega \in (\pi, 2\pi)$, the value of λ is explicitly known, $\lambda = \pi/\omega$. That means in particular $\lambda \in (1/2, 1)$.

In the more general case of the operator L from (1.6) we follow [17, Chap. 5] and consider the linear coordinate transformation $y_1 = x_1 + d_1 x_2$, $y_2 = d_2 x_2$, with $d_1 = -a_{12}/a_{22}$ and $d_2 = \sqrt{a_{11}a_{22} - a_{12}^2}/a_{22}$. In this way, the differential operator L_0 is transformed into a multiple of the Laplace operator, and the neighborhood of the corner, a circular sector with opening ω, into another sector with opening ω'. The quantity of interest is then $\lambda = \pi/\omega'$. Since $\omega' \in (\pi, 2\pi)$ for $\omega \in (\pi, 2\pi)$ we have also in the general case $\lambda \in (1/2, 1)$.

LEMMA 2.2. *Let $\lambda \in (1/2, 1)$ be the real number associated with the differential operator L and the domain Ω, as introduced in Remark 2.1. Let p and β be given real numbers with $p \in (1, \infty)$ and $\beta > 2 - \lambda - 2/p$. Moreover, let g be a function in $V_\beta^{0,p}(\Omega)$. Then the weak solution y of the boundary value problem (2.1) belongs to $H_0^1(\Omega) \cap V_\beta^{2,p}(\Omega)$. Moreover, there exists a positive constant c, such that*

$$\|y\|_{V_\beta^{2,p}(\Omega)} \le c\|g\|_{V_\beta^{0,p}(\Omega)}.$$

Proof. The regularity of elliptic boundary value problems is studied in many references. We follow here the outline by Sändig in [22]. With Remark 2.1 we have $\operatorname{Im} \lambda_- = -\lambda$, and Lemmata 1 and 2 of [22] state the assertion of our lemma. ☐

REMARK 2.3. *For $p < 2/(2 - \lambda)$ we can choose $\beta = 0$ in Lemma 2.2 and obtain that y is contained in the classical Sobolev space $W^{2,p}(\Omega)$. Since this space, for $p > 1$, is embedded in $C(\bar\Omega)$ we can conclude that the solution of (2.1) is bounded,*

$$y \in C(\bar\Omega) \quad \text{if } g \in L^p(\Omega),\ p > 1.$$

The assertion holds for all $p \in (1, \infty]$ because the data can, of course, be smoother than necessary.

Let us now return to our optimal control problem. Via (1.4), the operator S associates a state $y = Su$ to the control u. The solution operator S acts from $L^\infty(\Omega)$ into $L^2(\Omega)$. Even better, from Lemma 2.2 and Remark 2.3 we get $y = Su \in L^\infty(\Omega) \cap H_0^1(\Omega) \cap V_\beta^{2,p}(\Omega)$,

$$\|Su\|_{L^\infty(\Omega)} + \|Su\|_{H_0^1(\Omega)} + \|Su\|_{V_\beta^{2,p}(\Omega)} \le c\|u\|_{V_\beta^{0,p}(\Omega)} \le c\|u\|_{L^\infty(\Omega)}, \tag{2.2}$$

for all $p \in (1, \infty)$ and $\beta > 2 - \lambda - 2/p$. The last estimate makes use of the fact that a function u that is contained in $L^\infty(\Omega)$, is also contained in $V_\beta^{0,p}(\Omega)$ since

$$\|u\|^p_{V_\beta^{0,p}(\Omega)} = \int_\Omega r^{p\beta}|u|^p\, dx \le \|u\|^p_{L^\infty(\Omega)} \int_\Omega r^{p\beta}\, dx.$$

The integral is finite due to $p\beta > (2 - \lambda)p - 2 > -2$.

Next, we introduce the adjoint problem

$$L^*p = y - y_d \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma, \tag{2.3}$$

and denote by S^* the solution operator of this problem, that means we have

$$p = S^*(y - y_d).$$

Since we can also write

$$p = S^*(Su - y_d) = Pu$$

with an affine operator P we call the solution $p = Pu$ the associated adjoint state to u. The right hand side $y - y_d$ of (2.3) is contained in $L^\infty(\Omega)$. Due to Lemma 2.2, the adjoint equation admits a unique solution $Pu \in L^\infty(\Omega) \cap H_0^1(\Omega) \cap V_\beta^{2,p}(\Omega)$. We have

$$\begin{aligned} \|Pu\|_{L^\infty(\Omega)} + \|Pu\|_{H_0^1(\Omega)} + \|Pu\|_{V_\beta^{2,p}(\Omega)} &\le c\|Su - y_d\|_{L^\infty(\Omega)} \\ &\le c\left(\|u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \end{aligned} \tag{2.4}$$

for any $p \in (1, \infty)$ and $\beta > 2 - \lambda - 2/p$. By Remark 2.3, we have also $Pu \in C(\bar{\Omega})$.

Corollary 2.4. *For $u, y_d \in L^\infty(\Omega)$ and $\beta > 1-\lambda$, the adjoint state is also contained in $V_\beta^{1,\infty}(\Omega)$,*

$$\|Pu\|_{V_\beta^{1,\infty}(\Omega)} \le c\left(\|u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right). \tag{2.5}$$

Proof. For given β with $1 - \lambda < \beta < 2 - \lambda$ we define $\varepsilon = (\beta - 1 + \lambda)/2$ and observe that $0 < \varepsilon < \frac{1}{2}$ and $\beta > 1 - \lambda + \varepsilon$. There exists a $p > 2$, namely $p = 2/(1 - \varepsilon)$, such that the inequality $\beta > 1 - \lambda + \varepsilon$ can be rewritten as $\beta > 2 - \lambda - 2/p$. Of course, this inequality is also valid for arbitrary $p > 2$ if $\beta \ge 2 - \lambda$. We can summarize that we have shown the existence of a $p > 2$ such that $\beta > 2 - \lambda - 2/p$.

From (2.4) we find $Pu \in V_\beta^{2,p}(\Omega)$ with this choice of p and β. This is equivalent to $r^\beta Pu \in W^{2,p}(\Omega)$; by the Sobolev embedding theorem we conclude $r^\beta Pu \in W^{1,\infty}(\Omega)$, and this is equivalent to $Pu \in V_\beta^{1,\infty}(\Omega)$. This chain of conclusions can also be expressed in form of inequalities, $\|Pu\|_{V_\beta^{1,\infty}(\Omega)} \le c\|r^\beta Pu\|_{W^{1,\infty}(\Omega)} \le c\|r^\beta Pu\|_{W^{2,p}(\Omega)} \le c\|Pu\|_{V_\beta^{2,p}(\Omega)} \le c\left(\|u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right)$. ∎

Remark 2.5. *The regularity of the solution of an elliptic boundary value problem $Ly = f$ in Ω, $\ell y = g$ on $\partial\Omega$, is determined by the smoothness of the coefficients in the operators L and ℓ, the smoothness of the right hand sides f and g, and the geometrical properties of the domain Ω. If the coefficients of L and ℓ as well as the boundary of Ω are smooth, then the solution y is smoother than the right hand side (shift theorem). Therefore it is generally observed that the adjoint state p is more regular than the state y. In the case of nonconvex polygonal domains, however, the regularity is in general limited by singularities due to the corners of the domain such that the state y and the adjoint state p show the same regularity in the context of Sobolev spaces, as described above.*

We finish the section by recalling some results from the analysis of optimal control problems. Introducing the projection

$$\Pi_{[a,b]} f(x) := \max(a, \min(b, f(x))), \tag{2.6}$$

we can formulate the necessary and sufficient first-order optimality condition for the optimal control problem (1.1)–(1.5).

LEMMA 2.6. *The optimal control problem (1.1)–(1.5) has a unique solution $\bar{u}$. The variational inequality*

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Omega)} \geq 0 \quad \text{for all } u \in U_{\text{ad}} \tag{2.7}$$

is necessary and sufficient for the optimality of $\bar{u}$. This condition can be expressed equivalently by

$$\bar{u} = \Pi_{[a,b]} \left(-\frac{1}{\nu} \bar{p} \right). \tag{2.8}$$

Here, $\bar{p} = P\bar{u}$ denotes the corresponding adjoint state.

Proof. Since the optimal control problem is strictly convex and the objective is radially bounded, we obtain the existence of a unique optimal solution. The necessary optimality condition can be formulated as variational inequality (2.7). The strict convexity implies that the necessary condition is also sufficient. For the equivalence of this variational inequality with the above formulated projection formula, see, e. g. [14]. □

COROLLARY 2.7. *The optimal control $\bar{u}$ belongs to $C(\bar{\Omega})$.*

3. Discretization and superconvergence results. We will now discretize the optimal control problem by a finite element method. To this aim, we consider a family of graded triangulations $(T_h)_{h>0}$ of $\bar{\Omega}$. All meshes are admissible in Ciarlet's sense [8]. With h being the global mesh parameter, $\mu \in (0,1]$ being the grading parameter, and r_T being the distance of a triangle T to the corner,

$$r_T := \inf_{(x_1,x_2)\in T} \sqrt{x_1^2 + x_2^2},$$

we assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$\begin{array}{rcccll} c_1 h^{1/\mu} & \leq & h_T & \leq & c_2 h^{1/\mu} & \text{for } r_T = 0, \\ c_1 h r_T^{1-\mu} & \leq & h_T & \leq & c_2 h r_T^{1-\mu} & \text{for } r_T > 0. \end{array} \tag{3.1}$$

It has been proved that the number of elements of such a triangulation is of order h^{-2}, see, for example, [2]. Moreover, we set

$$\begin{aligned} U_h &:= \{u_h \in L^\infty(\Omega) : \ u_h|_T \in \mathcal{P}_0 \text{ for all } T \in T_h\}, \\ U_h^{\text{ad}} &:= U_h \cap U_{\text{ad}}, \\ V_h &:= \{y_h \in C(\bar{\Omega}) : \ y_h|_T \in \mathcal{P}_1 \text{ for all } T \in T_h \text{ and } y_h = 0 \text{ on } \Gamma\}, \end{aligned}$$

where $\mathcal{P}_k$, $k = 0, 1$, is the space of polynomials of degree less than or equal to k.

For each $u \in L^2(\Omega)$, we denote by $S_h u$ the unique element of V_h that satisfies

$$a(S_h u, v_h) = (u, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h, \tag{3.2}$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form defined by

$$a(y,v) := \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{2} a_i(x) \frac{\partial}{\partial x_i} y(x) v(x) + a_0(x) y(x) v(x) \right) dx.$$

In other words, $S_h u$ is the approximated state associated with a control u.

The finite dimensional approximation of the optimal control problem is defined by

$$\begin{aligned} J_h(\bar{u}_h) &= \min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h), \\ J_h(u_h) &:= \frac{1}{2} \|S_h u_h - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u_h\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.3}$$

The adjoint equation is discretized in the same way. We search $p_h = S_h^*(S_h u_h - y_d) = P_h u_h \in V_h$ such that

$$a(v_h, p_h) = (S_h u_h - y_d, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h. \tag{3.4}$$

Remark 3.1. *The optimal control problem (3.3) admits a unique solution $\bar{u}_h$. In the following, we use the notation $\bar{y}_h = S_h \bar{u}_h$ and $\bar{p}_h = P_h \bar{u}_h$ for the optimal discrete state and adjoint state. The variational inequality*

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \quad \text{for all } u_h \in U_h^{\mathrm{ad}} \tag{3.5}$$

is necessary and sufficient for the optimality of $\bar{u}_h$.

For our superconvergence result we need an additional assumption. The optimal control $\bar{u}$ is obtained by the projection formula (2.8). Therefore, we can classify the triangles $T \in T_h$ in two sets K_1 and K_2,

$$K_1 := \bigcup_{T \in T_h :\, \bar{u} \notin V_{2-2\mu}^{2,2}(T)} T, \qquad K_2 := \bigcup_{T \in T_h :\, \bar{u} \in V_{2-2\mu}^{2,2}(T)} T. \tag{3.6}$$

Clearly, the number of triangles in K_1 grows for decreasing h. Nevertheless, the assumption

$$\operatorname{meas} K_1 \leq ch \tag{3.7}$$

is fulfilled in many practical cases.

For continuous functions f we define now the projection into the space U_h of piecewise constant functions by

$$(R_h f)(x) := f(S_T) \quad \text{if } x \in T,$$

where S_T denotes the centroid of the triangle T. Note that $R_h \bar{u} \in U_h^{\mathrm{ad}}$. Now we are able to formulate our superconvergence results.

THEOREM 3.2. *Assume that the assumption (3.7) holds. Let $\bar{u}_h$ be the solution of (3.3) on a family of meshes with grading parameter $\mu < \lambda$. Then the estimate*

$$\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{3.8}$$

holds true. The proof of this theorem is postponed to Section 5.

Theorem 3.2 means that the values of the numerical solution $\bar{u}_h$ in the centroids have already quadratic convergence rate. This property allows us to prove the following error estimates for state and adjoint state. Note that the right hand side in the state equation is only first order accurate in the L^2-sense but we get second order accuracy in the state and the adjoint state.

COROLLARY 3.3. *Assume that the assumption (3.7) holds. Let $\bar{y}_h$ be the associated state and $\bar{y}_h$ be the associated adjoint state to the solution $\bar{u}_h$ of (3.3) on a family of meshes with grading parameter $\mu < \lambda$. Then the estimates*

$$\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{3.9}$$

$$\|\bar{p}_h - \bar{p}\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{3.10}$$

are valid. The proof is carried out in Section 5.

The idea is now to construct controls in a post-processing step. The control $\tilde{u}_h$ is calculated by a projection of the discrete adjoint state $\bar{p}_h := P_h\bar{u}_h$ to the admissible set U_{ad},

$$\tilde{u}_h := \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}_h\right).$$

Note that in general $\tilde{u}_h \notin U_h$ and $\tilde{u}_h \notin V_h$, but $\tilde{u}_h$ is still piecewise linear and continuous. The next theorem states, that it has superconvergence properties.

THEOREM 3.4. *Assume that the assumption (3.7) holds. Let $\tilde{u}_h$ be the control constructed above on a family of meshes with grading parameter $\mu < \lambda$. Then the inequality*

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{3.11}$$

is satisfied. The proof is also given in Section 5.

4. Results from finite element theory. In this section, we collect results from the finite element theory for elliptic equations and from numerical integration.

LEMMA 4.1. *The norms of the discrete solution operators S_h and S_h^* are bounded,*

$$\begin{aligned}
\|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c,\\
\|S_h\|_{L^2(\Omega)\to L^2(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} &\le c,\\
\|S_h\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c,\\
\|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c,
\end{aligned}$$

where c is, as always, independent of h.

Proof. With Remark 2.3 we see that S is a bounded operator from $L^2(\Omega)$ into $L^\infty(\Omega)$. By embedding theorems, we conclude that S is also bounded from $L^2(\Omega)$ into $L^2(\Omega)$ and from $L^\infty(\Omega)$ into $L^\infty(\Omega)$. Hence, we have only to prove that the operator $S - S_h$ is bounded in these pairs of spaces. Then the assertion for S_h follows by using the triangle inequality.

By standard interpolation arguments and the Aubin-Nitsche trick we find that the convergence of $\|Sf - I_hSf\|_{L^2(\Omega)}$ and $\|Sf - S_hf\|_{L^2(\Omega)}$ is better than first order even on quasi-uniform meshes [2]. This crude estimate allows to conclude

$$\|I_hSf - S_hf\|_{L^\infty(\Omega)} \le ch^{-1}\|I_hSf - S_hf\|_{L^2(\Omega)} \le c\|f\|_{L^2(\Omega)}$$

where we have used an inverse inequality and the triangle inequality. Since also

$$\|Sf - I_hSf\|_{L^\infty(\Omega)} \le c\|Sf\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)}$$

we conclude by using the triangle inequality again that

$$\|Sf - S_hf\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)}$$

holds. The other inequalities for S_h follow by embedding theorems. The proof for S^* can be carried out similarly. □

LEMMA 4.2. *Let $u \in L^2(\Omega)$ be any function. The discretization error can be estimated by*

$$\|Su - S_hu\|_{L^2(\Omega)} \le ch^2\|u\|_{L^2(\Omega)}, \tag{4.1}$$

$$\|S^*u - S_h^*u\|_{L^2(\Omega)} \le ch^2\|u\|_{L^2(\Omega)}, \tag{4.2}$$

$$\|Pu - P_hu\|_{L^2(\Omega)} \le ch^2\left(\|u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right), \tag{4.3}$$

provided that the mesh grading parameter satisfies $\mu < \lambda$.

Proof. The proof of (4.1) and (4.2) can be found in the literature, see [4, 17, 19]. For proving (4.3), we use

$$Pu - P_hu = S^*(Su - y_d) - S_h^*(S_hu - y_d) = (S^* - S_h^*)(Su - y_d) + S_h^*(S - S_h)u,$$

The assertion (4.3) follows with the approximation error estimate (4.1) and (4.2) in the form

$$\|S^* - S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} \le ch^2, \qquad \|S - S_h\|_{L^2(\Omega)\to L^2(\Omega)} \le ch^2,$$

and the boundedness of S and S_h^* as operators from $L^2(\Omega)$ into $L^2(\Omega)$. □

Next, we prove an estimate for the numerical integration. This estimate is the key for our superconvergence results. Numerical integration has second order accuracy although piecewise constants are used only.

LEMMA 4.3. *Let T_h be a triangulation with grading parameter μ. Then the estimate*

$$\left|\int_T (f - R_hf)\,dx\right| \le \begin{cases} ch^2|T|^{1/2}|f|_{V^{2,2}_{2-2\mu}(T)} & \text{if } r_T > 0 \\ ch^2|T|^{1/2}\|f\|_{V^{2,2}_{2-2\mu}(T)} & \text{if } r_T = 0 \\ ch^2\|f\|_{L^\infty(T)} & \text{if } r_T = 0 \end{cases}$$

holds for any triangle $T \in T_h$ and for any function $f \in V^{2,2}_{2-2\mu}(T)$ or $f \in L^\infty(T)$, respectively.

Proof. The key idea in the case $r_T > 0$ is that the integral vanishes for $f \in \mathcal{P}_1$ such that on the reference element $\hat{T}$ the estimate

$$\left| \int_{\hat{T}} (\hat{f} - \widehat{R_h f}) \, d\hat{x} \right| \le c |\hat{f}|_{W^{2,2}(\hat{T})}$$

is obtained by using the Bramble–Hilbert lemma. The transformation to the element T leads to

$$\left| \int_T (f - R_h f) \, dx \right| \le c h_T^2 |T|^{1/2} |f|_{W^{2,2}(T)}.$$

Here we have used that $f \in W^{2,2}(T)$ when $r_T > 0$. The assertion of the lemma is obtained by using $h_T \le c_2 h r_T^{1-\mu}$ and $r_T \le r$ in T.

In the case $r_T = 0$ we can proceed by using that R_h is a bounded operator from $L^\infty(T)$ to $L^\infty(T)$ with norm 1, and that $V^{2,2}_{2-2\mu}(T)$ is continuously embedded into $L^\infty(T)$ for all μ in the possible range $(0,1]$. This leads to the estimate

$$\left| \int_T (f - R_h f) \, dx \right| \le c|T| \|f\|_{L^\infty(T)} \le c|T|^{1/2} h_T^{2-(2-2\mu)} \|f\|_{V^{2,2}_{2-2\mu}(T)}.$$

The proof is finished by observing that $h_T^{2\mu} \le ch^2$ and $|T| \le ch^2$ for T with $r_T = 0$. ◻

Lemma 4.4. *On a mesh with grading parameter $\mu < \lambda$ the estimate*

$$(v_h, \bar{u} - R_h \bar{u})_{L^2(\Omega)} \le ch^2 \left(\|v_h\|_{L^\infty(\Omega)} + \|v_h\|_{H^1_0(\Omega)} \right) \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|\bar{y}_d\|_{L^\infty(\Omega)} \right) \quad (4.4)$$

can be proved for all $v_h \in V_h$, provided that Assumption (3.7) is fulfilled.

Proof. With the sets K_1 and K_2 introduced in (3.6) we have

$$(v_h, \bar{u} - R_h \bar{u})_{L^2(\Omega)} = \int_{K_1} v_h (\bar{u} - R_h \bar{u}) \, dx + \int_{K_2} v_h (\bar{u} - R_h \bar{u}) \, dx. \quad (4.5)$$

The K_1-part can be estimated as follows:

$$\left| \int_{K_1} v_h (\bar{u} - R_h \bar{u}) \, dx \right| \le \|v_h\|_{L^\infty(\Omega)} \sum_{T \in K_1} \int_T |\bar{u}(x) - R_h \bar{u}(x)| \, dx. \quad (4.6)$$

Next, we split K_1 into the union $K_{1,s}$ of triangles T with $r_T = 0$ and the union $K_{1,r}$ of triangles T with $r_T > 0$. The set $K_{1,s}$ contains only a finite number of triangles independent of h. Since R_h is a bounded operator from $L^\infty(\Omega)$ to $L^\infty(\Omega)$, we can continue by

$$\int_{K_{1,s}} |\bar{u}(x) - R_h \bar{u}(x)| \, dx \le |K_{1,s}| \|\bar{u} - R_h \bar{u}\|_{L^\infty(\Omega)} \le ch^2 \|\bar{u}\|_{L^\infty(\Omega)}. \quad (4.7)$$

On each triangle $T \subset K_{1,r}$ we have $\bar{u} = a$, $\bar{u} = b$, or $\bar{u} = -\bar{p}/\nu$ due to (2.8). This means in any case $|\bar u|_{W^{1,\infty}(T)} \le c|\bar p|_{W^{1,\infty}(T)}$, such that we can conclude

$$\begin{aligned}
\int_{K_{1,r}} |\bar u(x) - R_h\bar u(x)|\, dx &\le \sum_{T\subset K_{1,r}} |T|\, h_T |\bar u|_{W^{1,\infty}(T)} \\
&\le c \sum_{T\subset K_{1,r}} |T|\, h r_T^{1-\mu} |\bar p|_{W^{1,\infty}(T)} \\
&\le ch \sum_{T\subset K_{1,r}} |T|\, \|\bar p\|_{V^{1,\infty}_{1-\mu}(\Omega)} \\
&\le ch^2 \|\bar p\|_{V^{1,\infty}_{1-\mu}(\Omega)} \\
&\le ch^2 \left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right), \qquad (4.8)
\end{aligned}$$

using assumption (3.7) and estimate (2.5) with $\mu < \lambda$ in the last two steps. Combining (4.6)–(4.8), we obtain

$$\left| \int_{K_1} v_h(\bar u - R_h\bar u)\, dx \right| \le ch^2 \left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right) \|v_h\|_{L^\infty(\Omega)}. \qquad (4.9)$$

For a triangle T of the K_2-part we have

$$\int_T R_h\bar u\, v_h\, dx = \int_T \bar u(S_T) v_h\, dx = \int_T \bar u(S_T) v_h(S_T)\, dx.$$

This is a formula for the integration of $\bar u v_h$. Consequently, we obtain by Lemma 4.3 and the fact that $r_T = 0$ holds only for a finite number (independent of h) of triangles

$$\begin{aligned}
\left| \int_{K_2} (\bar u - R_h\bar u) v_h\, dx \right| &= \sum_{T\subset K_2} \left| \int_T (\bar u v_h - \bar u(S_T) v_h(S_T))\, dx \right| \\
&\le ch^2 \|\bar u v_h\|_{L^\infty(\Omega)} + ch^2 \sum_{T\subset K_2} |T|^{1/2} |\bar u v_h|_{V^{2,2}_{2-2\mu}(T)} \\
&\le ch^2 \|\bar u\|_{L^\infty(\Omega)} \|v_h\|_{L^\infty(\Omega)} + ch^2 \left(\sum_{T\subset K_2} |\bar u v_h|^2_{V^{2,2}_{2-2\mu}(T)} \right)^{1/2}. \qquad (4.10)
\end{aligned}$$

Next, we divide each triangle $T \subset K_2$ in an *active* part A_T and an *inactive* part I_T with $A_T \cup I_T = T$. In general, we will have for triangles $T \subset K_2$, that either $A_T = \emptyset$ or $I_T = \emptyset$, but we cannot exclude the case that both components are nonempty. The optimal control $\bar u$ is constant on the active component A_T ($\bar u = a$ or $\bar u = b$). Therefore, the seminorm is 0 on these parts. On the inactive parts I_T, we have $\bar u = -\bar p/\nu$. Therefore, we can estimate

$$\begin{aligned}
|\bar u v_h|_{V^{2,2}_{2-2\mu}(T)} &= |\bar u v_h|_{V^{2,2}_{2-2\mu}(I_T)} = \frac{1}{\nu} |\bar p v_h|_{V^{2,2}_{2-2\mu}(I_T)} \le c|\bar p v_h|_{V^{2,2}_{2-2\mu}(T)} \\
&\le c\|v_h\|_{L^\infty(T)} |\bar p|_{V^{2,2}_{2-2\mu}(T)} + c\|v_h\|_{H^1_0(T)} |\bar p|_{V^{1,\infty}_{2-2\mu}(T)}.
\end{aligned}$$

Consequently, we can conclude by means of (2.4) and (2.5)

$$\begin{aligned}
\left(\sum_{T\subset K_2} |\bar p v_h|^2_{V^{2,2}_{2-2\mu}(T)} \right)^{1/2} &\le c\|v_h\|_{L^\infty(\Omega)} |\bar p|_{V^{2,2}_{2-2\mu}(\Omega)} + c\|v_h\|_{H^1_0(\Omega)} |\bar p|_{V^{1,\infty}_{2-2\mu}(\Omega)} \\
&\le \left(\|v_h\|_{L^\infty(\Omega)} + \|v_h\|^2_{H^1_0(\Omega)} \right) \left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right).
\end{aligned}$$

Hence, we can continue the estimate in (4.10) by

$$\left|\int_{K_2} (\bar{u} - R_h\bar{u})v_h \, dx\right| \le ch^2 \left(\|v_h\|_{L^\infty(\Omega)} + \|v_h\|^2_{H^1_0(\Omega)}\right) \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right). \tag{4.11}$$

Inserting (4.9) and (4.11) into (4.5), we obtain the desired estimate (4.4). □

COROLLARY 4.5. *Consider meshes with grading parameter $\mu < \lambda$. Under Assumption (3.7) the estimates*

$$\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)} \le ch^2(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}) \tag{4.12}$$

$$\|P_h\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)} \le ch^2(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}). \tag{4.13}$$

hold.

Proof. We start with

$$\begin{aligned}
\|S_h\bar{u} - S_hR_h\bar{u}\|^2_{L^2(\Omega)} &= (S_h\bar{u} - S_hR_h\bar{u}, S_h\bar{u} - S_hR_h\bar{u})_{L^2(\Omega)} \\
&= a(S_h\bar{u} - S_hR_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} \\
&= (\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} \\
&\le ch^2 \left(\|P_h\bar{u} - P_hR_h\bar{u}\|_{L^\infty(\Omega)} + \|P_h\bar{u} - P_hR_h\bar{u}\|_{H^1_0(\Omega)}\right) \cdot \\
&\quad \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right),
\end{aligned} \tag{4.14}$$

where we have used Lemma 4.4 with $v_h = P_h\bar{u} - P_hR_h\bar{u}$. We benefit now from the fact that $P_h\bar{u}$ and $P_hR_h\bar{u}$ are the solutions of the discretized adjoint equation (3.4), that means $P_h\bar{u} - P_hR_h\bar{u} = S_h^*(S_h\bar{u} - y_d) - S_h^*(S_hR_h\bar{u} - y_d) = S_h^*(S_h\bar{u} - S_hR_h\bar{u})$. Hence, we have by using Lemma 4.1

$$\|P_h\bar{u} - P_hR_h\bar{u}\|_{L^\infty(\Omega)} + \|P_h\bar{u} - P_hR_h\bar{u}\|_{H^1_0(\Omega)} \le c\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}. \tag{4.15}$$

$$\|P_h\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)} \le c\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}. \tag{4.16}$$

Inserting (4.15) into (4.14) and dividing by $\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}$ we get inequality (4.12). By using (4.16) we obtain (4.13) from (4.12). □

COROLLARY 4.6. *Let $\bar{u}$ be the solution of our optimal control problem. If the problem is discretized on meshes with grading parameter $\mu < \lambda$, we have the error estimate*

$$\|\bar{p} - P_hR_h\bar{u}\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right). \tag{4.17}$$

Proof. We apply Lemma 4.2 with $u = \bar{u}$, the identity $\bar{p} = P\bar{u}$, and estimates (4.3) and (4.13) and get

$$\begin{aligned}
\|\bar{p} - P_hR_h\bar{u}\|_{L^2(\Omega)} &= \|P\bar{u} - P_h\bar{u}\|_{L^2(\Omega)} + \|P_h\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)} \\
&\le ch^2 \left(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + \|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right).
\end{aligned}$$

By using the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ we obtain the assertion. □

5. Superconvergence properties. We start with another auxiliary result.

LEMMA 5.1. *The inequality*

$$\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \tag{5.1}$$

is valid .

Proof. The optimality condition (2.7) is true for all $u \in U_{\rm ad}$. Therefore, we have pointwise a.e.

$$(\bar{p}(x) + \nu\bar{u}(x)) \cdot (u(x) - \bar{u}(x)) \ge 0 \qquad \forall u \in U_{\rm ad}.$$

Consider any triangle T with center of gravity S_T and apply this formula for $x = S_T$ and $u = \bar{u}_h$. This can be done because of the continuity of $\bar{u}$, $\bar{p}$, and $\bar{u}_h$ in these points. We arrive at

$$(\bar{p}(S_T) + \nu\bar{u}(S_T)) \cdot (\bar{u}_h(S_T) - \bar{u}(S_T)) \ge 0 \quad \text{for all } T.$$

Due to the definition of R_h, this is equivalent to

$$(R_h\bar{p}(S_T) + \nu R_h\bar{u}(S_T)) \cdot (\bar{u}_h(S_T) - R_h\bar{u}(S_T)) \ge 0 \quad \text{for all } T.$$

We integrate this formula over T, add over all T and get

$$(R_h\bar{p} + \nu R_h\bar{u}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \ge 0.$$

Moreover, we can test the optimality condition (3.5) for $\bar{u}_h$ with the function $R_h\bar{u}$ and get

$$(\bar{p}_h + \nu\bar{u}_h, R_h\bar{u} - \bar{u}_h)_{L^2(\Omega)} \ge 0.$$

We add these two inequalities and obtain

$$(R_h\bar{p} - \bar{p}_h + \nu(R_h\bar{u} - \bar{u}_h), \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \ge 0$$

which is equivalent to the formula (5.1). ∎

We are now ready to prove Theorem 3.2.

Proof. We begin by rewriting formula (5.1),

$$\begin{aligned}\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} &\le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\ &= (R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} + (\bar{p} - P_hR_h\bar{u}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\ &\quad +(P_hR_h\bar{u} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)}.\end{aligned} \tag{5.2}$$

The first term represents again a formula for the numerical integration, since $\bar{u}_h - R_h\bar{u}$ is piecewise constant. We obtain by using again Lemma 4.3 and estimate (2.4)

$$\begin{aligned}(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} &= \sum_{T\in T_h} \int_T (R_h\bar{p}(x) - \bar{p}(x))(\bar{u}_h(x) - R_h\bar{u}(x))\,dx \\ &= \sum_{T\in T_h} (\bar{u}_h(S_T) - R_h\bar{u}(S_T)) \int_T (R_h\bar{p} - \bar{p})\,dx \\ &\le \sum_{T\in T_h} ch^2|\bar{u}_h(S_T) - R_h\bar{u}(S_T)||T|^{1/2}\,\|\bar{p}\|_{V^{2,2}_{2-2\mu}(T)} \\ &\le ch^2\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Omega)}\left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right).\end{aligned} \tag{5.3}$$

The second term in (5.2) can be estimated by using the Cauchy–Schwartz inequality and formula (4.17)

$$(\bar{p} - P_h R_h \bar{u}, \bar{u}_h - R_h \bar{u})_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^2(\Omega)}\right) \|R_h \bar{u} - \bar{u}_h\|_{L^2(\Omega)}. \quad (5.4)$$

The third term can simply be omitted, since due to $\bar{p}_h = P_h u_h$ and the definition of P_h

$$(P_h R_h \bar{u} - \bar{p}_h, \bar{u}_h - R_h \bar{u})_{L^\infty(\Omega)} = (S_h R_h \bar{u} - S_h \bar{u}_h, S_h \bar{u}_h - S_h R_h \bar{u})_{L^2(\Omega)} \le 0. \quad (5.5)$$

Inserting (5.3)–(5.5) into (5.2), we get

$$\nu \|R_h \bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \|R_h \bar{u} - \bar{u}_h\|_{L^2(\Omega)}.$$

This formula is equivalent to the assertion of Theorem 3.2. □

Next, we prove Corollary 3.3.

Proof. We start with

$$\begin{aligned} \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &= \|S\bar{u} - S_h \bar{u}_h\|_{L^2(\Omega)} \\ &\le \|S\bar{u} - S_h \bar{u}\|_{L^2(\Omega)} + \|S_h \bar{u} - S_h R_h \bar{u}\|_{L^2(\Omega)} + \|S_h R_h \bar{u} - S_h \bar{u}_h\|_{L^2(\Omega)}. \end{aligned}$$

The first term was estimated in Lemma 4.2. Corollary 4.5 delivers an inequality for the second term. Theorem 3.2 contains the estimate of the third term. Consequently, we find

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right),$$

i.e., (3.9). The second inequality can be obtained similarly,

$$\begin{aligned} \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &= \|S^*(\bar{y} - y_d) - S_h^*(\bar{y}_h - y_d)\|_{L^2(\Omega)} \\ &\le \|S^*(\bar{y} - y_d) - S_h^*(\bar{y} - y_d)\|_{L^2(\Omega)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Omega)} \\ &\le ch^2 \|\bar{y} - y_d\|_{L^2(\Omega)} + ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \\ &\le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \end{aligned}$$

by means of Lemma 4.1 and (3.9). □

It remains to prove Theorem 3.4.

Proof. The projection operator $\Pi_{[a,b]}$ is Lipschitz continuous with constant 1 from $L^2(\Omega)$ to $L^2(\Omega)$. Therefore, we get

$$\begin{aligned} \nu \|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} &= \nu \left\| \Pi_{[a,b]} \left(-\frac{1}{\nu}\bar{p}\right) - \Pi_{[a,b]} \left(-\frac{1}{\nu}\bar{p}_h\right) \right\|_{L^2(\Omega)} \\ &\le \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \\ &\le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right). \end{aligned}$$

where we used (3.10) in the last step. The superconvergence result (3.11) is proved. □

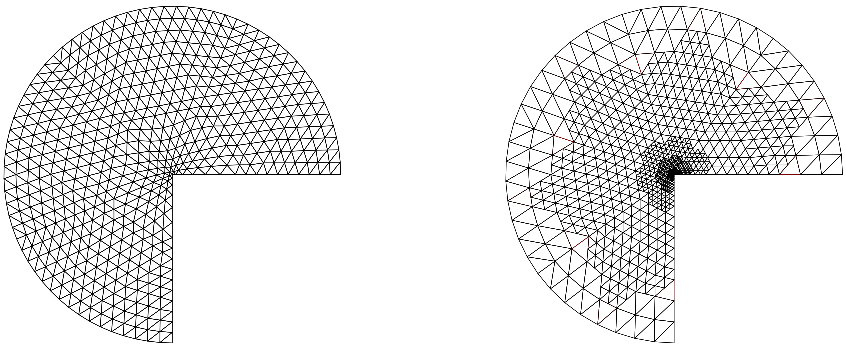

FIG. 6.1. *Ω with a graded mesh; left: $\mu = 0.6$, construction by mapping; right: $\mu = 0.3$, construction by dyadic partitioning*

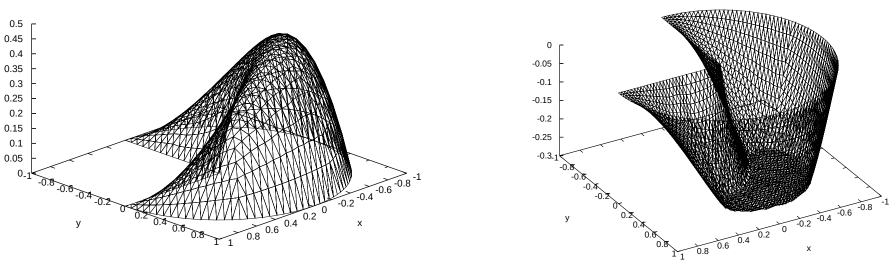

FIG. 6.2. *The exact state $\bar{y}$ and the optimal control function $\bar{u}$.*

6. Numerical example. In this section we present a numerical example showing the predicted convergence behavior. In order to construct meshes that fulfil the conditions (3.1) we transformed the mesh using the mapping

$$T(x) = x\|x\|^{\frac{1}{\mu}-1}$$

near the corner, see Figure 6.1, left hand side. An alternative is to use dyadic partitioning [10]: Starting with a coarse mesh the elements are divided until condition (3.1) is satisfied with suitable constants c_1 and c_2, as an example see Figure 6.1, right hand side.

Consider the optimal control problem in the form

$$\begin{aligned} -\Delta y + y &= u + f && \text{in } \Omega, \\ -\Delta p + p &= y - y_d && \text{in } \Omega, \\ u &= \Pi_{[a,b]}\left(-\tfrac{1}{\nu}\,p\right) && \text{in } \Omega \end{aligned} \tag{6.1}$$

with homogeneous Dirichlet boundary conditions for y and p. The data $f = L\bar{y} - \bar{u}$ and $y_d = \bar{y} - L^*\bar{p}$ are chosen such that the functions

$$\begin{aligned} \bar{y}(r,\varphi) &= (r^\lambda - r^\alpha)\sin\lambda\varphi \\ \bar{p}(r,\varphi) &= \nu(r^\lambda - r^\beta)\sin\lambda\varphi \end{aligned}$$

with $\lambda = \frac{2}{3}$ and $\alpha = \beta = \frac{5}{2}$ solve the optimal control problem exactly. Note that these functions have the typical singularity near the corner, as can also be seen in Figure 6.2. The control $\bar{u}$ is defined via (6.1) where we used $\nu = 10^{-4}$, $a = -0.3$ and $b = 1$.

	$\mu = 1$				$\mu = 0.6$			
	$\Vert\bar u - \bar u_h\Vert_{L^2(\Omega)}$		$\Vert\bar u - \tilde u_h\Vert_{L^2(\Omega)}$		$\Vert\bar u - \bar u_h\Vert_{L^2(\Omega)}$		$\Vert\bar u - \tilde u_h\Vert_{L^2(\Omega)}$	
ndof	value	rate	value	rate	value	rate	value	rate
18	1.54e+00		1.53e+00		1.54e+00		1.53e+00	
55	5.15e−01	1.58	5.91e−01	1.37	4.86e−01	1.67	5.47e−01	1.48
189	1.98e−01	1.38	2.04e−01	1.54	1.62e−01	1.59	1.65e−01	1.73
697	7.05e−02	1.49	6.92e−02	1.56	4.97e−02	1.70	4.74e−02	1.80
2673	2.70e−02	1.39	2.56e−02	1.44	1.64e−02	1.60	1.41e−02	1.75
10465	1.09e−02	1.30	9.94e−03	1.36	5.73e−03	1.52	3.75e−03	1.91
41409	4.55e−03	1.26	3.88e−03	1.36	2.37e−03	1.28	9.41e−04	2.00
164737	1.92e−03	1.25	1.54e−03	1.33	1.11e−03	1.09	2.12e−04	2.15

TABLE 6.1
L^2-error of the computed control $\bar u_h$ and of the postprocessed control $\tilde u_h$

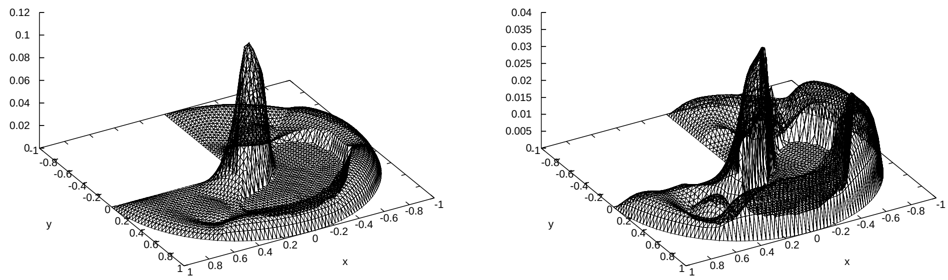

FIG. 6.3. *Plot of $I_h(\bar u - \tilde u_h)$; left: $\mu = 1.0$, right: $\mu = 0.6$.*

Table 6.1 shows the computed errors for various meshes differing in the numbers of degrees of freedom (ndof) and in the grading parameter μ. For the piecewise constant control $\bar u_h$ we see the predicted first order convergence only on the finest mesh computed with mesh grading. The superlinear convergence is therefore certainly an indication that we are not yet in the asymptotic range. We see also that mesh grading leads to a smaller error; the factor is about two. Moreover, the improved convergence rate for the postprocessed control $\tilde u_h$ is evident. On quasi-uniform meshes ($\mu = 1$) we get a rate of about 2λ, and on graded meshes ($\mu = 0.6$) we obtain convergence with second order. The error is significantly smaller on fine meshes.

Finally, we display in Figure 6.3 the error $\bar u - \tilde u_h$ (we actually plot $|I_h(\bar u - \tilde u_h)|$) for $\mu = 1$ and $\mu = 0.6$. We see that the error near the corner dominates the global error in the case $\mu = 1$, while it is smaller and more equilibrated for $\mu = 0.6$.

All solutions were computed with the finite element program of G. Winkler using a multigrid preconditioned conjugate gradient method for solving the systems of linear equations. The active sets were identified by a primal-dual active set strategy, see e.g. [13]. The moderate mesh grading slightly perturbed the mesh hierarchy. However, the number of iterations grew only by a factor of two compared to uniform meshes.

REFERENCES

[1] T. Apel, *Interpolation in h-version finite element spaces*, in Encyclopedia of Computational Mechanics, E. Stein, R. de Borst, and T. J. R. Hughes, eds., vol. 1 Fundamentals, Wiley, Chichester, 2004, pp. 55–72.

[2] T. Apel, A. Sändig, and J. Whiteman, *Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains.*, Math. Methods Appl. Sci., 19 (1996), pp. 63–85.

[3] N. Arada, E. Casas, and F. Tröltzsch, *Error estimates for a semilinear elliptic optimal control problem*, Computional Optimization and Approximation, 23 (2002), pp. 201–229.

[4] I. Babuška, R. Kellogg, and J. Pitkäranta, *Direct and inverse error estimates for finite elements with mesh refinements*, Numer. Math., 33 (1979), pp. 447–471.

[5] E. Casas, *Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems.* submitted.

[6] E. Casas, M. Mateos, and F. Tröltzsch, *Error estimates for the numerical approximation of boundary semilinear elliptic control problems*, Computational Optimization and Applications, 31 (2005), pp. 193–219.

[7] E. Casas and F. Tröltzsch, *Error estimates for linear-quadratic elliptic control problems*, in Analysis and Optimization of Differential Systems, V. B. et al, ed., Boston, 2003, Kluwer Academic Publishers, pp. 89–100.

[8] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978. Reprinted by SIAM, Philadelphia, 2002.

[9] R. Falk, *Approximation of a class of optimal control problems with order of convergence estimates*, J. Math. Anal. Appl., 44 (1973), pp. 28–47.

[10] R. Fritzsch, *Optimale Finite-Elemente-Approximationen für Funktionen mit Singularitäten*, PhD thesis, TU Dresden, 1990.

[11] T. Geveci, *On the approximation of the solution of an optimal control problem governed by an elliptic equation*, R.A.I.R.O. Analyse numeriqué, 13 (1979), pp. 313–328.

[12] M. Hinze, *A variational discretization concept in control constrained optimization: The linear-quadratic case*, Computational Optimization and Applications, 20 (2005), pp. 45–61.

[13] K. Kunisch and A. Rösch, *Primal-dual active set strategy for a general class of constrained optimal control problems*, SIAM Journal Optimization, 13 (2002), pp. 321–334.

[14] K. Malanowski, *Convergence of approximations vs. regularity of solutions for convex, control–constrained optimal control problems*, Appl.Math.Opt., 8 (1981), pp. 69–95.

[15] C. Meyer and A. Rösch, *Superconvergence properties of optimal control problems*, SIAM J. Control and Optimization, 43 (2004), pp. 970–985.

[16] ———, *L^∞-estimates for approximated optimal control problems*, SIAM J. Control and Optimization, (submitted).

[17] L. A. Oganesyan, L. Rukhovets, and V. Rivkind, *Variational-difference methods for solving elliptic equations, Part II*, vol. 8 of Differential equations and their applications, Izd. Akad. Nauk Lit. SSR, Vilnius, 1974. In Russian.

[18] L. A. Oganesyan and L. A. Rukhovets, *Variational-difference schemes for linear second-order elliptic equations in a two-dimensional region with piecewise smooth boundary*, Zh. Vychisl. Mat. Mat. Fiz., 8 (1968), pp. 97–114. In Russian. English translation in USSR Comput. Math. and Math. Phys., 9 (1968) 129–152.

[19] G. Raugel, *Résolution numérique de problèmes elliptiques dans des domaines avec coins*, PhD thesis, Université de Rennes (France), 1978.

[20] A. Rösch, *Error estimates for parabolic optimal control problems with control constraints*, ZAA, 23 (2004), pp. 353–376.

[21] ———, *Error estimates for linear-quadratic control problems with control constraints*, Optimization Methods and Software, (Accepted for publication).

[22] A.-M. Sändig, *Error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains*, Z. Anal. Anwend., 9 (1990), pp. 133–153.

[23] A. H. Schatz and L. B. Wahlbin, *Maximum norm estimates in the finite element method on plane polygonal domains. Part 2: Refinements*, Math. Comp., 33 (1979), pp. 465–492.

[24] G. Strang and G. Fix, *An analysis of the finite element method*, Prentice–Hall Inc., Englewood Cliffs, 1973.