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Convergence Analysis of Geometric Multigrid Methods for Solving Data-Sparse Boundary Element Equations *

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Abstract

The convergence analysis of multigrid methods for boundary element equations arising from negative-order pseudo-differential operators is quite different from the usual finite element multigrid analysis for elliptic partial differential equations. In this paper, we study the convergence of geometric multigrid methods for solving large-scale, data-sparse boundary element equations arising from the adaptive cross approximation to the single layer potential equations.

Keywords integral equations of first kind, single layer potential operator, boundary element method, adaptive cross approximation, geometric multigrid, preconditioners, iterative solvers.

1 Introduction

The spectral behavior of the matrices obtained from the Galerkin boundary element discretization of the single layer potential operator is quite different from the spectral behavior of the stiffness matrices which arise from the finite element discretization of elliptic boundary value problems: The high frequency modes belong to small eigenvalues whereas the smooth eigenfunctions correspond to large eigenvalues ! Therefore, a standard multigrid solver for the corresponding boundary element equation will fail. Rjasanow first proposed and analyzed an appropriate geometric two-grid method in [26]. Further contribution to the construction and analysis of geometric multigrid solvers for dense boundary element equations were made by Petersdorff and Stephan [30] and Bramble, Leyk

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and Pasciak [7]. Based on the approach presented in [7], Langer, Pusch and Reitzinger [24] proposed and studied algebraic versions.

The boundary element matrices are usually fully populated. In order to solve really large-scale systems of boundary element equations, we have to approximate the dense matrices by data-sparse representations. During the last two decades different data-sparse approximation techniques for boundary element matrices have been developed [27, 16, 19, 18, 1, 5, 13, 28]. In [23] Langer and Pusch proposed an Algebraic Multigrid (AMG) method for solving large-scale data-sparse boundary element equations arising from the the so-called Adaptive-Cross-Approximation (ACA) to the discrete single layer potential equations. The ACA technique was proposed by Bebendorf and Rjasanow [1, 5] and successfully used in many practical applications [21, 11]. A rigorous convergence analysis of AMG methods for data-sparse boundary element equations is quite difficult.

In this paper we provide uniform convergence rate estimates for data-sparse geometric multigrid (GMG) methods applied to large-scale data-sparse boundary element equations derived from the ACA to the single layer potential equation. The ACA matrices can be interpreted as certain perturbation of the dense boundary element matrices. Therefore, the multigrid convergence theory developed by Bramble, Goldstein and Pasciak [6] for perturbed bilinear forms (h-dependent bilinear forms) can be applied, see also [10]. From this theory we can derive appropriate conditions which must be imposed on the ACA in order to obtain uniform convergence even for the V-cycle.

The rest of the paper is organized as follows: In Section 2 we summerize the conditions which yield uniform convergence rates for the multigrid V-cycle algorithms under weak regularity assumptions. In the third section, we verify these conditions for discrete single layer potential equations, and proceed further with perturbations of the bilinear form in Section 4. Moreover, the special perturbation causing by the ACA to the single layer potential operator is treated in detail. Section 5 provide some numerical results supporting our theoretical analysis. Finally, we draw some conclusions and discuss some further research topics in Section 6.

2 Convergence Results for Multigrid Methods under Weak Regularity Assumptions

The standard convergence theory for multigrid methods is based on the smoothing and approximation properties, see e.g. [17, 10]. Following [8, 10], the so-called full regularity and approximation condition can be written in the form

$$\|(I - P_{k-1})v\|^2 \preceq \lambda_k^{-1} A(v,v) \quad \forall\, v \in M_k, \quad k = 2, \ldots, J, \tag{1}$$

where $M_1 \subset M_2 \subset \cdots \subset M_J = M$ is a nested sequence of finite dimensional vector spaces, P_{k-1} denotes the elliptic projection onto M_{k-1}, and λ_k is the largest eigenvalue of A_k. The operator A_k is defined by the identity

$$(A_k v, w) = A(v,w) \quad \forall\, v, w \in M_k, \tag{2}$$

where $(.,.)$ is some inner product on M and $\|.\|$ denotes the corresponding norm. $A(v,w)$ is the bilinear form of the underlaying variational problem that is supposed to be elliptic, bounded and symmetric. In order to avoid the introduction of many generic constants, we use the notation "$\preceq$" that stands for "$\leq C$", where C is some generic positive constant not depending on the number of levels J. Condition (1) together with an appropriate smoothing condition yield uniform multigrid convergence for problems that provides full elliptic regularity, see [8].

In many practically important cases, the full regularity and approximation condition (1) cannot be ensured. However, even under weaker conditions imposed on the bilinear form $A(.,.)$ and the multigrid smoothers R_k uniform V-cycle convergence can be shown. In the following, we want to recapitulate these conditions from [10].

Let us first formulate the conditions imposed on the smoother R_k acting on the subspace M_k for $k = 2, \ldots, J$. Basically, two suppositions are needed. The first one reads as

$$A((I - \bar{R}_k A_k)v, v) = A((I - R_k A_k)v, (I - R_k A_k)v) \leq A((I - \frac{\omega}{\lambda_k} A_k)v, v), \quad (3)$$

which is equivalent to

$$\frac{\omega}{\lambda_k}(v, v) \leq (\bar{R}_k v, v) \quad (4)$$

for all $v \in M_k$, where the operator $\bar{R}_k = R_k + R_k^t - R_k^t A_k R_k$ corresponds to the symmetric arrangement of the pre- and post-smoothing sweeps. That means, the smoother $\bar{R}_k$ acts comparable or better than the Richardson smoother $(\frac{\omega}{\lambda_k} I)$. The second condition

$$A(R_k v, R_k v) \leq \theta(R_k v, v) \quad v \in M_k \quad (5)$$

ensures a proper scaling of the smoothers R_k for all $k = 2, \ldots, J$, where the scaling parameter $\theta \in (0, 2)$.

The second group of conditions needed for a rigorous convergence analysis is connected with some properties of the bilinear form $A(.,.)$ and the spaces M_k, $k = 1, \ldots, J$. Let us assume that there exist positive constants C_a, C_{sc} and $\epsilon \in (0, 1)$ not depending on J such that

$$A(v, v) \leq C_a \left(A(P_1 v, v) + \sum_{k=2}^{J} \lambda_k^{-1} \|A_k P_k v\|^2 \right) \quad \forall\, v \in M \quad (6)$$

and

$$A(v_k, v_i) \leq C_{sc}\, \epsilon^{i-k} \|v_k\|_A (\lambda_i^{1/2} \|v_i\|) \quad \forall\, v_k \in M_k,\, v_i \in M_i \quad (7)$$

for all $1 \leq k \leq i \leq J$, where $\|.\|_A = (A(.,.))^{1/2}$ denotes the energy norm. Inequality (7) can alternatively be expressed by

$$\|A_i v\|^2 \leq (C_{sc} \epsilon^{i-k})^2 \lambda_i A(v, v) \quad \forall\, v \in M_k. \quad (8)$$

Condition (6) is nothing but the approximation property. Condition (7) resp. (8) is some kind of inverse property imposed on the spaces M_k.

Now we have all ingredients to formulate the main convergence result proved by Bramble and Pasciak in [9], see also Theorem 5.2 in [10]: *Assume that conditions (4), (5), (6) and (8) hold. Then there exists a positive constant C_M not depending on J such that*

$$A(\mathcal{E}v, v) \leq (1 - \frac{1}{C_M})A(v, v) \quad \forall\, v \in M, \tag{9}$$

where $\mathcal{E}$ denotes the error reduction operator for the V-cycle algorithm (see Algorithm 3.1 for the V-cycle and Algorithm 3.2 for the corresponding multigrid preconditioner in [10]).

3 The Single Layer Potential Operator

In this section we briefly present the application of the convergence theory to the single layer potential (SLP) operator. Bramble, Leyk and Pasciak [7] reformulated the general assumptions for this special class of operators. More generally, one can note that we are treating a multigrid approach for pseudodifferential operators of order *-1*, for which the use of weaker scalar products turned out to be a successful strategy.

Let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with the boundary $\Gamma = \partial\Omega$. The bilinear form induced by the single layer potential operator is given by the relation

$$V(u, v) = \int_\Gamma \int_\Gamma E(x, y)u(x)v(y)ds_x ds_y \tag{10}$$

with the fundamental solution $E(x, y) = (4\pi\|x-y\|)^{-1}$ of the Laplace operator. It is well known that the bilinear form $V(.,.)$ is elliptic, bounded and symmetric on $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, where $H^{-1/2}(\Gamma)$ denotes the usual Sobolev space with negative fractional index, see e.g. [29]. Let us construct a nested sequence of finite dimensional vector spaces

$$M_1 \subset M_2 \subset \cdots \subset M_J = M \subset H^{-1/2}(\Gamma).$$

In our numerical experiments presented in Section 5, we simply use piecewise constant approximations of the Neumann data on a sequence of triangulations that is obtained from a uniform (green) refinement of an initial triangulation of the surface Γ. Furthermore, we define discrete operators $V_k : M_k \mapsto M_k$ by the identity

$$(V_k u, v)_{-1} = V(u, v) \quad \forall\, u, v \in M_k, \tag{11}$$

where $(.,.)_{-1}$ denotes some inner product in the Hilbert space $H^{-1}(\Gamma)$ that is sometimes called the minus one inner product.

Following [7], we define the the smoothing operator $R_k : M_k \mapsto M_k$ by the identity

$$(R_k u, v)_{-1,k} = \frac{1}{\bar{\lambda}_k}(u, v)_{-1}, \quad \forall\, u, v \in M_k, \tag{12}$$

where $(.,.)_{-1,k}$ denotes a discrete minus one inner product on M_k and $\bar{\lambda}_k$ is an upper bound for the largest eigenvalue $\hat{\lambda}_k = \sup_{v \in M_k} \frac{V(v,v)}{(v,v)_{-1,k}}$. The smoothing conditions (4) and (5) can now be rewritten in the form

$$\frac{\omega}{\lambda_k} \|v\|_{-1}^2 \leq (R_k v, v)_{-1}, \quad \forall\, v \in M_k, \tag{13}$$

and

$$V(R_k v, R_k v) \leq \theta (R_k v, v)_{-1}, \quad \forall\, v \in M_k, \tag{14}$$

respectively, where ω is some positive relaxation parameter that has to be less than 1 and $\theta \in (0, 2)$. Note, that (13) would actually be demanded for $\bar{R}_k$. However, since R_k is symmetric this is automatically implied.

The conditions (6) and (8) imposed on the bilinear form must now be given in terms of the minus one inner product. Thus, condition (6) must be substituted by SLP approximation property

$$V(v,v) \leq C_a \left(V(P_1 v, v) + \sum_{k=2}^{J} \lambda_k^{-1} \|V_k P_k v\|_{-1}^2 \right), \quad \forall\, v \in M, \tag{15}$$

and the inverse property (8) by the inequalities

$$\|V_i v\|_{-1}^2 \leq (C_{sc} \epsilon^{i-k})^2 \lambda_i V(v,v), \quad \forall\, v \in M_k, 1 \leq k \leq i \leq J, \tag{16}$$

with positive constants $\epsilon < 1$, C_a and C_{sc} not depending on the number of levels.

If the conditions (13), (14), (15) and (16) are fulfilled, then uniform error reduction of the form

$$V(\mathcal{E} v, v) \leq (1 - \frac{1}{C}) V(v,v), \quad \forall v \in M, \tag{17}$$

follows from the general result stated in Section 2, where $\mathcal{E}$ again denotes the error reduction operator of the symmetric V-cycle, and C is a positive constant which is independent of the number J of levels.

The verification of the conditions (13), (14), (15) and (16) for the standard Galerkin approximation of the single layer potential operator was done by Bramble, Leyk and Pasciak [7], see also Section 13 in [10]. However, the standard Galerkin approximation of boundary integral operators leads to fully populated matrices. Therefore, the complexity of the multigrid methods based on dense matrices is far from being linear with respect to the number of unknowns. The main objective of this paper is to extend the convergence results to data-sparse versions obtained by the adaptive cross approximation (ACA) of the single layer potential operator. The ACA leads to perturbed bilinear forms to which the perturbation theory of [10] is applicable.

4 Data-Sparse Multigrid based on ACA to $V(.,.)$

If we apply the ACA to the SLP bilinear form $V(.,.)$ on M, then we obtain a perturbed bilinear form $\widetilde{V}(.,.)$ that is closely related to the original form $V(.,.)$

somehow. Let us now define perturbed forms at each level $k = 1, \ldots, J$ by the Galerkin projection

$$\widetilde{V}_k(u, v) = \widetilde{V}(u, v), \quad \forall\, u, v \in M_k. \tag{18}$$

Then it is possible to verify conditions (15) and (16) for $\widetilde{V}(.\,,.)$ under some additional assumptions. Alternatively, one can use different forms $\widetilde{V}_k(u, v)$ obtained by the ACA at each level and not by Galerkin projection of $\widetilde{V}_J(u, v) = \widetilde{V}(u, v)$, see Section 10 in [10] for this approach.

4.1 Multigrid Convergence Analysis for Perturbed Forms $\widetilde{V}(.\,,.)$

In [10], Bramble and Zhang provide conditions yielding the approximation property (15) and the inverse property (16) for the perturbed case.

First of all, $\widetilde{V}(.\,,.)$ has to be uniformly equivalent to $V(.\,,.)$ on M, i.e. there exist positive constants c_1 and c_2 independent of number of levels J such that

$$c_1 V(v, v) \leq \widetilde{V}(v, v) \leq c_2 V(v, v) \quad \forall\, v \in M. \tag{19}$$

In order to measure the approximation quality of the perturbed bilinear form, we require that there exists a positive constant C_p and an arbitrarily small constant $\beta > 0$ such that

$$|V(v, w) - \widetilde{V}(v, w)| \leq C_p \lambda_J^{-\beta/2} \|v\|_V \, \|w\|_V, \quad \forall\, v, w \in M, \tag{20}$$

where $\|.\|_V = (V(.\,,.))^{0.5}$ again denotes the energy norm and λ_J is the largest eigenvalue of V_J.

If condition (15) holds for $V(.\,,.)$ and if the perturbation is uniform in the sense of (19), then the approximation condition

$$\widetilde{V}(v, v) \leq \widetilde{C}_a \left(\widetilde{V}(\widetilde{P}_1 v, v) + \sum_{k=2}^{J} \widetilde{\lambda}_k^{-1} \|\widetilde{V}_k \widetilde{P}_k v\|_{-1}^2 \right), \quad \forall\, v \in M \tag{21}$$

is fulfilled for the perturbed form $\widetilde{V}(.\,,.)$ with some positive constant $\widetilde{C}_a$ not depending on J.

If condition (16) holds for $V(.\,,.)$ and if in addition to conditions (19) and (20) inequality

$$C\,(v, v)_{-1} \leq V(v, v), \quad \forall\, v \in M, \tag{22}$$

is fulfilled with some positive constant C not depending on J, then the inverse property

$$\|\widetilde{V}_i v\|_{-1}^2 \leq (\widetilde{C}_{sc} \widetilde{\epsilon}^{i-k})^2 \widetilde{\lambda}_i \widetilde{V}(v, v) \quad \forall\, v \in M_k \tag{23}$$

is valid for the perturbed form too, where $\widetilde{\epsilon} \in (0, 1)$ and $\widetilde{C}_{sc}$ is a positive constant not depending on J. Assumption (22) is true, because $V(v, v)$ is equivalent to $\|v\|_{-1/2}^2$. Moreover, constants $\gamma_1 \leq \gamma_2 \leq 1$ have to exist, so that the largest eigenvalues of V_k are related by

$$\gamma_1 \lambda_{k+1} \leq \lambda_k \leq \gamma_2 \lambda_{k+1}, \quad k = 1, \ldots, J-1, \tag{24}$$

which can be shown for single layer potentials on a sequence of nested meshes.

In the next subsection we will verify the conditions (19) and (20) for the ACA. Together with the smoothing properties, which are obviously valid for the perturbed case, we can then conclude uniform convergence of the V-cycle multigrid algorithm. In the case of separately defined bilinear forms $\widetilde{V}_k(.,.)$ at each level k, we can ensure similar convergence results provided that conditions (19) and (20) are valid at each level k, cf. Sections 7 and 10 in [10].

4.2 Verification of the Multigrid Convergence Conditions for the Adaptive Cross Approximation

The adaptive cross approximation of the single layer potential operator V on M results in a data-sparse matrix $\widetilde{\mathcal{V}}_h$ that can be represented in the form of a sum of the sparse near field matrix $\mathcal{V}_h^{near}$ the entries of which are evaluated exactly and N_B far field blocks that are described by low-rank matrices with rank rk_i:

$$\widetilde{\mathcal{V}}_h = \mathcal{V}_h^{near} + \sum_{i=1}^{N_B} \sum_{j=1}^{rk_i} \underline{u}_j^i \underline{v}_j^{i\top}. \tag{25}$$

The perturbed bilinear form $\widetilde{V}(.,.)$ is now defined by the identity

$$\widetilde{V}(v,w) = (\widetilde{\mathcal{V}}_h \underline{v}, \underline{w}), \quad \forall\, \underline{v}, \underline{w} \leftrightarrow v, w \in M, \tag{26}$$

where $\underline{v}$ and $\underline{w}$ are the vectors of coefficients with respect to the basis chosen in M. The bilinear form $(.,.)$ now denotes the usual Euclidean inner product in $\mathbb{R}^{N_h}$. As usual the discretization parameter $h = h_J$ is chosen in such a way that the number of (boundary) unknowns N_h at the finest level J is of the order $O(h^{-d}) = O(h^{-2})$ for the case $\Omega \subset \mathbb{R}^3$ considered here.

The accuracy provided by the ACA algorithm is usually measured in the Frobenius norm. More precisely, we can ensure the error estimate

$$\|\mathcal{V}_h - \widetilde{\mathcal{V}}_h\|_F \leq \epsilon \|\mathcal{V}_h\|_F \tag{27}$$

for any given fixed positive ϵ, see [5], where $\mathcal{V}_h$ denotes the dense boundary element matrix defined by the identity

$$(\mathcal{V}_h \underline{v}, \underline{w}) = V(v,w) \quad \forall \underline{v}, \underline{w} \leftrightarrow v, w \in M. \tag{28}$$

As we will see later, it is convenient and useful to have estimates in the spectral norm $\|.\|_2$. Using the well-known norm equivalence inequalities

$$\|\mathcal{A}\|_2 \leq \|\mathcal{A}\|_F \leq \sqrt{n}\|\mathcal{A}\|_2, \tag{29}$$

which are valid for real matrices $\mathcal{A}$ of the dimension $n \times n$, we immediately obtain the error estimate

$$\|\mathcal{V}_h - \widetilde{\mathcal{V}}_h\|_2 \leq \epsilon \sqrt{N_h} \|\mathcal{V}_h\|_2 \tag{30}$$

in the spectral norm from the corresponding estimate (27) in the Frobenius norm.

4.2.1 Spectral Equivalence of $V(.,.)$ and $\widetilde{V}(.,.)$ on M

The spectral equivalence inequalities (19) can be rewritten in the corresponding matrix formulation

$$c_1(\mathcal{V}_h\underline{v},\underline{v}) \leq (\widetilde{\mathcal{V}}_h\underline{v},\underline{v}) \leq c_2(\mathcal{V}_h\underline{v},\underline{v}) \quad \forall\,\underline{v}\in\mathbb{R}^{N_h}, \tag{31}$$

or briefly

$$c_1\mathcal{V}_h \leq \widetilde{\mathcal{V}}_h \leq c_2\mathcal{V}_h.$$

Let us remark that the discretized single layer potential $\mathcal{V}_h$ as well as its ACA approximation $\widetilde{\mathcal{V}}_h$ are symmetric and positive definite matrices (with the right scaling in 2D, cf. [29]). In particular, the matrices are invertible.

The following two lemmas provide the spectral equivalence inequalities for which we looking.

Lemma 4.1. *Let $V(.,.)$ be the single layer potential and $\widetilde{V}(.,.)$ the ACA induced perturbed bilinear forms. Then we obtain the inequality*

$$\widetilde{V}(v,v) \leq (1+\epsilon\sqrt{N_h}\kappa(\mathcal{V}_h))V(v,v) \quad \forall\, v\in M,$$

where $\kappa(\mathcal{V}_h)$ denotes the spectral condition number of the discretized single layer potential operator $\mathcal{V}_h$.

Proof. Due to the definitions $V(v,v)=(\mathcal{V}_h\underline{v},\underline{v})$ and $\widetilde{V}(v,v)=(\widetilde{\mathcal{V}}_h\underline{v},\underline{v})$ it is sufficient to find an appropriate constant c_2 for the corresponding matrix inequality. For the estimate of $\widetilde{\mathcal{V}}_h \leq c_2\mathcal{V}_h$, we investigate the equivalent upper inequality $\mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2} \leq c_2 I_h$ that provides the same spectral constant c_2. Thus, we have $\|\mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2}\|_2 = \|\mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2} - I_h + I_h\|_2 \leq 1 + \|I_h - \mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1}\|_2$. The last norm expression can be estimated further with $\|I_h-\mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2}\|_2 \leq \|\mathcal{V}_h-\widetilde{\mathcal{V}}_h\|_2\|\mathcal{V}_h^{-1/2}\|_2^2 \leq \epsilon\sqrt{N_h}\|\mathcal{V}_h\|_2\|\mathcal{V}_h^{-1}\|_2 = \epsilon\sqrt{N_h}\kappa(\mathcal{V}_h)$, which proves the lemma. □

Lemma 4.2. *Let $V(.,.)$ be the single layer potential and $\widetilde{V}(.,.)$ the ACA induced perturbed bilinear forms. If $\epsilon < 1/\sqrt{N_h}\kappa(\mathcal{V}_h)$, then we obtain the inequality*

$$(1-\epsilon\sqrt{N_h}\kappa(\mathcal{V}_h))V(v,v) \leq \widetilde{V}(v,v) \quad \forall\, v\in M.$$

Proof. Again we are concentrating on the corresponding matrix inequality, which in this case reads as $\widetilde{\mathcal{V}}_h^{-1/2}\mathcal{V}_h\widetilde{\mathcal{V}}_h^{-1/2} \leq 1/c_1 I_h$. We consider an estimate for the matrix $\mathcal{V}_h^{1/2}\widetilde{\mathcal{V}}_h^{-1}\mathcal{V}_h^{1/2}$ which has exactly the same eigenvalues (similarity transformation). Now we have the identity $\|\mathcal{V}_h^{1/2}\widetilde{\mathcal{V}}_h^{-1}\mathcal{V}_h^{1/2}\|_2 = \|(I_h - (I_h - \mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2}))^{-1}\|_2$. The inversion with the help of the Neumann series gives the estimate $\|\sum_{i=0}^{\infty}(I_h-\mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2})^i\|_2 \leq \frac{1}{1-\|I_h-\mathcal{V}_h^{-1/2}\widetilde{\mathcal{V}}_h\mathcal{V}_h^{-1/2}\|_2} \leq \frac{1}{1-\epsilon\sqrt{N_h}\kappa(\mathcal{V}_h)}$ which completes the proof. □

For sufficiently small ϵ, Lemma 4.1 and Lemma 4.2 ensure that the spectral equivalence inequalities (19) are valid. Moreover, we are able to make the spectral constants c_1 and c_2 arbitrarily close to 1 by decreasing the parameter ϵ. A similar result can be found in [4].

4.2.2 Approximation $\widetilde{V}(.,.)$ of $V(.,.)$

In order to verify the condition (20) for the ACA approximation of the discretized single layer potential we first have to assert several norm estimates.

The real method of interpolation provides estimates for some interpolation spaces between two Hilbert spaces. More precisely, if we consider $H^1(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma)$ one can establish the estimate

$$\|u_h\|_{1/2} \leq c\|u_h\|_0^{1/2}\|u_h\|_1^{1/2} \quad \forall\, u_h \in M \subset H^1(\Gamma). \tag{32}$$

The inverse inequality

$$\|u_h\|_1 \leq ch^{-1}\|u_h\|_0 \quad \forall\, u_h \in M \tag{33}$$

gives an estimate of H^1-norm by the L_2-norm of a function from M, where h denotes the usual typical mesh size of the fine grid space M. Later we need a lower estimate for the $H^{-1/2}$-norm of a function from M that can be found as follows:

$$\begin{aligned} \|u_h\|_{-1/2} &= \sup_{v\in H^{1/2}(\Gamma)} \frac{(u_h,v)_0}{\|v\|_{1/2}} \\ &\geq \frac{\|u_h\|_0^2}{\|u_h\|_{1/2}} \\ &\geq c\frac{\|u_h\|_0^2}{h^{-1/2}\|u_h\|_0} \\ &= ch^{1/2}\|u_h\|_0. \end{aligned} \tag{34}$$

Let us bear in mind that, in the case of pseudo-differential operators of order minus one, we are treating the whole multigrid analysis with respect to a weaker scalar product, namely the H^{-1} inner product $(.,.)_{-1}$. For the introduced operators $V_k : M_k \to M_k$ defined by (11) we have the corresponding maximal eigenvalues

$$\lambda_k = \sup_{v\in M_k} \frac{(V_k v, v)_{-1}}{\|v\|_{-1}^2}. \tag{35}$$

Furthermore, let us remember that the Euclidean vector norm and the L^2-norm are related by

$$\|u\|_0^2 = (\mathcal{M}_h \underline{u}, \underline{u}) \simeq h^d \|\underline{u}\|_{\mathbb{R}^{N_h}}^2, \tag{36}$$

where $\mathcal{M}_h$ denotes the mass matrix and d is the dimension of the parameterized boundary Γ of the computational domain Ω.
A relation between the matrix norm and the operator norm of the induced mappings is obtained by the following estimate

$$\begin{aligned} \|\mathcal{V}_h\|_2 &= \sup \tfrac{(\mathcal{V}_h \underline{v}_h, \underline{v}_h)}{(\underline{v}_h, \underline{v}_h)} \preceq \sup \tfrac{V(v_h,v_h)}{h^{-d}(v_h,v_h)_0} \\ &\preceq \sup \tfrac{(V_J v_h, v_h)_{-1}}{h^{-d}(v_h,v_h)_{-1}} = h^d\|V_J\|. \end{aligned} \tag{37}$$

Now we can state the approximation estimate (20) for the ACA representation of the single layer potential, i.e.

$$\begin{aligned} |V(v_h,w_h)-\widetilde{V}(v_h,w_h)| &\preceq \|\mathcal{V}_h-\widetilde{\mathcal{V}}_h\|_2\|\underline{v}_h\|_{\mathbb{R}^{N_h}}\|\underline{w}_h\|_{\mathbb{R}^{N_h}} \\ &\preceq \epsilon\sqrt{N_h}\|\mathcal{V}_h\|_2 h^{-d}\|v_h\|_0\|u_h\|_0 \\ &\preceq \epsilon\sqrt{N_h}\|\mathcal{V}_h\|_2 h^{-d-1}\|v_h\|_{-1/2}\|u_h\|_{-1/2} \\ &\preceq \epsilon h^{-d/2-1}\|V_J\|\|v_h\|_{-1/2}\|u_h\|_{-1/2}. \end{aligned} \tag{38}$$

Hold in mind, that $N_h = O(h^{-d})$ and $\lambda_J = \|V_J\| = O(h^{-1})$. Together with the above estimate and the norm equivalence

$$\|u_h\|^2_{-1/2} \simeq V(u_h,u_h) \equiv \|u_h\|^2_V, \quad u_h \in M, \tag{39}$$

we finally obtain the approximation result, which is given as

$$|V(v_h,w_h)-\widetilde{V}(v_h,w_h)| \preceq \epsilon h^{-(d+\beta)/2-2}\lambda_J^{-\beta/2}\|v_h\|_V \cdot \|w_h\|_V \tag{40}$$

with an arbitrarily positive constant β.
In order to get rid of the mesh parameter h, we can choose an appropriate ϵ that exactly cancels out the dependency. For instance in the case $\Omega \subset \mathbb{R}^3$ the boundary Γ is characterized by a 2-dimensional manifold, i.e. $d = 2$. Therefore, we would have $\epsilon \sim h^{3+\beta/2}$. As shown in [2] the effort for constructing an ACA approximation of a single layer potential behaves like $O(N_h^{1+\alpha}\epsilon^{-\alpha})$, with an arbitrarily small positive α.

We have shown that the ACA accuracy parameter ϵ can be chosen in such a way that the conditions (19) and (20) are fulfilled for ACA matrices originating from the discretized single layer potential operator. Therefore, we can state our main theorem.

Theorem 4.3. *Let the bilinear form $V(.,.)$ be defined by the single layer potential operator V and let $\mathcal{V}_h$ be the discrete analog defined by (28). Moreover, let $\widetilde{\mathcal{V}}_h$ be a corresponding data-sparse approximation provided by the ACA - algorithm and $\widetilde{V}(.,.)$ its induced bilinearform. Assume further (27) and appropriately defined smoothers satisfying (12)–(14). If the ACA accuracy $\epsilon = \epsilon_{aca}$ is of order $h^{(d+\beta)/2+2}$ (with d the dimension of $\Gamma = \partial\Omega$ the boundary of the computational domain and β an arbitrarily small positive number), then*

$$\widetilde{V}(\widetilde{\mathcal{E}}v,v) \le (1-\frac{1}{C})\widetilde{V}(v,v), \quad \forall\, v \in M, \tag{41}$$

where $\widetilde{\mathcal{E}}$ denotes the error reduction operator on the finest grid and C is a generic constant not depending on the number of levels.

Corollary 4.4. *In order to reduce the initial error via a multigrid V-cycle or a V-cycle preconditioned conjugate gradient method by the factor $\varepsilon_{it} = \varepsilon_{mg} = \varepsilon_{cg} \in (0,1)$, we need $O(N_h^{1+\gamma}\log\varepsilon_{it}^{-1})$ arithmetical operations, where γ is an arbitrarily small positive number. The memory demand is of almost optimal order $O(N_h^{1+\gamma})$ too.*

Proof. The complexity for constructing the ACA matrix can be estimated by $O(N_h^{1+\alpha}\epsilon_{aca}^{-\alpha})$ [5]. Summing up the arithmetical cost and the memory demand over all levels gives the same complexity [23]. The choice $\epsilon_{aca} = h^{(d+\beta)/2+2}$ and Theorem 4.3 yield the complexity estimate for the total cost of arithmetical operations. □

5 Numerical Studies

The solution of the interior Dirichlet problem for the Laplace equation

$$\begin{aligned} -\Delta u(x) &= 0, \qquad x \in \Omega \subset \mathbb{R}^3, \\ u(x) &= g(x), \qquad x \in \Gamma = \partial\Omega, \end{aligned} \tag{42}$$

can be represented by the so-called representation formula

$$u(y) = \int_\Gamma E(x,y)\frac{\partial u}{\partial n}(x) - \int_\Gamma \frac{\partial E}{\partial n_x}(x,y)g(x) \quad y \in \Omega. \tag{43}$$

The missing Neumann data $v = \partial u/\partial n$ on Γ can be found from the SLP equation

$$Vv = (\frac{1}{2}I + K)g, \tag{44}$$

where

$$(Kg)(y) = \int_\Gamma \frac{\partial E}{\partial n_x}(x,y)g(x)ds_x \tag{45}$$

denotes the double layer potential operator. The Galerkin discretization with piecewise constant basis functions $\{\varphi_k^j\}_{k=1,\ldots,N_j}$ on a regular sequence of triangulation of Γ obtained by dividing a given triangle into four smaller triangles (green refinement) results in a sequence of systems with dense matrices $\mathcal{V}_j$ defined by $(\mathcal{V}_j)_{kl} = V(\varphi_k^j, \varphi_l^j)$ at each level $j = 1, 2, \ldots, J$. To ensure efficiency with respect to memory consumption and computational time we use data-sparse ACA $\widetilde{\mathcal{V}}_j$ to the dense matrices $\mathcal{V}_j$ at each level $j = 1, 2, \ldots, J$. Alternatively, we can use the Galerkin projection of $\widetilde{\mathcal{V}}_J$ to obtain data-sparse matrices $\widetilde{\mathcal{V}}_j$ on each auxiliary grid level $j = J-1, J-2, \ldots, 1$ as was done in the algebraic version [23]. We mention that the ACA was also used for approximating the discrete single double potential operator when calculating the right-hand side $\widetilde{f}_J$ of the system

$$\widetilde{\mathcal{V}}_J \underline{v}_J = \widetilde{f}_J \tag{46}$$

on the finest grid. More precisely, we are using the ACA technique implemented in the software package AHMED developed by M. Bebendorf [3].

We now solve system (46) via the conjugate gradient method preconditioned by a symmetric V-cycle with one pre-smoothing and one post-smoothing step [20]. We mention that the dense matrices $\mathcal{V}_j$ as well as their data-sparse representations $\widetilde{\mathcal{V}}_J$ are symmetric and positive definite. Of course, it is possible to use multigrid as a solver, but in our experiments presented below it was used as a preconditioner within the conjugate gradient method. All calculations were done with a relative accuracy of $\varepsilon_{cg} = 10^{-6}$.

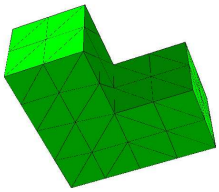

Figure 1: L-shaped Domain

For our numerical experiments, we take a 3D L-shaped computational domain Ω as is shown in Figure 1. The surface Γ of Ω is provided with an initial triangulation consisting of 1792 triangles, see also Figure 1. In Table 1, we present the effort for constructing the ACA system matrix with the accuracy $\epsilon_{aca} = 10^{-3}$ and the admissibility parameter $\eta = 1.2$.

3D L-Shape	System Matrix $\widetilde{\mathcal{V}}_h$		Matrix-Hierarchy	
Number of Unknowns	Assembling [sec]	Memory [MB]	Assembling [sec]	Memory [MB]
7168	77	16	15	2.5
28672	402	76	95	19
114688	1914	373	468	95

Table 1: Assembling $\widetilde{\mathcal{V}}_h$ and Setup Times.

One can observe an almost linear increase in the computational time. Furthermore, the corresponding demand for memory also grows almost like $O(N_h)$. The major part of setting up the multigrid hierarchy is spent for the system matrix, both the assembling time and the memory demand.

In Table 2 we are considering the numerical behavior of the preconditioned conjugate gradient (PCG) algorithm. The last column presents the time for

3D L-Shape			
Number of Unknowns	Number of Grids	PCG Iteration	Time per Iteration [sec]
7168	2	7	0.25
28672	3	8	1.0
114688	4	7	5.1

Table 2: Numerical features for the PCG solver.

one single PCG iteration. Obviously, the increase is almost linear. In order to confirm the quality of our multigrid preconditioner we recognize the constant number of iterations even for a large number of degrees of freedom.

6 Conclusions and Remarks

Starting from the abstract theoretical results of [10] we provided a rigorous convergence analysis for multigrid methods applied to the solution of data-sparse approximated single layer potential equations. Considering the adaptive cross approximated system matrix as a perturbed Galerkin matrix some additional conditions given in [10] have to be fulfilled in order to obtain uniform convergence. It turned out that the accuracy of the adaptive cross approximation on the different refinement levels can be controlled in such a way that the conditions for uniform convergence are fulfilled. Taking into account a proper accuracy on each level, the h-dependency of the convergence rate can be avoided. The analysis given in this paper carries over to hypersingular integral equations connected with the interior and exterior Neumann problems. Efficient preconditioners for the discrete data-sparse approximated single layer potential and hypersingular operators are needed in the primal and dual boundary element domain decomposition methods [22, 12, 25].

In general, it is much more difficult to establish theoretical convergence results for an algebraic multigrid methods, see e.g. [15, 14] for some recent results connected with finite element discretizations of second–order boundary value problems. However, in some cases the algebraic multigrid approach recovers the geometric version, e.g. for uniform refinement in 2D, see [24].

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